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On a mixed problem for a linear coupled system with variable coefficients *

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Abstract

We prove existence, uniqueness and exponential decay of solutions to the mixed problem

$$u''(x,t) - \mu(t)\Delta u(x,t) + \textstyle\sum_{i=1}^n \frac{\partial\theta}{\partial x_i}(x,t) = 0\,,$$
$$\theta'(x,t) - \Delta\theta(x,t) + \textstyle\sum_{i=1}^n \frac{\partial u'}{\partial x_i}(x,t) = 0\,,$$

with a suitable boundary damping, and a positive real-valued function μ.

1 Introduction

Let Ω be a bounded and open set in $\mathbb{R}^n$ $(n \geq 1)$ with boundary Γ of class C^2. Assumed that there exists a partition $\{\Gamma_0, \Gamma_1\}$ of Γ such that Γ_0 and Γ_1 each has positive induced Lebesgue measure, and that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is empty. We consider the linear system

$$u''(x,t) - \mu(t)\Delta u(x,t) + \textstyle\sum_{i=1}^n \frac{\partial\theta}{\partial x_i}(x,t) = 0 \quad \text{in} \quad \Omega\times]0,\infty[\tag{1.1}$$
$$\theta'(x,t) - \Delta\theta(x,t) + \textstyle\sum_{i=1}^n \frac{\partial u'}{\partial x_i}(x,t) = 0 \quad \text{in} \quad \Omega\times]0,\infty[\tag{1.2}$$
$$u(x,t) = 0, \quad \theta(x,t) = 0 \quad \text{on} \quad \Gamma_0\times]0,\infty[\tag{1.3}$$
$$\tfrac{\partial u}{\partial\nu}(x,t) + \alpha(x)u'(x,t) = 0 \quad \text{on} \quad \Gamma_1\times]0,\infty[\tag{1.4}$$
$$\tfrac{\partial\theta}{\partial\nu}(x,t) + \beta\theta(x,t) = 0 \quad \text{on} \quad \Gamma_1\times]0,\infty[\tag{1.5}$$
$$u(x,0) = u^0(x), \;\; u'(x,0) = u^1(x), \;\; \theta(x,0) = \theta^0(x) \quad \text{on} \quad \Omega, \tag{1.6}$$

where μ is a function of $W^{1,\infty}_{\rm loc}(0,\infty)$, such that $\mu(t) \geq \mu_0 > 0$. By α we represent a function of $W^{1,\infty}(\Gamma_1)$ such that $\alpha(x) \geq \alpha_0 > 0$, and by β a positive real number. The prime notation denotes time derivative, and $\frac{\partial}{\partial\nu}$ denotes derivative in the direction of the exterior normal to Γ.

The above system is physically meaningful only in one dimension. For which there exists an extensive literature on existence, uniqueness and stability when $\mu \equiv 1$. See the recent papers of Muñhoz Rivera [9], Henry, Lopes, Perisinotto [2], and Scott Hansen [10].

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The paper of Milla Miranda and L. A. Medeiros [8] on wave equations with variable coefficients has a particular relevance to this work. In that paper, due to the boundary condition of feedback type, the authors introduced a special basis necessary to apply the Galerkin method. This is the natural method solving problems with variable coefficients.

In this article, we show the existence of a strong global solution of (1.1)–(1.6), when u^0, u^1 and θ^0 satisfy additional regularity hypotheses. Then this result is used for finding a weak global solution to (1.1)–(1.6) in the general case. By the use of a method proposed in [4], we study the asymptotic behavior of an energy determined by solutions.

The paper is organized as follows: In §2 notation and basic results, in §3 strong solutions, in §4 weak solutions, and in §5 asymptotic behavior.

2 Notation and Basic Results

Let the Hilbert space

$$V = \{v \in H^1(\Omega); v = 0 \quad \text{on } \Gamma_0\}$$

be equipped with the inner product and norm given by

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, dx\,, \quad \|v\| = \left(\sum_{i=1}^{n} \int_\Omega \left(\frac{\partial u}{\partial x_i}(x) \right)^2 dx \right)^{1/2}.$$

While in $L^2(\Omega)$, $(.,.)$ and $|.|$ represent the inner product and norm, respectively.

Remark 2.1 *Milla Miranda and Medeiros [8] showed that in $V \cap H^2(\Omega)$ the norm $\left(|\Delta u|^2 + \|\frac{\partial u}{\partial \nu}\|^2_{H^{1/2}(\Gamma_1)}\right)^{1/2}$ is equivalent to the norm $\|.\|_{H^2(\Omega)}$.*

We assume that

$$\beta \geq \frac{n}{2\alpha_0 \mu_0}\,. \tag{2.1}$$

To obtain the strong solution and consequently weak solution for system (1.1)–(1.6), we need the following results.

Proposition 2.1 *Let $u_1 \in V \cap H^2(\Omega)$, $u_2 \in V$ and $\theta \in V \cap H^2(\Omega)$ satisfy*

$$\frac{\partial u_1}{\partial \nu} + \alpha(x) u_2 = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial \theta}{\partial \nu} + \beta\theta = 0 \quad \text{on } \Gamma_1\,. \tag{2.2}$$

Then, for each $\varepsilon > 0$, there exist w, y and z in $V \cap H^2(\Omega)$, such that

$$\|w - u_1\|_{V \cap H^2(\Omega)} < \varepsilon\,,\ \|z - u_2\| < \varepsilon\,,\ \|y - \theta\|_{V \cap H^2(\Omega)} < \varepsilon\,,$$

with

$$\frac{\partial w}{\partial \nu} + \alpha(x) z = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial y}{\partial \nu} + \beta y = 0 \quad \text{on } \Gamma_1\,.$$

Proof. We assume the conclusion of Proposition 3 in [8]. So, it suffices to prove the existence of y.

By the hypothesis $\Delta\theta \in L^2(\Omega)$, for each $\varepsilon > 0$ there exists $y \in \mathcal{D}(\Omega)$ such that $|y - \Delta\theta| < \varepsilon$. Let q be solution of the elliptic problem

$$\begin{gathered} -\Delta q = -y \quad \text{in } \Omega \\ q = 0 \quad \text{on } \Gamma_0 \\ \tfrac{\partial q}{\partial \nu} + \beta q = 0 \quad \text{on } \Gamma_1 \,. \end{gathered}$$

On the other hand, we observe that θ is the solution of the above problem with $y = \Delta\theta$. Using results of elliptic regularity, cf. H. Brezis [1], we conclude that $q - \theta \in V \cap H^2(\Omega)$ and that there exists a positive constant C such that

$$\|q - \theta\|_{V \cap H^2(\Omega)} \leq C\,|y - \Delta\theta|\,.$$

Proposition 2.2 *If $\theta \in V$, then for each $\varepsilon > 0$ there exists $q \in V \cap H^2(\Omega)$ satisfying $\frac{\partial q}{\partial \nu} + \beta q = 0$ on Γ_1 such that $\|\theta - q\| < \varepsilon$.*

Proof. Observe that the set

$$W = \left\{ q \in V \cap H^2(\Omega); \frac{\partial q}{\partial \nu} + \beta q = 0 \quad \text{on } \Gamma_1 \right\}$$

is dense in V. This is so because W is the domain of the operator $A = -\Delta$ determined by the triplet $\left\{ V, L^2(\Omega), a(u, v) \right\}$, where

$$a(u, v) = ((u, v)) + (\beta u, v)_{L^2(\Gamma_1)}\,.$$

See for example J. L. Lions [5]. Hence, the result follows.

3 Strong Solutions

In this section, we prove existence and uniqueness of a solution to (1.1)–(1.6) when u^0, u^1 and θ^0 are smooth. First, we have the following result.

Theorem 3.1 *Suppose that $u^0 \in V \cap H^2(\Omega)$, $u^1 \in V$, and $\theta^0 \in V \cap H^2(\Omega)$ satisfy*

$$\frac{\partial u^0}{\partial \nu} + \alpha(x) u^1 = 0 \quad \textit{on } \Gamma_1 \quad \textit{and} \quad \frac{\partial \theta^0}{\partial \nu} + \beta\theta^0 = 0 \quad \textit{on } \Gamma_1\,.$$

Then there exists a unique pair of real functions $\{u, \theta\}$ such that

$$u \in L^\infty_{\text{loc}}(0, \infty; V \cap H^2(\Omega)),\ u' \in L^\infty_{\text{loc}}(0, \infty; V), \tag{3.1}$$

$$u'' \in L^\infty_{\text{loc}}(0, \infty; L^2(\Omega)) \tag{3.2}$$

$$\theta \in L^\infty_{\text{loc}}(0, \infty; V \cap H^2(\Omega)),\ \ \theta' \in L^\infty_{\text{loc}}(0, \infty; V) \tag{3.3}$$

$$u'' - \mu\Delta u + \sum_{i=1}^{n} \frac{\partial\theta}{\partial x_i} = 0 \quad in \quad L^\infty_{\rm loc}(0,\infty;L^2(\Omega)) \tag{3.4}$$

$$\tfrac{\partial u}{\partial \nu} + \alpha u' = 0 \quad in \quad L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.5}$$

$$\theta' - \Delta\theta + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i} = 0 \quad in \quad L^\infty_{\rm loc}(0,\infty;L^2(\Omega)) \tag{3.6}$$

$$\tfrac{\partial \theta}{\partial \nu} + \beta\theta = 0 \quad in \quad L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.7}$$

$$u(0) = u^0, \;\; u'(0) = u^1, \;\; \theta(0) = \theta^0\,. \tag{3.8}$$

Proof. We use the Galerkin method with a special basis in $V \cap H^2(\Omega)$. Recall that from Proposition 2.1 there exist sequences $(u^0_\ell)_{\ell\in\mathbb{N}}$, $(u^1_\ell)_{\ell\in\mathbb{N}}$ and $(\theta^0_\ell)_{\ell\in\mathbb{N}}$ of vectors in $V \cap H^2(\Omega)$ such that:

$$u^0_\ell \longrightarrow u^0 \;\text{ strongly in }\; V \cap H^2(\Omega) \tag{3.9}$$

$$u^1_\ell \longrightarrow u^1 \;\text{ strongly in }\; V \tag{3.10}$$

$$\theta^0_\ell \longrightarrow \theta_0 \;\text{ strongly in }\; V \cap H^2(\Omega) \tag{3.11}$$

$$\tfrac{\partial u^0_\ell}{\partial\nu} + \alpha u^1_\ell = 0 \;\text{ on }\; \Gamma_1 \tag{3.12}$$

$$\tfrac{\partial \theta^0_\ell}{\partial\nu} + \beta \theta^0_\ell = 0 \;\text{ on }\; \Gamma_1\,. \tag{3.13}$$

For each $\ell \in \mathbb{N}$ pick u^0_ℓ, u^1_ℓ and θ^0_ℓ linearly independent, then define the vectors $w^\ell_1 = u^0_\ell$, $w^\ell_2 = u^1_\ell$ and $w^\ell_3 = \theta^0_\ell$, and then construct an orthonormal basis in $V \cap H^2(\Omega)$,

$$\{w^\ell_1, w^\ell_2, ..., w^\ell_j, ...\} \;\text{ for each } \ell \in \mathbb{N}\,.$$

For ℓ fixed and each $m \in \mathbb{N}$, we consider the subspace $W^\ell_m = [w^\ell_1, w^\ell_2, ..., w^\ell_m]$ generated by the m-first vectors of the basis. Thus for $u_{\ell m}(t)$, $\theta_{\ell m}(t) \in W^\ell_m$ we have

$$u_{\ell m}(t) = \sum_{j=1}^{m} g_{\ell jm}(t) w^\ell_j(x) \quad \text{and} \quad \theta_{\ell m}(t) = \sum_{j=1}^{m} h_{\ell jm}(t) w^\ell_j(x)\,.$$

For each $m \in \mathbb{N}$, we find pair of functions $\{u_{\ell m}(t), \theta_{\ell m}(t)\}$ in $W^\ell_m \times W^\ell_m$, such that for all $v \in W^\ell_m$ and all $w \in W^\ell_m$,

$$\begin{aligned} &(u''_{\ell m}(t), v) + \mu(t)((u_{\ell m}(t), v)) + \mu(t)\textstyle\int_{\Gamma_1} \alpha(x) u'_{\ell m}(t) v d\Gamma \\ &\qquad + \textstyle\sum_{i=1}^{n} \left(\frac{\partial \theta_{\ell m}}{\partial x_i}(t), v\right) = 0\,, \end{aligned} \tag{3.14}$$

$$(\theta'_{\ell m}(t), w) + ((\theta_{\ell m}(t), w)) + \beta \textstyle\int_{\Gamma_1} \theta_{\ell m}(t) w\, d\Gamma + \sum_{i=1}^{n} \left(\frac{\partial u'_{\ell m}}{\partial x_i}(t), w\right) = 0\,,$$

$$u_{\ell m}(0) = u^0_\ell, \quad u'_{\ell m}(0) = u^1_\ell \quad \text{and} \quad \theta_{\ell m}(0) = \theta^0\,.$$

The solution $\{u_{\ell m}(t), \theta_{\ell m}(t)\}$ is defined on a certain interval $[0, t_m[$. This interval will be extended to any interval $[0, T]$, with $T > 0$, by the use of the following a priori estimate.

Estimate I. In (3.14) we replace v by $u'_{\ell m}(t)$ and w by $\theta_{\ell m}(t)$. Thus

$$\begin{gathered}\tfrac{1}{2}\tfrac{d}{dt}|u'_{\ell m}(t)|^2 + \tfrac{1}{2}\tfrac{d}{dt}\left\{\mu(t)\|u_{\ell m}(t)\|^2\right\} + \mu(t)\int_{\Gamma_1}\alpha(x)(u'_{\ell m}(t))^2 d\Gamma \\ + \textstyle\sum_{i=1}^n \left(\tfrac{\partial\theta_{\ell m}}{\partial x_i}(t), u'_{\ell m}(t)\right) \;\le\; |\mu'(t)|\|u_{\ell m}(t)\|^2\,, \\ \tfrac{1}{2}\tfrac{d}{dt}|\theta_{\ell m}(t)|^2 + \|\theta_{\ell m}(t)\|^2 + \beta\textstyle\int_{\Gamma_1}(\theta_{\ell m}(t))^2 d\Gamma + \sum_{i=1}^n\left(\tfrac{\partial u'_{\ell m}}{\partial x_i}(t), \theta_{\ell m}(t)\right) = 0.\end{gathered}$$

Define

$$E_1(t) = \frac{1}{2}\left\{|u'_{\ell m}(t)|^2 + \mu(t)\|u_{\ell m}(t)\|^2 + |\theta_{\ell m}(t)|^2\right\}.$$

and we make use of the Gauss identity

$$\sum_{i=1}^n\left(\frac{\partial u'_{\ell m}}{\partial x_i}(t), \theta_{\ell m}(t)\right) = -\sum_{i=1}^n\left(u'_{\ell m}(t), \frac{\partial\theta_{\ell m}}{\partial x_i}(t)\right) + \sum_{i=1}^n\int_{\Gamma_1} u'_{\ell m}(t)\theta_{\ell m}(t)\nu_i d\Gamma$$

to obtain

$$\begin{aligned}&\frac{d}{dt}E_1(t) + \|\theta_{\ell m}(t)\|^2 + \mu(t)\int_{\Gamma_1}\alpha(x)(u'_{\ell m}(t))^2 d\Gamma \\ &+\sum_{i=1}^n\left(\frac{\partial\theta_{\ell m}}{\partial x_i}(t), u'_{\ell m}(t)\right) + \beta\int_{\Gamma_1}(\theta_{\ell m}(t))^2 d\Gamma \\ &\le \sum_{i=1}^n\int_{\Gamma_1} u'_{\ell m}(t)\theta_{\ell m}(t)\nu_i d\Gamma + \frac{|\mu'(t)|}{\mu(t)}E_1(t).\end{aligned}$$

By the Cauchy-Schwarz inequality it follows that

$$\sum_{i=1}^n\int_{\Gamma_1} u'_{\ell m}(t)\theta_{\ell m}(t)\nu_i d\Gamma \le \frac{n}{2\alpha_0\mu_0}\int_{\Gamma_1}(\theta_{\ell m}(t))^2 d\Gamma + \frac{\alpha_0\mu(t)}{2}\int_{\Gamma_1}(u'_{\ell m}(t))^2 d\Gamma,$$

and this yields

$$\begin{aligned}\tfrac{d}{dt}E_1(t) + \|\theta_{\ell m}(t)\|^2 + \mu(t)\tfrac{\alpha_0}{2}\textstyle\int_{\Gamma_1}(u'_{\ell m}(t))^2 d\Gamma + \left(\beta - \tfrac{n}{2\alpha_0\mu_0}\right)\int_{\Gamma_1}(\theta_{\ell m}(t))^2 d\Gamma \\ \le \tfrac{|\mu'(t)|}{\mu(t)}E_1(t)\,. \qquad (3.15)\end{aligned}$$

Integrating (3.15) over [0,t[, $0 \le t \le t_m$, using (2.1) and applying Gronwall inequality, we conclude that there is a positive constant $C > 0$, independent of ℓ and m, such that

$$E_1(t) + \int_0^t \|\theta_{\ell m}(s)\|^2 ds \le C. \qquad (3.16)$$

Then there exists a subsequence still denoted by $(u_{\ell m})_{m\in\mathbb{N}}$ and a subsequence still denoted by $(\theta_{\ell m})_{m\in\mathbb{N}}$, such that

$$(u_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^\infty_{loc}(0,\infty;V) \qquad (3.17)$$

$$(u'_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2(\Omega)) \qquad (3.18)$$

$$(\theta_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^2_{loc}(0,\infty;V). \qquad (3.19)$$

Estimate II. Differentiating in (3.14) with respect to t, taking $v = u''_{\ell m}(t)$ and $w = \theta'_{\ell m}(t)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}E_2(t) + \mu(t)\int_{\Gamma_1}\alpha(x)(u''_{\ell m}(t))^2 d\Gamma + \mu'(t)\int_{\Gamma_1}\alpha(x)u'_{\ell m}(t)u''_{\ell m}(t)\,d\Gamma \\
&+\|\theta'_{\ell m}(t)\|^2 + \beta\int_{\Gamma_1}(\theta'_{\ell m}(t))^2 d\Gamma \qquad (3.20)\\
&= \frac{1}{2}\mu'(t)\|u'_{\ell m}(t)\|^2 - \mu'(t)((u_{\ell m}(t), u''_{\ell m}(t))) + \sum_{i=1}^{n}\int_{\Gamma_1}\theta'_{\ell m}(t)u''_{\ell m}(t)\nu_i\,d\Gamma,
\end{aligned}$$

where

$$E_2(t) = \frac{1}{2}\left\{|u''_{\ell m}(t)|^2 + \mu(t)\|u'_{\ell m}(t)\|^2 + |\theta'_{\ell m}(t)|^2\right\}.$$

Put $v = \frac{\mu'(t)}{\mu(t)}u''_{\ell m}(t)$ in $(3.14)_1$, to obtain

$$\begin{aligned}
\mu'(t)((u_{\ell m}(t), u''_{\ell m}(t))) &= -\frac{\mu'(t)}{\mu(t)}|u''_{\ell m}(t)|^2 + \mu'(t)\int_{\Gamma_1}\alpha(x)u'_{\ell m}(t)u''_{\ell m}(t)d\Gamma \\
&\quad -\frac{\mu'(t)}{\mu(t)}\sum_{i=1}^{n}\left(\frac{\partial\theta_{\ell m}}{\partial x_i}(t), u''_{\ell m}(t)\right).
\end{aligned}$$

Replacing this last expression in (3.20) we obtain

$$\begin{aligned}
&\frac{d}{dt}E_2(t) + \mu(t)\int_{\Gamma_1}\alpha(x)(u''_{lm}(t))^2 d\Gamma + \|\theta'_{\ell m}(t)\|^2 + \beta\int_{\Gamma_1}(\theta'_{\ell m}(t))^2 d\Gamma \\
&= \frac{1}{2}\mu'(t)\|u'_{\ell m}(t)\|^2 + \frac{\mu'(t)}{\mu(t)}|u''_{\ell m}(t)|^2 + \frac{\mu'(t)}{\mu(t)}\sum_{i=1}^{n}\left(\frac{\partial\theta_{\ell m}}{\partial x_i}(t), u''_{\ell m}(t)\right) \\
&\quad + \sum_{i=1}^{n}\int_{\Gamma_1}\theta'_{\ell m}(t)u''_{\ell m}(t)\nu_i d\Gamma. \qquad (3.21)
\end{aligned}$$

Making use of the Cauchy-Schwarz inequality in the last two terms of the right-hand-side of (3.21), we obtain

$$\frac{\mu'(t)}{\mu(t)}\sum_{i=1}^{n}\left|\left(\frac{\partial\theta_{\ell m}}{\partial x_i}(t), u''_{\ell m}(t)\right)\right| \le \frac{1}{2}\frac{|\mu'(t)|}{\mu(t)}|u''_{\ell m}(t)|^2 + \frac{n}{2}\frac{|\mu'(t)|}{\mu(t)}\|\theta_{\ell m}(t)\|^2 \quad (3.22)$$

and

$$\sum_{i=1}^{n}\int_{\Gamma_1}\theta'_{\ell m}(t)u''_{\ell m}(t)\nu_i\,d\Gamma \le \frac{\mu_0\alpha_0}{2}\int_{\Gamma_1}(u''_{\ell m}(t))^2 d\Gamma + \frac{n}{2\mu_0\alpha_0}\int_{\Gamma_1}(\theta'_{\ell m}(t))^2 d\,\Gamma. \qquad (3.23)$$

Combining (3.21), (3.22) and (3.23) we obtain

$$\begin{aligned}
&\tfrac{d}{dt}E_2(t) + \mu(t)\tfrac{\alpha_0}{2}\textstyle\int_{\Gamma_1}(u''_{\ell m}(t))^2 d\Gamma + \|\theta'_{\ell m}(t)\|^2 + \left(\beta - \tfrac{n}{2\mu_0\alpha_0}\right)\textstyle\int_{\Gamma_1}(\theta'_{\ell m}(t))^2 d\Gamma \\
&\quad\le \tfrac{1}{2}\tfrac{|\mu'(t)|}{\mu_0}\mu(t)\|u'(t)\|^2 + \tfrac{3}{2}\tfrac{|\mu'(t)|}{\mu_0}|u''_{\ell m}(t)|^2 + \tfrac{n|\mu'(t)|}{2\mu_0}\|\theta_{\ell m}(t)\|^2. \qquad (3.24)
\end{aligned}$$

From (2.1) it follows that

$$\frac{d}{dt}E_2(t) + \|\theta'_{\ell m}(t)\|^2 + \leq 4\frac{|\mu'(t)|}{\mu_0}E_2(t) + \frac{n|\mu'(t)|}{2\mu_0}\|\theta_{\ell m}(t)\|^2.$$

To complete this estimate, we integrate the above inequality over [0,t], $t \leq T$. Now we show that $u''_{\ell m}(0)$ and $\theta'_{\ell m}(0)$ are bounded in $L^2(\Omega)$. For this end put $v = u''_{\ell m}(t)$, $w = \theta'_{\ell m}(t)$, and $t = 0$. Because of the choice of basis we have

$$\begin{aligned} &|u''_{\ell m}(0)|^2 \\ &\leq \left(\mu(0)|\Delta u^0_\ell| + \sum_{i=1}^n \left|\frac{\partial \theta^0_\ell}{\partial x_i}\right|\right)|u''_{\ell m}(0)| + \mu(0)\int_{\Gamma_1}\left(\frac{\partial u^0_\ell}{\partial \nu} + \alpha(x)u^1_\ell\right)u''_{\ell m}(0)d\Gamma \end{aligned}$$

and

$$|\theta'_{\ell m}(0)|^2 \leq \left(|\Delta\theta^0_\ell| + \sum_{i=1}^n \left|\frac{\partial u^1_\ell}{\partial x_i}\right|\right)|\theta'_{\ell m}(0)| + \int_{\Gamma_1}\left(\frac{\partial \theta^0_\ell}{\partial \nu} + \beta\theta^0_\ell\right)\theta'_{\ell m}(0)d\Gamma\,.$$

Since by hypothesis $\frac{\partial u^0_\ell}{\partial \nu} + \alpha(x)u^1_\ell = 0$ and $\frac{\partial \theta^0_\ell}{\partial \nu} + \beta\theta^0_\ell = 0$ in Γ_1, it follows that $(u''_{\ell m}(0))_{m\in\mathbb{N}}$ and $(\theta'_{\ell m}(0))_{m\in\mathbb{N}}$ are bounded in $L^2(\Omega)$. Consequently for a fixed ℓ,

$$(u'_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;V), \tag{3.25}$$
$$(u''_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)), \tag{3.26}$$
$$(\theta'_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)) \tag{3.27}$$
$$(\theta'_{\ell m})_{m\in\mathbb{N}} \text{ is bounded in } L^2_{\text{loc}}(0,\infty;V) \tag{3.28}$$

From (3.17)–(3.19) and (3.25)–(3.28), by induction and the diagonal process, we obtain subsequences, denoted with the same symbol as the original sequences, $(u_{\ell m_n})_{n\in\mathbb{N}}$ and $(\theta_{\ell m_n})_{n\in\mathbb{N}}$; and functions $u_\ell : \Omega\times]0,\infty[\longrightarrow \mathbb{R}$ and $\theta_\ell : \Omega\times]0,\infty[\longrightarrow \mathbb{R}$ such that:

$$u_{\ell m} \longrightarrow u_\ell \text{ weak star in } L^\infty_{\text{loc}}(0,\infty;V) \tag{3.29}$$
$$u'_{\ell m} \longrightarrow u'_\ell \text{ weak star in } L^\infty_{\text{loc}}(0,\infty;V) \tag{3.30}$$
$$u''_{\ell m} \longrightarrow u''_\ell \text{ weak star in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)) \tag{3.31}$$
$$u'_{\ell m} \longrightarrow u'_\ell \text{ weak star in } L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.32}$$
$$\theta_{\ell m} \longrightarrow \theta_\ell \text{ weakly in } L^2_{\text{loc}}(0,\infty;V) \tag{3.33}$$
$$\theta'_{\ell m} \longrightarrow \theta'_\ell \text{ weak star in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)) \tag{3.34}$$
$$\theta_{\ell m} \longrightarrow \theta_\ell \text{ weak star in } L^2_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)). \tag{3.35}$$

Next, we multiply both sides of (3.14) by $\psi \in \mathcal{D}(0,\infty)$ and integrate with respect to t. From (3.29)–(3.35), for all $v, w \in V^\ell_m$ we obtain

$$\int_0^\infty (u''_\ell(t),v)\psi(t)\,dt + \int_0^\infty \mu(t)((u_\ell(t),v))\psi(t)\,dt \tag{3.36}$$

$$+\int_0^\infty\int_{\Gamma_1}\alpha(x)u_\ell'(t)\,v\,\psi(t)\,d\Gamma\,dt+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial\theta_\ell}{\partial x_i}(t),v\right)\psi(t)\,dt=0,$$

$$\int_0^\infty(\theta_\ell',w)\psi(t)\,dt+\int_0^\infty((\theta_\ell(t),w))\psi(t)\,dt \tag{3.37}$$

$$+\beta\int_0^\infty\int_{\Gamma_1}\theta_\ell(t)w\psi(t)d\Gamma+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial u_\ell'}{\partial x_i}(t),w\right)\psi(t)\,dt=0.$$

Since $\{w_1^\ell,w_2^\ell,...\}$ is a basis of $V\cap H^2(\Omega)$, then by denseness it follows that the last two equalities are true for all v and w in $V\cap H^2(\Omega)$. Also notice that (3.17)–(3.19) and (3.25)–(3.28) hold for all $\ell\in\mathbb{N}$. Then by the same process used in obtaining of (3.29)–(3.35), we find diagonal subsequences denoted as the original sequences, $(u_{\ell_\ell})_{\ell\in\mathbb{N}}$ and $\theta_{\ell_\ell})_{\ell\in\mathbb{N}}$, and functions $u:\Omega\times]0,\infty[\longrightarrow\mathbb{R},\theta:\Omega\times]0,\infty[\longrightarrow\mathbb{R}$ such that:

$$u_\ell\longrightarrow u\ \text{ weak star in }\ L^\infty_{\rm loc}(0,\infty;V) \tag{3.38}$$
$$u_\ell'\longrightarrow u'\ \text{ weak star in }\ L^\infty_{\rm loc}(0,\infty;V) \tag{3.39}$$
$$u_\ell''\longrightarrow u''\ \text{ weak star in }\ L^\infty_{\rm loc}(0,\infty;L^2(\Omega)) \tag{3.40}$$
$$u_\ell'\longrightarrow u'\ \text{ weak star in }\ L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.41}$$
$$\theta_\ell\longrightarrow \theta\ \text{ weakly in }\ L^2_{\rm loc}(0,\infty;V) \tag{3.42}$$
$$\theta_\ell'\longrightarrow \theta'\ \text{ weak star in }\ L^\infty_{\rm loc}(0,\infty;L^2(\Omega)) \tag{3.43}$$
$$\theta_\ell\longrightarrow \theta\ \text{ weak star in }\ L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.44}$$

Taking limits in (3.36) and in (3.37), using the convergences showed in (3.38)–(3.44), and using the fact that $V\cap H^2(\Omega)$ is dense in V, we obtain that for all ψ in $\mathcal{D}(0,\infty)$ and $v,w\in V$,

$$\int_0^\infty(u''(t),v)\psi(t)\,dt+\int_0^\infty\mu(t)((u(t),v))\psi(t)\,dt \tag{3.45}$$

$$+\int_0^\infty\int_{\Gamma_1}\alpha(x)u'(t)v\psi(t)\,d\Gamma dt+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial\theta}{\partial x_i}(t),v\right)\psi(t)\,dt=0,$$

$$\int_0^\infty(\theta'(t),w)\psi(t)\,dt+\int_0^\infty((\theta(t),w))\psi(t)\,dt \tag{3.46}$$

$$+\beta\int_0^\infty\int_{\Gamma_1}\theta(t)w\psi(t)d\Gamma dt+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial u'}{\partial x_i}(t),w\right)\psi(t)\,dt=0\,.$$

Since $\mathcal{D}(\Omega)\subset V$, by (3.45) and (3.46) it follows that

$$u''-\mu\,\Delta u+\textstyle\sum_{i=1}^n\frac{\partial\theta}{\partial x_i}=0\ \text{ in }\ L^2_{\rm loc}(0,\infty;L^2(\Omega))\,, \tag{3.47}$$

$$\theta'-\Delta\theta+\textstyle\sum_{i=1}^n\frac{\partial u'}{\partial x_i}=0\ \text{ in }\ L^\infty_{\rm loc}(0,\infty;L^2(\Omega))\,. \tag{3.48}$$

Since $u\in L^\infty_{\rm loc}(0,\infty;V)$ and $\theta\in L^2_{\rm loc}(0,\infty;V)$, we take into account (3.47) and (3.48) to deduce that $\Delta u,\ \Delta\theta\in L^2_{\rm loc}(0,\infty;L^2(\Omega))$. Therefore

$$\frac{\partial u}{\partial\nu},\frac{\partial\theta}{\partial\nu}\in L^2_{\rm loc}(0,\infty;H^{-1/2}(\Gamma_1)) \tag{3.49}$$

Multiply (3.47) by $v\psi$ and (3.48) by $w\psi$ with v, $w \in V$ and $\psi \in \mathcal{D}(0,\infty)$. By integration and use of the Green's formula, we obtain

$$\begin{aligned} &\int_0^\infty (u''(t),v)\psi(t)\,dt + \int_0^\infty \mu(t)((u(t),v))\psi(t)\,dt \\ &-\int_0^\infty \langle \mu(t)\frac{\partial u}{\partial \nu}(t),v\rangle\psi(t)\,dt + \sum_{i=1}^n \int_0^\infty \left(\frac{\partial \theta}{\partial x_i}(t),v\right)\psi(t)\,dt = 0\,, \end{aligned} \tag{3.50}$$

$$\begin{aligned} &\int_0^\infty (\theta'(t),w)\psi(t)\,dt + \int_0^\infty ((\theta(t),w))\psi(t)\,dt \\ &-\int_0^\infty \langle \frac{\partial \theta}{\partial \nu}(t),w\rangle\psi(t)\,dt + \sum_{i=1}^n \int_0^\infty \left(\frac{\partial u'}{\partial x_i}(t),w\right)\psi(t)\,dt = 0, \end{aligned} \tag{3.51}$$

where $\langle .,. \rangle$ denotes the duality pairing of $H^{-1/2}(\Gamma_1)\times H^{1/2}(\Gamma_1)$.

Comparing (3.45) with (3.50) and (3.46) with (3.51), we obtain that for all ψ in $\mathcal{D}(0,\infty)$ and for all $v,w \in V$,

$$\int_0^\infty \langle \frac{\partial u}{\partial \nu}(t)+\alpha(x)u'(t),v\rangle\psi(t)\,dt = 0, \qquad \int_0^\infty \langle \frac{\partial \theta}{\partial \nu}(t)+\beta\theta(t),w\rangle\psi(t)\,dt = 0.$$

From (3.39), (3.44) and (3.49) it follows that

$$\begin{aligned} \tfrac{\partial u}{\partial \nu} + \alpha u' = 0 \ \text{ in } \ L^\infty_{\text{loc}}(0,\infty;H^{-1/2}(\Gamma_1)),\\ \tfrac{\partial \theta}{\partial \nu} + \beta\theta = 0 \ \text{ in } \ L^2_{\text{loc}}(0,\infty;H^{-1/2}(\Gamma_1)). \end{aligned}$$

Since $\alpha u' \in L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1))$ and $\beta\theta \in L^2_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1))$, it follows that

$$\frac{\partial u}{\partial \nu} + \alpha u' = 0 \quad \text{in} \quad L^2_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)) \tag{3.52}$$

$$\frac{\partial \theta}{\partial \nu} + \beta\,\theta = 0 \ \text{ in } \ L^2_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)). \tag{3.53}$$

To complete the proof of the Theorem 3.1, we shall show that u and θ are in $L^\infty_{\text{loc}}(0,\infty;H^2(\Omega))$. In fact, for all $T>0$ the pair $\{u,\theta\}$ is the solution to

$$\begin{aligned} -\Delta u = -\tfrac{1}{\mu}\left(u'' + \textstyle\sum_{i=1}^n \tfrac{\partial\theta}{\partial x_i}\right) \quad &\text{in} \quad \Omega\times]0,T[\\ -\Delta\theta = -\theta' - \frac{\partial u'}{\partial x_i} \quad &\text{in} \quad \Omega\times]0,T[\\ u=\theta=0 \quad &\text{on} \quad \Gamma_0\times]0,T[\\ \tfrac{\partial u}{\partial \nu} = -\alpha u' \quad &\text{on} \quad \Gamma_1\times]0,T[\\ \tfrac{\partial \theta}{\partial \nu} = -\beta\theta \quad &\text{on} \quad \Gamma_1\times]0,T[. \end{aligned} \tag{3.54}$$

In view of (3.40), (3.42) and (3.39) we have u'' and $\frac{\partial\theta}{\partial x_i}$ are in $L^\infty_{\text{loc}}(0,\infty;L^2(\Omega))$ and $\alpha u'$ is in $L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1))$. Thus by results on elliptic regularity, it follows that $u \in L^\infty_{\text{loc}}(0,\infty;V\cap H^2(\Omega))$. In the same manner it follows that $\theta \in L^\infty_{\text{loc}}(0,\infty;H^2(\Omega))$.

Uniqueness of the solution $\{u,\theta\}$ is showed by the standard energy method. The verification of the initial conditions is done through the convergences in (3.38)–(3.44). $\square$

Next, we establish a result on existence and uniqueness of global solutions.

Corollary 3.1 *Under the supplementary hypothesis $\mu' \in L^1(0,\infty)$, the pair of functions $\{u,\theta\}$ obtained by Theorem 3.1 satisfies*

$$\begin{gathered} u \in L^\infty(0,\infty; V\cap H^2(\Omega)),\quad u' \in L^\infty(0,\infty;V),\quad \theta \in L^\infty(0,\infty;V\cap H^2(\Omega)) \\ \tfrac{\partial u}{\partial \nu} + \alpha u' = 0 \text{ and } \tfrac{\partial \theta}{\partial \nu} + \beta\theta = 0 \text{ in } L^2(0,\infty;L^2(\Gamma_1)) \\ u(0)=u^0,\quad u'(0)=u^1 \text{ and } \theta(0)=\theta^0\,. \end{gathered}$$

4 Weak Solutions

In this section, we find a solution for the system (1.1)–(1.6) with initial data $u^0 \in V$, $u^1 \in L^2(\Omega)$ and $\theta^0 \in V$. To reach this goal we approximate u^0, u^1 and θ^0 by sequences of vectors in $V\cap H^2(\Omega)$, and we use the Theorem 3.1.

Theorem 4.1 *If $\{u^0,u^1,\theta^0\} \in V\times L^2(\Omega)\times V$, then for each real number $T>0$ there exists a unique pair of real functions $\{u,\theta\}$ such that:*

$$u \in C([0,T];V)\cap C^1([0,T];L^2(\Omega)),\quad \theta \in C([0,T];L^2(\Omega)) \tag{4.1}$$
$$u'' - \mu\,\Delta u + \textstyle\sum_{i=1}^n \frac{\partial\theta}{\partial x_i} = 0 \quad \text{in} \quad L^2(0,T;V') \tag{4.2}$$
$$\theta' - \Delta\theta + \textstyle\sum_{i=1}^n \frac{\partial u'}{\partial x_i} = 0 \quad \text{in} \quad L^2(0,T;V') \tag{4.3}$$
$$\tfrac{\partial u}{\partial\nu} + \alpha u' = 0 \quad \text{in} \quad L^2(0,T;L^2(\Gamma_1)) \tag{4.4}$$
$$\tfrac{\partial \theta}{\partial\nu} + \beta\theta = 0 \quad \text{in} \quad L^2(0,T;L^2(\Gamma_1)) \tag{4.5}$$
$$u(0)=u^0,\quad u'(0)=u^1,\quad \text{and } \theta(0)=\theta^0. \tag{4.6}$$

Proof. Let $(u_p^0)_{p\in\mathbb{N}}$, $(u_p^1)_{p\in\mathbb{N}}$, $(\theta_p^0)_{p\in\mathbb{N}}$ be sequences in $V\cap H^2(\Omega)$ such that

$$u_p^0 \longrightarrow u^0 \text{ in } V,\quad u_p^1 \longrightarrow u^1 \text{ in } L^2(\Omega) \text{ and } \theta_p^0 \longrightarrow \theta^0 \text{ in } V$$

with

$$\frac{\partial u_p^0}{\partial\nu} + \alpha(x)u_p^1 = 0 \text{ on } \Gamma_1 \quad\text{and}\quad \frac{\partial\theta_p^0}{\partial\nu} + \beta\theta_p^0 = 0 \text{ on } \Gamma_1.$$

Let $\{u_p,\theta_p\}_{p\in\mathbb{N}}$ be a sequence of strong solutions to (1.1)–(1.6) with initial data $\{u_p^0,u_p^1,\theta_p^0\}_{p\in\mathbb{N}}$. Using the same arguments as in the preceding section, we obtain the following estimates

$$(u_p)_{p\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;V) \tag{4.7}$$
$$(u_p')_{p\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;V) \tag{4.8}$$
$$(u_p')_{p\in\mathbb{N}} \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.9}$$

$$\left(\tfrac{\partial u_p}{\partial \nu}\right)_{p\in\mathbb{N}} \text{ is bounded in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.10}$$
$$(\theta_p)_{p\in\mathbb{N}} \text{ is bounded in } L^2_{\rm loc}(0,\infty;V) \tag{4.11}$$
$$(\theta_p)_{p\in\mathbb{N}} \text{ is bounded in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.12}$$
$$\left(\tfrac{\partial \theta_p}{\partial \nu}\right)_{p\in\mathbb{N}} \text{ is bounded in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)). \tag{4.13}$$

Note that (4.10) and (4.13) follow as a consequence of

$$\begin{aligned} &\tfrac{\partial u_p}{\partial \nu}+\alpha u_p'=0 \text{ in } L^\infty(0,\infty;H^{1/2}(\Gamma_1))\\ &\tfrac{\partial \theta_p}{\partial \nu}+\beta\theta_p=0 \text{ in } L^\infty(0,\infty;H^{1/2}(\Gamma_1)). \end{aligned} \tag{4.14}$$

From (4.7)–(4.13) there exist subsequences of $(u_p)_{p\in\mathbb{N}}$ and $(\theta_p)_{p\in\mathbb{N}}$, still denoted as the original sequences, and functions $u:\Omega\times]0,\infty[\to\mathbb{R}$, $\theta:\Omega\times]0,\infty[\to\mathbb{R}$, $\varphi_1 : \Gamma_1\times]0,\infty[\to \mathbb{R}$, $\varphi_2 : \Gamma_1\times]0,\infty[\to \mathbb{R}$, $\chi_1 : \Gamma_1\times]0,\infty[\to \mathbb{R}$, and $\chi_2 : \Gamma_1\times]0,\infty[\to\mathbb{R}$, such that

$$u_p\to u \text{ weak star in } L^\infty_{\rm loc}(0,\infty;V) \tag{4.15}$$
$$u_p'\to u' \text{ weak star in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega)) \tag{4.16}$$
$$u_p'\to \varphi_1 \text{ weakly in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.17}$$
$$\frac{\partial u_p}{\partial \nu}\to \varphi_2 \text{ weakly in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.18}$$
$$\theta_p\to \theta \text{ weakly in } L^2_{\rm loc}(0,\infty;V) \tag{4.19}$$
$$\theta_p\to \chi_1 \text{ weakly in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)) \tag{4.20}$$
$$\frac{\partial \theta_p}{\partial \nu}\to \chi_2 \text{ weakly in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)). \tag{4.21}$$

Moreover, from Theorem 3.1,

$$u_p''-\mu\Delta u_p+\textstyle\sum_{i=1}^n \frac{\partial\theta_p}{\partial x_i}=0 \text{ in } L^\infty_{loc}(0,\infty;L^2(\Omega)), \tag{4.22}$$
$$\theta_p'-\Delta\theta_p+\textstyle\sum_{i=1}^n \frac{\partial u_p'}{\partial x_i}=0 \text{ in } L^\infty_{loc}(0,\infty;L^2(\Omega))\,. \tag{4.23}$$

Multiplying (4.22) and (4.23) by $v\psi$ and $w\phi$ respectively, with v and w in V and ϕ in $\mathcal{D}(0,\infty)$, we deduce the equalities

$$\begin{aligned} &-\int_0^\infty (u_p'(t),v)\phi'(t)dt+\int_0^\infty \mu(t)((u_p(t),v))\phi(t)dt\\ &+\int_0^\infty\int_{\Gamma_1}\alpha(x)u_p'(t)v\phi(t)d\Gamma dt+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial\theta_p}{\partial x_i}(t),v\right)\phi dt=0\\ &-\int_0^\infty(\theta_p(t),w)\phi'(t)dt+\int_0^\infty((\theta_p(t),w))dt\\ &+\beta\int_0^\infty\int_{\Gamma_1}\theta_p(t)w\phi(t)d\Gamma dt+\sum_{i=1}^n\int_0^\infty\left(\frac{\partial u_p'}{\partial x_i}(t),w\right)\phi(t)dt=0. \end{aligned}$$

Taking the limit, as $p \longrightarrow \infty$, from (4.15)–(4.21) we conclude that

$$
\begin{aligned}
&-\int_0^\infty (u'(t),v)\phi'(t)dt + \int_0^\infty \mu(t)((u(t),v))\phi(t) \qquad (4.24)\\
&+\int_0^\infty \int_{\Gamma_1} \alpha(x)u'(t)v\phi(t)d\,\Gamma dt + \sum_{i=1}^n \int_0^\infty \left(\frac{\partial\theta}{\partial\nu}(t),v\right)\phi(t)dt = 0\\
&-\int_0^\infty (\theta(t),w)\phi'(t)dt + \int_0^\infty ((\theta(t),w))\phi(t)dt \qquad (4.25)\\
&+\beta\int_0^\infty \int_{\Gamma_1} \theta(t)w\phi(t)d\,\Gamma dt + \sum_{i=1}^n \int_0^\infty \left(\frac{\partial u'}{\partial x_i},w\right)\phi(t)dt = 0.
\end{aligned}
$$

In view of (4.24) and (4.25), for v and $w \in \mathcal{D}(\Omega)$, we obtain

$$
\begin{gathered}
u'' - \mu\Delta u + \textstyle\sum_{i=1}^n \frac{\partial\theta}{\partial x_i} = 0 \ \text{ in } \ H^{-1}_{loc}(0,\infty;L^2(\Omega))\\
\theta' - \Delta\theta + \displaystyle\sum_{i=1}^n \frac{\partial u'}{\partial x_i} = 0 \ \text{ in } \ H^{-1}_{loc}(0,\infty;L^2(\Omega)).
\end{gathered} \tag{4.26}
$$

As shown in M. Milla Miranda [7], from (4.8) follows that for $T > 0$

$$
u''_p \longrightarrow u'' \ \text{ weakly in } \ H^{-1}(0,T;L^2(\Omega)). \tag{4.27}
$$

Thus, from (4.19), (4.22) and (4.27) we conclude that

$$
\Delta u_p \longrightarrow \Delta u \ \text{ weakly in } \ H^{-1}(0,T;L^2(\Omega)). \tag{4.28}
$$

Furthermore, from (4.15) and (4.28) we obtain $\frac{\partial u}{\partial\nu}$ in $H^{-1}(0,T;H^{-1/2}(\Gamma_1))$ and

$$
\frac{\partial u_p}{\partial\nu} \to \frac{\partial u}{\partial\nu} \ \text{ weakly in } \ H^{-1}(0,T;H^{-1/2}(\Gamma_1)). \tag{4.29}
$$

To prove that $\varphi_1 = u'$ and $\varphi_2 = \frac{\partial u}{\partial\nu}$, we use (4.18) and the fact that

$$
\frac{\partial u_p}{\partial\nu} \to \varphi_2 \ \text{ weakly in } \ H^{-1}(0,T;H^{1/2}(\Gamma_1)). \tag{4.30}
$$

Whence we conclude that $\varphi_2 = \frac{\partial u}{\partial\nu}$ is in $L^2(0,T;L^2(\Gamma_1))$, for all $T > 0$. Also from (4.15), cf. M. Milla Miranda [7], we get

$$
u'_p \longrightarrow u' \ \text{ weakly in } \ H^{-1}(0,T;H^{1/2}(\Gamma_1)); \tag{4.31}
$$

and from (4.17) and (4.31) we have $u' = \varphi_1$ in $L^\infty(0,T;H^{1/2}(\Gamma_1))$.

Next, we shall prove that $\chi_1 = \theta$ and $\chi_2 = \frac{\partial\theta}{\partial\nu}$. In fact, from

$$
\begin{gathered}
\frac{\partial u'_p}{\partial x_i} \to \frac{\partial u}{\partial x_i} \ \text{ weakly in } \ H^{-1}(0,T;L^2(\Omega))\\
\theta'_p \to \theta' \ \text{ weakly in } \ H^{-1}(0,T;V)
\end{gathered} \tag{4.32}
$$

and (4.30) it follows that

$$\Delta\theta_p \longrightarrow \Delta\theta \text{ weakly in } H^{-1}(0,T;L^2(\Omega)). \tag{4.33}$$

From (4.19) and (4.33) it results that

$$\frac{\partial\theta_p}{\partial\nu} \longrightarrow \frac{\partial\theta}{\partial\nu} \text{ weakly in } H^{-1}(0,T;H^{-1/2}(\Gamma_1)).$$

On the other hand, by (4.21)

$$\frac{\partial\theta_p}{\partial\nu} \longrightarrow \chi_2 \text{ weakly in } H^{-1}(0,T;H^{-1/2}(\Gamma_1)),$$

whence we conclude that $\frac{\partial\theta}{\partial\nu} = \chi_2$. We deduce that $\chi_1 = \theta$ in $L^2(0,T;H^{1/2}(\Gamma_1))$ through of the convergences showed in (4.19) and (4.20). Therefore we obtain

$$\begin{aligned} &\frac{\partial u}{\partial\nu} + \alpha u' = 0 \text{ in } L^2(0,T;L^2(\Gamma_1)) \\ &\frac{\partial\theta}{\partial\nu} + \beta\theta = 0 \text{ in } L^2(0,T;L^2(\Gamma_1)). \end{aligned} \tag{4.34}$$

To prove (4.2) and (4.3) we remark that for all $v,\ w \in V$,

$$\begin{aligned} |\langle -\Delta u, v\rangle| &\le \|u\|.\|v\| + \left\|\tfrac{\partial u}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_1)}.\|v\|_{H^{1/2}(\Gamma_1)}, \\ |\langle -\Delta\theta, v\rangle| &\le \|\theta\|.\|w\| + \left\|\tfrac{\partial\theta}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_1)}.\|w\|_{H^{1/2}(\Gamma_1)} \end{aligned}$$

and by continuity of the trace operator we deduce to inequalities:

$$|\langle -\Delta u, v\rangle| \le C(u)\|v\| \text{ and } |\langle -\Delta\theta, w\rangle| \le C(\theta)\|w\|,$$

whence for all $T > 0$ we obtain that

$$-\Delta u \in L^2(0,T;V') \quad \text{and} \quad -\Delta\theta \in L^2(0,T;V'). \tag{4.35}$$

So, by (4.24), (4.25), (4.35) and Green's formula, for all ψ in $\mathcal{D}(0,T)$, for all v and w in V we get

$$\begin{aligned} &-\int_0^T (u'(t),v)\psi'(t)dt + \int_0^T \mu(t)\langle -\Delta u(t), v\rangle\psi(t)dt \\ &+\sum_{i=1}^n \int_0^T \left(\frac{\partial\theta}{\partial x_i}(t), v\right)\psi(t)dt = 0 \\ &-\int_0^T (\theta(t),w)\phi'(t)dt + \int_0^T \langle -\Delta\theta(t), w\rangle\psi(t)dt \\ &+\sum_{i=1}^n \int_0^T \left(\frac{\partial u'}{\partial x_i}(t), w\right)\psi(t)dt = 0\,. \end{aligned}$$

From these two inequalities and (4.35), we obtain that for each $T > 0$

$$\begin{aligned} u'' - \mu\Delta u + \textstyle\sum_{i=1}^{n} \frac{\partial\theta}{\partial x_i} = 0 \ \text{ in } \ L^2(0,T;V') \\ \theta' - \Delta\theta + \textstyle\sum_{i=1}^{n} \frac{\partial u'}{\partial x_i} = 0 \ \text{ in } \ L^2(0,T;V') \end{aligned}$$

The regularity in (4.1) follows from $\{u_p, \theta_p\}$ being a Cauchy sequence. The initial data considerations follow from the analysis of the Galerkin approximation. The uniqueness of the weak solution is proved by the method of Lions Magenes [6], see also Visik-Ladyzhenskaya [11]. □

Now, we give a result which assures the existence and uniqueness of a weak global solution for (1.1)–(1.6).

Corollary 4.1 *Under the supplementary hypothesis $\mu' \in L^1(0,\infty)$, the pair of functions $\{u, \theta\}$ obtained by Theorem 4.1 satisfies the following properties:*

$$\begin{gathered} u \in L^\infty(0,\infty;V), \quad \theta \in L^\infty(0,\infty;L^2(\Omega)) \\ \tfrac{\partial u}{\partial \nu} + \alpha u' = 0 \ \textit{ and } \ \tfrac{\partial\theta}{\partial\nu} + \beta\theta = 0 \ \textit{ in } \ L^2(0,\infty;L^2(\Gamma_1)) \\ u(0) = u^0, \ \ u'(0) = u^1 \quad \textit{and} \quad \theta(0) = \theta^0 \,. \end{gathered}$$

5 Asymptotic Behavior

This section concerns the behavior of the solutions obtained in the preceding sections, as $t \to +\infty$. First note that for strong solutions and weak solutions to (1.1)–(1.6), the energy

$$E(t) = \frac{1}{2}\left\{\mu(t)\|u(t)\|^2 + |u'(t)|^2 + |\theta(t)|^2\right\}. \tag{5.1}$$

does not increase. In fact, we can easily see that

$$\begin{aligned} E'(t) \quad = \quad & \frac{\mu'(t)}{2}\|u(t)\|^2 - \mu(t)\int_{\Gamma_1} \alpha(x)(u'(t))^2\,d\Gamma - \|\theta(t)\|^2 \\ & -\beta\int_{\Gamma_1}(\theta(t))^2\,d\Gamma - \sum_{i=1}^{n}\int_{\Gamma_1} u'(t)\theta(t)\nu_i\,d\Gamma\,. \end{aligned}$$

Also observe that

$$-\sum_{i=1}^{n}\int_{\Gamma_1} u'(t)\theta(t)\nu_i\,d\Gamma \le \frac{\mu(t)}{2}\int_{\Gamma_1}\alpha(x)(u'(t))^2\,d\Gamma + \frac{n}{2\mu(t)}\int_{\Gamma_1}\frac{1}{\alpha(x)}(\theta(t))^2\,d\Gamma\,.$$

Because $\mu'(t) \le 0$ and the hypothesis (2.1), we can conclude that

$$E'(t) \le -\frac{\mu(t)}{2}\int_{\Gamma_1}\alpha(x)(u'(t))^2\,d\Gamma - \|\theta(t)\|^2. \tag{5.2}$$

To estimate $E(t)$ we put $\alpha(x) = m(x).\nu(x)$ and use the representation

$$\Gamma_0 = \{x \in \Gamma;\ m(x).\nu(x) \le 0\}, \quad \Gamma_1 = \{x \in \Gamma;\ \ m(x).\nu(x) > 0\}\,,$$

where $m(x)$ is the vectorial function $x - x^0$, for $x \in \mathbb{R}^n$ and "." denotes scalar product in $\mathbb{R}^n$. We also use

$$R(x^0) = \|m\|_{L^\infty(\Omega)}\,, \tag{5.3}$$

and positive constants δ_0, δ_1, k such that

$$|v|^2 \le \delta_0 \|v\|^2, \quad \text{for all } \ v \in V \tag{5.4}$$

$$\|v\|^2 \le \delta_1 \|v\|^2_{V\cap H^2(\Omega)}, \quad \text{for all } \ v \in V \cap H^2(\Omega) \tag{5.5}$$

$$\textstyle\int_{\Gamma_1} (m.\nu) v^2 \, d\Gamma \le k\|v\|^2, \quad \text{for all } \ v \in V. \tag{5.6}$$

Theorem 5.1 *If* $\{u^0, u^1, \theta^0\} \in V \times L^2(\Omega) \times V$, $\mu \in W^{1,\infty}(0,\infty)$ *with* $\mu'(t) \le 0$ *on* $]0,\infty[$, *then there exists a positive constant* ω *such that*

$$E(t) \le 3E(0)e^{-\omega t}, \quad \textit{for all } \ t \ge 0. \tag{5.7}$$

Proof. As a first step, we consider the strong solution. Let

$$\rho(t) = 2(u'(t), m.\nabla u(t)) + (n-1)(u'(t), u(t)). \tag{5.8}$$

Then

$$|\rho(t)| \le (n-1)|u(t)|^2 + n|u'(t)|^2 + R^2(x^0)\|u(t)\|^2. \tag{5.9}$$

Let ε_1, ε_2, ε be positive real numbers such that

$$\varepsilon_1 \le \min\left\{\frac{1}{4n}, \frac{\mu_0}{12nR^2(x^0) + 12n^3\delta_0}\right\} \tag{5.10}$$

$$\varepsilon_2 \le \min\left\{\frac{1}{2\left(R^2(x^0) + \dfrac{1}{\mu_0} + 6kn^2\right)}, \frac{2}{\delta_0}\right\} \tag{5.11}$$

$$\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}\,. \tag{5.12}$$

Also let the perturbed energy given by

$$E_\varepsilon(t) = E(t) + \varepsilon\rho(t). \tag{5.13}$$

Then from (5.13), (5.4), and (5.9) we get

$$E_\varepsilon(t) \le E(t) + \left(\varepsilon n\delta_0 + \varepsilon R^2(x^0)\right)\|u(t)\|^2 + \varepsilon n|u'(t)|^2,$$

whence by (5.12) it follows that

$$E_\varepsilon(t) \le E(t) + \varepsilon_1\left(n\delta_0 + R^2(x^0)\right)\|u(t)\|^2 + \varepsilon_1 n|u'(t)|^2.$$

By (5.1) and (5.10) we obtain $E_\varepsilon \le \frac{3}{2}E(t)$. On the other hand, using similar arguments, from (5.9) and (5.13) we deduce that $\frac{1}{2}E(t) \le E_\varepsilon$. In summary,

$$\frac{1}{2}E(t) \le E_\varepsilon \le \frac{3}{2}E(t), \quad \text{for all } t \ge 0. \tag{5.14}$$

To estimate $E'_\varepsilon(t)$ we differentiate $\rho(t)$,

$$\begin{aligned} \rho'(t) &= 2(u''(t), m.\nabla(t)) + 2(u'(t), m.\nabla u'(t)) \\ &\quad +(n-1)(u''(t), u(t)) + (n-1)|u'(t)|^2 . \end{aligned} \tag{5.15}$$

Since $u'' = \mu\Delta u - \sum_{i=1}^n \frac{\partial\theta}{\partial x_i}(t)$ we have

$$\begin{aligned} \rho'(t) &= 2\mu(t)(\Delta u(t), m.\nabla u(t)) - 2\sum_{i=1}^n \left(\frac{\partial\theta}{\partial x_i}(t), m.\nabla u(t)\right) \\ &\quad +2\,(u'(t), m.\nabla u'(t)) + (n-1)\mu(t)(\Delta u(t), u(t)) \\ &\quad -(n-1)\sum_{i=1}^n \left(\frac{\partial\theta}{\partial x_i}(t), u(t)\right) + (n-1)|u'(t)|^2. \end{aligned} \tag{5.16}$$

our next objective is to find bounds for the right-hand-side terms of the equation above.

Remark 5.1 *For all $v \in V \cap H^2(\Omega)$,*

$$2\,(\Delta v, m.\nabla v) \le (n-2)\|v\|^2 + R^2(x^0)\int_{\Gamma_1} \frac{1}{m.\nu}\left|\frac{\partial v}{\partial\nu}\right|^2 d\,\Gamma\,. \tag{5.17}$$

In fact, the Rellich's identity, see V. Komornik and E. Zuazua [4], gives

$$2\,(\Delta v, m.\nabla v) = (n-2)\|v\|^2 - \int_\Gamma (m.\nu)|\nabla v|^2 d\,\Gamma + 2\int_\Gamma \frac{\partial v}{\partial\nu} m.\nabla v\, d\,\Gamma. \tag{5.18}$$

Note that

$$\begin{aligned} -\int_\Gamma (m.\nu)|\nabla v|^2\, d\,\Gamma &= -\int_{\Gamma_0} (m.\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 d\,\Gamma - \int_{\Gamma_1} (m.\nu)|\nabla v|^2 d\,\Gamma \\ &\le -\int_{\Gamma_0} (m.\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 d\,\Gamma\,, \end{aligned} \tag{5.19}$$

because $\frac{\partial v}{\partial x_i} = \nu_i \frac{\partial v}{\partial\nu}$ on Γ_0 and $m.\nu > 0$ on Γ_1. Also note that

$$2\int_\Gamma \frac{\partial v}{\partial\nu} m.\nabla v\, d\,\Gamma = 2\int_{\Gamma_0} (m.\nu)\left(\frac{\partial v}{\partial\nu}\right)^2 d\,\Gamma + 2\int_{\Gamma_1} \frac{\partial v}{\partial\nu} m.\nabla v\, d\,\Gamma, \tag{5.20}$$

and by (5.3)

$$\begin{aligned} 2\int_{\Gamma_1} \frac{\partial v}{\partial\nu} m.\nabla v\, d\,\Gamma &\le 2\int_{\Gamma_1} \left|\frac{\partial v}{\partial\nu}\right| R(x^0)|\nabla v|\, d\,\Gamma \\ &\le R^2(x^0)\int_{\Gamma_1} \frac{1}{m.\nu}\left(\frac{\partial v}{\partial\nu}\right)^2 d\,\Gamma + \int_{\Gamma_1} (m.\nu)|\nabla v|^2 d\,\Gamma. \end{aligned}$$

This inequality with (5.20) yields

$$
\begin{aligned}
&2\int_\Gamma \frac{\partial v}{\partial \nu} m.\nabla v \, d\Gamma \qquad (5.21)\\
&\leq \; 2\int_{\Gamma_0} (m.\nu)\left(\frac{\partial v}{\partial \nu}\right)^2 d\Gamma + R^2(x^0)\int_{\Gamma_1} \frac{1}{m.\nu}\left(\frac{\partial v}{\partial \nu}\right)^2 d\Gamma + \int_{\Gamma_1} (m.\nu)|\nabla v|^2 d\Gamma.
\end{aligned}
$$

Combining (5.18), (5.19), and (5.21), we come to the inequality

$$
\begin{aligned}
2(\Delta v, m.\nabla v) \;\; &\leq \;\; (n-2)\|v\|^2 + \int_{\Gamma_0} (m.\nu)\left(\frac{\partial v}{\partial \nu}\right)^2 d\Gamma \\
&\quad + R^2(x^0)\int_{\Gamma_1} \frac{1}{m.\nu}\left(\frac{\partial v}{\partial \nu}\right)^2 d\Gamma.
\end{aligned}
$$

Recall that $m.\nu \leq 0$ on Γ_0; therefore, (5.17) follows. Now, we shall analyze each term in (5.17).

Analysis of $2\mu(t)(\Delta u(t), m.\nabla u(t))$**:** Thanks to Remark 5.1 and (3.5) we have

$$
2\mu(t)(\Delta u(t), m.\nabla u(t)) \leq \mu(t)(n-2)\|u(t)\|^2 + \mu(t)R^2(x^0)\int_{\Gamma_1} (m.\nu)(u'(t))^2 d\Gamma. \tag{5.22}
$$

Analysis of $-2\sum_{i=1}^n \left(\frac{\partial\theta}{\partial x_i}(t), m.\nabla u(t)\right)$**:**

$$
\begin{aligned}
-2\sum_{i=1}^n \left(\frac{\partial\theta}{\partial x_i}(t), m.\nabla u(t)\right) \;\; &\leq \;\; 2\sum_{i=1}^n \left|\frac{\partial\theta}{\partial x_i}(t)\right| R(x^0)\|u(t)\| \\
&\leq \;\; \sum_{i=1}^n \frac{6nR^2(x^0)}{\mu_0}\left|\frac{\partial\theta}{\partial x_i}(t)\right|^2 + \sum_{i=1}^n \frac{1}{6n}\mu_0\|u(t)\|^2.
\end{aligned}
$$

Thus

$$
-2\sum_{i=1}^n \left(\frac{\partial\theta}{\partial x_i}(t), m.\nabla u(t)\right) \leq \frac{6nR^2(x^0)}{\mu_0}\|\theta(t)\|^2 + \frac{\mu(t)}{6}\|u(t)\|^2. \tag{5.23}
$$

Analysis of $2(u'(t), m.\nabla u'(t))$**:**

$$
\begin{aligned}
2(u'(t), m.\nabla u'(t)) \;\; &= \;\; 2\int_\Omega u'(t) m_j \frac{\partial u'}{\partial x_j}(t)\, dx \\
&= \;\; \int_\Omega m_j \frac{\partial (u')^2}{\partial x_j}(t)\, dx \\
&= \;\; -\int_\Omega \frac{\partial m_j}{\partial x_j}(u'(t))^2 dx + \int_{\Gamma_1} (m_j\nu_j)(u'(t))^2 d\Gamma \qquad (5.24)\\
&= \;\; -n|u'(t)|^2 + \int_{\Gamma_1} (m.\nu)(u'(t))^2 d\Gamma\,.
\end{aligned}
$$

Analysis of $\mu(t)(n-1)(\Delta u(t), u(t))$: Applying Green's theorem and (3.5), we get

$$\mu(t)(n-1)(\Delta u(t), u(t)) = -\mu(t)(n-1)\left[\|u(t)\|^2 + \int_{\Gamma_1}(m.\nu)u'(t)u(t)\,d\Gamma\right].$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}\mu(t)(n-1)(\Delta u(t), u(t)) &\leq -\mu(t)(n-1)\|u(t)\|^2\\ &\quad +6k\mu(t)(n-1)^2\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma\\ &\quad +\frac{\mu(t)}{6k}\int_{\Gamma_1}(m.\nu)(u(t))^2 d\Gamma,\end{aligned}$$

and by (5.6)

$$\begin{aligned}\mu(t)(n-1)(\Delta u(t), u(t)) &\leq -\mu(t)(n-1)\|u(t)\|^2\\ &\quad +6k\mu(t)(n-1)^2\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma + \frac{\mu(t)}{6}\|u(t)\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\mu(t)(n-1)(\Delta u(t), u(t)) &\leq -\mu(t)(n-\frac{7}{6})\|u(t)\|^2 \qquad (5.25)\\ &\quad +6k\mu(t)(n-1)^2\int_{\Gamma_1}(m.\nu)(u'(t)^2 d\Gamma.\end{aligned}$$

Analysis of $-(n-1)\left(\sum_{i=1}^n \frac{\partial\theta}{\partial x_i}, u(t)\right)$:

$$\begin{aligned}-(n-1)\sum_{i=1}^n\left(\frac{\partial\theta}{\partial x_i}(t), u(t)\right) &\leq (n-1)\sum_{i=1}^n\left|\frac{\partial\theta}{\partial x_i}(t)\right||u(t)|\\ &\leq \frac{6n\delta_0(n-1)^2}{\mu_0}\|\theta(t)\|^2 + \sum_{i=1}^n\frac{\mu_0}{6n\delta_0}|u(t)|^2,\end{aligned}$$

whence by (5.4)

$$-(n-1)\sum_{i=1}^n\left(\frac{\partial\theta}{\partial x_i}(t), u(t)\right) \leq \frac{6n\delta_0(n-1)^2}{\mu_0}\|\theta(t)\|^2 + \frac{\mu(t)}{6}\|u(t)\|^2. \qquad (5.26)$$

Using (5.22)–(5.6) in (5.17) we conclude that

$$\begin{aligned}\rho'(t) &\leq -\frac{\mu(t)}{2}\|u(t)\|^2 + \left[\frac{6nR^2(x^0)+6n^3\delta_0}{\mu_0}\right]\|\theta(t)\|^2 - |u'(t)|^2\\ &\quad +\mu(t)\left[R^2(x^0)+\frac{1}{\mu_0}+6kn^2\right]\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma. \qquad (5.27)\end{aligned}$$

Combining (5.2), (5.13) and (5.27), we get

$$\begin{aligned} E'_\varepsilon(t) &\leq -\|\theta(t)\|^2 - \frac{\mu(t)}{2}\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma \\ &\quad -\frac{\varepsilon}{2}\mu(t)\|u(t)\|^2 + \varepsilon\left[\frac{6nR^2(x^0)+6n^3\delta_0}{\mu_0}\right]\|\theta(t)\|^2 - \varepsilon|u'(t)|^2 \\ &\quad +\varepsilon\mu(t)\left[R^2(x^0)+\frac{1}{\mu_0}+6kn^2\right]\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma\,. \end{aligned}$$

Then, by (5.4) and (5.12), it results that

$$\begin{aligned} E'_\varepsilon(t) &\leq -\|\theta(t)\|^2 - \frac{\mu(t)}{2}\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma \\ &\quad -\frac{\varepsilon}{2}\mu(t)\|u(t)\|^2 + \varepsilon_1\left[\frac{6nR^2(x^0)+6n^3\delta_0}{\mu_0}\right]\|\theta(t)\|^2 - \varepsilon|u'(t)|^2 \\ &\quad +\varepsilon_2\mu(t)\left[R^2(x^0)+\frac{1}{\mu_0}+6kn^2\right]\int_{\Gamma_1}(m.\nu)(u'(t))^2 d\Gamma. \end{aligned}$$

Using (5.10) and (5.11) we obtain

$$E'_\varepsilon(t) \leq -\frac{1}{2}\|\theta(t)\|^2 - \frac{\varepsilon}{2}\mu(t)\|u(t)\|^2 - \frac{\varepsilon}{2}|u'(t)|^2.$$

Also, from (5.4), (5.11) and (5.12) we obtained

$$E'_\varepsilon(t) \leq -\frac{1}{\delta_0}|\theta(t)|^2 - \frac{\varepsilon}{2}\mu(t)\|u(t)\|^2 - \frac{\varepsilon}{2}|u'(t)|^2.$$

By (5.11) and (5.12) we have $-\frac{\varepsilon}{2} \geq -\frac{1}{\delta_0}$, then

$$\begin{aligned} E'_\varepsilon(t) &\leq -\frac{\varepsilon}{2}|\theta(t)|^2 - \frac{\varepsilon}{2}\mu(t)\|u(t)\|^2 - \frac{\varepsilon}{2}|u'(t)|^2 \\ &= -\frac{\varepsilon}{2}E(t)\,. \end{aligned} \tag{5.28}$$

From (5.14), we obtain $E'_\varepsilon(t) \leq -\frac{2\varepsilon}{3}E_\varepsilon(t)$. In turn this inequality implies $E_\varepsilon(t) \leq E_\varepsilon(0)e^{-\frac{2}{3}\varepsilon t}$. From (5.14), we obtain exponential decay for strong solutions

$$E(t) \leq 3E(0)e^{-\frac{2}{3}\varepsilon t}, \quad \text{for all } t \geq 0.$$

Remark Using a denseness argument, we prove the same behavior for weak solutions.

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