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RENORMALIZED SOLUTIONS TO A CHEMOTAXIS SYSTEM WITH CONSUMPTION OF CHEMOATTRACTANT

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Abstract. This article concerns the high-dimensional chemotaxis system with consumption of chemoattractant

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - uv, \end{gathered}$$

under homogeneous boundary conditions of Neumann type, in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 4)$ with smooth boundary. We prove that that if the initial data satisfy $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ for some $q > n$, this model possesses at least one global renormalized solution.

1. Introduction

In this article, we consider the existence of renormalized global solutions to the chemotaxis system with consumption of chemoattractant

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 4)$ with smooth boundary, where the scalar functions $u = u(x,t)$ and $v = v(x,t)$ denote bacterial density and the oxygen concentration, respectively. u_0 and v_0 are given functions. $\frac{\partial}{\partial \nu}$ denotes the differentiation with respect to the outward normal derivative on $\partial\Omega$. Model (1.1) was initially introduced by Keller and Segel [11] to describe the traveling band behavior of chemotactic bacteria, that is, the biased movement of bacteria to the oxygen concentration gradient. It can be regarded as the 'fluid-free' version of the coupled chemotaxis-fluid model which was first presented in [19]. Aerobic bacteria such as *Bacillus subtilis* often live in thin fluid layers near solid-air-water contact line, in which the biology of chemotaxis, metabolism, and cell-cell signaling is intimately connected to the physics of buoyancy, diffusion, and mixing [19].

In the previous few years, model (1.1) has been studied by some authors. Tao [17] showed that (1.1) admits global classical bounded solutions under the assumption that $n \geq 2$ and $\|v_0\|_{L^\infty(\Omega)}$ are sufficiently small. Tao and Winkler [18] proved that

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if $n = 2$, (1.1) possesses a unique global classical solution which is bounded and satisfies $u(x,t) \to \bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$ and $v(x,t) \to 0$ as $t \to \infty$; in the case $n = 3$, for arbitrary large initial data, this problem possesses at least one global weak solution which becomes eventually smooth and also satisfies $(u, v) \to (\bar{u}_0, 0)$ as $t \to \infty$. Furthermore, Zhang and Li [28] obtained that if either $n \le 2$ or $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)}, n \ge 3$, the global classical solution of (1.1) converges to $(\bar{u}_0, 0)$ exponentially in the large time limit. The chemotaxis fluid system has been investigated by some authors, we refer the reader to [7, 14, 15, 22, 23, 27] for further reading.

The concept of renormalized solutions was introduced by DiPerna and Lions [4, 5, 6]. Fischer [8] established the existence of global renormalized solutions to reaction-diffusion systems with entropy-dissipating reactions. Chen and Jüngel [3] proved the existence of global-in-time renormalized solutions to reaction-cross-diffusion systems for an arbitrary number of variables in bounded domains with no-flux boundary conditions. For the existence of global renormalized solutions of the Landau equation and Boltzmann equation, see for example [1, 2, 20]. Recently, it was shown in [26] that the Keller-Segel system with singular sensitivity and signal absorption admits renormalized radial solutions (u, v) which are continuous in $(\overline{\Omega}\backslash\{0\})\times[0,\infty)$ and smooth in $(\overline{\Omega}\backslash\{0\})\times(0,\infty)$, and which solve the corresponding initial-boundary value problem in an appropriate generalized sense.

To the best of our knowledge, for arbitrarily large initial data, whether any kind of solution to (1.1) in high-dimensional domains exists globally is an open problem. The difficulty mainly arises from the cross-diffusive term in the first equation when considering the existence of global weak solutions. The known energy estimates are not sufficient to guarantee the boundedness of $u\nabla v$ in $L^s(\Omega \times [0, T])(s > 1)$. Therefore, we consider renormalized solutions.

Main results. As usual, we shall assume that the initial data u_0 and v_0 satisfy

$$\begin{gathered} u_0 \in C^0(\overline{\Omega}), \quad u_0 > 0 \quad \text{in } \overline{\Omega}, \\ v_0 \in W^{1,q}(\Omega) \cap (W^{2,\frac{n+2}{n}}(\Omega), L^{\frac{n+2}{n}}(\Omega))_{\frac{n}{n+2},\frac{n+2}{n}} \quad \text{for some } q > n, \\ v_0 > 0 \quad \text{in } \overline{\Omega}. \end{gathered} \tag{1.2}$$

Remark 1.1. To obtain local existence of classical solution to regularized problem, we suppose that v_0 belongs to $W^{1,q}(\Omega)$. In addition, the choice of v_0 depends on the maximal Sobolev regularity [9] which plays an important role in the proof of Lemma 4.4. Here, $(W^{2,\frac{n+2}{n}}(\Omega), L^{\frac{n+2}{n}}(\Omega))_{\frac{n}{n+2},\frac{n+2}{n}}$ is an intermediate space between $W^{2,\frac{n+2}{n}}(\Omega)$ and $L^{\frac{n+2}{n}}(\Omega)$.

Our main result reads as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 4$ be a bounded domain with smooth boundary, and assume that u_0 and v_0 satisfy* (1.2). *Then there exists a global renormalized solution of* (1.1) *in the sense of Definition 4.1 below.*

The rest of this paper is organized as follows. In Section 2, we introduce a family of regularized problems and give some preliminary properties. In addition, we obtain the global existence of the regularized problems. Based on an energy-type inequality, a priori estimates are given in Section 3. Finally, we give the proof of the main result in Section 4.

2. Approximate problems

Following an idea from [18], we consider the approximate problems

$$
\begin{gathered}
u_{\varepsilon t}=\Delta u_{\varepsilon}-\nabla\cdot(u_{\varepsilon}F_{\varepsilon}'(u_{\varepsilon})\nabla v_{\varepsilon}),\quad x\in\Omega,\ t>0,\\
v_{\varepsilon t}=\Delta v_{\varepsilon}-F_{\varepsilon}(u_{\varepsilon})v_{\varepsilon},\quad x\in\Omega,\ t>0,\\
\frac{\partial u_{\varepsilon}}{\partial\nu}=\frac{\partial v_{\varepsilon}}{\partial\nu}=0,\quad x\in\partial\Omega,\ t>0,\\
u_{\varepsilon}(x,0)=u_0(x),\quad v_{\varepsilon}(x,0)=v_0(x),\quad x\in\Omega,
\end{gathered}
\tag{2.1}
$$

where $\varepsilon\in(0,1)$.

The approximate function F_ε in (2.1) can be chosen as

$$
F_\varepsilon(s):=\frac{1}{\varepsilon}\ln(1+\varepsilon s),\quad \forall s\geq 0.
$$

Note that our choice of F_ε ensures that

$$
0\leq {F_\varepsilon}'(s)=\frac{1}{1+\varepsilon s}\leq 1,\quad\text{and}\quad 0\leq F_\varepsilon(s)\leq s\quad \forall s\geq 0, \tag{2.2}
$$

$$
s{F_\varepsilon}'(s)=\frac{s}{1+\varepsilon s}\leq\frac{1}{\varepsilon},\quad \forall s\geq 0, \tag{2.3}
$$

$$
{F_\varepsilon}'(s)\nearrow 1\quad\text{and}\quad F_\varepsilon(s)\nearrow s\quad\text{as}\quad \varepsilon\searrow 0\quad \forall s\geq 0. \tag{2.4}
$$

All the above approximate problems admit local-in-time smooth solutions:

Lemma 2.1. *Suppose that u_0 and v_0 satisfy (1.2), then for any $\varepsilon\in(0,1)$, there exist $T\max,\varepsilon\in(0,\infty]$ and a classical solution $(u_\varepsilon,v_\varepsilon)$ of (2.1) in $\Omega\times(0,T\max,\varepsilon)$. Moreover, if $T\max,\varepsilon<\infty$, then*

$$
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|v_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)}\to\infty,\ \text{as } t\nearrow T\max,\varepsilon. \tag{2.5}
$$

Proof. For convenience, we omit the subscript ε. Existence: With $R>0$ and $T\in(0,1)$ to be fixed below, in the Banach space

$$
X:=L^\infty((0,T);C^0(\overline{\Omega})\times W^{1,q}(\Omega)),
$$

we consider the closed set

$$
S:=\{(u,v)\in X|\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,q}(\Omega)}\leq R\text{for a.e. } t\in(0,T)\}
$$

and introduce the map $\Psi=(\Psi_1,\Psi_2)$ on S by defining

$$
\begin{gathered}
\Psi_1(u,v)(\cdot,t):=e^{t\Delta}u_0-\int_0^t e^{(t-s)\Delta}\nabla\cdot(uF'(u)\nabla v)ds,\\
\Psi_2(u,v)(\cdot,t):=e^{t(\Delta-1)}v_0+\int_0^t e^{(t-s)(\Delta-1)}v(1-F(u))ds
\end{gathered}
$$

for $(u,v)\in S$ and $t\in(0,T)$.

Since $q>n$, we can pick $\beta\in(0,1)$ such that $\frac{n}{2q}<\beta<\frac12$, so that $D(B^\beta)\hookrightarrow C^0(\overline{\Omega})$, where B stands for the sectorial operator $-\Delta+1$ in $L^q(\Omega)$ with homogeneous Neumann boundary conditions [10], we have

$$
\begin{aligned}
\|\Psi_1(u,v)(\cdot,t)\|_{C^0(\overline{\Omega})}&\leq\|e^{t\Delta}u_0\|_{C^0(\overline{\Omega})}+\int_0^t\|e^{(t-s)\Delta}\nabla\cdot(uF'(u)\nabla v)\|_{C^0(\overline{\Omega})}ds\\
&\leq\|u_0\|_{C^0(\overline{\Omega})}+c\int_0^t\|B^\beta e^{(t-s)\Delta}\nabla\cdot(uF'(u)\nabla v)\|_{L^q(\Omega)}ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \|u_0\|_{C^0(\overline{\Omega})} + c\int_0^t (t-s)^{-\beta-\frac{1}{2}-\delta}\|uF'(u)\nabla v\|_{L^q(\Omega)}ds \\
&\le \|u_0\|_{C^0(\overline{\Omega})} + cR\int_0^t (t-s)^{-\beta-\frac{1}{2}-\delta}ds \\
&\le \|u_0\|_{C^0(\overline{\Omega})} + cRT^{\frac{1}{2}-\beta-\delta}
\end{aligned}
$$

where $0<\delta<\frac{1}{2}-\beta$. Similarly, we fix $\gamma\in(\frac{1}{2},1)$ and estimate

$$
\begin{aligned}
&\|\Psi_2(u,v)(\cdot,t)\|_{W^{1,q}(\Omega)} \\
&\le \|e^{t(\Delta-1)}v_0\|_{W^{1,q}(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}v(1-F(u))\|_{W^{1,q}(\Omega)}ds \\
&\le c\|v_0\|_{W^{1,q}(\Omega)} + c\int_0^t \|B^\gamma e^{(t-s)(\Delta-1)}v(1-F(u))\|_{L^q(\Omega)}ds \\
&\le c\|v_0\|_{W^{1,q}(\Omega)} + c\int_0^t (t-s)^{-\gamma}\|v(1-F(u))\|_{L^q(\Omega)}ds \\
&\le c\|v_0\|_{W^{1,q}(\Omega)} + cR^2T^{1-\gamma}
\end{aligned}
$$

Considering the above estimates, we prove that Ψ maps S into itself if we pick $R>0$ large enough and $T>0$ sufficiently small. In order to show that Ψ is a contraction on S, we first estimate

$$
\begin{aligned}
&\|uF'(u)\nabla v-\bar{u}F'(\bar{u})\nabla\bar{v}\|_{L^q(\Omega)} \\
&= \|(u-\bar{u})F'(u)\nabla v+\bar{u}(F'(u)-F'(\bar{u}))\nabla v+\bar{u}F'(\bar{u})(\nabla v-\nabla\bar{v})\|_{L^q(\Omega)} \\
&\le R\|u-\bar{u}\|_{C^0(\overline{\Omega})} + \varepsilon R^2\|u-\bar{u}\|_{C^0(\overline{\Omega})} + R\|\nabla(v-\bar{v})\|_{L^q(\Omega)} \\
&\le R^2\|(u,v)-(\bar{u},\bar{v})\|_X
\end{aligned}
$$

for all $(u,v),(\bar{u},\bar{v})\in S$. Consequently,

$$
\begin{aligned}
&\|\Psi_1(u,v)(t)-\Psi_1(\bar{u},\bar{v})(t)\|_{C^0(\overline{\Omega})} \\
&\le \int_0^t \|e^{(t-s)\Delta}\nabla\cdot(uF'(u)\nabla v-\bar{u}F'(\bar{u})\nabla\bar{v})\|_{C^0(\overline{\Omega})}ds \\
&\le c\int_0^t (t-s)^{-\beta-\frac{1}{2}-\delta}\|uF'(u)\nabla v-\bar{u}F'(\bar{u})\nabla\bar{v}\|_{L^q(\Omega)}ds \\
&\le cR^2T^{\frac{1}{2}-\beta-\delta}\|(u,v)-(\bar{u},\bar{v})\|_X
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Psi_2(u,v)(t)-\Psi_2(\bar{u},\bar{v})(t)\|_{W^{1,q}(\Omega)} \\
&\le \int_0^t \|e^{(t-s)(\Delta-1)}[v(1-F(u))-\bar{v}(1-F(\bar{u}))]\|_{W^{1,q}(\Omega)}ds \\
&\le c\int_0^t (t-s)^{-\gamma}\|v(1-F(u))-\bar{v}(1-F(\bar{u}))\|_{L^q(\Omega)}ds \\
&\le cRT^{1-\gamma}\|(u,v)-(\bar{u},\bar{v})\|_X.
\end{aligned}
$$

Then Ψ is a contraction on S if T is sufficiently small. In view of standard bootstrap arguments including the regularity theories for parabolic equations and semigroup techniques [9], (u,v) is a classical solution of (2.1).

Positivity: By comparison we obtain $u \geq 0$ and $v \geq 0$. Since both u and v are classical solutions of their respective equations, we now apply the strong maximum principle to derive that both functions are strictly positive in $\overline{\Omega} \times [0, T_{max})$.

Uniqueness: We assume that (u, v) and $(\hat{u}, \hat{v})$ are two solutions of (2.1) in $\Omega \times (0, T)$ for some $T > 0$. Letting $w := u - \hat{u}$, $z := v - \hat{v}$, we specify $T_0 \in (0, T)$ and multiply the difference of the equations satisfied by u and $\hat{u}$ by w to derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 &= \int_\Omega uF'(u)\nabla v \cdot \nabla w - \int_\Omega \hat{u}F'(\hat{u})\nabla \hat{v} \cdot \nabla w \\
&= \int_\Omega (u - \hat{u})F'(u)\nabla v \cdot \nabla w + \int_\Omega \hat{u}(F'(u) - F'(\hat{u}))\nabla v \cdot \nabla w \\
&\quad + \hat{u}F'(\hat{u})\nabla z \cdot \nabla w \\
&:= I_1 + I_2 + I_3 \quad \text{for all } t \in (0, T_0).
\end{aligned}$$

In combination with Young's inequality, Hölder's inequality, Ehrling's lemma and the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q}{q-2}}(\Omega)$, we have

$$\begin{aligned}
I_1 &\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\int_\Omega w^2 F'(u)^2 |\nabla v|^2 \\
&\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\int_\Omega w^2 |\nabla v|^2 \\
&\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\Big(\int_\Omega |\nabla v|^q\Big)^{2/q}\Big(\int_\Omega w^{\frac{2q}{q-2}}\Big)^{\frac{q-2}{q}} \\
&\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\|w\|^2_{L^{\frac{2q}{q-2}}(\Omega)} \\
&\leq \frac{1}{10}\int_\Omega |\nabla w|^2 + c\int_\Omega w^2
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\int_\Omega \hat{u}^2 [F'(u) - F'(\hat{u})]^2 |\nabla v|^2 \\
&\leq \frac{1}{20}\int_\Omega |\nabla w|^2 + c\int_\Omega w^2 |\nabla v|^2 \\
&\leq \frac{1}{10}\int_\Omega |\nabla w|^2 + c\int_\Omega w^2
\end{aligned}$$

as well as

$$\begin{aligned}
I_3 &\leq \frac{1}{10}\int_\Omega |\nabla w|^2 + c\int_\Omega \hat{u}^2 F'(\hat{u})^2 |\nabla z|^2 \\
&\leq \frac{1}{10}\int_\Omega |\nabla w|^2 + c\int_\Omega |\nabla z|^2
\end{aligned}$$

for all $t \in (0, T_0)$. Altogether,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 \leq c\int_\Omega w^2 + c\int_\Omega |\nabla z|^2$$

for all $t \in (0, T_0)$. Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 + \int_\Omega |\nabla z|^2 = -\int_\Omega F(u)vz + \int_\Omega F(\hat{u})\hat{v}z$$

$$
\begin{aligned}
&= -\int_\Omega (F(u)-F(\hat{u}))vz - \int_\Omega F(\hat{u})(v-\hat{v})z \\
&= -\int_\Omega (F(u)-F(\hat{u}))vz - \int_\Omega F(\hat{u})z^2 \\
&\le c\int_\Omega w^2 + c\int_\Omega z^2 + \frac{1}{2}\int_\Omega |\nabla z|^2
\end{aligned}
$$

for all $t \in (0, T_0)$. All in all, we infer that $y(t) := \int_\Omega w^2 + \int_\Omega z^2$ satisfies $y'(t) \le cy(t)$. On integration, this yields $y \equiv 0$ and thereby proves the claim. $\square$

The following estimates of u_ε and v_ε are basic but important in the proof of our result.

Lemma 2.2. *For each $\varepsilon \in (0,1)$, we have*

$$
\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \quad \forall t \in (0, T\max,\varepsilon), \tag{2.6}
$$

$$
\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \quad \textit{in } \Omega \times (0, T\max,\varepsilon). \tag{2.7}
$$

Proof. Integrating the first equation in (2.1), we obtain (2.6). And an application of the maximum principle to the second equation in (2.1) gives (2.7). $\square$

Now we are in position to show that the solution of the approximate problem (2.1) is actually global in time. The idea of the proof is based on the argument in [22]. All constants appearing in the proof of Lemma 2.3 possibly depend on ε.

Lemma 2.3. *For each $\varepsilon \in (0,1)$, we have $T\max,\varepsilon = \infty$; that is, the solutions of* (2.1) *are global in time.*

Proof. Assume that $T\max,\varepsilon < \infty$ for some $\varepsilon \in (0,1)$. Now pick $p > \frac{nq}{n+q}$ ensures that $\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) > 0$. Moreover,

$$
F_\varepsilon(s) \le \frac{p}{\varepsilon e}(1+\varepsilon s)^{1/p} \quad \forall s \ge 0.
$$

This entails that there exists a constant $c_1 > 0$ such that

$$
\|F_\varepsilon(u_\varepsilon)v_\varepsilon\|_{L^p(\Omega)} \le c_1 \quad \forall t \in (\frac{1}{2}T\max,\varepsilon, T\max,\varepsilon) \tag{2.8}
$$

because of (2.6) and (2.7). As a consequence of (2.8), the variation-of-constants formula and well-known smoothing estimates for the Neumann heat semigroup [21, Lemma 1.3] yield the estimate

$$
\begin{aligned}
&\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \\
&\le \|\nabla e^{t\Delta} v_\varepsilon(\frac{1}{2}T\max,\varepsilon)\|_{L^q(\Omega)} + \int_{\frac{1}{2}T\max,\varepsilon}^t \|\nabla e^{(t-s)\Delta} F_\varepsilon(u_\varepsilon)v_\varepsilon\|_{L^q(\Omega)} ds \\
&\le c_2\Big(1 + \int_{\frac{1}{2}T\max,\varepsilon}^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big)\|F_\varepsilon(u_\varepsilon)v_\varepsilon\|_{L^p(\Omega)} ds\Big)^{2/q} \\
&\le c_3 \quad \forall t \in (\frac{3}{4}T\max,\varepsilon, T\max,\varepsilon)
\end{aligned} \tag{2.9}
$$

with certain positive constants c_2 and c_3.

We next use (2.9) to estimate $\|u_\varepsilon\|_{L^\infty(\Omega)}$. Now taking any $\beta \in \big(\frac{n}{2q}, \frac{1}{2}\big)$ and letting B denote the operator $-\Delta + 1$ in $L^q(\Omega)$ with homogeneous Neumann data, we have

$D(B^\beta) \hookrightarrow L^\infty(\Omega)$ (see for example [10]) and hence we find positive constants c_4, c_5 and c_6 such that

$$
\begin{aligned}
&\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\\
&\le \|u_\varepsilon(\cdot,\frac{3}{4}T\max,\varepsilon)\|_{L^\infty(\Omega)}\\
&\quad + c_4\int_{\frac{3}{4}T\max,\varepsilon}^{t} \|B^\beta e^{-(t-s)(B-1)}\nabla\cdot\big(u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon\big)\|_{L^q(\Omega)}ds\\
&\le c_5\Big(1+\int_{\frac{3}{4}T\max,\varepsilon}^{t}(t-s)^{-\beta-\frac{1}{2}}\big\|u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon\big\|_{L^q(\Omega)}ds\Big)\\
&\le c_6 \quad \forall t\in(\frac{7}{8}T\max,\varepsilon, T\max,\varepsilon).
\end{aligned}
\tag{2.10}
$$

This combined with (2.9) contradicts (2.5) and thereby proves that $T\max,\varepsilon = \infty$. $\square$

3. A priori estimates

This section is devoted to establishing an energy-type inequality which will play a key role in the derivation of further estimates.

Lemma 3.1 ([16, Lemma 4.2])). *Suppose that Ω is bounded and $w \in C^2(\overline{\Omega})$ satisfying $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$, then we have*

$$
\frac{\partial|\nabla w|^2}{\partial\nu} \le 2\kappa|\nabla w|^2 \quad \text{on } \partial\Omega,
$$

where $\kappa = \kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$.

Next, we can use Lemma 3.1 to deal with boundary term.

Lemma 3.2. *For each $\varepsilon \in (0,1)$, the solution of (2.1) satisfies*

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon\ln u_\varepsilon + 2\int_\Omega|\nabla\sqrt{v_\varepsilon}|^2\Big\} + \int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\\
&+\frac{1}{2}\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\\
&\le C\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.1}
$$

for all $t > 0$.

Proof. Testing the first equation of (2.1) by $\ln u_\varepsilon$ and integrating by parts, we obtain

$$
\frac{d}{dt}\int_\Omega u_\varepsilon\ln u_\varepsilon + \int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} = \int_\Omega F_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon
$$

Multiplying the second equation of (2.1) by $\Delta\ln v_\varepsilon$ and $\frac{|\nabla v_\varepsilon|^2}{2v_\varepsilon^2}$ and then integrating by parts, we have

$$
\begin{aligned}
-\int_\Omega\frac{\nabla v_\varepsilon\cdot\nabla v_{\varepsilon t}}{v_\varepsilon} &= -\int_\Omega v_\varepsilon\nabla\ln v_\varepsilon\cdot\nabla\Delta\ln v_\varepsilon + \int_\Omega\frac{\nabla v_\varepsilon}{v_\varepsilon}\cdot\nabla(F_\varepsilon(u_\varepsilon)v_\varepsilon)\\
&= -\int_\Omega v_\varepsilon\Big(\frac{1}{2}\Delta|\nabla\ln v_\varepsilon|^2 - |D^2\ln v_\varepsilon|^2\Big) + \int_\Omega F_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad + \int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_\Omega \Delta v_\varepsilon \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} - \frac{1}{2}\int_{\partial\Omega} v_\varepsilon \frac{\partial|\nabla \ln v_\varepsilon|^2}{\partial\nu} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \\
&\quad + \int_\Omega F_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&= -\frac{1}{2}\int_\Omega \Delta v_\varepsilon \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} - \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon} \frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \\
&\quad + \int_\Omega F_\varepsilon'(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}
\end{aligned}
$$

and

$$
\frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} = \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon - \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}.
$$

Hence,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u_\varepsilon \ln u_\varepsilon + 2\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2\Big) + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \\
&\quad + \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&= \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \\
&\le \kappa \int_{\partial\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&= 4\kappa \int_{\partial\Omega} |\nabla\sqrt{v_\varepsilon}|^2
\end{aligned}
$$

where we use Lemma 3.1. By the trace theorem [13, Theorem I.9.4], it holds

$$
\int_{\partial\Omega} |\nabla\sqrt{v_\varepsilon}|^2 \le C\|\sqrt{v_\varepsilon}\|^2_{H^{\frac{3+s}{2}}(\Omega)}, \quad \forall s \in (0, \frac{1}{2}).
$$

Moreover, we infer from the interpolation inequality [13, Remark I.9.6] that

$$
\begin{aligned}
\|\sqrt{v_\varepsilon}\|^2_{H^{\frac{3+s}{2}}(\Omega)} &\le C\|\sqrt{v_\varepsilon}\|^{\frac{3+s}{2}}_{H^2(\Omega)}\|\sqrt{v_\varepsilon}\|^{\frac{1-s}{2}}_{L^2(\Omega)} \\
&\le C\|\Delta\sqrt{v_\varepsilon}\|^{\frac{3+s}{2}}_{L^2(\Omega)}\|\sqrt{v_\varepsilon}\|^{\frac{1-s}{2}}_{L^2(\Omega)} + C\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)}
\end{aligned}
$$

By direct computation, we have

$$
\Delta\sqrt{v_\varepsilon} = \frac{\Delta v_\varepsilon}{2\sqrt{v_\varepsilon}} - \frac{|\nabla v_\varepsilon|^2}{4v_\varepsilon^{3/2}}, \quad \Delta \ln v_\varepsilon = \frac{\Delta v_\varepsilon}{v_\varepsilon} - \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}.
$$

Hence,

$$
\Delta\sqrt{v_\varepsilon} = \frac{1}{2}\sqrt{v_\varepsilon}\Delta \ln v_\varepsilon + \frac{|\nabla v_\varepsilon|^2}{4v_\varepsilon^{3/2}}.
$$

Then it follows that

$$
\|\Delta\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)} \le \frac{1}{2}\int_\Omega v_\varepsilon|\Delta \ln v_\varepsilon|^2 + \frac{1}{8}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.
$$

Along with the above estimates and Young's inequality, we have

$$
\frac{d}{dt}\Big(\int_\Omega u_\varepsilon \ln u_\varepsilon + 2\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2\Big) + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2
$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&\le \delta_1\|\Delta\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)} + C(\delta_1)\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)} \\
&\le \delta_1\int_\Omega v_\varepsilon|\Delta\ln v_\varepsilon|^2 + \delta_1\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + C(\delta_1)\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)} \\
&\le \delta_1 n\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + \delta_1(2+\sqrt{n})^2\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + C(\delta_1)\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)} \\
&= \delta_1[n+(2+\sqrt{n})^2]\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 + C(\delta_1)\|\sqrt{v_\varepsilon}\|^2_{L^2(\Omega)}
\end{aligned}
$$

where we use the pointwise inequality $|\Delta z|^2 \le n|D^2 z|^2$ for $z \in C^2(\overline{\Omega})$ and [22, Lemma 3.3]. Pick $0 < \delta_1 < \frac{1}{2[n+(2+\sqrt{n})^2]}$, we can obtain (3.1). □

We next collect some consequences of the above energy inequality which are convenient for our purpose.

Corollary 3.3. *For each $T > 0$ and all $\varepsilon \in (0,1)$, there exists $C(T) > 0$ such that the solution of* (2.1) *satisfies*

$$
\int_\Omega u_\varepsilon(\cdot,t)|\log u_\varepsilon(\cdot,t)| \le C(T), \quad \forall t \in (0,T), \tag{3.2}
$$

$$
\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le C(T), \tag{3.3}
$$

$$
\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 \le C(T), \quad \forall t \in (0,T), \int_0^T\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 \le C(T). \tag{3.4}
$$

Proof. Integrating (3.1) over $t \in (0,\infty)$ we obtain

$$
\begin{aligned}
&\int_\Omega u_\varepsilon\ln u_\varepsilon + 2\int_\Omega|\nabla\sqrt{v_\varepsilon}|^2 + \int_0^t\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{1}{2}\int_0^t\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 \\
&+ \frac{1}{2}\int_0^t\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&\le \int_\Omega u_0\ln u_0 + 2\int_\Omega|\nabla\sqrt{v_0}|^2 + C\|v_0\|_{L^\infty(\Omega)}t \\
&\le \int_\Omega u_0\ln u_0 + 2\int_\Omega|\nabla\sqrt{v_0}|^2 + Ct
\end{aligned}
$$

for all $\varepsilon \in (0,1)$. Since $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi \ge 0$ and $|\nabla\sqrt{v_\varepsilon}|^2 = \frac{|\nabla v_\varepsilon|^2}{4v_\varepsilon}$, this shows that

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_\Omega u_\varepsilon\ln u_\varepsilon + \frac{1}{2}\sup_{t\in(0,T)}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_0^T\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \frac{1}{2}\int_0^T\int_\Omega v_\varepsilon|D^2\ln v_\varepsilon|^2 \\
&+ \frac{1}{2}\int_0^T\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&\le \int_\Omega u_0\ln u_0 + 2\int_\Omega|\nabla\sqrt{v_0}|^2 + CT.
\end{aligned}
$$

Therefore recalling (2.7), we obtain the desired results. □

Lemma 3.4 ([22, Lemma 4.1])**.** *Suppose that $n \geq 1$ and that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $p > 1$ and $r \geq 1$ be such that*

$$p \leq \frac{n}{(n-2)_+}, \quad r \leq \frac{2p}{n(p-1)}. \tag{3.5}$$

Then, for all $T > 0$ and each $M > 0$, there exists $C(T, M) > 0$ such that if $\varphi \in L^2((0,T); W^{1,2}(\Omega))$ is nonnegative with

$$\int_\Omega \varphi(\cdot, t) \leq M \quad \text{for all } t \in (0,T), \tag{3.6}$$

then

$$\int_0^T \|\varphi\|_{L^p(\Omega)}^r dt \leq C(T,M) \Big\{ \int_0^T \int_\Omega \frac{|\nabla \varphi|^2}{\varphi} + 1 \Big\}^{\frac{n(p-1)r}{2p}}. \tag{3.7}$$

In view of (2.6), Lemma 3.4 implies the following result.

Corollary 3.5. *Suppose that $n \geq 4$. Then for all $T > 0$, there exists $C > 0$ such that for any $\varepsilon \in (0,1)$, the solution of (2.1) satisfies*

$$\int_0^T \int_\Omega u_\varepsilon^{\frac{n+2}{n}} \leq C. \tag{3.8}$$

Proof. It is a consequence of Corollary 3.3 and of Lemma 3.4 applied to $p := \frac{n+2}{n}$ and $r := \frac{n+2}{n}$. $\square$

4. Existence of renormalized solutions

Having established the existence of solutions for our approximate problem, we turn to the proof of the existence of renormalized solutions to the original equations (1.1). Before going into detail, let us first give the definition of renormalized solutions.

Definition 4.1. Assume that $n \geq 1$, that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ are nonnegative. Then a pair (u, v) of functions

$$u \in L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)), \quad v \in L^\infty_{\rm loc}(\overline{\Omega} \times [0,\infty)),$$

satisfying $u \geq 0$ and $v \geq 0$ almost everywhere in $\Omega \times (0,\infty)$, will be called a global renormalized solution of (1.1) if for all $\xi \in C^\infty([0,\infty))$ with $\xi' \in C_0^\infty([0,\infty))$ we have

$$\begin{aligned} &- \int_0^\infty \int_\Omega \xi(u)\psi_t - \int_\Omega \xi(u_0)\psi(\cdot, 0) \\ &= - \int_0^\infty \int_\Omega \xi''(u)|\nabla u|^2 \psi - \int_0^\infty \int_\Omega \xi' \nabla u \cdot \nabla \psi \\ &\quad + \int_0^\infty \int_\Omega u\xi''(u)(\nabla u \cdot \nabla v)\psi + \int_0^\infty \int_\Omega u\xi'(u) \nabla v \cdot \nabla \psi \end{aligned} \tag{4.1}$$

for all $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, and moreover

$$\int_0^\infty \int_\Omega v\psi_t + \int_\Omega v_0 \psi(\cdot, 0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla \psi + \int_0^\infty \int_\Omega uv\psi \tag{4.2}$$

for any $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

Remark 4.2. Renormalized solution, which is very weak, partially resembles the notion of generalized solution [24, 25]. In the definition of the renormalized solution, we use some truncations of u, so if u is bounded, renormalized solution is a weak solution. If nonnegative functions u and v belong to $C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$ and such that (u,v) is a generalized solution in the sense of [24, 25], then (u,v) is a classical solution.

To construct renormalized solutions, we use the notation from [8]. Let $\varphi_E : \mathbb{R}_0^+ \to \mathbb{R}_0^+, E \in \mathbb{N}$, be truncation function subject to the following conditions:

- (E1) Let $\varphi_E \in C^2(\mathbb{R}_0^+)$.
- (E2) Assume that there exists $K_1 > 0$ so that $v|\varphi_E''(v)| \leq K_1$ holds for all E and all $v \in \mathbb{R}_0^+$.
- (E3) Suppose that for every E the set $\operatorname{supp} D\varphi_E$ is bounded.
- (E4) Assume that $\lim_{E\to\infty} \varphi_E'(v) = 1$ holds for all $v \in \mathbb{R}_0^+$.
- (E5) Suppose that there exists $K_2 > 0$ such that $|\varphi_E'(v)| \leq K_2$ holds for every $v \in \mathbb{R}_0^+$ and every E.
- (E6) Assume that $\varphi_E(v) = v$ holds for any $v \in \mathbb{R}_0^+$ with $v < E$.
- (E7) Suppose that we have $\lim_{E\to\infty} \sup_{|v|\leq K} |\varphi_E''(v)| = 0$ for every $K \in \mathbb{R}^+$.

Indeed truncations φ_E satisfying (E1)–(E7) can be constructed. Let $\phi \in C^\infty(\mathbb{R})$ be a smooth nonincreasing function taking values in $[0,1]$ with $\phi \equiv 1$ for $x < 0$ and $\phi \equiv 0$ for $x > 1$. Define

$$\varphi_E(v) := v\phi\Big(\frac{v-E}{E}\Big) + 3E\Big(1 - \phi\Big(\frac{v-E}{E}\Big)\Big). \tag{4.3}$$

Then one verifies readily that φ_E satisfy conditions (E1)-(E7). Note that we shall also use the same family of truncations in the construction of our renormalized solutions below.

In the first step, we show that a subsequence of the solutions u_ε to the approximate problems (2.1) converges to some limit u as $\varepsilon \to 0$.

Lemma 4.3. *Consider a sequence u_ε of solutions to the approximate problems, with ε converging to zero. Then there exists a subsequence (still denoted by u_ε) which converges almost everywhere on $\Omega \times [0,\infty)$ to some $u \in L^\infty_{\rm loc}([0,\infty); L^1(\Omega))$ with $u|\log u| \in L^\infty_{\rm loc}([0,\infty); L^1(\Omega))$. Moreover, the convergence $\sqrt{u_\varepsilon} \rightharpoonup \sqrt{u}$ weakly in $L^2([0,T], H^1(\Omega))$ holds for all $T > 0$.*

Proof. As a consequence of Corollary 3.3 and (2.6), $\sqrt{u_\varepsilon}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega))$. Let φ_E be as in (4.3). Noting that

$$\nabla[\varphi_E(u_\varepsilon)] = \varphi_E'(u_\varepsilon)\nabla u_\varepsilon$$

and that $\operatorname{supp} \varphi_E'(v)$ is a compact subset of $\mathbb{R}_0^+$, we obtain that $\varphi_E(u_\varepsilon)$ is uniformly bounded with respect to ε in $L^2([0,T]; H^1(\Omega))$ for every fixed $T > 0$ and every fixed $E \in \mathbb{N}$.

Let $\psi \in C^\infty(\overline{\Omega} \times [0,\infty))$. Testing the first equation of (2.1) by $\psi\varphi_E'(u_\varepsilon)$ and integrating by parts, we have

$$\begin{aligned} &\int_\Omega \varphi_E(u_\varepsilon(\cdot,T))\psi(\cdot,T) - \int_\Omega \varphi_E(u_0)\psi(\cdot,0) - \int_0^T \int_\Omega \varphi_E(u_\varepsilon)\psi_t \\ &= \int_0^T \int_\Omega \frac{d}{dt}\varphi_E(u_\varepsilon)\psi \end{aligned}$$

$$\begin{aligned}
&= -\int_0^T \int_\Omega \varphi_E''(u_\varepsilon)|\nabla u_\varepsilon|^2\psi - \int_0^T \int_\Omega \varphi_E'(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\psi \\
&\quad + \int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)\varphi_E''(u_\varepsilon)(\nabla u_\varepsilon \cdot \nabla v_\varepsilon)\psi \\
&\quad + \int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)\varphi_E'(u_\varepsilon)\nabla v_\varepsilon \cdot \nabla\psi \\
&=: I + II + III + IV.
\end{aligned} \tag{4.4}$$

Then we estimate each term.

$$\begin{aligned}
|I| &\le \int_0^T \int_\Omega |\varphi_E''(u_\varepsilon)||\nabla u_\varepsilon|^2|\psi| \\
&= \int_0^T \int_\Omega u_\varepsilon|\varphi_E''(u_\varepsilon)|\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}|\psi| \\
&\le K_1\|\psi\|_{L^\infty(\Omega\times[0,T])}\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le C\,,
\end{aligned}$$

where we use Corollary 3.3 and (E2).

$$\begin{aligned}
|II| &\le \int_0^T \int_\Omega |\nabla\varphi_E(u_\varepsilon)||\nabla\psi| \\
&\le \|\nabla\varphi_E(u_\varepsilon)\|_{L^2(\Omega\times[0,T])}\|\nabla\psi\|_{L^2(\Omega\times[0,T])} \le C,
\end{aligned}$$

$$\begin{aligned}
|III| &\le \int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)|\varphi_E''(u_\varepsilon)||\nabla u_\varepsilon||\nabla v_\varepsilon||\psi| \\
&\le K_1\|\psi\|_{L^\infty(\Omega\times[0,T])}\int_0^T \int_\Omega F_\varepsilon'(u_\varepsilon)|\nabla u_\varepsilon||\nabla v_\varepsilon| \\
&\le C\Big(\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\Big)^{1/2}\Big(\int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)^2|\nabla v_\varepsilon|^2\Big)^{1/2} \\
&\le C\Big(\int_0^T \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\Big)^{1/2} \le C
\end{aligned}$$

where we use (E2), Corollary 3.3 and the fact that $K := \sup_{\xi\ge 0}\frac{\xi}{(1+\xi)^2\ln(1+\xi)}$ is positive and finite. Furthermore,

$$\begin{aligned}
|IV| &\le \int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)|\varphi_E'(u_\varepsilon)||\nabla v_\varepsilon||\nabla\psi| \\
&\le K_2\|\nabla\psi\|_{L^\infty(\Omega\times[0,T])}\int_0^T \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)|\nabla v_\varepsilon| \\
&\le C\Big(\int_0^T \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\Big)^{1/2}\Big(\int_0^T \int_\Omega \frac{u_\varepsilon^2 F_\varepsilon'(u_\varepsilon)^2}{F_\varepsilon(u_\varepsilon)}\Big)^{1/2} \\
&\le C\Big(\int_0^T \int_\Omega u_\varepsilon\Big)^{1/2} \le C
\end{aligned}$$

Along with (E5), (2.6), Corollary 3.3 and the definition of K, the above inequality holds. Therefore, $\frac{d}{dt}\varphi_E(u_\varepsilon)$ is bounded uniformly in $L^1([0,T];(W^{1,\infty}(\Omega))')$ for every $T > 0$ and every fixed E.

In view of $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,\infty}(\Omega))'$, the boundedness of $(\varphi_E(u_\varepsilon))_{\varepsilon\in(0,1)}$ in $L^2([0,T];H^1(\Omega))$ and the boundedness of $\big(\frac{d}{dt}\varphi_E(u_\varepsilon)\big)_{\varepsilon\in(0,1)}$ in the space $L^1([0,T];(W^{1,\infty}(\Omega))')$, we can obtain from the Aubin-Lions Lemma (see for example [12]) that the sequence $\varphi_E(u_\varepsilon)$ is relatively compact in $L^2([0,T];L^2(\Omega))$ for every fixed $T>0$ and fixed $E\in\mathbb{N}$. By a diagonal sequence argument (we do not relabel the subsequence), we may assume that for every $E\in\mathbb{N}$ the sequence $(\varphi_E(u_\varepsilon))_\varepsilon$ converges almost everywhere to some measurable limit w_E. According to Corollary 3.3, $u_\varepsilon|\log u_\varepsilon|$ is uniformly bounded in $L^\infty([0,T];L^1(\Omega))$ for every fixed $T>0$. Combining this with (E6), we deduce that $\varphi_E(u_\varepsilon)|\log\varphi_E(u_\varepsilon)|$ is also bounded uniformly in $L^\infty([0,T];L^1(\Omega))$ for every fixed $T>0$. Moreover, the boundedness is also uniform with respect to E. Thus, by Fatou's Lemma we know that w_E is almost everywhere finite and $w_E|\log w_E|$ is bounded uniformly (with respect to E) in $L^\infty([0,T];L^1(\Omega))$.

We now prove that the pointwise limit $\lim_{E\to\infty} w_E$ exists almost everywhere and define a measurable function u with $u|\log u|\in L^\infty([0,T];L^1(\Omega))$. If for some (x,t) and some E we have $w_E(x,t)=\lim_{\varepsilon\to0}\varphi_E(u_\varepsilon(x,t))<E$, then $w_{\tilde E}(x,t)=w_E(x,t)$ holds for all $\tilde E>E$. If we have $w_E(x,t)<E$, then for ε small enough it holds that $\varphi_E(u_\varepsilon(x,t))<E$. By our choice of φ_E we know that $\varphi_E(u_\varepsilon(x,t))=\varphi_{\tilde E}(u_\varepsilon(x,t))$, therefore we get $w_E(x,t)=w_{\tilde E}(x,t)$ for $\tilde E>E$. Since $s\le s\log s+1$ for all $s>0$, w_E is bounded uniformly in $L^\infty([0,T];L^1(\Omega))$. Based on this, we have

$$\begin{aligned}
&\mathcal{L}^{n+1}\Big(\Big\{(x,t)\in\Omega\times[0,T]: w_E(x,t)\ge E\Big\}\Big)E\\
&\le \int_0^T\int_\Omega w_E(x,t)\\
&\le \|w_E\|_{L^\infty([0,T];L^1(\Omega))}T\le C,
\end{aligned}$$

and

$$\mathcal{L}^{n+1}\Big(\big\{(x,t)\in\Omega\times[0,T]: w_E(x,t)\ge E\big\}\Big)\le\frac{C}{E}\to0,\quad\text{as } E\to\infty.$$

Consequently, the limit $\lim_{E\to\infty}w_E(x,t)$ exists for almost every $(x,t)\in\Omega\times[0,T]$ and defines a measurable function u. By Fatou's Lemma, we have $u|\log u|\in L^\infty([0,T];L^1(\Omega))$.

The function u is now the natural candidate for being a renormalized solution of (1.1).

First we notice that (after possibly passing to another subsequence) u_ε converges almost everywhere to u. By uniform boundedness of u_ε in $L^1(\Omega\times[0,T])$, the measure of the set of points (x,t) with $u_\varepsilon(x,t)\ge E$ tends to zero as $E\to\infty$, uniformly in ε; thus the measure of the set of points (x,t) for which $\varphi_E(u_\varepsilon(x,t))\ne u_\varepsilon(x,t)$ holds tends to zero as $E\to\infty$, uniformly in ε. We have for any $\delta>0$

$$\begin{aligned}
&\mathcal{L}^{n+1}\Big(\big\{(x,t)\in\Omega\times[0,T]: |u_\varepsilon(x,t)-u(x,t)|>\delta\big\}\Big)\\
&\le \mathcal{L}^{n+1}\Big(\big\{(x,t)\in\Omega\times[0,T]: u_\varepsilon(x,t)\ne\varphi_E(u_\varepsilon)(x,t)\big\}\Big)\\
&\quad+\mathcal{L}^{n+1}\Big(\big\{(x,t)\in\Omega\times[0,T]: |\varphi_E(u_\varepsilon)(x,t)-w_E(x,t)|>\frac{\delta}{2}\big\}\Big)\\
&\quad+\mathcal{L}^{n+1}\Big(\big\{(x,t)\in\Omega\times[0,T]: |w_E(x,t)-u(x,t)|>\frac{\delta}{2}\big\}\Big),
\end{aligned}$$

where by the previous considerations the first term on the right-hand side converges to zero as $E \to \infty$, uniformly in $\varepsilon > 0$. The last term tends to zero as $E \to \infty$ by the definition of u; it is independent of ε. The penultimate term converges to zero as $\varepsilon \to 0$ for fixed E. To sum up, we have shown that u_ε converges to u in measure, which implies convergence almost everywhere for a subsequence.

As u_ε is bounded uniformly in $L^\infty([0,T];L^1(\Omega))$ for every $T > 0$, we deduce that u_ε converges to u strongly in $L^p([0,T];L^1(\Omega))$ for every $T > 0$ and $p \geq 1$. This, in particular, implies convergence of $\sqrt{u_\varepsilon}$ to $\sqrt{u}$ in the sense of distribution, and we obtain that $\sqrt{u} \in L^2([0,T];H^1(\Omega))$ with

$$\int_0^T \int_\Omega |\nabla \sqrt{u}|^2 \leq \liminf_{\varepsilon \to 0} \int_0^T \int_\Omega |\nabla \sqrt{u_\varepsilon}|^2$$

[the latter lim inf being finite due to Corollary 3.3]. In particular, $\sqrt{u_\varepsilon}$ converges to $\sqrt{u}$ weakly in $L^2([0,T];H^1(\Omega))$ for every $T > 0$. □

In the second step, as a preparation for the proof of Theorem 1.2, we show that a subsequence of the solutions v_ε to the approximate problems (2.1) converges to some limit v as $\varepsilon \to 0$.

Lemma 4.4. *Consider a sequence v_ε of solutions to the approximate problems, with ε converging to zero. Then there exists a subsequence v_ε (not relabeled)which converges almost everywhere on $\Omega \times [0,\infty)$ to some limit $v \in L^\infty_{\rm loc}(\overline{\Omega} \times [0,\infty))$. Moreover, v satisfies* (4.2)*.*

Proof. Firstly, we show that v_ε is strongly precompact in $L^1([0,T];W^{1,1}(\Omega))$. Then since $0 \leq F_\varepsilon(u_\varepsilon) \leq u_\varepsilon$, in view of (2.7) and Corollary 3.5 we can pick positive constant C such that

$$\int_0^T \int_\Omega |F_\varepsilon(u_\varepsilon)v_\varepsilon|^{\frac{n+2}{n}} \leq C \int_0^T \int_\Omega u_\varepsilon^{\frac{n+2}{n}} \leq C$$

for all $\varepsilon \in (0,1)$. This shows that $(F_\varepsilon(u_\varepsilon)v_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^{\frac{n+2}{n}}(\Omega \times (0,T))$, so that standard results on Sobolev regularity for the heat equation [9] assert the boundedness of both $(v_{\varepsilon t})_{\varepsilon \in (0,1)}$ in $L^{\frac{n+2}{n}}(\Omega \times (0,T))$ and of $(v_\varepsilon)_{\varepsilon \in (0,1)}$ in $L^{\frac{n+2}{n}}((0,T);W^{2,\frac{n+2}{n}}(\Omega))$. Again by the Aubin-Lions lemma, this shows that $(v_\varepsilon)_{\varepsilon \in (0,1)}$ is relatively compact in $L^1([0,T];W^{1,1}(\Omega))$. It is possible to pick a sequence of numbers $(0,1) \ni \varepsilon_j \searrow 0$ such that as $\varepsilon = \varepsilon_j \searrow 0$, the solutions v_ε of (2.1) satisfy

$$\begin{gathered} v_\varepsilon \to v \quad \text{in } L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)) \ \text{ and a.e. in } \ \Omega \times (0,\infty), \\ \nabla v_\varepsilon \to \nabla v \quad \text{in } L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \ \Omega \times (0,\infty), \end{gathered}$$

for some limit function v. To see that v satisfies (4.2), we fix $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$. Multiplying the second equation in (2.1) by ψ, on integrating by parts we obtain

$$\int_0^\infty \int_\Omega v_\varepsilon \psi_t + \int_\Omega v_0 \psi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \psi + \int_0^\infty \int_\Omega F_\varepsilon(u_\varepsilon) v_\varepsilon \psi.$$

Combined with the boundedness of $F_\varepsilon(u_\varepsilon)v_\varepsilon$ in $L^{\frac{n+2}{n}}(\Omega \times (0,T))$, we derive that (4.2) by letting $\varepsilon \to 0$ and thereby completes the proof. □

In the third step we prove the existence of renormalized solutions, we show that the "truncations" $\varphi_E(u)$ of the limit u, which has been constructed in the first step, satisfy a certain PDE.

Lemma 4.5. *Let u be the function constructed in the Lemma 4.3. Let φ_E be the function defined in (4.3). Let $\psi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$. Then $\varphi_E(u)$ satisfies*

$$\begin{aligned} &-\int_0^\infty \int_\Omega \varphi_E(u) \frac{d}{dt}\psi \, dx\, dt - \int_\Omega \varphi_E(u_0)\psi(\cdot,0) dx \\ &= -\int_{\overline{\Omega}\times[0,\infty)} \psi d\mu^E(x,t) - \int_0^\infty \int_\Omega \varphi_E'(u)\nabla u \cdot \nabla \psi \, dx\, dt \\ &\quad + \int_0^\infty \int_\Omega u\varphi_E'(u)\nabla v \cdot \nabla\psi \, dx\, dt\,, \end{aligned} \tag{4.5}$$

where μ^E denotes a sequence of signed Radon measures satisfying

$$\lim_{E\to\infty} |\mu^E|(\overline{\Omega}\times[0,T)) = 0 \tag{4.6}$$

for all $T > 0$.

Proof. Let $T > 0$ and $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$. For fixed $E \in \mathbb{N}$ we pass to the limit $\varepsilon \to 0$ in (4.4). Convergence of the left-hand side and of the terms II, IV is immediate by the convergence properties proven in Lemma 4.3 and by the fact that supp $D\varphi_E$ is compact.

Two terms whose convergence cannot be ensured are term I, III. In order to deal with them, we intend to show that they vanish in the limit $E \to \infty$. Consider the signed measures

$$\begin{aligned} \mu_\varepsilon^E &:= \big(\varphi_E''(u_\varepsilon)|\nabla u_\varepsilon|^2 - u_\varepsilon F_\varepsilon'(u_\varepsilon)\varphi_E''(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v_\varepsilon\big)\ dx\, dt \\ &= \Big(4u_\varepsilon\varphi_E''(u_\varepsilon)|\nabla\sqrt{u_\varepsilon}|^2 - 2u_\varepsilon^{3/2}F_\varepsilon'(u_\varepsilon)\varphi_E''(u_\varepsilon)\nabla\sqrt{u_\varepsilon}\cdot\nabla v_\varepsilon\Big)\ dx\, dt. \end{aligned} \tag{4.7}$$

Note that we have

$$|\mu_\varepsilon^E|(\overline{\Omega}\times[0,T)) \le C\int_0^T \int_\Omega |\nabla\sqrt{u_\varepsilon}|^2 \,dx\,dt + \int_0^T\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\,dx\,dt,$$

which follows from the definition of μ_ε^E using (E2) and Corollary 3.3 as well as Young's inequality. The uniform boundedness of $\sqrt{u_\varepsilon}$ in $L^2([0,T];H^1(\Omega)$ for any $T > 0$ implies that after passing to a subsequence we may assume that μ_ε^E weak-$*$ converges on $\overline{\Omega}\times[0,\infty)$ to some limit μ^E as ε tends to 0.

It remains to prove (4.6). We now consider the measures

$$\begin{aligned} \nu_\varepsilon^K &:= \chi_{\{|u_\varepsilon|\in[K-1,K)\}}|\nabla\sqrt{u_\varepsilon}|^2\,dx\,dt \\ \gamma_\varepsilon^K &:= \chi_{\{|u_\varepsilon|\in[K-1,K)\}}F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\,dx\,dt \end{aligned}$$

on $\overline{\Omega}\times[0,\infty)$. Using (E2) and Corollary 3.3 we deduce from (4.7) that

$$\begin{aligned} |\mu_\varepsilon^E|(\overline{\Omega}\times[0.T)) &\le C\sum_{K=1}^\infty \int_0^T\int_\Omega \chi_{\{|u_\varepsilon|\in[K-1,K)\}}u_\varepsilon|\varphi_E''(u_\varepsilon)||\nabla\sqrt{u_\varepsilon}|^2\,dx\,dt \\ &\quad + C\sum_{K=1}^\infty \int_0^T\int_\Omega \chi_{\{|u_\varepsilon|\in[K-1,K)\}}u_\varepsilon|\varphi_E''(u_\varepsilon)|F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\,dx\,dt \\ &\le C\sum_{K=1}^\infty \nu_\varepsilon^K(\overline{\Omega}\times[0,T))\sup_{|v|\in[K-1,K)} v|\varphi_E''(v)| \end{aligned}$$

$$+ C \sum_{K=1}^{\infty} \gamma_\varepsilon^K(\overline{\Omega} \times [0,T)) \sup_{|v| \in [K-1,K)} v|\varphi_E''(v)|$$

By (E3), for fixed $E \in \mathbb{N}$ only finitely many terms in the series do not vanish. We may therefore pass to the limit $\varepsilon \to 0$. Using the fact that the measure of open sets is lower semicontinuous with respect to weak-$*$ convergence of measures, we obtain, after passing to a subsequence (the passage to a subsequence in particular ensuring that the limits in the last line of the next formula exist),

$$\begin{aligned} |\mu^E|(\overline{\Omega} \times [0,T)) &\le \liminf_{\varepsilon\to 0} |\mu_\varepsilon^E|(\overline{\Omega} \times [0,T)) && (4.8)\\ &\le C \sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \nu_\varepsilon^K(\overline{\Omega} \times [0,T)) \sup_{|v| \in [K-1,K)} v|\varphi_E''(v)| && (4.9)\\ &\quad + C \sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \gamma_\varepsilon^K(\overline{\Omega} \times [0,T)) \sup_{|v| \in [K-1,K)} v|\varphi_E''(v)|. && (4.10) \end{aligned}$$

However, we have

$$\sum_{K=1}^{\infty} \nu_\varepsilon^K(\overline{\Omega} \times [0,T)) = \int_0^T \int_\Omega |\nabla \sqrt{u_\varepsilon}|^2 \,dx\,dt$$
$$\sum_{K=1}^{\infty} \gamma_\varepsilon^K(\overline{\Omega} \times [0,T)) = \int_0^T \int_\Omega F_\varepsilon(u_\varepsilon) |\nabla v_\varepsilon|^2 \,dx\,dt.$$

As the latter quantities are bounded uniformly with respect to ε, we obtain, using Fatou's lemma (for the counting measure on $\mathbb{N}$; recall that the limits in the next formula actually exist since we have passed to an appropriate subsequence),

$$\sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \nu_\varepsilon^K(\overline{\Omega} \times [0,T)) < \infty,$$
$$\sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \gamma_\varepsilon^K(\overline{\Omega} \times [0,T)) < \infty.$$

By dominated convergence applied to the counting measure on $\mathbb{N}$ (which is possible by (E2) and (E7) as well as the previous estimate), we deduce from (4.8)

$$\begin{aligned} \limsup_{E\to\infty} |\mu^E|(\overline{\Omega} \times [0,T)) &\le C \sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \nu_\varepsilon^K(\overline{\Omega} \times [0,T)) \lim_{E\to\infty} \sup_{|v| \in [K-1,K)} v|\varphi_E''(v)| \\ &\quad + C \sum_{K=1}^{\infty} \lim_{\varepsilon\to 0} \gamma_\varepsilon^K(\overline{\Omega} \times [0,T)) \lim_{E\to\infty} \sup_{|v| \in [K-1,K)} v|\varphi_E''(v)| \\ &= 0. \end{aligned}$$

This completes the proof. □

We can now prove our main result.

Proof of Theorem 1.2. To show that u is a renormalized solution, we apply [8, Lemma 4] to the map $v := \varphi_E(u)$ in order to approximately identify the weak time derivative of $\xi(\varphi_E(u))$; then we pass to the limit $E \to \infty$ to deduce the equation for $\xi(u)$.

More precisely, we choose $T > 0$ arbitrary but fixed; we then prove that u is a renormalized solution on $[0, T)$. Let ξ be a smooth function with compactly supported derivatives. Recall that $\varphi_E(u)$ satisfies (4.5), we see that in [8, Lemma 4] we need to choose

$$\begin{gathered}\rho = \varphi_E(u), \quad \nu = -\mu^E, \quad q = 0, \\ w = 0, \quad z = u\varphi_E'(u)\nabla v - \varphi_E'(u)\nabla u.\end{gathered}$$

Obviously, they satisfy the assumptions of [8, Lemma 4]. Thus, we infer that for any $\psi \in C_0^\infty(\overline{\Omega} \times [0, T))$ the function $\xi(\varphi_E(u))$ must satisfy the estimate

$$\begin{aligned}&\Big| -\int_0^T \int_\Omega \xi(\varphi_E(u)) \frac{d}{dt}\psi \,dx\,dt - \int_\Omega \xi(\varphi_E(u_0))\psi(\cdot, 0) dx \\ &- \int_0^T \int_\Omega \xi'(\varphi_E(u)) u\varphi_E'(u)\nabla v \cdot \nabla\psi \,dx\,dt \\ &+ \int_0^T \int_\Omega \xi'(\varphi_E(u))\varphi_E'(u)\nabla u \cdot \nabla\psi \,dx\,dt \\ &- \int_0^T \int_\Omega \psi\xi''(\varphi_E(u)) u\varphi_E'(u)\nabla v \cdot \nabla\varphi_E(u) \,dx\,dt \\ &+ \int_0^T \int_\Omega \psi\xi''(\varphi_E(u))\varphi_E'(u)\nabla u \cdot \nabla\varphi_E(u) \,dx\,dt \Big| \\ &\le C(\Omega)\|\psi\|_{L^\infty} \sup_v |D\xi(v)| \mu^E|(\overline{\Omega} \times [0, T)).\end{aligned}$$

To obtain the desired equation for $\xi(u)$, we now pass to the limit $E \to \infty$. To do so, we use (4.3) as well as (4.6); note that due to (4.6), the left-hand side must be zero in the limit, that is, we obtain an exact equation in the limit. Convergence of the terms in the first line is immediate, as is convergence of the terms in the second and the third line [observe that $\varphi_E(u)$ converges pointwise almost everywhere to u and that the φ_E' is bounded by a constant by (E5)].

It remains to deal with the fourth and the fifth line. To show convergence of the two terms, besides the fact $\nabla\sqrt{u} \in L^2([0, T]; L^2(\Omega))$ we need the following assertion: there exists a constant r such that for all $E > r$ the estimate $u \ge r$ implies $\xi'(\varphi_E(u(x, t))) = \xi'(u(x, t)) = 0$ and $\xi''(\varphi_E(u(x, t))) = \xi''(u(x, t)) = 0$. Given this assertion, convergence of the remaining terms in the previous formula as $E \to \infty$ is also immediate since one factor in the integrals will be zero as soon as $u(x, t)$ becomes too large.

To show this assertion, choose r so large that supp $D\xi \subset B_r(0)$. Let $E > r$. Then $u(x, t) \ge r$ implies $\varphi_E(u(x, t)) \ge r$ and therefore $\xi'(\varphi_E(u(x, t))) = 0$ as well as $\xi''(\varphi_E(u(x, t))) = 0$. Combining with Lemma 4.4, we finish the proof of Theorem 1.2. □

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