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A RESONANCE PROBLEM FOR THE P-LAPLACIAN IN $\mathbb{R}^N$

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Abstract. We show the existence of a weak solution for the problem

$$-\Delta_p u = \lambda_1 h(x)|u|^{p-2}u + a(x)g(u) + f(x), \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

where, $2 < p < N$, λ_1 is the first eigenvalue of the p-Laplacian on $\mathcal{D}^{1,p}(\mathbb{R}^N)$ relative to the radially symmetric weight $h(x) = h(|x|)$. In this problem, $g(s)$ is a bounded function for all $s \in \mathbb{R}$, $a \in L^{(p^*)'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f \in L^{(p^*)'}(\mathbb{R}^N)$. To establish an existence result, we employ the Saddle Point Theorem of Rabinowitz [9] and an improved Poincaré inequality from an article of Alziary, Fleckinger and Takáč [2].

1. Introduction

Resonance problems for divergence operators have been of interest since the 197O's. For the ordinary Laplacian on bounded domains there are a number of classical papers and some recent papers explore resonant problems in $\mathbb{R}^N$. For the p-Laplacian, a family of resonant problems in $\mathbb{R}^N$ has been studied just recently in [2] among others. In this paper, we study the family of p-Laplacian equations:

$$-\Delta_p u = \lambda h(x)|u|^{p-2}u + a(x)g(u) + f(x) \quad \text{in } \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $2 < p < N$, $(N \geqslant 3)$; $f \in L^{(p^*)'}(\mathbb{R}^N)$, $p^* = \frac{Np}{N-p}$ and $(p^*)'$ denotes the conjugate of p^*; $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function ($|g(s)| \leqslant M$) for all $s \in \mathbb{R}$; the function $h \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $h \geqslant 0$ a.e. is a weight function and $a \in L^{(p^*)'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. As usual, the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int |\nabla u|^p\Big)^{1/p}.$$

¿From here and henceforth the integrals and all the spaces are taken over $\mathbb{R}^N$ unless otherwise specified.

The term resonance is well known in the literature, and refers to the case in which λ is an eigenvalue of the problem

$$\begin{gathered} -\Delta_p u = \lambda h(x)|u|^{p-2}u, \\ u \in \mathcal{D}^{1,p}. \end{gathered} \tag{1.2}$$

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In [1], Allegreto et al. show that the eigenvalue problem (1.2) possesses a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant, \dots$ and a corresponding sequence of eigenfunctions $\{\varphi_j\}$, where φ_1 can be chosen to be positive a.e.. Moreover, we have the Rayleigh quotient characterization:

$$\lambda_1 = \inf\Big\{ \int |\nabla u|^p : u \in \mathcal{D}^{1,p} \text{ with } \int h|u|^p = 1 \Big\}. \tag{1.3}$$

We consider the function φ_1 to be normalized; i.e., $\int h|\varphi_1|^p = 1$ and we decompose any function $u \in \mathcal{D}^{1,p}$ as a direct sum

$$\begin{gathered} u = \alpha\varphi_1 + w \text{ where} \\ \alpha = \int h|\varphi_1|^{p-2}\varphi_1 u \text{ and } \int h|\varphi_1|^{p-2}\varphi_1 w = 0. \end{gathered} \tag{1.4}$$

Hence, we introduce the spaces

$$\begin{gathered} V \stackrel{\text{def}}{=} \operatorname{span}\{\varphi_1\}, \\ W \stackrel{\text{def}}{=} \big\{ w \in \mathcal{D}^{1,p} : \int h|\varphi_1|^{p-2}\varphi_1 w = 0 \big\} \end{gathered} \tag{1.5}$$

In order to prove our main result we use some of the results introduced by Alziary, Fleckinger and Takáč in [2] where the cases $1 < p < 2$ and $2 < p < N$ are treated separately. The case $2 < p < N$ requires the use of the so called "Improved Poincaré inequality" ([2, lemma 3.7 p.8]):

$$\begin{gathered} \int |\nabla u|^p - \lambda_1 \int h|u|^p \geqslant c\Big(|\alpha|^{p-2} \int |\nabla\varphi_1|^{p-2}|\nabla w|^2 + \int |\nabla w|^p \Big), \\ c > 0, \quad 2 < p < N \end{gathered} \tag{1.6}$$

where h satisfies the hypothesis:

(H) The function h is radially symmetric, $h(x) = h(|x|)$. There exist constants $\delta > 0$ and $C > 0$ such that

$$0 < h(r) \leqslant \frac{C}{(1+r)^{p+\delta}} \quad \text{for almost all } 0 \leqslant r < \infty \ \ (r = |x|). \tag{1.7}$$

Following [2] we define:

$$\begin{gathered} \mathcal{C}_\gamma \stackrel{\text{def}}{=} \big\{ u = \alpha\varphi_1 + w \in \mathcal{D}^{1,p} : \|w\| \leqslant \gamma|\alpha| \big\}, \\ \mathcal{C}'_\gamma \stackrel{\text{def}}{=} \big\{ u \in \mathcal{D}^{1,p} : \|w\| \geqslant \gamma|\alpha| \big\}, \text{ for } 0 < \gamma < \infty, \\ \mathcal{C}'_\infty \stackrel{\text{def}}{=} \big\{ u \in \mathcal{D}^{1,p} : |\alpha| = 0 \big\} \end{gathered}$$

The next two lemmas are borrowed from [2] (Lemma 6.2 in page 18, and Lemma 6.3 in page 19). They play important roles in the proof of our main result.

Lemma 1.1. *If h satisfies (H), $1 < p < N$, and $0 < \gamma \leqslant \infty$ then*

$$\lambda_1 < \Lambda_\gamma \stackrel{def}{=} \inf\Big\{ \frac{\int |\nabla u|^p}{\int h|u|^p} : u \in \mathcal{C}'_\gamma \setminus \{0\} \Big\}. \tag{1.8}$$

For the case in which $\|w\|/|\alpha|$ is small the following lemma is needed.

Lemma 1.2. *If h satisfies (H) and $2 \leqslant p < N$, then*

$$\lambda_1 < \tilde{\Lambda} \stackrel{def}{=} \liminf_{\|\phi\|\to 0,\ \phi\in W} \Big\{ \frac{\int |\nabla(\varphi_1 + \phi)|^p}{\int h|\varphi_1 + \phi|^p} : u \in \mathcal{C}'_\gamma \setminus \{0\} \Big\}. \tag{1.9}$$

The main result of this paper is the following.

Theorem 1.3. *Let h satisfy (H), $\lambda = \lambda_1$, $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be bounded, $G(s) = \int_0^s g(t)dt$ and $a \in L^{(p^*)'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If*

$$\lim_{|t|\to\infty} \Big\{ \int a(x)G(t\varphi_1) + t \int f(x)\varphi_1 \Big\} = +\infty, \tag{1.10}$$

then, problem (1.1) has a weak solution for $2 < p < N$.

Note that condition (1.10) is a Landesman-Lazer type condition (see [6]). Related problems for the p-Laplacian near Resonance had been studied by To Fu Ma et al. [8], with a different settings for the function $F(x, u) := a(x)g(u)$. The study for the case $p = 2$, without the hypothesis (H), is treated by López and Rumbos in [7]. The existence of weak solutions for (1.1) is an extension of previous results for bounded domains and the ordinary Laplacian by Ahmad, Lazer and Paul [3].
Remark: Even though the hypothesis (H) is required for the proof of our main result, several steps use only that $h \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

2. Variational Setting

The solutions to (1.1) are the critical points of the functional

$$J_\lambda(u) = \frac{1}{p}\int |\nabla u|^p - \frac{\lambda}{p}\int h(x)|u|^p - \int aG(u) - \int fu \tag{2.1}$$

where $G(s) = \int_0^s g(t)dt$, $s \in \mathbb{R}$. It is known (see for instance [5]) that the functional J_λ belongs to $\mathcal{C}^1(\mathcal{D}^{1,p}, \mathbb{R}^N)$ for $u \in \mathcal{D}^{1,p}$ with Fréchet derivative given by:

$$\langle J'_\lambda(u), v\rangle = \int |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda \int h|u|^{p-2}uv - \int ag(u)v - \int fv \tag{2.2}$$

for all $u, v \in \mathcal{D}^{1,p}$.

To prove theorem 1.3 we use the Minimax Methods introduced by Rabinowitz [9]. We recall here for the convenience of the reader some previous definitions and theorems.
Palais-Smale condition. Suppose that E is a real Banach space. A functional $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted $(PS)_c$, if any sequence $(u_n) \subset E$ for which

(i) $I(u_n) \to c$ as $n \to \infty$ and
(ii) $I'(u_n) \to 0$ as $n \to \infty$,

possesses a convergent subsequence. If $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the $(PS)_c$ for every $c \in \mathbb{R}$, we say that (u_n) satisfies the (PS) condition. Any sequence for which (i) and (ii) hold is called a $(PS)_c$ sequence for I.

Now we establish a preliminary result.

Proposition 2.1. *Let $J_\lambda : \mathcal{D}^{1,p} \to \mathbb{R}$ be defined as 2.1 where $\lambda \in \mathbb{R}$. Suppose that g is a continuous function with $|g(s)| \leqslant M$ for all $s \in \mathbb{R}$, $f \in L^{(p^*)'}(\mathbb{R}^N)$, $2 < p < N$, $h \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $h \geqslant 0$ a.e. and $a \in L^{(p^*)'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then if every $(PS)_c$ sequence for J_λ is bounded, J_λ satisfies the $(PS)_c$ condition.*

Proof. In the first place we note that if h satisfies (H) then $h \in L^{N/p}(\mathbb{R}^N)$. In fact,

$$\int h(x)^{N/p}dx = \int_0^\infty h(r)^{N/p} r^{N-1} dr \leqslant \int_0^\infty \big(\frac{C}{(1+r)^{p+\delta}}\big)^{\frac{N}{p}} r^{N-1} dr < \infty.$$

Let (u_n) be a $(PS)_c$ sequence for J_λ. Thus, by assumption (u_n) is bounded, therefore there exists a subsequence, which we also denote by (u_n) such that $u_n \to u$ weakly in $\mathcal{D}^{1,p}$ as $n \to \infty$, in particular we have

$$\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \to \int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi, \quad \forall \varphi \in \mathcal{D}^{1,p}. \tag{2.3}$$

Passing to a subsequence if necessary, we see that $\int |\nabla(u_n-u)|^p \to 0$ as $n \to \infty$. Now since (u_n) satisfies the $(PS)_c$ condition, $\lim_{n\to\infty}\langle J'_\lambda(u_n), \varphi\rangle = 0$. That is,

$$\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi - \lambda \int h|u_n|^{p-2}u_n\varphi - \int ag(u_n)\varphi - \int f\varphi = o(1) \tag{2.4}$$

as $n \to \infty$. On the other hand, by weak convergence, we obtain

$$\lim_{n\to\infty} \int |\nabla u|^{p-2}\nabla u \cdot \nabla(u_n - u) = 0.$$

For $p > 2$ (see [1] inequality (7) p.237 and subsequent inequalities)

$$\begin{aligned} \int |\nabla u_n - \nabla u|^p \leqslant C \Big\{ \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_n - u) \Big\} \\ \times \Big(\int |\nabla u_n|^p + \int |\nabla u|^p \Big). \end{aligned} \tag{2.5}$$

Thus it is sufficient to show that $\lim_{n\to\infty} \int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u) = 0$. To this aim, taking $\varphi = u_n - u$, in (2.4) we have

$$\begin{aligned} &\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u) \\ &= \lambda \int h|u_n|^{p-2}u_n(u_n - u) + \int ag(u_n)(u_n - u) + \int f(u_n - u) + o(1) \end{aligned} \tag{2.6}$$

as $n \to \infty$. For the first integral in the right hand side, using the Hölder's inequality we have

$$\Big| \int h|u_n|^{p-2}u_n(u_n - u) \Big| \leqslant \Big(\int h|u_n|^p \Big)^{1/p'} \Big(\int h|u_n - u|^p \Big)^{1/p}.$$

Noting that $h \in L^{N/p}(\mathbb{R}^N) = L^{(p^*/p)'}(\mathbb{R}^N)$, for $1 \leqslant q < p^*$ the functional $u \mapsto \int h|u|^q$ is weakly continuous in $\mathcal{D}^{1,p}$ (see [4, Prop. 2.1 p. 826]). Consequently,

$$\lim_{n\to\infty} \int h|u_n|^{p-2}u_n(u_n - u) = 0. \tag{2.7}$$

For the integral $\int ag(u_n)|u_n - u|$ we consider the ball $B_r(0)$. Since a, g are bounded we have

$$\Big| \int_{B_r(0)} ag(u_n)(u_n - u) \Big| \leqslant C \int_{B_r(0)} |u_n - u| \to 0 \quad \text{as } n \to \infty,$$

since $u_n \to u$ strongly in $L^1(B_r(0))$ due to the Relich-Kondrachov theorem. Now, together with the assumption that u_n and g are bounded, we obtain

$$\Big| \int_{\mathbb{R}^N \setminus B_r(0)} ag(u_n)|u_n - u| \Big| \leqslant C \Big(\int_{\mathbb{R}^N \setminus B_r(0)} |a|^{\frac{Np}{N-p}} \Big)^{\frac{N+p}{Np}}$$

So, by taking r big enough it follows that

$$\limsup_{n\to\infty} \Big| \int ag(u_n)|u_n - u| \Big| \leqslant C\varepsilon$$

For arbitrary ε. Finally, since $f \in L^{(p^*)'}(\mathbb{R}^N)$ we can use similar arguments as above to show that $\lim_{n\to\infty} \int f(u_n - u) = 0$. □

3. Proof of Theorem 1.3

In this section we consider the problem

$$\begin{gathered} -\Delta_p u = \lambda_1 h(x)|u|^{p-2}u + a(x)g(u) + f(x) \\ u \in \mathcal{D}^{1,p} \end{gathered} \tag{3.1}$$

where h satisfies (H). To prove the main theorem of this section we require the Saddle Point Theorem of Rabinowitz [9], which we introduce here for the convenience of the reader.

Theorem 3.1 (Saddle Point Theorem)**.** *Let $E = V \oplus W$, where E is a real Banach space and $V \neq \{0\}$ is finite dimensional. Suppose $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the (PS) condition and*

- (I1) *there is a constant α and a bounded neighborhood D of 0 in V such that $I\big|_{\partial D} \leqslant \alpha$, and*
- (I2) *there is a constant $\beta > \alpha$ such that $I|_W \geqslant \beta$.*

Then, I possesses a critical value $c \geqslant \beta$. Moreover c can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in\overline{D}} I(h(u)),$$

where $\Gamma = \{h \in \mathcal{C}(\overline{D}, E) : h = id \text{ on } \partial D\}$.

Now, we can show the existence of weak solutions for J_{λ_1}.

Proof of Theorem 1.3. First, we show that the functional J_{λ_1} corresponding to problem (3.1) satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$, and thereafter we verify that J_{λ_1} satisfies the other hypotheses of the Theorem 3.1.

Let (u_n) be a $(PS)_c$ sequence for the functional J_{λ_1}. We claim that (u_n) is bounded. For each $n \in \mathbb{N}$ write

$$u_n \stackrel{\text{def}}{=} v_n + w_n = \alpha_n\varphi_1 + w_n \quad \text{with } \alpha_n \in \mathbb{R} \text{ and } w_n \in W.$$

Since (u_n) is a $(PS)_c$ sequence we have $|J_{\lambda_1}(u_n)| < c$, i.e.

$$\Big|\frac{1}{p}\int |\nabla u_n|^p - \frac{\lambda_1}{p}\int h|u_n|^p - \int aG(u_n) - \int fu_n\Big| \leqslant C_1. \tag{3.2}$$

By inequality (1.6), we have

$$\frac{c}{p}\int |w_n|^p \leqslant \big|\frac{1}{p}\int |\nabla u_n|^p - \frac{\lambda_1}{p}\int h|u_n|^p\big| \tag{3.3}$$

with $c > 0$. By standard calculations (see for instance [7, p.16]), we have

$$\big|\int a(G(v_n + w_n) - G(v_n))\big| \leqslant M\int a|w_m| \leqslant C_2\|w_n\|. \tag{3.4}$$

Consequently, using (3.2), (3.3) and (3.4) we have

$$\big|\int aG(v_n) + \int fv_n\big| \leqslant C_1 + C_2\|w_n\| + \frac{c}{p}\|w_n\|^p. \tag{3.5}$$

So, given that $\int aG(v_n) + \int f v_n \to \infty$ as $\|v_n\| = |\alpha_n| \to \infty$, we have shown that (v_n) is bounded if (w_n) is bounded. We show now that (w_n) is bounded. In fact, note that

$$\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w_n \geqslant \|u_n\|^p - \int |\nabla u_n|^{p-2} u_n \cdot \nabla v_n. \tag{3.6}$$

On the other hand, since $\langle J'_{\lambda_1}(u_n), v_n\rangle \xrightarrow{n} 0$, there exists m_0 such that if $n \geqslant m_0$ then,

$$\Big| \int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w_n - \int z_n \cdot w_n \Big| \leqslant C\|w_n\|, \tag{3.7}$$

where $z_n = \lambda_1 h |u_n|^{p-2}u_n + a g(u_n) w_n + f$. Adding and subtracting $\lambda_1 \int h|u_n|^{p-2}u_n \cdot v_n$ and $\int ag(u_n)v_n + \int f(x)v_n$, and substituting (3.6) in (3.7)

$$\begin{aligned} \|u_n\|^p - \lambda_1 \int h|u_n|^p &\leqslant C\|w_n\| + \langle J'_{\lambda_1}(u_n), v_m\rangle + \int ag(u_n)v_n + \int f(x)v_n \\ &\leqslant C\|w_n\| + \langle J'_{\lambda_1}(u_n), v_m\rangle + C'|\alpha_n| \end{aligned} \tag{3.8}$$

Again, since $J'_{\lambda_1} \to 0$ as $n \to \infty$, there exist m_1 such that if $n \geqslant m_1$ then $\langle J'_{\lambda_1}(u_n), v_m\rangle \leqslant C\|v_n\| = C|\alpha_n|$, taking $n \geqslant \max\{m_0, m_1\}$

$$\|u_n\|^p - \lambda_1 \int h|u_n|^p \leqslant C\|w_n\| + C'|\alpha_n|. \tag{3.9}$$

Now fix $\gamma > 0$, and suppose that $(u_n) \in \mathcal{C}'_\gamma$ for all n. Then we have, $|\alpha_n| \leqslant (1/\gamma)\|w_n\|$ and $\int h|u_n|^p \leqslant (1/\Lambda_\gamma)\int |\nabla u_n|^p$. Thus, by Lemma 1.1

$$\big(1 - \frac{\lambda_1}{\Lambda_\gamma}\big)\|u_n\|^p \leqslant C\|w_n\| \tag{3.10}$$

Since the projection $u \mapsto w$ is bounded in $\mathcal{D}^{1,p}$ we obtain

$$\|w_n\|^p \leqslant C_\gamma \|w_n\|, \tag{3.11}$$

given that $\lambda_1/\Lambda_\gamma < 1$ by Lemma 1.1.

Hence by Lemma 1.1, $\Lambda_\gamma > \lambda_1$; therefore, (w_n) is bounded if $(u_n) \in \mathcal{C}'_\gamma$. Now, set $\gamma_n = \|w_n\|/|\alpha_n|$ and define

$$\gamma \stackrel{\text{def}}{=} \liminf_n \gamma_n.$$

We have two cases: (i) $\gamma \in (0, \infty]$ and (ii) $\gamma = 0$. By the above argument, if $\gamma \in (0,\infty]$ then (w_n) is bounded and the proof is concluded. If $\gamma = 0$, take $\varepsilon > 0$ arbitrarily small, such that $\|w_n\| \leqslant \varepsilon|\alpha_n|$. Using inequality (3.9), Lemma 1.2 with $\phi = \phi_n \stackrel{\text{def}}{=} (\|w_n\|/|\alpha_n|) \cdot w_n/\|w_n\|$, and the fact that the projection $u \mapsto \alpha$ is bounded in $\mathcal{D}^{1,p}$ we obtain

$$\begin{gathered} |u_n|^p\big(1 - \frac{\lambda_1}{\tilde\Lambda}\big) \leqslant C\varepsilon|\alpha_n| + C'\|v_n\|, \\ |\alpha_n|^p \leqslant c_\gamma|\alpha_n|. \end{gathered}$$

Therefore, $|\alpha_n|$ is bounded, and since $\|w_n\| \leqslant \varepsilon|\alpha_n|$ we have that (u_n) is bounded as wanted.

To verify the geometric hypotheses of the Saddle Point Theorem we note that since λ_1 is isolated (see [1]) we have

$$\lambda_2 \stackrel{\text{def}}{=} \inf\big\{\|w\|^p : w \in W, \int h|w|^p = 1\big\}, \tag{3.12}$$

which satisfies $\lambda_1 < \lambda_2$. As a consequence of (3.12) we have

$$\int |\nabla w|^p \geqslant \lambda_2 \int h|w|^p, \quad \forall w \in W. \tag{3.13}$$

Now, if $w \in W$,

$$\int |\nabla w|^p - \lambda_1 \int h|w|^p \geqslant \big(1 - \frac{\lambda_1}{\lambda_2}\big). \tag{3.14}$$

Moreover, since $|g(s)| \leqslant M$ for all $s \in \mathbb{R}$, we have that for all $w \in \mathcal{D}^{1,p}$,

$$\Big| \int aG(w)\Big| \leqslant M \int |a||w| \leqslant C\|w\|.$$

Therefore, J_{λ_1} is bounded from below on W; i.e. (I2) in Theorem 3.1 holds.

Finally, if $v \in V$ we have

$$J_{\lambda_1}(v) = -\int aG(v) - \int fv.$$

Since $\int aG(v) + \int fv \to \infty$ as $\|v\| \to \infty$ by (1.10) and, therefore, (I1) in the Saddle Point Theorem also holds. Hence, J_{λ_1} has a critical point and the proof is concluded. $\square$

Remark. Suppose $\lim_{s\to\infty} g(s) = g_\infty$ and $\lim_{s\to-\infty} g(s) = g_{-\infty}$ exist. Then, if $g_\infty > 0$ and $g_{-\infty} < 0$, $G(s) = \int_0^s g(t)dt \to \infty$ as $|s| \to \infty$. Consequently, by L' Hôspital's rule, the Lebesgue dominated convergence theorem and the fact that $\varphi_1 > 0$ a.e. in $\mathbb{R}^N$ we have that

$$\lim_{|t|\to\infty} \frac{1}{t} \int a(x)G(t\varphi_1) = \lim_{|t|\to\infty} \int ag(t\varphi_1)\varphi_1 = \begin{cases} g_\infty \int a\varphi_1 & \text{as } t \to \infty, \\ g_{-\infty} \int a\varphi_1 & \text{as } t \to -\infty. \end{cases}$$

Thus, the condition (1.10) in the resonance Theorem 1.3 holds if

$$g_\infty \int a\varphi_1 + \int f\varphi_1 > 0 \quad \text{and} \quad g_{-\infty} \int a\varphi_1 + \int f\varphi_1 < 0,$$

or

$$g_{-\infty} \int a\varphi_1 < -\int f\varphi_1 < g_\infty \int a\varphi_1. \tag{3.15}$$

This is the original Landesman-Lazer condition in [6] for the case of resonance around the first eigenvalue.

It can be shown that if

$$g_{-\infty} < g(s) < g_{+\infty} \quad \text{for all } s \in \mathbb{R},$$

then (3.15) is necessary and sufficient for the solvability of (3.1). If $g_{-\infty} = g_{+\infty}$, then the Landesman-Lazer condition (3.15) cannot hold, and if $g_{-\infty}$ and $g_{+\infty}$ are both zero, then condition (1.10) might not hold in general.

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