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EXISTENCE OF VIABLE SOLUTIONS FOR NONCONVEX DIFFERENTIAL INCLUSIONS

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Abstract. We show the existence result of viable solutions to the differential inclusion

$$\dot{x}(t) \in G(x(t)) + F(t, x(t))$$
$$x(t) \in S \quad \text{on } [0, T],$$

where $F : [0, T] \times H \to H$ $(T > 0)$ is a continuous set-valued mapping, $G : H \to H$ is a Hausdorff upper semi-continuous set-valued mapping such that $G(x) \subset \partial g(x)$, where $g : H \to \mathbb{R}$ is a regular and locally Lipschitz function and S is a ball, compact subset in a separable Hilbert space H.

1. Introduction

Let $T > 0$. It is well known that the solution set of the differential inclusion

$$\dot{x}(t) \in G(x(t)) \quad \text{a.e.} \quad [0, T]$$
$$x(0) = x_0 \in \mathbb{R}^d,$$

can be empty when the set-valued mapping G is upper semicontinuous with nonempty nonconvex values. Bressan, Cellina and Colombo [8], proved an existence result fo the above equation by assuming that the set-valued mapping G is included in the subdifferential of a convex lower semicontinuous (l.s.c.) function $g : \mathbb{R}^d \to \mathbb{R}$. This result has been extended in many ways by many authors; see for example [1, 2, 3, 4, 11, 12, 13]. The recent extension of the above equation was studied by Bounkhel [4], in which the author proved an existence result of viable solutions in the finite dimensional case for the differential inclusion

$$\begin{gathered}\dot{x}(t) \in G(x(t)) + F(t, x(t)) \quad \text{a.e. } [0, T] \\ x(t) \in S, \quad \text{on } [0, T].\end{gathered} \tag{1.1}$$

This extension covers all the other extensions given in the finite dimensional case. In the present paper we extend this result to the infinite dimensional setting. A function $x(\cdot)$ is called a viable solution if it satisfies the differential inclusion and $x(t) \in S$ for all $t \in [0, T]$ and for some closed set S.

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2. Uniformly regular functions

Let H be a real separable Hilbert space. Let us recall the concept of regularity that will be used in the sequel [4].

Definition 2.1 ([4])**.** Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and let $O \subset \operatorname{dom} f$ be a nonempty open subset. We will say that f is uniformly regular over O if there exists a positive number $\beta \geq 0$ such that for all $x \in O$ and for all $\xi \in \partial^P f(x)$ one has

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \beta \|x' - x\|^2 \quad \text{for all } x' \in O. \tag{2.1}$$

Here $\partial^P f(x)$ denotes the proximal subdifferential of f at x (for its definition the reader is refereed for instance to [6]). We will say that f is uniformly regular over closed set S if there exists an open set O containing S such that f is uniformly regular over O. The class of functions that are uniformly regular over sets is so large. For more details and examples we refer the reader to [4]. The following proposition summarizes some important properties for uniformly regular locally Lipschitz functions over sets needed in the sequel. For the proof of these results we refer the reader to [4, 5].

Porposition 2.2. *Let $f : H \to \mathbb{R}$ be a locally Lipschitz function and S a nonempty closed set. If f is uniformly regular over S, then the following hold:*

- (i) *The proximal subdifferential of f is closed over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ with $\xi_n \in \partial^P f(x_n)$ one has $\xi \in \partial^P f(x)$*
- (ii) *The proximal subdifferential of f coincides with $\partial^C f(x)$ the Clarke subdifferential for any point x (see for instance [6] for the definition of $\partial^C f$)*
- (iii) *The proximal subdifferential of f is upper hemicontinuous over S, that is, the support function $x \mapsto \langle v, \partial^P f(x) \rangle$ is u.s.c. over S for every $v \in H$*
- (iv) *For any absolutely continuous map $x : [0, T] \to S$ one has*

$$\frac{d}{dt}(f \circ x)(t) = \langle \partial^C f(x(t)); \dot{x}(t) \rangle.$$

Now we are in position to state and prove our main result in this paper.

Theorem 2.3. *Let $g : H \to \mathbb{R}$ be a locally Lipschitz function and β-uniformly regular over $S \subset H$. Assume that*

- (i) *S is nonempty ball compact subset in H, that is, the set $S \cap r\mathbb{B}$ is compact for any $r > 0$;*
- (ii) *$G : H \to H$ is a Hausdorff u.s.c set valued map with compact values satisfying $G(x) \subset \partial^C g(x)$ for all $x \in S$;*
- (iii) *$F : [0, T] \times H \to H$ is a continuous set valued map with compact values;*
- (iv) *For any $(t, x) \in I \times S$, the following tangential condition holds*

$$\liminf_{h \to 0} \frac{1}{h} e\big(x + h\,[G(x) + F(t, x)]\,; S\big) = 0\,, \tag{2.2}$$

where $e(A; S) := \sup_{a \in A} d_S(a)$.

Then, for any $x_0 \in S$ there exists $a \in]0, T[$ such that the differential inclusion (2.26) has a viable solution on $[0, a]$.

Proof. Let $\rho > 0$ such that $K_0 := S \cap (x_0 + \rho B)$ is compact and g is L-Lipschitz on $x_0 + \rho B$. Since F and G are continuous and the set $I \times K_0$ is compact, there exists a positive scalar M such that

$$\|G(x)\| + \|F(t, x)\| \leq M, \tag{2.3}$$

for all $(t,x) \in I \times K_0$. Since $(t_0, x_0) \in I \times K_0$, then (by (2.2))

$$\liminf_{h\to 0} \frac{1}{h} e\Big(x_0 + h\big[G(x_0) + F(t_0,x_0)\big]; S\Big) = 0.$$

Put $\alpha := \min\{T, \frac{\rho}{M+1}, 1\}$. Hence for every $m \geq 1$ there exists $0 < \xi < \frac{\alpha}{2}$ such that

$$e\Big(x_0 + \xi\left[G(x_0) + F(t_0,x_0)\right]; S\Big) < \frac{\xi}{m}. \tag{2.4}$$

Let $b_0 \in G(x_0) + F(t_0, x_0)$ and put

$$\lambda_0^m := \max\big\{\xi \in (0, \frac{\alpha}{2}] : \xi \leq T - t_0 \text{ and } d_S(x_0 + \xi b_0) < \frac{\xi}{m}\big\}.$$

Since $x_0 \in S$, we have

$$d_S(x_0 + \lambda_0^m b_0) \leq \lambda_0^m \|b_0\| \leq \lambda_0^m M < M.$$

So, there exists $\Psi_0^m \in S \cap \mathbb{B}(x_0 + \lambda_0^m b_0, M+1)$ such that

$$\|\Psi_0^m - x_0 - \lambda_0^m b_0\| = d_S(x_0 + \lambda_0^m b_0),$$

and so

$$\|\frac{1}{\lambda_0^m}[\Psi_0^m - x_0] - b_0\| = \frac{1}{\lambda_0^m} d_S(x_0 + \lambda_0^m b_0) < \frac{1}{m}$$

by (2.4) and the definition of λ_0^m. Let $w_0^m := \frac{\Psi_0^m - x_0}{\lambda_0^m}$ and $x_1^m := x_0 + \lambda_0^m w_0^m \in S$. Thus, we obtain

$$\begin{gathered} w_0^m \in G(x_0) + F(t_0,x_0) + \frac{1}{m} B, \\ \|x_1^m - x_0\| = \lambda_0^m \|w_0^m\| < \lambda_0^m (M + \frac{1}{m}) < \lambda_0^m (M+1). \end{gathered} \tag{2.5}$$

We can choose, a priori, $a < \alpha$ and find $\lambda_0^m < a$ such that $0 < \lambda_0^m < a < T$. Then $\|x_1^m - x_0\| < \rho$, that is, $x_1^m \in (x_0 + \rho B)$ and so (2.5) ensures $x_1^m \in S \cap (x_0 + \rho B) = K_0$. We reiterate this process for constructing sequences $\{w_i^m\}_i$, $\{t_i^m\}_i$, $\{\lambda_i^m\}_i$, and $\{x_i^m\}_i$ satisfying for some rank $\nu_m \geq 1$ the following assertions:

(a) $0 = t_0^m$, $t_{\nu_m}^m \leq a < T$ with $t_i^m = \sum_{k=0}^{i-1} \lambda_k^m$ for all $i \in \{1, \dots, \nu_m\}$;

(b) $x_i^m = x_0 + \sum_{k=0}^{i-1} \lambda_k^m w_k^m$ and $(t_i^m, x_i^m) \in [0,T] \times K_0$ for all $i \in \{0, \dots, \nu_m\}$;

(c) $w_i^m \in G(x_i^m) + F(t_i^m, x_i^m) + \frac{1}{m} B$ with $w_i^m = \frac{\Psi_i^m - x_i^m}{\lambda_i^m}$ and $\Psi_i^m \in S \cap \mathbb{B}(x_i^m + \lambda_i^m b_i^m, M+1)$ for all $i \in \{0, \dots, \nu_m - 1\}$, where

$$\lambda_i^m := \max\{\xi \in (0, \frac{\alpha}{2}] : \xi \leq T - t_i^m \text{ and } d_S(x_i^m + \xi b_i) < \frac{1}{m}\xi\} (\forall\, i = 1, \dots, \nu_m - 1).$$

It is easy to see that for $i = 1$ the assertions (a), (b), and (c) are fulfilled. Let now $i \geq 2$. Assume that (a), (b), and (c) are satisfied for any $j = 1, \dots, i$. If, $a < t_{i+1}^m$, then we take $\nu_m = i$ and so the process of iterations is stopped and we get (a), (b), and (c) satisfied with

$$t_{\nu_m}^m \leq a < t_{\nu_m+1}^m < T.$$

In the other case, i.e., $t_{i+1}^m \leq a$, we define x_{i+1}^m as follows

$$x_{i+1}^m := x_i^m + \lambda_i^m w_i^m = x_0 + \sum_{k=0}^{i} \lambda_k^m w_k^m$$

and so

$$\|x_{i+1}^m - x_0\| \le \sum_{k=0}^{i} \lambda_k^m \|w_k^m\| \le (M+1) \sum_{k=0}^{i} \lambda_k^m \le t_{i+1}^m (M+1) \le a(M+1) < \rho,$$

which ensures that $x_{i+1}^m \in K_0$. Thus the conditions (a), (b), and (c) are satisfied for $i+1$. Now we have to prove that this iterative process is finite, i.e., there exists a positive integer ν_m such that

$$t_{\nu_m}^m \le a < t_{\nu_m+1}^m.$$

Suppose the contrary that is,

$$t_i^m \le a, \quad \text{for all } i \ge 1.$$

Then the bounded increasing sequence $\{t_i^m\}_i$ converges to some $\bar{t}$ such that $\bar{t} \le a < T$. Hence

$$\|x_i^m - x_j^m\| \le (M+1)|t_i^m - t_j^m| \to 0 \quad \text{as } i, j \to \infty.$$

Therefore, the sequence $\{x_i\}_i$ is a Cauchy sequence and hence, it converges to some $\bar{x} \in K_0$. As $(\bar{t}, \overline{x}) \in [0, T] \times K_0$, by (2.2) and the Hausdorff upper semi-continuity of $G + F$, there exist $\lambda \in (0, T - \bar{t})$, and an integer $i_0 \ge 1$ such that for all $i \ge i_0$,

$$e\Big(\overline{x} + \lambda \left[G(\overline{x}) + F(\bar{t}, \overline{x})\right]; S\Big) \le \frac{\lambda}{6m} \tag{2.6}$$

$$e\Big(G(x_i^m) + F(t_i^m, x_i^m); G(\overline{x}) + F(\bar{t}, \overline{x})\Big) \le \frac{1}{12m} \tag{2.7}$$

$$\|x_i^m - \overline{x}\| \le \frac{\lambda}{6m} \tag{2.8}$$

$$\bar{t} - t_i^m \le \frac{\lambda}{2}. \tag{2.9}$$

Therefore, for any $b_i \in G(x_i^m) + F(t_i^m, x_i^m)$, there exists (by the definition of the distance function) an element $\bar{b}$ in $G(\overline{x}) + F(\bar{t}, \overline{x})$ such that

$$\|b_i - \bar{b}\| \le d(b_i, G(\overline{x}) + F(\bar{t}, \overline{x})) + \frac{1}{12m}.$$

Hence this inequality and (2.7) yield

$$\|b_i - \bar{b}\| \le e\Big(G(x_i^m) + F(t_i^m, x_i^m); G(\overline{x}) + F(\bar{t}, \overline{x})\Big) + \frac{1}{12m} \le \frac{1}{6m}.$$

This last inequality and the relations (2.6) and (2.8) ensure

$$\begin{aligned} d_S(x_i^m + \lambda b_i) &\le \|x_i^m - \overline{x}\| + d_S(\overline{x} + \lambda \bar{b}) + \lambda \|b_i - \bar{b}\| \\ &\le \frac{\lambda}{6m} + e\Big(\overline{x} + \lambda \left[G(\overline{x}) + F(\bar{t}, \overline{x})\right]; S\Big) + \frac{\lambda}{6m} \le \frac{\lambda}{2m}. \end{aligned}$$

On the other hand, by construction and by (2.9), we obtain

$$t_{i+1}^m \le \bar{t} < t_i^m + \lambda \le T, \text{ and hence } \lambda > t_{i+1}^m - t_i^m = \lambda_i^m.$$

Thus, there exists some $\lambda > \lambda_i^m$ such that $0 < \lambda < T - \bar{t} \le T - t_i^m$ (for all $i \ge i_0$) and $d_S(x_i^m + \lambda b_i) \le \frac{\lambda}{2m} < \frac{\lambda}{m}$. This contradicts the definition of λ_i^m. Therefore, there is an integer $\nu_m \ge 1$ such that $t_{\nu_m} \le a < t_{\nu_m+1}$ and for which the assertions (a), (b), and (c) are fulfilled.

According to what precedes, we have (by (c))

$$\begin{aligned}\|\Psi_i^m\| &\leq \|\Psi_i^m - (x_i^m + \lambda_i^m b_i^m)\| + \|x_i^m + \lambda_i^m b_i^m\| \\ &\leq (M+1) + \|x_0 - (x_0 - x_i^m) + \lambda_i^m b_i^m\| \\ &\leq \|x_0\| + \|x_0 - x_i^m\| + \lambda_i^m \|b_i^m\| + (M+1) \\ &\leq \|x_0\| + \rho + 2M + 1.\end{aligned}$$

This implies $\Psi_i^m \in K_1 := S \cap \mathbb{B}(0,R)$, with $R := \|x_0\| + \rho + 2M + 1$. Note that the ball-compactness of S ensures the compactness of K_1.

On the other hand, it follows from the assertion (c) that

$$w_i^m - f_i^m - c_i^m \in G(x_i^m), \text{ where } c_i^m \in \frac{1}{m}B \quad \text{and} \quad f_i^m \in F(t_i^m, x_i^m), \tag{2.10}$$

for all $i \in \{0, \dots, \nu_m\}$.

Approximate Solutions. Using the sequences $\{x_i^m\}_i$, $\{t_i^m\}_i$, $\{f_i^m\}_i$, and $\{c_i^m\}_i$ constructed previously to construct the step functions $x_m(\cdot)$, $f_m(\cdot)$, $c_m(\cdot)$, and $\theta_m(\cdot)$ with the following properties:

(1) $x_m(t) = x_i^m + (t - t_i^m)w_i^m$ on $[t_i^m, t_{i+1}^m]$ for all $i \in \{0, \dots, \nu_m\}$;
(2) $f_m(t) = f_m(\theta_m(t)) \in F(\theta_m(t), x_m(\theta_m(t)))$ on $[0,a]$ with

$$\theta_m(t) = t_i^m \text{ if } t \in [t_i^m, t_{i+1}^m[, \quad \text{for all } i \in \{0, \dots, \nu_m\},\ \theta_m(a) = a;$$

(3) $c_m(t) = c_i^m \in \frac{1}{m}B$ if $t \in [t_i^m, t_{i+1}^m]$, for all $i \in \{0, \dots, \nu_m\}$ and

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|c_m(t)\| = 0. \tag{2.11}$$

Then

$$\|x_m(t_{i+1}^m) - x_m(t_i^m)\| = (t_{i+1}^m - t_i^m)\|w_i^m\| \leq (1+M)(t_{i+1}^m - t_i^m),$$

and so, for all $i, j \in \{0, \dots, \nu_m - 1\}(i > j)$, we have

$$\begin{aligned}\|x_m(t_i^m) - x_m(t_j^m)\| &\leq \sum_{k=j+1}^{i} \|x_m(t_k^m) - x_m(t_{k-1}^m)\| \\ &\leq (M+1) \sum_{k=j+1}^{i} (t_k^m - t_{k-1}^m) = (M+1)|t_i^m - t_j^m|.\end{aligned}$$

Also, we have by construction for a.e. $t \in [t_i^m, t_{i+1}^m]$ and for all $i \in \{0, \dots, \nu_m\}$

$$\|\dot{x}_m(t)\| = \|w_i^m\| \leq M + 1. \tag{2.12}$$

Convergence of approximate solutions. We note that the sequence f_m can be constructed with the relative compactness property in the space of bounded functions (see [13]). Therefore, without loss of generality we can suppose that there is a bounded function f such that

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|f_m(t) - f(t)\| = 0\,. \tag{2.13}$$

Now, we prove that the approximate solutions $x_m(.)$ converge to a viable solution of (1.1).

It is clear by construction that $\{x_m\}_m$ are Lipschitz continuous with constant $M+1$ and

$$x_m(t) = x_i^m + (t - t_i^m)w_i^m = x_i^m + (\frac{t - t_i^m}{\lambda_i^m})(\Psi_i^m - x_i^m).$$

On the other hand, we have $0 \leq t - t_i^m \leq t_{i+1}^m - t_i^m = \lambda_i^m$ and so $0 \leq \frac{t-t_i^m}{\lambda_i^m} \leq 1$, and hence we get

$$(\frac{t - t_i^m}{\lambda_i^m})(\Psi_i^m - x_i^m) \in \overline{co}[\{0\} \cup (K_1 - K_0)]. \tag{2.14}$$

Thus,

$$x_m(t) \in K := K_0 + \overline{co}[\{0\} \cup (K_1 - K_0)]. \tag{2.15}$$

Therefore, since the set K is compact (because K_0 and K_1 are compact), then the assumptions of the Arzela-Ascoli theorem are satisfied. Hence a subsequence of x_m my be extracted (still denoted x_m) that converges to an absolutely continuous mapping $x : [0, a] \to H$ such that

$$\begin{gathered} \lim_{m\to\infty} \max_{t\in[0,a]} \|x_m(t) - x(t)\| = 0 \\ \dot{x}_m(.) \rightharpoonup \dot{x}(.) \text{ in the weak topology of } L^2([0, a], H). \end{gathered} \tag{2.16}$$

Recall now that f_m converges pointwise a.e. on $[0, a]$ to f. Then the continuity of the set-valued mapping F and the closedness of the set $F(t, x(t))$ entail $f(t) \in F(t, x(t))$. Now, it remains to prove that

$$\begin{gathered} x(t) \in S; \\ -f(t) + x'(t) \in G(x(t)) \quad \text{a.e. on} \quad [0, a]. \end{gathered} \tag{2.17}$$

By construction we have $x_i^m \in K_0$ (for all $i \in \{0, \dots, \nu_m - 1\}$). This ensures

$$d_{K_0}(x(t)) \leq \|x_m(t) - x_i^m\| + \|x_m(t) - x(t)\| \leq \frac{1+M}{m} + \|x_m(t) - x(t)\|$$

which approaches 0 as m approaches ∞. The closedness of K_0 yields $d_{K_0}(x(t)) = 0$ and so $x(t) \in K_0 \subset S$.

By construction, we have for a.e. $t \in [0, a]$

$$\dot{x}_m(t) - f_m(t) - c_m(t) \in G(x_m(\theta_m(t))) \subset \partial^C g(x_m(\theta_m(t))) = \partial^P g(x_m(\theta_m(t))), \tag{2.18}$$

where the above equality follows from the uniform regularity of g over C and the part (ii) in Proposition 2.2. We can thus apply Castaing techniques (see for example [9]). The weak convergence (by (2.16)) in $L^2([0, a], H)$ of $\dot{x}_m(\cdot)$ to $\dot{x}(\cdot)$ and Mazur's Lemma entail

$$\dot{x}(t) \in \bigcap_m \overline{co}\{\dot{x}_k(t) : k \geq m\}, \quad \text{for a.e. on } [0, a].$$

Fix any such t and consider any $\xi \in H$. Then, the last relation above yields

$$\langle \xi, \dot{x}(t) \rangle \leq \inf_m \sup_{k\geq m} \langle \xi, \dot{x}_m(t) \rangle$$

and hence Proposition 2.2 part (iii) and (2.18) yield

$$\begin{aligned} \langle \xi, \dot{x}(t) \rangle &\leq \limsup_m \sigma(\xi, \partial^P g(x_m(\theta_m(t))) + f_m(t) + c_m(t)) \\ &\leq \sigma(\xi, \partial^P g(x(t)) + f(t)) \quad \text{for any } \xi \in H, \end{aligned}$$

So, the convexity and the closedness of the set $\partial^P g(x(t))$ ensure

$$-f(t) + \dot{x}(t) \in \partial^P g(x(t)). \tag{2.19}$$

Now, since g is uniformly regular over C and $x : [0, a] \to C$ we have

$$\begin{aligned}\frac{d}{dt}(g \circ x)(t) &= \langle \partial^P g(x(t)), \dot{x}(t)\rangle \\ &= \langle -f(t) + \dot{x}(t), \dot{x}(t)\rangle \\ &= \|\dot{x}(t)\|^2 - \langle f(t), \dot{x}(t)\rangle.\end{aligned}$$

Consequently,

$$g(x(a)) - g(x_0) = \int_0^a \|\dot{x}(s)\|^2 ds - \int_0^a \langle f(s), \dot{x}(s)\rangle\, ds \tag{2.20}$$

On the other hand, by (2.18) and Definition 2.1 we have for all $i \in \{0, \dots \nu_m - 1\}$

$$\begin{aligned} g(x_{i+1}^m) - g(x_i^m) &\geq \langle \dot{x}_m(t) - f_i^m - c_i^m, x_{i+1}^m - x_i^m\rangle - \beta\|x_{i+1}^m - x_i^m\|^2 \\ &= \Big\langle \dot{x}_m(t) - f_m(t) - c_i^m, \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s)ds\Big\rangle - \beta\|x_{i+1}^m - x_i^m\|^2 \\ &\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2 ds - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s)\rangle ds \\ &\quad - \langle c_i^m, \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s)ds\rangle - \beta(M+1)^2(t_{i+1}^m - t_i^m)^2 \\ &\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2 ds - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s)\rangle ds \\ &\quad - \langle c_i^m, \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s)ds\rangle - \frac{\beta(M+1)^2}{m}(t_{i+1}^m - t_i^m). \end{aligned}$$

By adding, we obtain

$$g(x_m(t_{\nu_m}^m)) - g(x_0) \geq \int_0^{t_{\nu_m}^m} \|\dot{x}_m(s)\|^2 ds - \int_0^{t_{\nu_m}^m} \langle f_m(s), \dot{x}_m(s)\rangle\, ds - \varepsilon_{1,m} \tag{2.21}$$

with

$$\varepsilon_{1,m} = \sum_{i=0}^{\nu_m - 1} \langle c_i^m, \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s)ds\rangle + \frac{\beta(M+1)^2 t_{\nu_m}^m}{m}$$

and

$$g(x_m(a)) - g((t_{\nu_m}^m)) \geq \int_{t_{\nu_m}^m}^a \|\dot{x}_m(s)\|^2 ds - \int_{t_{\nu_m}^m}^a \langle \dot{x}_m(s), f_m(s)\rangle\, ds - \varepsilon_{2,m} \tag{2.22}$$

with

$$\varepsilon_{2,m} = \langle c_{\nu_m}^m, \int_{t_{\nu_m}^m}^a \dot{x}(s)ds\rangle + \frac{\beta(M+1)^2(a - t_{\nu_m}^m)}{m}.$$

Therefore, we get

$$g(x_m(a)) - g((x_0)) \geq \int_0^a \|\dot{x}_m(s)\|^2 ds - \int_0^a \langle f_m(s), \dot{x}_m(s)\rangle ds - \varepsilon_m \tag{2.23}$$

where

$$\varepsilon_m = \varepsilon_{1,m} + \varepsilon_{2,m} = \sum_{i=0}^{\nu_m - 1} \langle c_i^m, \int_{t_i^m}^{t_{i+1}^m} \dot{x}(s)ds \rangle + \langle c_{\nu_m}^m, \int_{t_{\nu_m}^m}^{a} \dot{x}(s)ds \rangle + \frac{\beta a(M+1)^2}{m}.$$

Using our construction we get

$$\begin{aligned} |\varepsilon_m| &\le \sum_{i=0}^{\nu_m - 1} \|c_i^m\| \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}(s)\| ds + \|c_{\nu_m}^m\| \int_{t_{\nu_m}^m}^{a} \|\dot{x}(s)\| ds + \frac{\beta(M+1)^2 a}{m} \\ &\le \sum_{i=0}^{\nu_m - 1} \frac{1}{m}(t_{i+1}^m - t_i^m)(M+1) + \frac{1}{m}(a - t_{\nu_m}^m)(M+1) + \frac{\beta(M+1)^2 a}{m} \\ &= \frac{(M+1)}{m} \Big[\sum_{i=0}^{\nu_m - 1} (t_{i+1}^m - t_i^m) + (a - t_{\nu_m}^m) \Big] + \frac{\beta(M+1)^2 a}{m} \\ &= \frac{(M+1)a}{m} + \frac{\beta(M+1)^2 a}{m} \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

We have also

$$\lim_{m\to\infty} \int_0^a \langle f_m(s), \dot{x}_m(s) \rangle ds = \int_0^a \langle f(s), \dot{x}(s) \rangle ds.$$

Taking the limit superior in (2.23) when $m \to \infty$ we obtain

$$g(x(a)) - g(x_0) \ge \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 ds - \int_0^a \langle f(s), \dot{x}(s) \rangle ds. \tag{2.24}$$

This inequality compared with (2.20) yields

$$\int_0^a \|\dot{x}(s)\|^2 ds \ge \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 ds,$$

that is,

$$\|\dot{x}\|^2_{L^2([0,a],H)} \ge \limsup_m \|\dot{x}_m\|^2_{L^2([0,a],H)}. \tag{2.25}$$

On the other hand the weak $l.s.c$ of the norm ensures

$$\|\dot{x}\|^2_{L^2([0,a],H)} \le \liminf_m \|\dot{x}_m\|^2_{L^2([0,a],H)}$$

Consequently, we get

$$\|\dot{x}\|_{L^2([0,a],H)} = \lim_m \|\dot{x}_m\|_{L^2([0,a],H)}.$$

Hence there exists a subsequence of $\{\dot{x}_m\}_m$ (still denoted $\{\dot{x}_m\}_m$) converges poitwisely a.e on $[0.a]$ to $\dot{x}$.

Since

$$(x_m(t), \dot{x}_m(t) - f_m(t) - c_m(t)) \in gphG, \quad \text{a.e. on } [0.a],$$

and as G has a closed graph, we obtain

$$(x(t), \dot{x}(t) - f(t)) \in gphG \quad \text{a.e. on } [0.a],$$

and so

$$\dot{x}(t) \in G(x(t)) + F(t, x(t)) \quad \text{a.e. on } [0.a]$$

The proof is complete. □

Remark 2.4. An inspection of the proof of Theorem 2.3 shows that the uniformity of the constant β was needed only over the set K_0 and so it was not necessary over all the set S. Indeed, it suffices to take the uniform regularity of g locally over S, that is, for every point $\bar{x} \in S$ there exist $\beta \geq 0$ and a neighborhood V of x_0 such that g is uniformly regular over $S \cap V$.

We conclude the paper with two corollaries of our main result in Theorem 2.3.

Corolloray 2.5. *Let $K \subset H$ be a nonempty uniformly prox-regular closed subset of a finite dimensional space H and $F : [0,T] \times H \to H$ be a continuous set-valued mapping with compact values. Then, for any $x_0 \in K$ there exists $a \in]0,T[$ such that the following differential inclusion*

$$\begin{gathered} \dot{x}(t) \in -\partial^C d_K(x(t)) + F(t, x(t)) \quad \text{a.e. on } [0,a] \\ x(0) = x_0 \in K, \end{gathered}$$

has at least one absolutely continuous solution on $[0,a]$.

Proof. In [7, Theorem 3.4] (see also [4, theorem 4.1]) it is shown that the function $g := d_K$ is uniformly regular over K and so it is uniformly regular over some neighborhood V of $x_0 \in K$. Thus, by Remark 2.4, we apply Theorem 2.3 with $S = H$ (hence the tangential condition (2.2) is satisfied), $K_0 := V \cap S = V$, and the set-valued mapping $G := \partial^C d_K$ which satisfies the hypothesis of Theorem 2.3. $\square$

Our second corollary concerns the following differential inclusion

$$\begin{aligned} &\dot{x}(t) \in -N^C(S; x(t)) + F(t, x(t)) \quad \text{a.e.} \\ &x(t) \in S, \quad \text{for all } t \text{ and } x(0) = x_0 \in S. \end{aligned} \tag{2.26}$$

This type of differential inclusion has been introduced in [10] for studying some economic problems.

Corolloray 2.6. *Let H be a separable Hilbert space. Assume that*

(1) *$F : [0,T] \times H \to H$ is a continuous set-valued mapping with compact values;*
(2) *S is a nonempty uniformly prox-regular closed subset in H;*
(3) *For any $(t,x) \in I \times S$ the tangential condition*

$$\liminf_{h \downarrow 0} h^{-1} e\big(x + h(\partial^C d_S(x) + F(t,x)); S\big) = 0,$$

for any $(t,x) \in I \times S$ holds.

Then, for any $x_0 \in S$, there exists $a \in]0,T[$ such that the differential inclusion (2.26) has at lease one absolutely continuous solution on $[0,a]$.

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