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Existence and regularity results for the gradient flow for p-harmonic maps *

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Abstract

We establish existence and regularity for a solution of the evolution problem associated to p-harmonic maps if the target manifold has a non-positive sectional curvature.

1 Introduction

Let M and N be compact, smooth Riemannian manifolds without boundary, of dimensions m and k, with metrics g and γ, respectively. Since N is compact, by Nash's embedding theorem we can regard N as being isometrically embedded in a Euclidean space $\mathbb{R}^n$ for some n. For a C^1-map $u : M \to N \subset R^n$, we define the p-energy $E(u)$ by

$$E(u) = \int_M \tfrac{1}{p}|Du|^p dM, \quad p \geq 2, \tag{1.1}$$

where, in local coordinates on M,

$$dM = \sqrt{|g|}dx, \quad |Du|^2 = \sum_{\alpha,\beta=1}^{m} \sum_{i=1}^{n} g^{\alpha\beta} D_\alpha u^i D_\beta u^i,$$

with $\left(g^{\alpha\beta}\right) = (g_{\alpha\beta})^{-1}$, $|g| = |\det(g_{\alpha\beta})|$ and $D_\alpha = \partial/\partial x^\alpha$, $\alpha = 1, \cdots, m$.

The Euler-Lagrange equation of the p-energy is

$$-\triangle_p u + A_p(u)(Du, Du) = 0\,, \tag{1.2}$$

where $\triangle_p$ denotes the p-Laplace operator

$$\triangle_p u = \tfrac{1}{\sqrt{|g|}}\, D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} |Du|^{p-2} D_\beta u \right)$$

on M, which is a degenerate elliptic operator, and where $A_p(u)(Du, Du)$ is given by

$$A_p(u)(Du, Du) = |Du|^{p-2} g^{\alpha\beta} A(u)(D_\alpha u, D_\beta u)$$

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in terms of the second fundamental form $A(u)(Du, Du)$ of N in $\mathbb{R}^n$ at u.

Here and in what follows, the summation notation over repeated indices is adopted.

We call (weak) solutions of (1.2) (weakly) p-harmonic maps.

One method to look for p-harmonic maps is to exploit the gradient flow related to the p-energy, which is called p-harmonic flow. The gradient flows are described by a system of second order nonlinear degenerate parabolic partial differential equations

$$\partial_t u - \triangle_p u + A_p(u)(Du, Du) = 0 \quad \text{in} \quad (0, \infty) \times M, \tag{1.3}$$

$$u(0, x) = u_0(x) \quad \text{for } x \in M. \tag{1.4}$$

For $p = 2$, Eells and Sampson showed in [12] that there exists a global smooth solution provided that the target manifold N has nonpositive sectional curvature and that the solution converges to a harmonic map suitably as $t_k \to \infty$. This result concerns the homotopy problem, that is, to find a harmonic map homotopic to a given map. When the target manifold N is of non-positive sectional curvature and $p > 2$, the homotopy problem was solved by Duzzar and Fuchs [11] by applying the direct method in the calculus of variations for the regularized p-energy functional (see (2.2) below) and using $C^1_\alpha-$estimates for solutions of the Euler-Lagrange equation (1.2). In this paper we establish the global existence and $C^{0,1}_\alpha-$regularity of a weak solution to the p-harmonic flow provided that the target manifold N has non-positive sectional curvature. The regularity of weak solutions of degenerate parabolic systems with only principal terms was discussed and the $C^{0,1}_\alpha-$regularity of solutions was established in[2, 7, 8, 9]. (Also see [4, 5, 28, 29] for corresponding elliptic systems.) The global existence of a weak solution to the p-harmonic flow was shown when the target manifold is a sphere in [1], and, more generally, a homogeneous space in [18, 19]. For $p = m$, the global existence of a partial $C^{0,1}_\alpha-$ weak solution was established in [20]. For the regularity of harmonic maps and flows, we refer to [25, 14, 27, 3].

To state our results, we need some preliminaries. Let us define the metric δ_q, $q \geq 1$, by

$$\delta_q(z_1, z_2) = \max\{|t_1 - t_2|^{1/q}, |x_1 - x_2|\}$$

for any $z_i = (t_i, x_i) \in (0, \infty) \times R^m$, $i = 1, 2$. If $q = 2$, the metric δ_2 is the usual parabolic metric. For a bounded domain $\Omega \subset R^m$, we use the usual function spaces $C^k_\alpha(\Omega, R^n)$, $L^q(\Omega, R^n)$ and $W^1_q(\Omega, R^n)$. For any $T > 0$, denote by $C^{\alpha/q,\alpha}([0, T] \times \Omega, R^n)$ the space of functions defined on $[0, T] \times \Omega$ with values in $\mathbb{R}^n$, Hölder continuous with respect to the metric δ_q with an exponent α, $0 < \alpha < 1$. In particular, $C^{1/q,1}([0, T] \times \Omega, R^n)$ is the space of functions with values in $\mathbb{R}^n$ that are Lipschitz continuous with respect to the metric δ_q. We also use the notation

$$C^{1,2}_\alpha([0, T] \times \Omega, R^n) = C^0_{\alpha/2}([0, T]; C^2_\alpha(\Omega, R^n)) \cap C^1_{\alpha/2}([0, T]; C^0_\alpha(\Omega, R^n)),$$

$$C^{0,1}_\alpha([0, T] \times \Omega, R^n) = C^0_{\alpha/2}([0, T]; C^1_\alpha(\Omega, R^n)).$$

If the domain is a compact, smooth Riemannian manifold M, then, for $z_i = (t_i, x_i) \in (0,\infty) \times M$, $i = 1, 2$, we replace the metric δ_q, $q \geq 1$, by

$$\max\left\{ |t_1 - t_2|^{1/q}, \mathrm{dist}_M\,(x_1, x_2) \right\},$$

where $\mathrm{dist}_M\,(x_1, x_2)$ means the geodesic distance of $x_1, x_2 \in M$ with respect to the metric g on the manifold M, and we define $C^k_\alpha(M, R^n)$, $C^{1/q,1}_\alpha([0,T] \times M, R^n)$, $C^{\alpha/q,\alpha}_\alpha([0,T] \times M, R^n)$, $C^{1,2}_\alpha([0,T] \times M, R^n)$ and $C^{0,1}_\alpha([0,T] \times M, R^n)$ to be the spaces of functions belonging to the corresponding spaces above with $\Omega = U$ for any local coordinate neighborhood U on M. We now define a set of Sobolev mappings from M to N, which is called the energy space:

$$W^{1,p}(M, N) = \{u \in W^{1,p}(M, R^n) : u(x) \in N \quad \text{for almost all } x \in M\},$$

equipped with the topology inherited from the one of the linear Sobolev spaces $W^{1,p}(M, R^n)$.

We are interested in a global weak solution $u \in L^\infty((0,\infty); W^{1,p}\ (M, N)) \cap W^{1,2}((0,\infty); L^2(M, R^n))$ of (1.3) and (1.4), satisfying, for all $\phi \in L^{p'}((0,\infty); W^{1,p'}(M, R^n)) \cap L^\infty(\ (0,\infty) \times M, R^n)$ with p' the dual exponent of p, the support of which is compactly contained in $(0,\infty) \times U$ for a coordinate chart U on M,

$$\int_{(0,\infty)\times M} \left\{ \phi \cdot \partial_t u + |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + \phi \cdot A_p(u)(Du, Du) \right\} dM\, dt = 0, \tag{1.5}$$

and satisfying the initial condition

$$|u(t) - u_0|_{L^2(M)} \to 0, \quad t \to 0. \tag{1.6}$$

Our main theorem is the following:

Theorem 1.1 *Assume that the sectional curvature of the target manifold N is nonpositive. Let $u_0 \in C^2_\beta(M, N)$ with $0 < \beta < 1$, the image of which is contained in a geodesic ball $\mathcal{B}\,(a_0)$ in N around a point $a_0 \in N$. Then there exists a global weak solution $u \in L^\infty((0,\ \infty); W^{1,p}(M, N)) \cap W^{1,2}((0,\infty); L^2(M, R^n))$ with the energy inequality*

$$\int_{(0,T)\times M} |\partial_t u|^2 dM dt + \sup_{0\leq t\leq T} E(u(t)) \leq E(u_0) \quad \textit{for all } T > 0. \tag{1.7}$$

Moreover, for a positive number $\alpha, 0 < \alpha < 1$, $u \in C^{\alpha/p,\alpha}_{\mathrm{loc}}((0,\infty) \times M, R^n)$ and $Du \in C^{\alpha/2,\alpha}_{\mathrm{loc}}((0,\infty) \times M, R^n)$.

2 The regularized p-energy

First we will make a special isometric embedding of (N^k, γ) in (R^n, h). (Refer to [20].) Let us define a metric h as follows. Since N is compact, we can use the standard Nash embedding of N in $\mathbb{R}^n$ and choose a tubular neighborhood $\mathcal{O}_{2\delta}(N) \subset R^n$ of N such that $\mathcal{O}_{2\delta}(N) = \{x \in R^n : \mathrm{dist}(x, N) < 2\delta\}$, where δ is a sufficiently small positive constant, and dist is the usual Euclidean distance. Then let us put $(\widetilde{\gamma}_{ij}) = (\gamma_{ij}) \otimes (\delta_{ij})$ locally on $N \times B^{n-k}_{2\delta}$, where $B^{n-k}_{2\delta}$ is a ball in $\mathbb{R}^{n-k}$ with a radius 2δ. We can extend $\widetilde{\gamma}_{ij}$ smoothly to $\mathbb{R}^n$ by defining $h_{ij} = \phi\widetilde{\gamma}_{ij} + (1-\phi)\delta_{ij}$ for $\phi \in C^\infty_0(R^n, R)$ with support in $\mathcal{O}_{2\delta}(N)$ and $\phi \equiv 1$ on $\mathcal{O}_\delta(N)$. By such an embedding of N into $\mathbb{R}^n$, we have an involutive isometry π from a tubular neighborhood $\mathcal{O}_\delta$ to itself, which has exactly the target manifold N for its fixed points.

For $u \in R^n$, let

$$\Gamma^l_{ik}(u) = \tfrac{1}{2}h^{ij}\left(\tfrac{dh_{jk}}{du^i}(u) - \tfrac{dh_{ik}}{du^j}(u) + \tfrac{dh_{ij}}{du^k}(u)\right), \quad \left(h^{ij}\right) = \left(h_{ij}\right)^{-1}, \tag{2.1}$$

be the Christoffel symbol for the metric (h_{ij}). For $\epsilon > 0$, the regularized p-energy (refer to [11], [20]) of a map $u : (M, g) \to (R^n, h)$ is defined by

$$E_\epsilon(u) = \int_M e_\epsilon(u)dM, \quad e_\epsilon(u) = \tfrac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}}, \tag{2.2}$$

where, in local coordinates (x^α) of M and (u^i) of $\mathbb{R}^n$,

$$|Du|^2 = g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j. \tag{2.3}$$

We consider the gradient flow for E_ϵ, described by the parabolic system

$$\partial_t u - \Delta^\epsilon_p u - \Gamma^\epsilon_p(u)(Du, Du) = 0, \tag{2.4}$$

where, in local coordinates of M and $\mathbb{R}^n$,

$$\begin{aligned} \Delta^\epsilon_p u &= \tfrac{1}{\sqrt{|g|}} D_\alpha\left((\epsilon + |Du|^2)^{\frac{p}{2}-1}\sqrt{|g|}g^{\alpha\beta}D_\beta u\right), \\ \Gamma^\epsilon_p(u)(Du, Du) &= (\epsilon + |Du|^2)^{\frac{p}{2}-1}g^{\alpha\beta}\Gamma^l_{ij}(u)D_\alpha u^i D_\beta u^j. \end{aligned} \tag{2.5}$$

Recall that u_0 is a member of $C^2_\beta(M, N)$, $0 < \beta < 1$, and has image in the geodesic ball $\mathcal{B}(a_0) \subset N$ around the point $a_0 \in N$. Let us consider the initial value problem for the equation (2.4) with (1.4). We apply the Leray-Schauder fixed point theorem to show the existence of a solution u_ϵ to the problem for any ϵ, $0 < \epsilon < 1$.

For this purpose we introduce the linearized parabolic system: Let us take $T > 0$ arbitrarily. For any τ, $0 \le \tau \le 1$, and $w \in C^{0,1}_\alpha([0, T] \times M, R^n)$, we find a classical solution $u \in C^{1,2}_\alpha([0, T] \times M, R^n)$ of the linear parabolic system

$$\begin{aligned} \partial_t u^i = A^{\alpha\beta}_{ij}(t, x)D_\alpha D_\beta u^j + B^\beta_{ij}(t, x)D_\beta u^j \quad &\text{in } (0, T) \times M, \quad i = 1, \cdots, n, \\ u = \exp_{a_0}\left(\tau \exp^{-1}_{a_0}(u_0)\right) \quad &\text{on } \{t = 0\} \times M, \end{aligned} \tag{2.6}$$

where $\exp_{a_0}(\cdot)$ is the exponential map defined on a Euclidean ball $B(0) \subset R^k$ around the origin with values in $\mathcal{B}(a_0) \subset N$, and the coefficients are, in local coordinates of M and $\mathbb{R}^n$,

$$\begin{aligned} A_{ij}^{\alpha\beta}(t,x) &= (pe_\epsilon(w))^{1-\frac{2}{p}}\left(g^{\alpha\beta}\delta_{ij} + (p-2)\frac{g^{\beta\nu}D_\nu w^k h_{jk}(w) g^{\alpha\mu}D_\mu w^i}{(pe_\epsilon(w))^{\frac{2}{p}}}\right), \\ B_{ij}^{\beta}(t,x) &= \delta_{ij}(pe_\epsilon(w))^{1-\frac{2}{p}}\left\{\frac{1}{\sqrt{|g|}}\, D_\alpha\left(\sqrt{|g|}g^{\alpha\beta}\right)\right. \\ &\quad \left. + \left(\frac{p}{2}-1\right)\frac{g^{\alpha\beta}D_\mu w^k D_\nu w^l}{(pe_\epsilon(w))^{\frac{2}{p}}}\left(\frac{dg^{\mu\nu}}{dx^\alpha}(x)\, h^{kl}(w) + g^{\mu\nu}D_\alpha w\cdot \frac{dh^{kl}}{du}(w)\right)\right\} \\ &\quad + (pe_\epsilon(w))^{1-\frac{2}{p}}\, g^{\alpha\beta}\Gamma_{jk}^i(w) D_\alpha w^k. \end{aligned} \tag{2.7}$$

The equation (2.6) is written as

$$h_{il}(w)\partial_t u^i = h_{il}(w)A_{ij}^{\alpha\beta}(t,x)D_\alpha D_\beta u^j + h_{il}(w)B_{ij}^{\beta}(t,x)D_\beta u^j, \tag{2.8}$$

in which

$$\begin{aligned} &h_{il}(w)A_{ij}^{\alpha\beta}(t,x) \\ &\quad = (pe_\epsilon(w))^{1-\frac{2}{p}}\left(g^{\alpha\beta}h_{jl}(w) + (p-2)\frac{g^{\beta\nu}D_\nu w^k h_{jk}(w) g^{\alpha\mu}D_\mu w^i h_{il}(w)}{(pe_\epsilon(w))^{\frac{2}{p}}}\right), \end{aligned}$$

which is a positive definite matrix. Here we note the relation for the principal term of (2.4) with $0 \le \epsilon < 1$:

$$\begin{aligned} &\left(\Delta_p u^j + \left(\Gamma_p^\epsilon(u)(Du,Du)\right)^j\right) h_{ij}(u) \\ &\quad = \frac{1}{\sqrt{|g|}}\, D_\alpha\left((pe_\epsilon(u))^{1-\frac{2}{p}}\, \sqrt{|g|}g^{\alpha\beta}h_{ij}(u)D_\beta u^j\right) \\ &\qquad -\frac{1}{2}\,(pe_\epsilon(u))^{1-\frac{2}{p}}\, g^{\alpha\beta}\frac{dh_{jk}}{du^i}(u)\, D_\alpha u^j D_\beta u^k. \end{aligned}$$

We fix an "approximating number" ϵ, $0 < \epsilon < 1$. We define an operator P: $[0,1]\times C_\alpha^{0,1}([0,T]\times M, R^n) \ni (\tau,w) \mapsto u = P(\tau,w) \in C_\alpha^{0,1}([0,T]\times M, R^n)$ such that $u = P(\tau,w)$ is a classical solution to (2.6). The exponent α, $0<\alpha<1$, will be stipulated later.

To exploit the Leray-Schauder fixed point theory, we have to verify the following conditions:

1. There exists a unique classical solution to (2.6), which implies that the operator P is well-defined.
2. The operator P is continuous and compact on $[0,1]\times C_\alpha^{0,1}([0,T]\times M, R^n)$.
3. If $\tau = 0$, there exists a unique solution determined uniformly on all $w \in C_\alpha^{0,1}([0,T]\times M, R^n)$.
4. Fixed points u_τ of the operator $P(\tau,\cdot)$, which are solutions to the equation with $w = u_\tau$ in (2.6), are uniformly bounded in $C_\alpha^{0,1}([0,T]\times M, R^n)$ with respect to τ, $0 \le \tau \le 1$ (and ϵ, $0<\epsilon<1$).

In the following sections, we will show the validity of the above statements.

3 Linearized parabolic system

In this section, we prove the existence of a classical solution to the linearized parabolic system (2.6), and show that the corresponding operator P is continuous and compact.
Let the exponent α be $0 < \alpha \leq \beta$, where β is a Hölder exponent of the initial value u_0.

Lemma 3.1 *There exists a unique classical solution to the linearized parabolic system (2.6).*

Noting (2.7), we immediately see that the coefficients $A_{ij}^{\alpha\beta}$ and B_{ij}^{α}, $\alpha, \beta = 1, \cdots, m$; $i, j = 1, \cdots, n$, are Hölder continuous in $[0,T] \times M$ with the exponent α and the Hölder constant depending only on $\left(g^{\alpha\beta}\right)$, (h_{ij}), ϵ, p and $|w|_{C_\alpha^{0,1}}$, and that

$$\epsilon^{\frac{p}{2}-1} |\xi|^2 \leq A_{ij}^{\alpha\beta} \xi_\beta^j \xi_\alpha^k h_{ki}(w) \leq \left(\epsilon + \sup_{[0,T]\times M} |Dw|^2\right)^{\frac{p}{2}-1} |\xi|^2 \tag{3.1}$$

holds for any $(t,x) \in [0,T] \times M$ and $\xi = \left(\xi_\alpha^i\right) \in R^{mn}$, where

$$|\xi|^2 = \sum_{\alpha=1}^{m} \sum_{i=1}^{n} (\xi_\alpha^i)^2 .$$

The parabolic system of the same type as (2.6) is investigated in [22] and the maximum principal for a classical solution is obtained. By combination of it with the Schauder estimates in [23](see [22]), we have the uniform boundedness in $C_\alpha^{1,2}([0,T] \times M, R^n)$ for classical solutions u:

$$|u|_{C_\alpha^{1,2}} \leq \gamma \left(|f|_{C^{\alpha/2,\alpha}} + |u_0|_{C_\alpha^2}\right), \tag{3.2}$$

where a positive constant γ depends only on the Hölder constant of $\left(A_{jl}^{\alpha\gamma}\right)$ and $\left(B^\beta\right)$ and hence γ depends on p, ϵ and $|w|_{C_\alpha^{0,1}}$. Thus we conclude the following result.

Lemma 3.2 *Let $u \in C_\alpha^{1,2}([0,T] \times M, R^n)$ be a solution to the parabolic system (2.6). Then there exists a positive constant γ depending only on $|w|_{C^{\alpha/2,\alpha}}$, $|u_0|_{C_\alpha^2}$, ϵ, p, $(g_{\alpha\beta})$ and (h_{ij}) such that*

$$|u|_{C_\alpha^{1,2}} \leq \gamma. \tag{3.3}$$

As in [22], we can prove the existence of a classical solution of (2.6).
Now we prove the continuity and compactness of the operator P.

Lemma 3.3 *The operator P is continuous and compact in $[0,1] \times C_\alpha^{0,1}([0,T] \times M, R^n)$.*

Proof. (**Compactness**) For all $w \in X := C^{0,1}_{\alpha}([0,T] \times M, R^n)$ such that $|w|_X \leq U$ with a uniform positive constant U, and all τ, $0 \leq \tau \leq 1$, let $u = P(\tau, w)$. Then, by Lemma 3.2, we have

$$|u, Du, D^2u, \partial_t u|_{C^{\alpha/2,\alpha}} \leq \gamma, \tag{3.4}$$

with a positive constant γ depending only on U, $|u_0|_{C^2_\alpha}$, ϵ and p. Here we note that the coefficients in (2.7) are Lipschitz continuous in w and Dw with a Lipschitz constant depending on ϵ. By the uniform boundedness of D^2u and $\partial_t u$, we can apply Lemma 3.1 in [21, pp.78-9] with $\alpha = \beta = 1$ to find that $|Du|_{C^{1/2,1}([0,T]\times M)}$ is uniformly bounded. The family $\{u\}$ of such functions is actually a compact set in X, since $\alpha < 1$. Consequently, the operator $P(\tau, \cdot)$, $0 \leq \tau \leq 1$, maps a bounded set in X into a compact set in X.
(**Continuity**) Take $w_1, w_2 \in X$ satisfying, for $\delta > 0$,

$$|w_1 - w_2|_X \leq \delta \tag{3.5}$$

and let $u_1 = P(\tau, w_1)$ and $u_2 = P(\tau, w_2)$ for any τ, $0 \leq \tau \leq 1$. Subtract the equation for u_1 from the one for u_2 to obtain, for $u = u_2 - u_1$,

$$\partial_t u = A(x, w_2, Dw_2) \cdot D^2u + B(x, w_2, Dw_2) \cdot Du + F(t, x), \tag{3.6}$$

where $A(x, v, Dv)$ and $B(x, v, Dv)$ are $\left(A^{\alpha\gamma}_{jl}\right)$ and $\left(B^{\beta}\right)$ in (2.7) with $w = v$, respectively, and

$$\begin{aligned} F(t,x) \;\; = \;\; & (A(x, w_2, Dw_2) - A(x, w_1, Dw_1)) \cdot D^2u_1 \\ & + (B(x, w_2, Dw_2) - B(x, w_1, Dw_1)) \cdot Du_1 \,. \end{aligned}$$

Noting the Lipschitz continuity in the variables w, Dw of the coefficients $A(x, w, Dw)$ and $B(x, w, Dw)$, we obtain, from (3.2),

$$|u|_{C^{1,2}_\alpha} \leq \gamma |F|_{C^{\alpha/2,\alpha}}, \tag{3.7}$$

where we note that $u = 0$ on $\{t = 0\} \times M$, and that the positive constant γ is determined by $|A|_{C^{\alpha/2,\alpha}}$ and $|B|_{C^{\alpha/2,\alpha}}$, and hence γ depends only on $|w_2|_{C^{0,1}_\alpha}$, ϵ, $(g_{\alpha\beta})$ and (h_{ij}). F is estimated from above by

$$|F|_{C^{\alpha/2,\alpha}} \leq \gamma |w_1 - w_2|_X, \tag{3.8}$$

where the positive constant γ depends only on $|Du_1|_{C^{\alpha/2,\alpha}}$, $|D^2u_1|_{C^{\alpha/2,\alpha}}$, ϵ, $(g_{\alpha\beta})$ and (h_{ij}). Thus, we choose a positive constant γ depending only on $|w_1|_X$, $|u_0|_{C^2_\alpha}$, ϵ, $(g_{\alpha\beta})$ and (h_{ij}) such that

$$|u_1 - u_2|_X \leq |u|_{C^{1,2}_\alpha} \leq \gamma\delta. \tag{3.9}$$

As above, we can verify that $P(\tau, w)$ is continuous on τ for each $w \in X$: For τ_1, τ_2, $0 \leq \tau_1, \tau_2 \leq 1$, we put $u_1 = P(\tau_1, w)$ and $u_2 = P(\tau_2, w)$ for fixed $w \in X$. Then $u = u_2 - u_1$ satisfies the equation

$$\begin{aligned} \partial_t u = A(x, w, Dw) \cdot D^2u + B(x, w, Dw) \cdot Du \quad \text{in } [0,T] \times M, \\ u(0) = \exp_{a_0}\left(\tau_2 \exp^{-1}_{a_0}(u_0)\right) - \exp_{a_0}\left(\tau_1 \exp^{-1}_{a_0}(u_0)\right). \end{aligned} \tag{3.10}$$

Noting the definition of the exponential map $\exp_{a_0}(\cdot)$, we have, with a positive constant γ depending only on (h_{ij}),

$$|u(0)|_{C^2_\alpha} \le \gamma\, |\tau_2 - \tau_1|\, |u_0|_{C^2_\alpha}\,. \tag{3.11}$$

Applying Schauder estimates (3.2) and (3.11) for (3.10), we obtain

$$|u|_{C^{1,2}_\alpha} \le \gamma\, |\tau_2 - \tau_1|\, |u_0|_{C^2_\alpha}\,, \tag{3.12}$$

where the positive constant γ depends only on p, ϵ, $|w|_{C^{0,1}_\alpha}$ and (h_{ij}). Consequently, we find that the operator P is continuous in $[0,1]\times X$.
We now consider the case $\tau = 0$. If $\tau = 0$, then, for any $w \in X$, $u = P(0,w)$ is a solution of (2.6) with the initial condition

$$u = a_0 \quad \text{on } \{t=0\}\times M. \tag{3.13}$$

By the uniqueness of the solution of (2.6) with this initial condition, $P(0,w) = a_0$ for all $w \in X$. Thus, $P(0,\cdot)$ maps all $w \in X$ into the constant map a_0.

4 Uniform boundedness of Du

Now we consider a priori estimates for fixed points of the operator $P(\tau,\cdot)$, $0 \le \tau \le 1$, which are solutions to the parabolic system

$$\begin{aligned} \partial_t u &= \tfrac{1}{\sqrt{|g|}}\, D_\alpha\left((pe_\epsilon(u))^{1-\frac{2}{p}}\sqrt{|g|}g^{\alpha\beta}D_\beta u\right) \\ &\quad + (pe_\epsilon(u))^{1-\frac{2}{p}} g^{\alpha\beta}\Gamma_{ij}(u)D_\alpha u^i D_\beta u^j \quad \text{in } (0,T]\times M, && (4.1)\\ u &= \exp_{a_0}\left(\tau \exp_{a_0}^{-1}(u_0)\right) \quad \text{on } \{t=0\}\times M. && (4.2)\end{aligned}$$

First we establish an energy inequality for solutions of (4.1).

Lemma 4.1 *Let $u \in C^{1,2}_0([0,T]\times M, R^n)$ be a solution to (4.1). Then the energy inequality*

$$\int_{(t_0,t_1)\times M} |\partial_t u|^2 dM dt + E_\epsilon(u(t_1)) \le E_\epsilon(u(t_0)) \tag{4.3}$$

holds for all t_0, t_1, $0 \le t_0 < t_1 \le T$.

Proof. We multiply (4.1) by $h_{ij}(u)\partial_t u^i$. For the right hand side of the resulting equality, we use (refer to [26, pp.558-9, pp.564-5])

$$\begin{aligned} &\tfrac{1}{\sqrt{|g|}}\, D_\alpha\left((pe_\epsilon(u))^{1-\frac{2}{p}}\sqrt{|g|}g^{\alpha\beta}D_\beta u^j \partial_t u^i h_{ij}(u)\right)\\ &= \tfrac{1}{\sqrt{|g|}}\, D_\alpha\left((pe_\epsilon(u))^{1-\frac{2}{p}}\sqrt{|g|}g^{\alpha\beta}D_\beta u^j\right)\partial_t u^i h_{ij}(u) \qquad (4.4)\\ &\quad + (pe_\epsilon(u))^{1-\frac{2}{p}} g^{\alpha\beta}D_\beta u^j D_\alpha\left(\partial_t u^i h_{ij}(u)\right)\\ &= \partial_t e_\epsilon(u) + \left(\Delta^\epsilon_p u^j + (pe_\epsilon(u))^{1-\frac{2}{p}}\Gamma^j(u)(Du,Du)\right)\partial_t u^i h_{ij}(u).\end{aligned}$$

Integrate (4.4) on $[t_0, t_1] \times M$ to obtain

$$\int_{(t_0,t_1)\times M} h_{ij}(u)\partial_t u^i \partial_t u^j \, dM dt + \int_M \{e_\epsilon(u(t_1)) - e_\epsilon(u(t_0))\} \, dM = 0$$

and hence the desired estimate. In particular, noting that $Du(0) = \tau Du_0$ in M, we have obtained (4.3) with $E_\epsilon(u(t_0))$ replaced by $E_\epsilon(\tau u_0)$ for all t_1, $0 \le t_1 \le T$.

Lemma 4.2 *Let $u \in C_0^{1,2}([0,T] \times M, R^n)$ be a solution to* (4.1). *Suppose that the image of u is contained in the target manifold N. Then we have, with a positive constant γ depending only on M, N, T and $\sup_M |Du_0|$,*

$$\sup_{(0,T)\times M} |Du| \le \gamma = \gamma\left(M, N, T, \sup_M |Du_0|\right). \tag{4.5}$$

For solutions to (4.1), we have the Bochner formula (refer to [10, pp.134-135] and [15, pp.128-131]): Put $v = (\epsilon + |Du|^2)/2$. Then we have, in $(0,T) \times M$,

$$\begin{aligned}
&\partial_t v - \tfrac{1}{\sqrt{|g|}} D_\alpha \left((2v)^{\frac{p}{2}-1} a^{\alpha\beta} D_\beta v\right) + (p-2)(2v)^{\frac{p}{2}-2} g^{\alpha\beta} D_\alpha v D_\beta v \\
&+(2v)^{\frac{p}{2}-1} g^{\gamma\bar\gamma} g^{\beta\bar\beta} D_\gamma D_\beta u^i D_{\bar\gamma} D_{\bar\beta} u^j h_{ij}(u) + (2v)^{\frac{p}{2}-1} R_M^{\alpha\beta} D_\alpha u^i D_\beta u^j h_{ij}(u) \\
&= \ (2v)^{\frac{p}{2}-1} g^{\alpha\bar\alpha} g^{\beta\bar\beta} R^N_{ijkl} D_\alpha u^i D_\beta u^j D_{\bar\alpha} u^k D_{\bar\beta} u^l,
\end{aligned} \tag{4.6}$$

where we put

$$a^{\alpha\beta}(t,x) = \sqrt{|g|}\left(g^{\alpha\beta} + (p-2)\,\tfrac{g^{\alpha\mu} g^{\beta\nu} D_\mu u^i D_\nu u^j h_{ij}(u)}{2v}\right).$$

Since we assume that the sectional curvature of N is nonpositive, we have

$$g^{\alpha\bar\alpha} g^{\beta\bar\beta} R^N_{ijkl} D_\alpha u^i D_\beta u^j D_{\bar\alpha} u^k D_{\bar\beta} u^l \le 0. \tag{4.7}$$

Thus we obtain, from (4.7) and (4.6), with a positive constant γ depending only on $(g_{\alpha\beta})$ and the derivative,

$$\partial_t v - \tfrac{1}{\sqrt{|g|}} \, D_\alpha\left((2v)^{\frac{p}{2}-1} a^{\alpha\beta} D_\beta v\right) \le \gamma (2v)^{\frac{p}{2}} \quad \text{in } (0,T) \times M. \tag{4.8}$$

For brevity, we assume that $(g_{\alpha\beta}) = Id$. (We can argue similarly in the general case.) Then the formula (4.8) becomes

$$\partial_t v - D_\alpha\left((2v)^{\frac{p}{2}-1} a^{\alpha\beta} D_\beta v\right) \le \gamma v^{p/2}. \tag{4.9}$$

Let k be $k \ge \hat k = \max\{1, \sup_M |Du_0|^2\}$ and put $M_t = (0,t) \times M$ for $0 < t < T$. Then we substitute a test function $\phi = (v-k)^+ = \max\{v-k, 0\}$ into the formula (4.9) to obtain

$$\int_{M_t} \left\{\partial_t v (v-k)^+ + (2v)^{\frac{p}{2}-1} \, a^{\alpha\beta} D_\beta v D_\alpha (v-k)^+\right\} dz \le \gamma \int_{M_t} v^{p/2} (v-k)^+ dz. \tag{4.10}$$

Now we estimate $\int_{M_t} v^{p/2}(v-k)^+dz$. First we deform $v^{p/2}(v-k)^+$ as

$$((v-k)^+)^{\frac{p}{2}+1}+k^{\frac{p}{2}+1}.$$

We estimate the quantity $\int_{M_t}((v-k)^+)^{p/2+1}dz$ by using the Hölder and Sobolev inequalities. Set $V=(v-k)^+$. Then

$$\begin{aligned}
&\int_{M_t} V^{\frac{p}{2}+1}dz \\
&\quad\le \sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M} V^2dx\right)^{1/a}\sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M} V^{\frac{p}{2}}dx\right)^{1/b}\times \\
&\qquad \int_0^t\left(\int_{\{\tau\}\times M} V^{\frac{m}{m-2}(\frac{p}{2}+1)}dx\right)^{\frac{1}{c}}d\tau \\
&\quad\le \sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M} V^2dx\right)^{1/a}\sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M} V^{p/2}dx\right)^{1/b}\times \\
&\qquad \gamma\left(m,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{(c-1)m-2c}}\left(\int_{M_t}\left(V^{\frac{p}{2}+1}+|DV^{\frac{1}{2}(\frac{p}{2}+1)}|^2\right)dz\right)^{\frac{m}{c(m-2)}},
\end{aligned}$$

where the exponents a,b and c satisfy

$$\frac{1}{a}=\frac{2p}{m(p-2)+2p},\quad \frac{1}{b}=\frac{2(p-2)}{m(p-2)+2p},\quad \frac{1}{c}=\frac{(m-2)(p-2)}{m(p-2)+2p}. \tag{4.11}$$

Noting that $1/a+m/c(m-2)=1$, we have

$$\begin{aligned}
\int_{M_t} V^{\frac{p}{2}+1}dz \quad\le\quad & \gamma\left(m,p,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{(c-1)m-2c}}\sup_{0\le\tau\le t}\left(\int_{M_t} V^{p/2}dx\right)^{1/b}\times \\
&\left\{\sup_{0\le\tau\le t}\int_{\{\tau\}\times M} V^2dx+\int_{M_t}\left(V^{\frac{p}{2}+1}+\left|DV^{\frac{p+2}{4}}\right|^2\right)dz\right\}.
\end{aligned}$$

Using the energy inequality (4.3) and choosing $t>0$ to be small, we estimate

$$\begin{aligned}
&\gamma\left(m,p,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{(c-1)m-2c}}\sup_{0\le\tau\le t}\left(\int_{M_t} V^{p/2}dx\right)^{\frac{1}{b}} \\
&\quad\le\quad \gamma\left(m,p,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{m(c-1)-2c}}\left(\int_{M_t}|Du_0|^pdx\right)^{1/b}\le\frac{1}{2},
\end{aligned}$$

where we note that $c(m-2)/(m(c-1)-2c)>0$ and that the positive number t depends only on $E(u_0)$ and $\gamma(m,p,|M|^{-1/m})$. Thus we have

$$\begin{aligned}
\int_{M_t} V^{\frac{p}{2}+1}dz \quad\le\quad & \widetilde{\gamma}\left(m,p,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{(c-1)m-2c}}\sup_{0\le\tau\le t}\left(\int_{M_t} V^{p/2}dx\right)^{1/b}\times \\
&\left\{\sup_{0\le\tau\le t}\int_{M} V^2dx+\int_{M_t}\left|DV^{\frac{p+2}{4}}\right|^2dz\right\}.
\end{aligned}\tag{4.12}$$

Next we treat $k^{p/2+1}|M_t \times \{v > k\}|$. By Hölder's inequality, we have

$$k^{\frac{p+2}{2}}|M_t \times \{v > k\}| \le k^{2\delta} \sup_{0\le\tau\le t}\left(\int_M v^{p/2}dx\right)^{1/b}\int_0^t |\{v > k\}|^{\frac{1}{a}+\frac{1}{c}}d\tau, \quad (4.13)$$

where the exponent δ is determined by

$$2\delta = \frac{(p-2)(p+2)+8p}{2(m(p-2)+2p)}. \quad (4.14)$$

Now we note that, if we take the exponents κ, q and r to satisfy

$$\frac{2(1+\kappa)}{r} = 1, \quad \frac{r}{q} = \frac{1}{a}+\frac{1}{c}, \quad \frac{1}{r}+\frac{m}{2q} = \frac{m}{4}, \quad (4.15)$$

then

$$\kappa > 0, \quad 0 < \delta < 1+\kappa.$$

Combining (4.12) with (4.13) and substituting the resulting inequalities into (4.10), we have

$$\begin{aligned} &\sup_{0\le\tau\le t}\int_{M_\tau}((v-k)^+)^2dx + \int_{M_t} v^{\frac{p}{2}-1}|D(v-k)^+|^2dz \\ \le\ & \gamma\left(m,p,|M|^{-\frac{1}{m}}\right)t^{\frac{c(m-2)}{m(c-1)-2c}}\sup_{0\le\tau\le t}\int_{\{\tau\}\times M}((v-k)^+)^{\frac{p}{2}}dx\Big)^{1/b} \times \\ & \left(\sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M}((v-k)^+)^2dx + \int_{M_t}\left|D((v-k)^+)^{\frac{p+2}{4}}\right|^2dz\right)\right. \quad (4.16) \\ & +\gamma(m,p)\sup_{0\le\tau\le t}\left(\int_{\{\tau\}\times M}v^{p/2}dx\right)^{1/b}k^{2\delta}\int_0^t|\{v>k\}|^{\frac{1}{a}+\frac{1}{c}}dt, \end{aligned}$$

where we used the facts that the matrix $\left(a^{\alpha\beta}\right)$ is positive definite and that $v \le \max\{1, \sup_M|Du_0|^2\}$ on $\{t=0\}\times M$.
Using (4.3) and noting that $c(m-2)/(m(c-1)-2c) > 0$, we choose $t_1 = t > 0$ to satisfy

$$t^{\frac{c(m-2)}{m(c-1)-2c}}\left(\frac{p+2}{4}\right)^2\gamma\left(m,p,|M|^{\frac{1}{m}}\right)E_1(u_0)^{\frac{1}{b}} \le \tfrac{1}{2}. \quad (4.17)$$

Then we obtain, from (4.16), with a positive constant γ depending only on m and p,

$$\begin{aligned} &\sup_{0\le\tau\le t_1}\int_{\{\tau\}\times M}((v-k)^+)^2dx + \int_{M_{t_1}}|D(v-k)^+|^2dz \quad (4.18) \\ \le\ & \gamma(m,p)\sup_{0\le\tau\le t_1}\left(\int_{\{\tau\}\times M}v^{p/2}dx\right)^{1/b}k^{2\delta}\int_0^{t_1}|\{v>k\}|^{\frac{1}{a}+\frac{1}{c}}dt, \end{aligned}$$

where we used that $k \ge 1$ and

$$\int_{M_{t_1}}\left|D((v-k)^+)^{\frac{p+2}{4}}\right|^2dz \le (\tfrac{p+2}{4})^2\int_{M_{t_1}}v^{\frac{p}{2}-1}|D(v-k)^+|^2dz.$$

Now apply Theorem 6.1 in [21, pp.102-103] for (4.18) to obtain

$$\sup_{M_{t_1}} v \le \gamma(m,p) \max\left\{1, \sup_M |Du_0|^2\right\}.$$

Noting that, by (4.17), the positive number t_1 depends on $E_1(u_0), |M|, m$ and p, and arguing as in [21, p.186], we have

$$\sup_{(0,T)\times M} v \le \gamma(m,p) \max\left\{1, \sup_M |Du_0|^2\right\}.$$

Once we have the uniform boundedness (4.5), we can argue as in [6, p.245, Theorem 1.1; p.291, 14, pp.217–218] (also see [5]) to arrive at the following:

Lemma 4.3 *Let $u \in C_0^{1,2}([0,T]\times M, R^n)$ be a solution of (4.1). We can choose positive constants γ, depending only on $M, N, p, \sup_{(0,T)\times M}|Du|$, and $\widetilde{\alpha}, 0 < \widetilde{\alpha} < 1$, depending only on m and p, such that*

$$|u|_{C^{\tilde{\alpha}/p,\tilde{\alpha}}} + |Du|_{C^{\tilde{\alpha}/2,\tilde{\alpha}}} \le \gamma. \tag{4.19}$$

We now specify the value of the exponent α, $0 < \alpha \le \beta$, which has not yet been determined. We set $\alpha = \min\{\widetilde{\alpha}, \beta\}$, where $\widetilde{\alpha}$ is selected in Lemma 4.3.
Now we prove the uniqueness of a solution of (4.1).

Lemma 4.4 *Let $u_1, u_2 \in C_0^{1,2}([0,T]\times, R^n)$ be two solutions to (4.1) with the same initial value $\exp_{a_0}\left(\tau \exp_{a_0}^{-1}(u_0)\right)$. Then $u_1 \equiv u_2$ in $[0,T]\times M$.*

Proof. We consider only the case $\tau = 1$, since $u(0) = \exp_{a_0}\left(\tau \exp_{a_0}^{-1}(u_0)\right) \in N$ on M and the case $0 \le \tau < 1$ is investigated similarly. Let $u \in C_\alpha^{1,2}([0,T]\times M, R^n)$ be a solution to (4.1) with $\tau = 1$. Then $u(0) = u_0$ in M.
Since the image of u_0 is contained in the target manifold N, we can choose a positive number $\widetilde{T} = \widetilde{T}(u)$ such that $u \in \mathcal{O}_\delta(N)$ in $[0,\tilde{T}]\times M$. Then, by the definition of the metric (h_{ij}) of $\mathbb{R}^n$, we find that

$$g^{\alpha\bar{\alpha}} g^{\beta\bar{\beta}} R^N_{ijkl}(u) D_\alpha u^i D_\beta u^j D_{\bar{\alpha}} u^k D_{\bar{\beta}} u^l \le 0 \quad \text{in } [0,\widetilde{T}]\times M, \tag{4.20}$$

since the sectional curvature of N is nonpositive. Thus, by Lemma 4.2, we have (4.5) with replacing T by $\widetilde{T}$. Let $u_1, u_2 \in C_\alpha^{1,2}([0,T]\times M, R^n)$ be two solutions to (4.1) with $\tau = 1$. Set $\widetilde{T} = \min\{\widetilde{T}(u_1), \widetilde{T}(u_2)\}$. Subtract the equation for u_1 from the one for u_2 and take a test function $u_2 - u_1$ in the resulting equation for t, $0 \le t \le \widetilde{T}$ to obtain, with $v = u_2 - u_1$,

$$\begin{aligned}
&\int_{M_t} v \cdot \partial_t v \, dM\, dt \\
&+ \int_{M_t} \left\{(pe_\epsilon(u_2))^{1-\frac{2}{p}} h_{ij}(u_2) D_\beta u_2^j - (pe_\epsilon(u_1))^{1-\frac{2}{p}} h_{ij}(u_1) D_\beta u_1^j\right\} g^{\alpha\beta} D_\alpha v^i dM dt \\
&\quad = \int_{M_t} g^{\alpha\beta}\Big\{(pe_\epsilon(u_2))^{1-\frac{2}{p}} \Gamma_{ij}(u_2)(D_\alpha u_2^i, D_\beta u_2^j) \\
&\qquad\qquad - (pe_\epsilon(u_1))^{1-\frac{2}{p}} \Gamma_{ij}(u_1)(D_\alpha u_1^i, D_\beta u_1^j)\Big\} \cdot v \, dM\, dt\,.
\end{aligned}$$

We estimate each term of this equality. Put $w(s) = (1-s)u_1 + su_2$ for s, $0 \le s \le 1$. Then

$$\begin{aligned}
&\Big((pe_\epsilon(u_2))^{1-\frac{2}{p}} h_{ij}(u_2) Du_2^j - (pe_\epsilon(u_1))^{1-\frac{2}{p}} h_{ij}(u_1) Du_1^j\Big) g^{\alpha\beta} D_\alpha v^i \\
&= \int_0^1 \Big\{ (pe_\epsilon(w(s)))^{1-\frac{2}{p}} |Dv|^2 + (p-2)(pe_\epsilon(w(s)))^{1-\frac{4}{p}} \langle Dv, Dw(s)\rangle^2 \\
&\quad + (pe_\epsilon(w(s)))^{1-\frac{2}{p}} g^{\alpha\beta} D_\beta v^j D_\alpha w^i(s) \tfrac{dh^{ij}}{du}(w(s)) \cdot v \\
&\quad + \tfrac{p-2}{2} (pe_\epsilon(w(s)))^{1-\frac{4}{p}} g^{\alpha\beta} D_\beta w^j(s) D_\alpha w^i(s) v \cdot \frac{dh^{ij}}{du}(w(s)) \langle Dw(s), Dv\rangle \Big\} ds.
\end{aligned}$$

The third and fourth terms on the right hand side are bounded from above by

$$\begin{aligned}
&\gamma\Big(p, N, \sup_{M_{\tilde T}} |Du_1|, \sup_{M_{\tilde T}} |Du_2|\Big) \int_0^1 |v|^2 ds \\
&\quad + \tfrac{1}{2} \int_0^1 (pe_\epsilon(w(s)))^{1-\frac{2}{p}} \Big(|Dv|^2 + (p-2) \tfrac{\langle Dw(s), Dv\rangle^2}{(pe_\epsilon(w(s)))^{\frac{2}{p}}} \Big) ds\,.
\end{aligned}$$

As above, we have

$$\begin{aligned}
&g^{\alpha\beta} \Big((pe_\epsilon(u_2))^{1-\frac{2}{p}} \Gamma_{ij}(u_2)(D_\alpha u_2^i, D_\beta u_2^j) - (pe_\epsilon(u_1))^{1-\frac{2}{p}} \Gamma_{ij}(u_1)(D_\alpha u_1^i, D_\beta u_1^j) \Big) \cdot v \\
&\le \gamma\Big(p, M, N, \sup_{M_{\tilde T}} |Du_1|, \sup_{M_{\tilde T}} |Du_2|\Big) \int_0^1 |v|^2 ds \\
&\quad + \tfrac{1}{2} \int_0^1 (pe_\epsilon(w(s)))^{1-\frac{2}{p}} \Big(|Dv|^2 + (p-2) \tfrac{\langle Dw(s), Dv\rangle^2}{(pe_\epsilon(w(s)))^{\frac{2}{p}}} \Big) ds.
\end{aligned}$$

As a result we have

$$\begin{aligned}
&\int_{M_t} \Big\{ v \cdot \partial_t v + \tfrac{1}{2} \int_0^1 (pe_\epsilon(w(s)))^{1-\frac{2}{p}} \Big(|Dv|^2 + (p-2) \tfrac{\langle Dw(s), Dv\rangle^2}{(pe_\epsilon(w(s)))^{\frac{2}{p}}} \Big) ds \Big\} dM dt \\
&\le \gamma\Big(p, M, N, \sup_{M_{\tilde T}} |Du_1|, \sup_{M_{\tilde T}} |Du_2|\Big) \int_{M_t} |v|^2 \, dM \, dt\,. \qquad (4.21)
\end{aligned}$$

Putting $F(t) = \int_{M_t} |v|^2 dM dt$ for any t, $0 \le t \le \widetilde{T}$, and noting $v(0) = 0$, we find from (4.21) that

$$\tfrac{d}{dt} F(t) \le \gamma\Big(p, M, N, \sup_{M_{\tilde T}} |Du_1|, \sup_{M_{\tilde T}} |Du_2|\Big) F(t)$$

for all $0 \le t \le \tilde{T}$, from which it follows that $\exp(-\gamma t)F(t) \le 0$ for all $t \in [0, \tilde{T}]$. Therefore we have $F(\tilde{T}) = 0$, which implies that $v = 0$ in $[0, \widetilde{T}] \times M$. Now we observe that the images of u_1 and u_2 are in the target manifold N. We consider $u = u_1$. Take a positive number $\tilde{T} = \widetilde{T}(u)$ such that $u \in \mathcal{O}_\delta(N)$ in $[0, \widetilde{T}] \times M$.

We use the involutive isometry π from $\mathcal{O}_\delta(N)$ to itself such that the fixed point set of π is exactly the target manifold N. Compare $\pi(u)$ with u: Since the image of u_0 is imposed on N, $\pi(u)(0) = u(0)$ in M. Noting that the operator $\pi : \mathcal{O}_\delta(N) \to \mathcal{O}_\delta(N)$ is isometry, we know that $\pi(u)$ satisfies (4.1) with $\tau = 1$, of which u is also a solution. By the arguments above, we find that $\pi(u) \equiv u$ in $[0, \widetilde{T}] \times M$ and that the image of u in $[0, \widetilde{T}] \times M$ is on the fixed point set N of π. Therefore we have verified that $u_1 = u_2 \in N$ in $[0, \tilde{T}] \times M$.
Replacing an initial value u_0 with $u_1(\widetilde{T})(= u_2(\widetilde{T}))$ and repeating the above argument, we conclude our uniqueness assertion: $u_1 \equiv u_2$ in $[0, T] \times M$. In addition, we have proven the following:

Lemma 4.5 *Let $u \in C^{1,2}_0([0, T] \times M, R^n)$ be a solution to (4.1). Then $u \in N$ in $[0, T] \times M$.*

By combination of Lemmata 4.2, 4.3 with Lemma 4.5, we conclude that (4.5) and (4.19) hold uniformly for all solutions $u \in C^{1,2}_0([0, T] \times M, R^n)$ of (4.1).

5 The limit $\epsilon \to 0$

First we claim the existence and uniqueness of the regularized p-harmonic flow, which is a solution of (4.1) with $\tau = 1$. By the arguments in Sect.3 and Sect.4, we can apply the Leray-Schauder fixed point theorem and obtain a unique fixed point u_ϵ in $C^{1,2}_\alpha([0, T] \times M, N)$ of the operator P_1.

Lemma 5.1 *For any ϵ, $0 < \epsilon < 1$, there exists a unique solution u_ϵ in $C^{1,2}_\alpha([0, T] \times M, N)$ of (2.4) with the initial value (1.4).*

We now explain how to pass to the limit $\epsilon \to 0$ and show the validity of Theorem 1.1.
By Lemma 4.1, we choose a subsequence $\{u_k\}$ with $u_k = u_{\epsilon_k}$, $0 < \epsilon_k < 1$, and a function u defined on $(0, T) \times M$ with value in $\mathbb{R}^n$ such that, as $\epsilon_k \to 0$,

$$\begin{aligned} Du_k \to Du \quad &\text{weakly* in } L^\infty((0, T); L^p(M)), \\ \partial_t u_k \to \partial_t u \quad &\text{weakly in } L^2((0, T) \times M), \end{aligned} \tag{5.1}$$

Noting Lemmata 4.2 and 4.3, we apply the Ascoli-Arzela theorem to obtain

$$u_k \to u \quad \text{strongly in } C^{0,1}_0([0, T] \times M, R^n). \tag{5.2}$$

By Lemma 4.5 and (5.2), we know that

$$u \in N \quad \text{in } [0, T] \times M. \tag{5.3}$$

By (5.1) and (5.2), we can take the limit $\epsilon_k \to 0$ in the weak form of the equation (2.4) with a test function $\phi \in C^\infty([0, T] \times M, R^n)$:

$$\begin{aligned} \int_{(0,T)\times M} \{ &\phi \cdot \partial_t u_k + (pe_{\epsilon_k}(u_k))^{1-\frac{2}{p}} g^{\alpha\beta} D_\beta u_k \cdot D_\alpha \phi \\ &-(pe_{\epsilon_k}(u_k))^{1-\frac{2}{p}} g^{\alpha\beta} \Gamma_{ij}(u_k) D_\alpha u^i_k \cdot D_\beta u^j_k \} \, dM \, dt = 0 \end{aligned}$$

and find that the limit function u satisfies (1.5), where we note (5.3). Using (5.1) in the energy inequality (4.3) with $\epsilon = \epsilon_k$ and $u = u_k$, we have (1.7). Lemma 4.3 with (5.2) implies the Hölder continuity of u and Du in the statement of Theorem 1.1 with the Hölder exponent $\alpha = \min\{\widetilde{\alpha}, \beta\}$.
Finally, we use the energy inequality (4.3) to make the estimate

$$\int_M |u_k(t) - u_0|^2 \, dM \leq t \int_{(0,t)\times M} |\partial_t u_k|^2 \, dM dt \leq t E_1(u_0). \tag{5.4}$$

By (5.2), we take the limit $k \to \infty$ in (5.4) to show the validity of (1.6).

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