

Bifurcations for semilinear elliptic equations with convex nonlinearity *

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Abstract

We investigate the exact number of positive solutions of the semilinear Dirichlet boundary value problem $\Delta u + f(u) = 0$ on a ball in \mathbb{R}^n where f is a strictly convex C^2 function on $[0, \infty)$. For the one-dimensional case we classify all strictly convex C^2 functions according to the shape of the bifurcation diagram. The exact number of positive solutions may be 2, 1, or 0, depending on the radius. This full classification is due to our main lemma, which implies that the time-map cannot have a minimum. For the case $n > 1$ we prove that for sublinear functions there exists a unique solution for all R . For other convex functions estimates are given for the number of positive solutions depending on R . The proof of our results relies on the characterization of the shape of the time-map.

1 Introduction

We investigate the exact number of positive solutions of the semilinear boundary value problem

$$\begin{aligned} \Delta u + f(u) &= 0 \\ u|_{\partial B_R} &= 0 \end{aligned} \tag{1}$$

where $B_R \subset \mathbb{R}^n$ is the ball centered at the origin with radius R and f is a strictly convex C^2 function on $[0, \infty)$ (i.e. $f'' \geq 0$ and f'' does not vanish identically on any interval). According to the well-known result of Gidas, Ni and Nirenberg [9] every positive solution of (1) is radially symmetric, hence it satisfies

$$\begin{aligned} ru''(r) + (n-1)u'(r) + rf(u(r)) &= 0 \\ u'(0) = 0, u(R) &= 0; \end{aligned} \tag{2}$$

further, there holds

$$u'(r) < 0 \quad (0 < r < R). \tag{3}$$

Our aim is to determine the bifurcation diagram of positive solutions versus the radius R for every strictly convex C^2 function, and to classify these functions according to the shape of the bifurcation diagram.

* 1991 Mathematics Subject Classifications: 35J60.

Key words and phrases: semilinear elliptic equations, time-map, bifurcation diagram.

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Submitted June 22, 1999. Published October 18, 1999.

A brief survey of some known results

The number of positive solutions of (1) has been widely studied for different types of f on a general bounded domain using variational and topological methods, see e.g. [2, 3, 18, 24]. In the general case, even for convex functions f [1, 3], the exact number of positive solutions is hard to determine and is only possible under restrictive assumptions on the function f .

If the domain is a ball, then ODE techniques can be applied to get more information on the number of solutions [5, 22]. The essence of the frequently used shooting method is the investigation of the initial value problem

$$\begin{aligned} ru''(r, p) + (n-1)u'(r, p) + rf(u(r, p)) &= 0 \\ u(0, p) = p, \quad u'(0, p) &= 0. \end{aligned} \tag{4}$$

It has a unique C^2 solution $u(\cdot, p)$ for any $p > 0$ [21]. The time-map, associating the first zero of $u(\cdot, p)$ to p , determines the number of positive solutions of (2).

There are several results concerning the number of positive solutions of (2) for special convex functions f , but the problem is far from being solved for a general convex function. Joseph and Lundgren [10] determined the number of solutions in the case $f(u) = e^u$ and $f(u) = (1 + \alpha u)^\beta$ for $\alpha, \beta > 0$. They used Emden's transformation, because for these special functions (2) can be transformed to a two-dimensional autonomous system and phase plane techniques can be applied. It can be shown by Pohozaev's formula [18] that for $f(u) = u^p$ there exists a positive solution (on a star-shaped domain) if and only if $p < (n+2)/(n-2)$, and by Emden's transformation or Sturm's theorem that it is unique (on the ball). Several authors studied the case $f(u) = u^p + \lambda u^q$, especially when p is near to the critical value $(n+2)/(n-2)$ [4, 6, 7, 8, 17]. McLeod's result [16] on the uniqueness of positive solutions is valid for a certain class of convex functions, typically for $f(u) = u^p - u$ (if $1 < p < (n+2)/(n-2)$), see also [11].

A more detailed description of the exact number of positive solutions is available in the case $n = 1$, because then (2) is a Hamiltonian system on the phase plane. However, according to our knowledge, the whole classification of the possible bifurcation diagrams for the convex functions f is not known even in the one-dimensional case. Schaaf's book [19] is an excellent summary of the results on the time-map for large function classes for $n = 1$. Certain convex functions (e.g. polynomials with not purely imaginary complex roots) are not contained in these classes. In [20] and [25] the monotonicity of the time-map is investigated for cubic-like functions f . For convex positive functions f Laetsch [12, 13] gave a detailed description, also for the boundary condition of the third type. In these works the specialities of the case $n = 1$, i.e. the phase plane and the integral formula for the time-map, are applied, hence neither the results nor the methods can be applied for the case $n > 1$.

The main results and the method

For the one-dimensional case we classify all strictly convex C^2 functions according to the shape of the bifurcation diagram and thus determine the exact number of positive solutions. We prove that for asymptotically linear or superlinear functions (at infinity) the bifurcation diagram is determined by the behaviour of f at infinity and at 0, further, by the sign changes of f . We essentially prove that there exist numbers $0 \leq R_\infty < R_{sup} \leq \infty$, depending on $\lim_{u \rightarrow \infty} \frac{f(u)}{u}$ and the number of zeros of f , respectively, which divide the interval $(0, \infty)$ according to the number of positive solutions of (1). As it is sketched in Table 1, these numbers are influenced by the sign of $f(0)$. (The exact formulation of these results is given in Theorems 1 and 2.)

| | $R \leq R_\infty$ | $R_\infty < R < R_{sup}$ | $R_{sup} < R$ |
|---------------|-------------------|--------------------------|---------------|
| $f(0) > 0$ | 1 | 2 | 0 |
| $f(0) \leq 0$ | 0 | 1 | 0 |

Table 1. Number of positive solutions

The above table shows the number of positive solutions of (1) in the superlinear and asymptotically linear case for $n = 1$. In the superlinear case $R_\infty = 0$. If f has two zeros or a double zero, then $R_{sup} = \infty$.

For sublinear functions there exists a unique solution for all R ; this result is considered in the general case $n > 1$. This full classification in one dimension is due to our main lemma (Lemma 1), which implies that the time-map cannot have a minimum. Otherwise, where it is possible, we prove our results for any dimension $n \geq 1$. Especially, we avoid the application of the integral formula for the time-map, except for determining the limit of the time map at infinity, where the result is indeed not true for all n (see e.g. [10]).

For the case $n > 1$ we prove that for sublinear functions there exists a unique solution for all R (Theorem 3). For other convex functions estimates are given for the number of positive solutions (Theorem 4).

The proof of our results relies on the characterization of the shape of the time-map. The following three characteristic properties determine the exact number of positive solutions of (2):

- the domain of the time-map
- the limit of the time-map at the boundary points of its domain
- the monotonicity of the time-map on the maximal subintervals of its domain.

The main tools during our studies are Sturm's theorems, Pohozaev's formula and the application of auxiliary functions, e.g. the Hamiltonian of the 1-dimensional case as a Liapunov function for the case $n > 1$.

Definition 1 The *time-map* associated to the above initial value problem is the following function T :

$$T(p) = \min\{r > 0 : u(r, p) = 0\} ; \quad D(T) = \{p > 0 : \exists r > 0 \ u(r, p) = 0\}.$$

The function T is determined by the implicit equation

$$u(T(p), p) \equiv 0 \tag{5}$$

and the assumption $u(r, p) > 0$ if $r \in [0, T(p))$.

Differentiating (5) one gets the following equations for the derivatives of T :

$$\partial_r u(T(p), p)T'(p) + \partial_p u(T(p), p) \equiv 0, \tag{6}$$

$$\partial_r^2 u(T(p), p)T'(p)^2 + 2\partial_{rp} u(T(p), p)T'(p) + \partial_r u(T(p), p)T''(p) + \partial_p^2 u(T(p), p) \equiv 0. \tag{7}$$

Differentiating (4) with respect to p and introducing the notations $h(r, p) = \partial_p u(r, p)$ and $z(r, p) = \partial_p^2 u(r, p)$ we get

$$\begin{aligned} rh''(r, p) + (n-1)h'(r, p) + rf'(u(r, p))h(r, p) &= 0 \tag{8} \\ h(0, p) &= 1, \quad h'(0, p) = 0; \end{aligned}$$

$$\begin{aligned} rz''(r, p) + (n-1)z'(r, p) + rf'(u(r, p))z(r, p) + rf''(u(r, p))h^2(r, p) &= 0 \tag{9} \\ z(0, p) &= 0, \quad z'(0, p) = 0. \end{aligned}$$

Differentiating (4) with respect to r and introducing the notation $v(r, p) = \partial_r u(r, p)$ we get

$$\begin{aligned} v''(r, p) + \frac{n-1}{r}v'(r, p) + (f'(u(r, p)) - \frac{n-1}{r^2})v(r, p) &= 0 \tag{10} \\ v(0, p) &= 0, \quad v'(0, p) = -\frac{f(p)}{n}. \end{aligned}$$

Using the above notations, (6) is written as

$$v(T(p), p)T'(p) + h(T(p), p) \equiv 0. \tag{11}$$

2 General results on the shape of the time-map

In this section we formulate results on the domain $D(T)$ of the time-map, the limit of the time-map at the boundary points of $D(T)$ and the monotonicity of the time-map. These will enable us to cover the possible cases of the shape of the time-map for strictly convex f .

The results are proved for all dimensions n where it is possible. Most of the one-dimensional results exploit the first main result (Lemma 1). This is given in the first subsection.

Main lemma in one dimension

Lemma 1 *Let $n = 1$, $f \in C^2$ be strictly convex. If $T'(p) = 0$, then $T''(p) < 0$.*

PROOF. Let $T'(p) = 0$. Then (11) implies $h(T(p), p) = 0$. This means that $h > 0$ in the interval $[0, T(p))$. Otherwise, using Sturm comparison, v should have a root between the previous root of h and $T(p)$ since v and h are two solutions of the same equation (8). This cannot occur by (3).

From this we obtain $z(T(p), p) < 0$ for the solution of (9). Namely, assume first that $f''(p) > 0$. Then $z''(0) = -f''(p) < 0$ implies $z < 0$ in some right neighbourhood of 0. Assume that $z(r_1) = 0$ for some $r_1 \in (0, T(p)]$. Then comparison of the equations

$$h'' + f'(u)h = 0 \quad \text{and} \quad z'' + \left(f'(u) + \frac{f''(u)h^2}{z} \right) z = 0$$

shows that h should have a root in $(0, r_1)$. I.e. $z < 0$ in $(0, T(p)]$.

If $f''(p) = 0$ then we can use the above argument for all other values \tilde{p} where $f''(\tilde{p}) > 0$ holds to show that $z(r, \tilde{p}) < 0$ until the first root of $h(r, \tilde{p})$. Then by continuity this extends to p .

Finally, $z(T(p), p) < 0$ yields $T''(p) < 0$, using (7) (and $v(T(p), p) \leq 0$). \diamond

Corollary 1 *If $n = 1$ and $f \in C^2$ is strictly convex then T may have at most one local extremum in any subinterval of $D(T)$, namely, a local maximum.*

The domain $D(T)$ of the time-map

In this subsection we do not assume first that f is convex. Necessary conditions for $p \in D(T)$ can be obtained using the Hamiltonian of the 1-dimensional case

$$H(r) := \frac{u'(r)^2}{2} + F(u(r)) \tag{12}$$

(where $F(u) := \int_0^u f$) as a Liapunov function, because $H'(r) = -\frac{n-1}{r}u'^2(r)$ is negative (see (3)). Namely:

- Proposition 1** 1. *If $p \in D(T)$, then $f(p) > 0$.*
 2. *$D(T) \subset \{p > 0 : F(p) > F(s) \forall s \in I_p \text{ and } f(p) \neq 0\} =: \mathcal{P}_f$ where $I_p := (0, p)$ if both $n = 1$ and $f(0) < 0$ and $I_p := [0, p)$ otherwise.*

PROOF. 1. If $f(p) = 0$, then $u(r) \equiv p$. In case $f(p) < 0$ we have $u''(0) > 0$, hence u initially increases. This implies $u(r) \geq u(0)$ for all $r > 0$, because $u(r_0) = u(0)$ would imply $H(r_0) \geq H(0)$. This is impossible for $n > 1$, and in the case $n = 1$ there holds $u''(r_0) > 0$. (We note that the end of the proof also follows from (3).)

2. Let $p \in D(T)$. According to 1. we have $f(p) > 0$. Let $s \in (0, p)$, then there exists $r \in (0, T(p))$ with $u(r) = s$. Then

$$F(s) = F(u(r)) = H(r) - \frac{u'(r)^2}{2} < H(r) \leq H(0) = F(p).$$

It is easy to see that if $n > 1$ or $f(0) \geq 0$, then the above inequality also holds for $s = 0$ (i.e. $r = T(p)$). \diamond

The sufficient condition may be a difficult question (cf. e.g. Pohozaev's identity [18]). The characterization of $D(T)$ can only be given for $n = 1$. Besides this, we mention special cases for $n > 1$.

Proposition 2 *If $n = 1$ then $D(T) = \mathcal{P}_f$.*

PROOF. Let $p \in \mathcal{P}_f$, $u(r) = u(r, p)$. We argue by contradiction, so assume that $u(r) > 0$ for all $r > 0$. Using that H is constant and $p \in \mathcal{P}_f$, (12) shows that $u'(r) < 0$ ($r > 0$). Hence there exists $\lim_{\infty} u =: c \in [0, p]$, and (12) implies $F(c) = F(p)$. In case $f(0) \geq 0$ this contradicts $F(s) < F(p)$ ($s < p$). In case $f(0) < 0$ $F(s) < F(p)$ implies $c = 0$. From (4) $\lim_{\infty} u'' > 0$, which contradicts $\lim_{\infty} u = 0$. \diamond

If f is strictly convex, then \mathcal{P}_f clearly consists of at most two intervals (around 0 and ∞). First we consider the unbounded component. (The bounded one can be studied for any n .)

Corollary 2 *Let $n = 1$ and $\alpha > 0$. Let f be strictly convex, $f(\alpha) = 0$, $f > 0$ on $(\alpha, +\infty)$. Then there exists $\beta \geq \alpha$ such that $(\beta, +\infty)$ or $[\beta, +\infty)$ is a connected component of $D(T)$ (in the cases $f(0) \geq 0$ and $f(0) < 0$, respectively). Further, β is the solution of $F(\beta) = F(\alpha_1)$ in $[\alpha, \infty)$, where α_1 is the first root of f in $[0, \alpha]$ if $f(0) \geq 0$ and $\alpha_1 = 0$ otherwise. Therefore, if $f'(\alpha) = 0$ ($f'(\alpha) > 0$), then $\beta = \alpha$ ($\beta > \alpha$), resp.*

Proposition 3 *Let $n \leq 2$. If $f(u) > 0$ ($u \in (0, +\infty)$), then $D(T) = (0, +\infty)$.*

PROOF. It is a consequence of the integral form of (4)

$$-r^{n-1}u'(r) = \int_0^r \rho^{n-1}f(u(\rho))d\rho \quad (13)$$

since this implies $u'(r) \leq -\frac{K}{r^{n-1}}$ ($r > r_1$) for some $r_1 > 0$ and $K > 0$, hence u attains 0 if $n \leq 2$. \diamond

Remark 1 We obtain similarly that for $n \leq 2$ and for nonnegative f we have $D(T) = \mathcal{P}_f$.

Remark 2 Propositions 2 and 3 cannot be extended for all $n \geq 1$. An important restriction is imposed by Pohozaev's formula [18] on the growth of f depending on n .

Proposition 4 *Let $n \geq 1$, $\alpha \in (0, +\infty]$. If $f > 0$ in $(0, \alpha)$ and $\liminf_{u \rightarrow 0} \frac{f(u)}{u} > 0$, then $(0, \alpha) \subset D(T)$. Consequently, if $f(\alpha) = 0$ then $(0, \alpha)$ is a maximal subinterval of $D(T)$.*

PROOF.

- (i) In case $f(0) > 0$ the statement is an easy consequence of (13). Namely, let $p \in (0, \alpha)$ and let $M := \min f|_{[0,p]}$. Then (13) yields $u(r, p) \leq p - \frac{M}{2n}r^2$, hence $u(\cdot, p)$ has a root, i.e. $p \in D(T)$.
- (ii) In case $f(0) = 0$ we use a linear lower estimate for f , namely, we choose $\delta > 0$ and $m > 0$ such that $f(u) \geq mu$ for $u \in [0, \delta]$. Let $p \in (0, \alpha)$. If $p > \delta$, then let $M := \min f|_{[\delta,p]} > 0$. Then applying (13) again we get that there exists $r_1 > 0$ such that $u(r_1) = \delta$. In case $p \leq \delta$ let $r_1 := 0$. Comparison of

$$u'' + \frac{n-1}{r}u' + \frac{f(u)}{u}u = 0 \quad \text{and} \quad w'' + \frac{n-1}{r}w' + mw = 0$$

with $w(r_1) = \delta$, $w'(r_1) = 0$ shows that u attains 0 before the first root of w which exists since the equation for w is linear and $m > 0$.

◇

Corollary 3 *Let $n \geq 1$. Let $f(u) \geq mu$ ($u \in [0, +\infty)$) for some $m > 0$. Then $D(T) = (0, +\infty)$, and T is bounded.*

2.1 The limit of the time-map at the boundary points of $D(T)$

Proposition 5 *Let $n \geq 1$. Let $0 \in \partial D(T)$.*

- (a) *If $f(0) > 0$, then $\lim_0 T = 0$.*
- (b) *If $f(0) = 0$ and $f'(0) > 0$, then $\lim_0 T \in (0, +\infty)$.*
- (c) *If $f(0) = 0$ and $f'(0) = 0$, then $\lim_0 T = +\infty$.*

PROOF.

- (a) The proof of Proposition 4 (i) is used: with suitable $M > 0$ for small enough p we have $f(u(r)) \geq M$ in $[0, T(p)]$, hence u attains 0 before $w(r) = p - \frac{M}{2n}r^2$ does. The root of the latter tends to 0 as $p \rightarrow 0$.
- (b) Similarly, for any $\varepsilon > 0$ comparison to $u''_{\pm} + \frac{n-1}{r}u'_{\pm} + (f'(0) \pm \varepsilon)u_{\pm} = 0$ shows that for small enough p the first root of u lies between those of u_{\pm} , hence $\lim_0 T$ coincides with the root of the linearized equation at 0.
- (c) Similarly, for any $\delta > 0$ comparison to $w''_{\delta} + \frac{n-1}{r}w'_{\delta} + \delta w_{\delta} = 0$ shows that for small enough p the first root of u lies after that of w_{δ} which tends to $+\infty$ as $\delta \rightarrow 0$.

◇

Proposition 6 Let $n \geq 1$. Let $c > 0$ belong to $\partial D(T) \setminus D(T)$. Then $\lim_c T = +\infty$.

PROOF. Let $R > 0$. Let $\varepsilon > 0$ such that $u(r, c) \geq \varepsilon$ ($r \in [0, R]$). The continuous dependence of $u(r, p)$ on p is uniform on compact intervals, hence for small enough $\delta > 0$ we have $u(r, p) > 0$ ($r \in [0, R]$) for all $|p - c| < \delta$. \diamond

Proposition 7 Let $n = 1$. Let $+\infty \in \partial D(T)$.

(a) If $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty$ then $\lim_{+\infty} T = 0$.

(b) If $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = L \in (0, +\infty)$ then $\lim_{+\infty} T = \frac{\pi}{2\sqrt{L}}$.

(c) If $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = 0$ then $\lim_{+\infty} T = +\infty$.

PROOF.

(a) Let $k(u) = \inf\{\frac{f(s)}{s} : s \geq u\}$ ($u > 0$) and $K(u) = \int_0^u k(s)s ds$. Then k increases to $+\infty$ and $f(u) \geq k(u)u$. For large enough p we have

$$\begin{aligned} T(p) &= \frac{1}{\sqrt{2}} \int_0^p \frac{1}{\sqrt{F(p) - F(s)}} ds \\ &\leq \frac{1}{\sqrt{2}} \int_0^p \frac{1}{\sqrt{K(p) - K(s)}} ds \\ &= \frac{p}{\sqrt{2K(p)}} \int_0^1 \frac{1}{\sqrt{1 - \frac{K(pt)}{K(p)}}} dt. \end{aligned} \quad (14)$$

Here $K(pt) = t^2 \int_0^p k(vt)v dv \leq t^2 K(p)$, hence the integral is bounded by $\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}$. Thus

$$T(p) \leq \frac{p}{\sqrt{2K(p)}} \cdot \frac{\pi}{2}. \quad (15)$$

Further, $K(p) \geq \int_{p/2}^p k(s)s ds \geq k(p/2) \frac{3p^2}{8}$, hence we have

$$T(p) \leq \text{const.} \cdot \frac{1}{\sqrt{k(p/2)}} \rightarrow 0 \quad (p \rightarrow +\infty).$$

(b) Let $\varepsilon, \delta > 0$ be fixed. Then for sufficiently large p

$$K(p) \geq \int_{\delta p}^p k(s)s ds \geq k(\delta p) \frac{p^2}{2} (1 - \delta^2).$$

Here $\lim_{p \rightarrow +\infty} k(\delta p) = L$, hence for sufficiently large p we have

$$\frac{p}{\sqrt{2K(p)}} \leq \frac{1}{\sqrt{L(1-\delta^2)}} + \varepsilon. \quad (16)$$

In order to obtain a lower estimate for $T(p)$, we also introduce $g(u) = \sup\{\frac{f(s)}{s} : s \geq u\}$ ($u > 0$) and $G(u) = \int_0^u g(s)s ds$. Then g decreases to L . Exchanging K to G in (14), the estimate is reversed, and similarly we obtain

$$T(p) \geq \frac{p}{\sqrt{2G(p)}} \cdot \frac{\pi}{2}. \quad (17)$$

Let $p_\delta > 0$ such that there holds $g(p) < L + \delta$ ($p > p_\delta$). Then

$$G(p) = \int_0^{p_\delta} g(s)s ds + \int_{p_\delta}^p g(s)s ds \leq g(0)\frac{p_\delta^2}{2} + (L + \delta)\frac{p^2}{2}.$$

Hence for sufficiently large p we have

$$\frac{p}{\sqrt{2G(p)}} \geq \frac{1}{\sqrt{L + \delta}} - \varepsilon. \quad (18)$$

Summarizing (15)–(18), we obtain for sufficiently large p

$$\left(\frac{1}{\sqrt{L + \delta}} - \varepsilon \right) \frac{\pi}{2} \leq T(p) \leq \left(\frac{1}{\sqrt{L(1-\delta^2)}} + \varepsilon \right) \frac{\pi}{2}.$$

- (c) Similar to (b), now using g and G only to get the suitable lower estimate for $T(p)$. \diamond

Remark 3 (a) Proposition 7 is proved for $f > 0$ in [12].

- (b) The whole proposition cannot be extended to all $n \geq 1$ as the examples $f(u) = e^u$ or $(1 + \alpha u)^\beta$ show (see [10]). It is possible for last part.

Proposition 8 Part (c) of Proposition 7 holds for all $n \geq 1$.

PROOF. The integral formula (13) implies $u(r, p) \geq p - \frac{M_p}{2n} r^2$ where $M_p = \max f|_{[0, p]}$. Then $T(p) \geq \sqrt{\frac{2np}{M_p}} \rightarrow +\infty$ since $\lim_{p \rightarrow +\infty} \frac{M_p}{p} = 0$. \diamond

2.2 The monotonicity of the time-map

Proposition 9 Let $n \geq 1$. Let $\alpha \in (0, +\infty]$, $f \in C^2$ be convex and positive on $(0, \alpha)$. Then for any $p \in (0, \alpha)$ $T'(p) = 0$ implies $f'(p) \geq 0$.

PROOF. (8), (3) and the convexity of f imply that the function

$$H(r) = f'(u(r))\frac{h^2(r)}{2} + \frac{h'^2(r)}{2}$$

is strictly decreasing (here $h = \partial_p u$). According to (11) $T'(p) = 0$ implies $h(T(p)) = 0$. Hence

$$0 \leq \frac{h'^2(T(p))}{2} = H(T(p)) \leq H(0) = \frac{f'(p)}{2}.$$

◇

Corollary 4 *Let $n \geq 1$. Let $\alpha \in (0, +\infty]$, $f \in C^2$ be convex and positive on $(0, \alpha)$. If $\lim_{\alpha} f = 0$ then T is strictly increasing on $(0, \alpha)$.*

Proposition 10 *Let $n = 1$ and $f \in C^2$ be strictly convex. If $f(0) \leq 0$ then T is strictly decreasing.*

PROOF. Let $l(u) = uf'(u) - f(u)$ ($u > 0$). The strict convexity of f implies that l is strictly increasing, thus, owing to $l(0) = -f(0) \geq 0$, we have $l(u) > 0$ ($u > 0$), i.e. $\frac{f(u)}{u} < f'(u)$. Thus comparison of $u'' + \frac{f(u)}{u}u = 0$ and $h'' + f'(u)h = 0$ shows that h has a root in $(0, T(p))$, and comparison of h and v (as in Lemma 1) shows that h has no other roots, i.e. $h(T(p), p) < 0$, hence $T'(p) < 0$. ◇

Remark 4 This property is open for $n > 1$. Under a certain set of assumptions it follows from [16] in the case when f has a positive root and $f(0) = 0$.

Proposition 11 *Let $n = 1$ and $f \in C^2$ be strictly convex. If one of the end-points of a maximal subinterval of $D(T)$ does not belong to $D(T)$, then T is strictly monotonic on this subinterval.*

PROOF. It is the consequence of Lemma 1 and Proposition 6 ◇

3 Classification of the number of positive solutions in one dimension

The results of Section 2 enable us to give a complete classification of strictly convex C^2 functions according to the number of positive solutions. Thus we obtain the extension of the results in [12] concerning the case $f(u) > 0$ ($u > 0$). It is easy to see that for u large enough the function $\frac{f(u)}{u}$ is monotone, therefore the limit $\lim_{u \rightarrow +\infty} \frac{f(u)}{u}$ exists. The value of this limit serves as the first level of our classification. The superlinear ($\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty$), asymptotically linear ($\lim_{u \rightarrow +\infty} \frac{f(u)}{u} \in (0, +\infty)$) and sublinear ($\lim_{u \rightarrow +\infty} \frac{f(u)}{u} \leq 0$) cases are considered in Theorems 1, 2, 3, respectively.

Theorem 1 Let $n = 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f \in C^2$ be strictly convex, $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty$.

- (i) If $f(u) > 0$ ($u \in [0, \infty)$) then there exists $R_{sup} > 0$ such that (1) has two solutions for $R < R_{sup}$, one solution for $R = R_{sup}$, and no solution for $R > R_{sup}$.
- (ii) If $f(0) > 0$ and f has a root in $(0, \infty)$ then (1) has two solutions for all $R > 0$.
- (iii) If $f(0) = 0$ and $f'(0) > 0$ then there exists $R_{sup} > 0$ such that (1) has one solution for $R < R_{sup}$ and no solution for $R \geq R_{sup}$.
- (iv) If $f(0) = 0$ and $f'(0) \leq 0$ then (1) has one solution for all $R > 0$.
- (v) If $f(0) < 0$ then there exists $R_{sup} > 0$ such that (1) has one solution for $R \leq R_{sup}$ and no solution for $R > R_{sup}$.

PROOF.

- (i) Propositions 5 and 7 imply that $\lim_0 T = \lim_{+\infty} T = 0$. The maximum of T is a unique extremum owing to Corollary 1, hence T increases from 0 to some $R_{sup} > 0$, then it decreases again to 0.
- (ii) Proposition 4 and Corollary 2 imply that $D(T)$ consists of two maximal subintervals $(0, \alpha)$ and (β, ∞) where the endpoint(s) α and β are not in $D(T)$. Using Propositions 5, 6, and 7 we conclude that $\lim_0 T = \lim_{+\infty} T = 0$ and $\lim_{\alpha^-} T = \lim_{\beta^+} T = +\infty$. From Proposition 11 T is strictly monotonic on both subintervals, hence T attains each value twice.
- (iii) Corollary 3 and Propositions 7 and 5 yield $D(T) = (0, \infty)$, further, that $\lim_{+\infty} T = 0$ and $\lim_0 T = R_{sup}$ where $R_{sup} > 0$ is the first root of the linearized equation $u'' + f'(0)u = 0$. Proposition 10 implies that T is strictly decreasing, hence it attains each value $R \in (0, R_{sup})$ once.
- (iv) If $f'(0) = 0$, then propositions 3 and 5 yield that $D(T) = (0, \infty)$ and $\lim_0 T = +\infty$. In the case $f'(0) < 0$ Corollary 2 and Proposition 6 yield that for some $\beta > 0$ we have $D(T) = (\beta, \infty)$ and $\lim_{\beta^+} T = +\infty$. In both cases we have $\lim_{+\infty} T = 0$. Proposition 10 implies that T strictly decreases to 0.
- (v) Corollary 2 yields that $D(T) = [\beta, \infty)$ for some $\beta > 0$. Proposition 10 and $\lim_{+\infty} T = 0$ imply again that T strictly decreases to 0.

◇

Theorem 2 Let $n = 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f \in C^2$ be strictly convex, $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = L \in (0, +\infty)$ and $R_\infty := \frac{\pi}{2\sqrt{L}}$.

- (i) If $f(u) > 0$ ($u \in [0, \infty)$) then there exists $R_{sup} > R_\infty$ such that (1) has one solution for $R \leq R_\infty$ and $R = R_{sup}$, two solutions for $R_\infty < R < R_{sup}$ and no solution for $R > R_{sup}$.
- (ii) If $f(0) > 0$ and f has a root in $(0, \infty)$ then (1) has one solution for $R \leq R_\infty$ and two solutions for $R > R_\infty$.
- (iii) If $f(0) = 0$ and $f'(0) > 0$ then there exists $R_{sup} > R_\infty$ such that (1) has no solution for $R \leq R_\infty$, one solution for $R_\infty < R < R_{sup}$ and no solution for $R \geq R_{sup}$.
- (iv) If $f(0) = 0$ and $f'(0) \leq 0$ then (1) has no solution for $R \leq R_\infty$ and one solution for $R > R_\infty$.
- (v) If $f(0) < 0$ then there exists $R_{sup} > R_\infty$ such that (1) has no solution for $R \leq R_\infty$, one solution for $R_\infty < R \leq R_{sup}$ and no solution for $R > R_{sup}$.

PROOF. The proof proceeds just in the same way as in Theorem 1, now using $\lim_{+\infty} T = R_\infty$ from Proposition 7. (The existence of the maximum of T in case (i) can be found in [12]). \diamond

In the sublinear case the result is independent of n , therefore it is dealt with in the next section.

4 The n -dimensional case

For the general case ($n \geq 1$) the whole classification of bifurcation diagrams cannot be extended from the case $n = 1$. In the sublinear case the next theorem gives the answer.

Theorem 3 *Let $n \geq 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f \in C^2$ be strictly convex, and $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} \leq 0$. If f has a positive value then for any $R > 0$ (1) has a unique solution. (If $f \leq 0$ then (1) has obviously no positive solution for any $R > 0$.)*

PROOF. Due to its convexity, f is strictly decreasing. In the nontrivial case $f(0) > 0$ there are two possibilities: either f has a single positive root $\alpha > 0$ (and then $u > 0$ on $(0, \alpha)$ and $u < 0$ on (α, ∞)) or $f > 0$ on $(0, \infty)$ (in which case we define $\alpha := \infty$). Proposition 4 implies $D(T) = (0, \alpha)$. Propositions 5, 6 and 8 imply that $\lim_0 T = 0$ and $\lim_\alpha T = +\infty$, and from Corollary 4 T is strictly increasing, hence it attains each value once. \diamond

Remark 5 This result can be extended to general bounded domains using monotone operator theory, because f is strictly decreasing.

In the superlinear and asymptotically linear case ($\lim_{u \rightarrow \infty} \frac{f(u)}{u} > 0$) the problem is much more complicated. The known results thus usually concern given special functions, as described in the Introduction. First, we briefly list some

of the reasons for the difficulties concerning the domain, the limits and the monotonicity of the time-map.

The domain of the time-map depends sensitively on the dimension n , as is shown by example $f(u) = u^k$: according to the Pohozaev identity [18] $D(T) = \emptyset$ if $k \geq \frac{n+2}{n-2}$, and using variational methods [24] or Emden's transformation [10] one can prove that $D(T) = (0, \infty)$ if $k < \frac{n+2}{n-2}$. The limit of the time map at infinity may also change as the dimension n increases: in [10] it is shown for the example $f(u) = e^u$ that for $n \geq 3$ $\lim_{\infty} T \neq 0$. We note that in case $n \leq 2$, $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty$ implies $\lim_{\infty} T = 0$. The monotonicity properties of the time-map also change for higher dimensions. Our main lemma (Lemma 1) is not true generally, since e.g. for $f(u) = e^u$ and for $3 \leq n \leq 10$ the time-map has infinitely many maxima and minima, see [10]. For the case $f(0) \leq 0$ McLeod proved a uniqueness result for a sophisticatedly chosen function class with main typical member $f(u) = u^p - u$. Unfortunately, many convex functions are not contained in this class, e.g. $f(u) = u^2 + au - b$ for $a, b > 0$. In the class $f(0) = 0$, $f'(0) > 0$ Srikanth [23] and Zhang [26] have proved uniqueness for $f(u) = u^p + u$ for $n \geq 3$. The difficulties concerning the monotonicity of the time-map generally arise from the fact that in case $n > 1$ the Sturm comparison of h and v is not possible; from equations (8) and (10) one can see that v oscillates more slowly than h .

Now we summarize the bifurcation results that follow from our investigations on the domain, limits and monotonicity of the time-map. Of course, this theorem for a general convex function f does not contain the strong results concerning the special functions mentioned above.

Theorem 4 *Let $n \geq 1$, $f : [0, \infty) \rightarrow \mathbb{R}$, $f \in C^2$ be strictly convex, $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} > 0$.*

- (i) *If $f(u) > 0$ ($u \in [0, \infty)$) then there exists $R_{sup} > 0$ such that (1) has at least one solution for $R \leq R_{sup}$ and no solution for $R > R_{sup}$.*
- (ii) *If $f(0) > 0$ and f has a root in $(0, \infty)$ then (1) has at least one solution for all $R > 0$.*
- (iii) *If $f(0) = 0$ and $f'(0) > 0$ then there exist $R_{sup} > 0$ and $R_{inf} \in [0, R_{sup}]$ such that (1) has no solution for $R > R_{sup}$ or $R < R_{inf}$ and it has at least one solution for $R_{inf} < R < R_{sup}$.*

PROOF.

- (i), (iii) Corollary 3 yields that T is bounded, hence we can set $R_{sup} = \max T$. In case (i) Proposition 5 implies that $\lim_0 T = 0$, hence T attains all values between 0 and R_{sup} . In case (iii) the proofs of Propositions 4 and 5 imply that R_{sup} coincides with the root of the linearized equation, i.e. with $\lim_0 T$.
- (ii) Let us denote by α the first root of f . Proposition 4 implies that $D(T)$ contains the interval $(0, \alpha)$ and $\alpha \notin D(T)$. Proposition 5 yields that

$\lim_0 T = 0$ and Proposition 6 yields that $\lim_{\alpha^-} T = +\infty$, hence T attains all positive values. We note that from Corollary 4 T is strictly monotonic on $(0, \alpha)$, hence the solution is unique under the restriction $u(r) < \alpha$.

◇

Finally, we list some open problems and conjectures concerning the time-map.

CONJECTURES

1. If $1 \leq n \leq 2$ then $D(T) = \mathcal{P}_f$.
2. Let $1 \leq n \leq 2$, $f \in C^2$ be strictly convex. If $T'(p) = 0$, then $T''(p) < 0$, i.e. Lemma 1 holds for $1 \leq n \leq 2$.
3. Let $n > 1$, $f \in C^2$ be strictly convex. If $f(0) \leq 0$, then (1) has at most one solution for any $R > 0$.

OPEN PROBLEMS

1. Let $n > 2$ and $f \geq 0$. Determine the domain $D(T)$.
2. Let $n > 2$, $f(0) \leq 0$. Assume that f has at most one positive root and f is positive after this root, and $\lim_{u \rightarrow \infty} \frac{f(u)}{u^p} = 0$ for some $p < \frac{n+2}{n-2}$. Prove that there exists b such that $D(T) = (b, \infty)$.
3. Let $n > 2$, $f(0) \leq 0$. Determine the limit $\lim_{\infty} T$.

Acknowledgement The authors are grateful to Prof. L.A. Peletier for the useful discussion. This work was supported by OTKA grants F-022228 and T-019460.

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