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# THE KOLMOGOROV EQUATION WITH TIME-MEASURABLE COEFFICIENTS

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Abstract. Using both probabilistic and classical analytic techniques, we investigate the parabolic Kolmogorov equation

$$L_t v + \tfrac{\partial v}{\partial t} \equiv \frac{1}{2} a^{ij}(t) v_{x^i x^j} + b^i(t) v_{x^i} - c(t)v + f(t) + \tfrac{\partial v}{\partial t} = 0$$

in $H_T := (0,T) \times E_d$ and its solutions when the coefficients are bounded Borel measurable functions of $t$. We show that the probabilistic solution $v(t,x)$ defined in $\bar{H}_T$, is twice differentiable with respect to $x$, continuously in $(t,x)$, once differentiable with respect to $t$, a.e. $t \in [0,T)$ and satisfies the Kolmogorov equation $L_t v + \frac{\partial v}{\partial t} = 0$ a.e. in $\bar{H}_T$. Our main tool will be the Aleksandrov-Busemann-Feller Theorem. We also examine the probabilistic solution to the fully nonlinear Bellman equation with time-measurable coefficients in the simple case $b \equiv 0$, $c \equiv 0$. We show that when the terminal data function is a paraboloid, the payoff function has a particularly simple form.

## 1. Introduction

It is well-known in the theory of diffusion processes [2, 3] that when $g \in C^2(E_d)$ and the coefficients $a(t,x)$, $b(t,x)$, $c(t,x)$ and free term $f(t,x)$ are sufficiently smooth in $(t,x)$ and satisfy certain growth conditions, with $c(t,x) \geq 0$, then the function

$$\begin{gathered} v(t,x) = \mathbf{E}\Big[\int_t^T f(r,\xi_r(t,x)) e^{-\varphi_r(t,x)}\,dr + e^{-\varphi_T(t,x)} g(\xi_T(t,x))\Big], \\ \varphi_s(t,x) = \int_t^s c(r,\xi_r(t,x))\,dr \end{gathered} \tag{1.1}$$

belongs to $C^{1,2}(H_T)$ and satisfies the Kolmogorov equation $Lv(t,x) + \frac{\partial v}{\partial t}(t,x) = 0, \forall\,(t,x) \in \bar{H}_T$, where $Lv := \frac{1}{2} a^{ij}(t,x) v_{x^i x^j} + b^i(t,x) v_{x^i} - c(t,x)v + f(t,x)$, with $v(T,x) = g(x)$. In (1.1), for fixed $(t,x) \in \bar{H}_T$, $\omega \in \Omega$ and $s \geq t$, $\xi_s(t,x) = \xi_s(\omega,t,x)$ is the solution of the stochastic equation $\xi_s = x + \int_t^s \sigma(r,\xi_r)\,d\mathbf{w}_r + \int_t^s b(r,\xi_r)\,dr$, where $(\Omega,\mathcal{F},P)$ is a complete probability space on which $(\mathbf{w}_t,\mathcal{F}_t)$ is a $d_1$-dimensional Wiener process, defined for $t \geq 0$. Furthermore, $\sigma(t,x)$ and $b(t,x)$ are assumed continuous in $(t,x)$ and have values in the set of $d \times d_1$ matrices, $E_d$ respectively, with $a = \sigma\sigma^*$. The fact that the probabilistic solution $v$ satisfies the

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Kolmogorov equation throughout $\bar{H}_T$ is proved using Itô's formula and relies heavily on the continuity in $t$ of the coefficients to establish the existence and continuity in $(t,x)$ of $\frac{\partial v}{\partial t}$ [3, Chapter 5]. In this paper, we show that if the coefficients are only bounded Borel measurable functions of $t$, the second derivatives $v_{x^ix^j}(t,x)$ exist and are continuous in $(t,x)$ (Theorem 2.1) but in general, $\frac{\partial v}{\partial t}$ exists only in the generalized sense (Theorem 2.3) and the Kolmogorov equation will be satisfied only in the almost everywhere sense (Theorem 2.5). For example, consider the function $v(t,x) = |x|^2 + 2d(\frac{1}{2} - t)_+$. For $t \neq \frac{1}{2}$, $\frac{\partial v}{\partial t}(t,x)$ exists and equals $-2d\, I_{0\le t<\frac{1}{2}}$ and hence for $t \neq \frac{1}{2}$, $v$ is a solution of the degenerate equation $I_{0\le t<\frac{1}{2}}\Delta v + \frac{\partial v}{\partial t} = 0$ in $[0,1) \times E_d$. Note $\frac{\partial v}{\partial t}(t,x)$ is discontinuous in $t$.

When the coefficients and free term are independent of $x$, the right hand side of our stochastic equation is independent of $\xi_{.}$ and the probabilistic solution (1.1) takes a decidedly more convenient form (see (3.4)). Since the other terms in (3.4) are independent of $x$ and their derivatives with respect to $t$ can be explicitly calculated (almost everywhere) it suffices to investigate the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$.

We do this in two ways. In section 1, we use probabilistic arguments to show that for $g \in C^2(E_d)$, the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ is twice differentiable with respect to $x$, continuously in $(t,x)$ and once differentiable with respect to $t$, a.e. $t \in [0,T)$. We then apply the Aleksandrov-Busemann-Feller theorem to a variant of $v$ to show that $v$ satisfies the Kolmogorov equation $\frac{1}{2}a^{ij}(t)v_{x^ix^j} + b^i(t)v_{x^i} + \frac{\partial v}{\partial t} = 0$ a.e. in $H_T$. From this it follows (by our previous remark) that the simplified version of (1.1), given by (3.4) satisfies the more general Kolmogorov equation a.e. in $H_T$. In section 2, we use the fact that $\xi_T(t,x)$ is a Gaussian vector to express $v$ as a convolution (in $x$) of $g$ with a kernel $p$ which is the fundamental solution of the Kolmogorov equation (a.e. $t$). Our proof that this convolution satisfies the Kolmogorov equation amounts to showing that we can differentiate the kernel under the integral sign. Here we assume only that $g$ is continuous and slowly increasing, that is $|g(x)| \le C_1 e^{C_2|x|^2}$. Our derivative estimates are done under the assumption that the coefficient matrix $a(t)$ is non-degenerate. This assumption was not needed in section 1, (due to the assumption $g \in C^2(E_d)$) yet we do get a slightly more refined result here, namely $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ satisfies the Kolmogorov equation for almost every $t \in [0,T)$ and any $x \in E_d$. Finally in section 4, we examine the payoff function for the fully nonlinear Bellman equation in the simple case $b \equiv 0$, $c \equiv 0$. It turns out that when $g$ is a paraboloid, the probabilistic solution of the Bellman equation has a particularly simple form.

## 2. The Probabilistic Approach

Throughout this section, we assume the following.

Let $g \in C^2(E_d)$ and assume that for all $x, y \in E_d$, $|g(x)|$, $|g_{(y)}(x)|$, $|g_{(y)(y)}(x)| \le K(1 + |x|^m)$, where for any twice differentiable function $u(x)$ and $l \in E_d$, $u_{(l)}(x) = |l|^{-1}u_x(x) \cdot l$, $u_{(l)(l)}(x) = |l|^{-2}l^* u_{xx}(x)l$. For $t \in [0,T]$ and $x \in E_d$, we define, for $s \in [t,T]$, the diffusion process $\xi_s(t,x) = x + \int_t^s \sigma(r)\, d\mathbf{w}_r + \int_t^s b(r)\, dr$, where the Borel measurable coefficients $\sigma(t)$, $b(t)$ are defined on $[0,T]$, independent of $\omega \in \Omega$ and satisfy

$$\int_0^T \left[\|\sigma(t)\|^2 + |b(t)|\right] dt < \infty. \tag{2.1}$$

Under these assumptions, we prove our first theorem.

**Theorem 2.1.** *For $(t,x) \in \bar{H}_T$, the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ is twice differentiable with respect to $x$, continuously in $(t,x)$ and for any $y, \bar{y} \in E_d$, $v_{y\bar{y}}(t,x) = \mathbf{E}g_{y\bar{y}}(\xi_T(t,x))$.*

*Proof.* We show that $v(t,x)$ is differentiable with respect to $x$. Writing $\xi_T(t,x) = x + \eta_T(t)$, where $\eta_T(t) := \int_t^T \sigma(r)\, d\mathbf{w}_r + \int_t^T b(r)\, dr$, note that for any $y \in E_d$ and any sequence $h_n \to 0$ as $n \to \infty$

$$\Delta^1_{h_n,y} v(t,x) := \frac{v(t,x+h_n y) - v(t,x)}{h_n} = \mathbf{E}\Delta^1_{h_n,y} g(x+\eta_T(t)) = \mathbf{E}\Delta^1_{h_n,y} g(\xi_T(t,x)).$$

Since $g_y$ is continuous, the Mean Value Theorem yields

$$\Delta^1_{h_n,y} g(\xi_T(t,x)) = \int_0^1 g_y(\xi_T(t,x) + rh_n y)\, dr = g_y(\xi_T(t,x)) + \theta h_n y),$$

for some $\theta \in [0,1]$. Since $g \in C^1(E_d)$, $\Delta^1_{h_n,y} g(\xi_T(t,x)) \to g_y(\xi_T(t,x))$ as $n \to \infty$. Furthermore, as $n \to \infty$

$$\mathbf{E}\Delta^1_{h_n,y} g(\xi_T(t,x)) \to \mathbf{E}g_y(\xi_T(t,x)). \tag{2.2}$$

To see this observe that

$$\begin{aligned} |\Delta^1_{h_n,y} g(\xi_T(t,x))| &= |g_y(\xi_T(t,x) + \theta h_n y)| \\ &\le |y| K \left(1 + |\xi_T(t,x) + \theta h_n y|^m\right) \\ &\le 2^m K |y| \left(1 + |\xi_T(t,x)|^m + |\theta h_n y|^m\right) \\ &\le N|y| \Big(1 + |x|^m + \Big|\int_t^T \sigma(r) d\mathbf{w}_r\Big|^m + \Big|\int_t^T b(r)\, dr\Big|^m + |y|^m\Big), \end{aligned}$$

where $N = N(m,K)$. By (2.1), the Burkholder-Davis-Gundy inequalities and the fact that $\sigma, b$ are independent of $\omega$, the last expression above has finite expectation. Hence by [3, Lemma III.6.13 (f)], (2.2) holds. Since $\{h_n\}$ was an arbitrary sequence converging to 0 as $n \to \infty$, we conclude

$$\lim_{h\to 0} \mathbf{E}\Delta^1_{h,y} g(\xi_T(t,x)) = \mathbf{E}g_y(\xi_T(t,x)).$$

Thus $v(t,x)$ is differentiable with respect to $x$ and for any $y \in E_d$, $v_y(t,x) = \lim_{h\to 0} \mathbf{E}\Delta^1_{h,y} g(\xi_T(t,x)) = \mathbf{E}g_y(\xi_T(t,x))$. We now show that $v(t,x)$ is twice differentiable with respect to $x$. By the above expression for $v_y(t,x)$, we have, for any $\bar{y} \in E_d$

$$\frac{v_y(t,x+h\bar{y}) - v_y(t,x)}{h} = \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)). \tag{2.3}$$

But since $g_{y\bar{y}}$ is continuous, for any sequence $h_n \to 0$ as $n \to \infty$, the Mean Value Theorem yields

$$\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x)) = \int_0^1 g_{y\bar{y}}(\xi_T(t,x) + rh_n y)\, dr = g_{y\bar{y}}(\xi_T(t,x)) + \theta h_n y),$$

for some $\theta \in [0,1]$. Since $g \in C^2(E_d)$, $\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x)) \to g_{y\bar{y}}(\xi_T(t,x))$ as $n \to \infty$. By the argument immediately following (2.2), except with $|y|^2 + |\bar{y}|^2$ in place of $|y|$ and using the growth condition on $|g_{(y)(y)}(x)|$, we see that $|\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x))|$ is bounded above (independently of $n$) by a random variable which has finite expectation. Hence

$$\mathbf{E}\Delta^1_{h_n,\bar{y}} g_y(\xi_T(t,x)) \to \mathbf{E}g_{y\bar{y}}(\xi_T(t,x)) \quad \text{as } n \to \infty.$$

Since $\{h_n\}$ was an arbitrary sequence converging to 0 as $n \to \infty$,

$$\lim_{h\to 0} \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)) = \mathbf{E}g_{y\bar{y}}(\xi_T(t,x)).$$

Thus by (2.3), $v_{y\bar{y}}(t,x)$ exists and since $y, \bar{y} \in E_d$ were arbitrary, $v(t,x)$ is twice differentiable with respect to $x$ and

$$v_{y\bar{y}}(t,x) = \lim_{h\to 0} \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)) = \mathbf{E}g_{y\bar{y}}(\xi_T(t,x)).$$

We now show the continuity of $v_{y\bar{y}}(t,x)$ in $(t,x)$. To this end, fix $(t,x)$ and let $t^n \to t^+, x^n \to x$. It suffices to show $v_{y\bar{y}}(t^n,x^n) \to v_{y\bar{y}}(t,x)$. We have

$$|v_{y\bar{y}}(t^n,x^n) - v_{y\bar{y}}(t,x)| \le \mathbf{E}|g_{y\bar{y}}(\xi_T(t^n,x^n)) - g_{y\bar{y}}(\xi_T(t,x))|. \tag{2.4}$$

Observe that $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$ and since $g_{y\bar{y}}$ is continuous, $g_{y\bar{y}}(\xi_T(t^n,x^n)) \xrightarrow{P} g_{y\bar{y}}(\xi_T(t,x))$. Since $|g_{y\bar{y}}(\xi_T(t^n,x^n))| \le \eta$ with $\mathbf{E}\eta < \infty$, the right hand side of (2.5) tends to zero as $n \to \infty$. The details are as follows. To see that $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$, observe that

$$\begin{aligned} &|\xi_T(t^n,x^n) - \xi_T(t,x)| \\ &\le |x^n - x| + \Big|\int_{t^n}^T \sigma(r)d\mathbf{w}_r - \int_t^T \sigma(r)d\mathbf{w}_r\Big| + \Big|\int_{t^n}^T b(r)\,dr - \int_t^T b(r)\,dr\Big|. \end{aligned} \tag{2.5}$$

The middle summand tends to zero in probability as $n \to \infty$ by [3, Theorem III.6.6] and the fact that

$$\int_0^T \|I_{t^n \le r}\sigma(r) - I_{t\le r}\sigma(r)\|^2\,dr = \int_0^T \|\sigma(r)\|^2 I_{t\le r<t^n}\,dr \to 0 \quad \text{as } n \to \infty$$

by (2.1) and the Dominated Convergence Theorem. The third summand on the right hand side of (2.5) tends to zero by the Dominated Convergence Theorem. Since $x^n \to x$, we have $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$. Since $g_{y\bar{y}}(x)$ is continuous,

$$g_{y\bar{y}}(\xi_T(t^n,x^n)) \xrightarrow{P} g_{y\bar{y}}(\xi_T(t,x)),$$

by [3, Theorem III.6.13 (c)]. Finally,

$$\begin{aligned} &|g_{y\bar{y}}(\xi_T(t^n,x^n))| \\ &\le K(|y|^2 + |\bar{y}|^2)(1 + |\xi_T(t^n,x^n)|^m) \\ &\le 3^m K(|y|^2 + |\bar{y}|^2)\Big\{1 + |x^n|^m + \Big|\int_{t^n}^T \sigma(r)\,d\mathbf{w}_r\Big|^m + \Big|\int_{t^n}^T b(r)\,dr\Big|^m\Big\}. \end{aligned} \tag{2.6}$$

Since

$$\Big|\int_{t^n}^T \sigma(r)\,d\mathbf{w}_r\Big|^m \le 2^m \sup_s \Big|\int_0^{s\wedge T} \sigma(r)\,d\mathbf{w}_r\Big|^m$$

as $x^n \to x$ and $\big|\int_{t^n}^T b(r)\,dr\big|^m \le \big(\int_0^T |b(r)|\,dr\big)^m$, the right hand side of (2.6) is bounded uniformly in $n$ by a random variable, which, by the Burkholder-Davis-Gundy inequalities and (2.1), has finite expectation. Hence, by [3, Theorem III.6.13 (f)],

$$\mathbf{E}|g_{y\bar{y}}(\xi_T(t^n,x^n)) - g_{y\bar{y}}(\xi_T(t,x))| \to 0 \quad \text{as } n \to \infty$$

and hence by (2.4), $v_{y\bar{y}}(t^n,x^n) \to v_{y\bar{y}}(t,x)$. $\square$

The proof that $v(t,x)$ and $v_y(t,x)$ are continuous in $\bar{H}_T$ follow same the technique shown here, except we use the respective assumptions $|g(x)|$, $|g_{(y)}(x)| \le K(1+|x|^m)$. Observe that by (2.6) and the Burkholder-Davis-Gundy inequalities, we obtain the following estimate, which holds for $(t,x) \in \bar{H}_T$

$$\begin{aligned} &\|v_{xx}(t,x)\| \\ &\le N(d,m,K)\Big\{1+|x|^m+\Big(\int_t^T \|\sigma(r)\|^2\,dr\Big)^{m/2}+\Big(\int_t^T |b(r)|\,dr\Big)^m\Big\}. \end{aligned} \tag{2.7}$$

If in addition, $\sigma, b$ satisfy $\sup_{t\le T}(\|\sigma(t)\|+|b(t)|) \le K$, inequality (2.7) yields, with $N_1 = N_1(d,m,K)$

$$\begin{aligned} \|v_{xx}(t,x)\| &\le 2N(1\vee K^m)(1+|x|^m)\left\{1+(T-t)^m\right\} \\ &\le 4N(1\vee K^m)(1+|x|^m)e^{(T-t)m} \\ &\le N_1(1+|x|)^m e^{N_1(T-t)}. \end{aligned} \tag{2.8}$$

The following lemma appears in [3, p. 195]. We will use this lemma and the fact that $v, v_x, v_{xx}$ are continuous in $(t,x)$ to show that when $\sigma(t), b(t)$ are bounded, $v(t,x)$ is differentiable with respect to $t$ for almost every $t \in [0,T]$.

**Lemma 2.2.** *Let $\xi_s(t,x) = x + \int_t^s \sigma(r)\,d\mathbf{w}_r + \int_t^s b(r)\,dr$, where $\sup_{t\le T}(\|\sigma(t)\| + |b(t)|) \le K$. For $\epsilon > 0$ and $(t,x) \in Q$, let*

$$\tau_\epsilon(t,x) = \inf\{s \ge t : (s, \xi_s(t,x)) \notin Q_\epsilon(t,x)\},$$

*where $Q_\epsilon(t,x) = (t-\epsilon^3, t+\epsilon^3) \times B_\epsilon(x)$. Then for any compact set $\Gamma \subset Q_+ := Q \cap \{t \ge 0\}$,*

$$\epsilon^{-3}P\{\tau_\epsilon(t,x) - t < \epsilon^3\} \to 0, \quad \epsilon^{-3}\mathbf{E}[\tau_\epsilon(t,x)-t] \to 1,$$

*uniformly in $(t,x) \in \Gamma$, as $\epsilon \to 0^+$.*

**Theorem 2.3.** *Under the hypotheses of Theorem 2.1 suppose that $\sup_{t\le T}(\|\sigma(t)\|+|b(t)|) \le K$. Then for any $x \in E_d$, the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ is differentiable with respect to $t$ for almost every $t \in [0,T)$.*

*Proof.* Fix any $(t,x) \in H_T$ and choose $\epsilon$ so small that $t+\epsilon^3 < T$. Since absolutely continuous functions of a single real variable are differentiable almost everywhere, it suffices to show that $v(t,x)$ is Lipschitz in $t$. By the strong Markov property we can write

$$v(t,x) = \mathbf{E}v(\tau_\epsilon(t,x), \xi_{\tau_\epsilon(t,x)}(t,x)), \tag{2.9}$$

which we henceforth abbreviate as $\mathbf{E}v(\tau_\epsilon, \xi_{\tau_\epsilon})$. By Itô's formula applied to the $C^2$ function (of $x$) $v(t+\epsilon^3, \cdot)$, we have

$$\begin{aligned} &v(t,x) - v(t+\epsilon^3, x) \\ &= \mathbf{E}[v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})] + \mathbf{E}[v(t+\epsilon^3, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_t)] \\ &= \mathbf{E}\, I_{\tau_\epsilon < t+\epsilon^3}[v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})] + \mathbf{E}\int_t^{\tau_\epsilon} L_r v(t+\epsilon^3, \xi_r)\,dr \end{aligned} \tag{2.10}$$

Certainly $|v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})| \le 2\sup_{[t,t+\epsilon^3]\times\overline{B_\epsilon(x)}}|v|$. We recall that $v, v_x, v_{xx}$ are continuous and hence bounded in any compact set. By definition, $L_r v(t+$

$\epsilon^3, \xi_r) = \frac{1}{2}\operatorname{tr}[a(r)\, v_{xx}(t+\epsilon^3, \xi_r)] + b(r) \cdot v_x(t+\epsilon^3, \xi_r)$. From the elementary inequality $|\operatorname{tr}[a \cdot m]| \le \|a\|\,\|m\|$ and the fact that $\|\sigma(t)\| + |b(t)| \le K$, we get, for $r \in [t, \tau_\epsilon]$,

$$\begin{aligned} \big|L_r v(t+\epsilon^3, \xi_r)\big| &\le \frac{K^2}{2}\|v_{xx}(t+\epsilon^3, \xi_r)\| + K|v_x(t+\epsilon^3, \xi_r)| \\ &\le N(K)\big(\sup_{B_\epsilon(x)} \|v_{xx}(t+\epsilon^3, \cdot)\| + \sup_{B_\epsilon(x)} |v_x(t+\epsilon^3, \cdot)|\big). \end{aligned} \tag{2.11}$$

So in any small closed cylinder $\widetilde{Q} \supset [t, t+\epsilon^3] \times \overline{B_\epsilon(x)}$, we have, by (2.10) and Lemma 2.2, for sufficiently small $\epsilon$,

$$\begin{aligned} &|v(t,x) - v(t+\epsilon^3, x)| \\ &\le 2\sup_{\widetilde{Q}} |v| \cdot P\{\tau_\epsilon - t < \epsilon^3\} + N(K)(\sup_{\widetilde{Q}} \|v_{xx}\| + \sup_{\widetilde{Q}} |v_x|)\mathbf{E}[\tau_\epsilon - t] \\ &\le N_1(K)(\sup_{\widetilde{Q}} |v| + \sup_{\widetilde{Q}} |v_x| + \sup_{\widetilde{Q}} \|v_{xx}\|)\epsilon^3. \end{aligned}$$

Since $t, \epsilon$ were arbitrary (such that $t + \epsilon^3 < T$), we get, for any $s, t \in [0, T)$ and any fixed $x \in E_d$,

$$|v(t,x) - v(s,x)| \le N_2|t-s|,$$

where $N_2$ is independent of $s, t, x$. Hence the generalized derivative $\frac{\partial v}{\partial t}$ exists and $\big|\frac{\partial v}{\partial t}(t,x)\big| \le N_2$. □

We will now show that the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ satisfies the Kolmogorov equation almost everywhere in $H_T$, under the assumptions of Theorem 2.3. Our main tool will be the Aleksandrov-Busemann-Feller (ABF) theorem (see [4, Theorem 1.1]) for continuous functions which are convex in $x$ and non-increasing in $t$.

**Theorem 2.4** (Aleksandrov-Busemann-Feller). *Let $u(t,x)$ be convex in $x$, non-increasing in $t$ and continuous in $\bar{H}_T$. Let $P(s,x,t,y) = u(s,x) + u_s^{(0)}(s,x)t + u_x(s,x) \cdot y + \frac{1}{2}y^* u_{xx}^{(0)}(s,x)y$, where $u_s^{(0)}, u_{x^i x^j}^{(0)}$ denote generalized derivatives. Then for almost all $(s,x) \in E_{d+1}$, $u(s+t, x+y) = P(s,x,t,y) + o(|t| + |y|^2)$ as $(t,y) \to (0,0)$.*

Equivalently, for almost all $(t_0, x_0) \in E_{d+1}, u(t,x) = P_{(t_0,x_0)}(t,x) + o(|t-t_0| + |x-x_0|^2)$ as $(t,x) \to (t_0,x_0)$, where $P_{(t_0,x_0)}(t,x) = u(t_0,x_0) + u_t^{(0)}(t_0,x_0)(t-t_0) + u_x(t_0,x_0) \cdot (x-x_0) + \frac{1}{2}(x-x_0)^* u_{xx}^{(0)}(t_0,x_0)(x-x_0)$. We want to apply the ABF theorem to a variant of $v$. To this end, note that by (2.6), for any $l \in E_d$, we have

$$\big|v_{(l)(l)}(t,x)\big| \le \mathbf{E}\left|g_{(l)(l)}(\xi_T(t,x))\right| \le Ne^{N(T-t)}(1+|x|)^m,$$

where $N = N(m,K)$. Direct calculation shows that for any $l, x \in E_d$, $(m+2)2^{-\frac{m}{2}}(1+|x|)^m \le \big[(1+|x|^2)^{\frac{m}{2}+1}\big]_{(l)(l)}$. Hence

$$\big|v_{(l)(l)}(t,x)\big| \le \frac{Ne^{N(T-t)}2^{\frac{m}{2}}}{m+2}\left[(1+|x|^2)^{\frac{m}{2}+1}\right]_{(l)(l)} \le Ne^{N(T-s)}\left[(1+|x|^2)^{\frac{m}{2}+1}\right]_{(l)(l)}$$

which yields

$$0 \le \Big(v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}\Big)_{(l)(l)} \quad \forall (t,x) \in H_T,\, l \in E_d.$$

That is, the function $v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}$ is convex in $x$. We may also consider this function to be decreasing in $t$ by the following argument. By Lemma

2.2, the first summand on the right hand side of (2.10) is $o(\epsilon^3)$ as $\epsilon \to 0$. By the continuity of $v_{xx}(t,x), v_x(t,x)$, the last factor on the right hand side of (2.11) tends to $\|v_{xx}(t,x)\|+|v_x(t,x)|$ as $\epsilon \to 0$. Since the estimate $|v_x(t,x)| \le Ne^{N(T-t)}(1+|x|)^m$ also holds, dividing (2.10) by $\epsilon^3$, letting $\epsilon \to 0$, using (2.8) and applying the second result in Lemma 2.2, we get for almost every $t \in [0,T)$ and any $x \in E_d$

$$\Big|\frac{\partial v}{\partial t}(t,x)\Big| \le Ne^{N(T-t)}(1+|x|)^m \le Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}, \tag{2.12}$$

where $N = N(d,m,K)$. From (2.12) it follows, as before, that for some $N = N(d,m,K)$, $v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1} := v + v_0$ is decreasing in $t$.

**Theorem 2.5.** *Under the assumptions of Theorem 2.3, the function* $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ *satisfies the Kolmogorov equation almost everywhere in* $H_T$.

*Proof.* Since the ABF theorem holds for the function $v + v_0$ and $v_0$ is smooth, the ABF theorem also holds for $v$. Since $v$ has continuous second derivatives (by Theorem 2.1), $v_{xx}^{(0)} = v_{xx}$ almost everywhere. So fix any $(t,x) \in H_T$ for which the assertion of the ABF theorem holds for $v$, $v_{xx}^{(0)}(t,x) = v_{xx}(t,x)$ and $t$ is in the Lebesgue set of the operator $L_s \equiv \frac{1}{2}a^{ij}(s)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(s)\frac{\partial}{\partial x^i}$. By the strong Markov property, $v(t,x) = \mathbf{E}v(\tau_\epsilon, \xi_{\tau_\epsilon})$, where $\tau_\epsilon(t,x)$ is as in Lemma 2.2. By the ABF theorem, $v(\tau_\epsilon,\xi_{\tau_\epsilon}) = P_{(t,x)}(\tau_\epsilon,\xi_{\tau_\epsilon}) + o(|\tau_\epsilon - t| + |\xi_{\tau_\epsilon} - x|^2)$ as $\epsilon \to 0$. Since $\xi_t(t,x) = x$ and $P_{(t,x)}(t,x) = v(t,x)$, applying Itô's formula to the paraboloid $P_{(t,x)}$ yields

$$0 = \mathbf{E}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr + \mathbf{E}[o(|\tau_\epsilon - t| + |\xi_{\tau_\epsilon} - x|^2)]. \tag{2.13}$$

Since $0 \le \tau_\epsilon - t \le \epsilon^3$, the estimates $\mathbf{E}|\xi_{\tau_\epsilon} - x|^p \le N(p,K)\epsilon^{\frac{3p}{2}}(1+\epsilon^{\frac{3p}{2}})$ and $|v(t,x)| \le N(T,m,K)(1+|x|)^m$ imply that the second summand on the right of (2.13) is $o(\epsilon^3)$. Let us write the first summand on the right of (2.13) as

$$\mathbf{E}I_{\tau_\epsilon < t+\epsilon^3}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr + \mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\int_t^{t+\epsilon^3} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr. \tag{2.14}$$

Since the coefficients of $L_r$ are uniformly bounded and $r \in [t,\tau_\epsilon]$ implies $|\xi_r - x| < \epsilon$, the integrand in the first summand of (2.14) satisfies

$$\big|\big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\big| \le (K^2 + K\epsilon)\|v_{xx}(t,x)\| + K|v_x(t,x)| + |v_t^{(0)}(t,x)|.$$

Since $\epsilon \in (0,1)$,

$$\begin{aligned} &\Big|\mathbf{E}I_{\tau_\epsilon < t+\epsilon^3}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr\Big| \\ &\le N(K)(\|v_{xx}(t,x)\| + |v_x(t,x)| + |v_t^{(0)}(t,x)|)\cdot P\{\tau_\epsilon < t+\epsilon^3\}, \end{aligned}$$

and hence the first expectation in (2.14) is $o(\epsilon^3)$ by Lemma 2.2. Dividing the second expectation in (2.14) by $\epsilon^3$ and evaluating it explicitly yields

$$\begin{aligned} &\mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} \Big(L_r v(t,x) + v_t^{(0)}(t,x)\Big)\,dr \\ &+ \mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} b(r)^* v_{xx}(t,x)(\xi_r - x)\,dr. \end{aligned} \tag{2.15}$$

Since $t$ is a Lebesgue point for $L_s$, we have (almost surely)

$$I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3}\Big(L_r v(t,x)+v_t^{(0)}(t,x)\Big)\,dr \to L_t v(t,x)+v_t^{(0)}(t,x) \quad \text{as } \epsilon\to 0$$

and since

$$\begin{aligned}&\Big|I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3}\Big(L_r v(t,x)+v_t^{(0)}(t,x)\Big)\,dr\Big|\\ &\le K^2\|v_{xx}(t,x)\|+K|v_x(t,x)|+|v_t^{(0)}(t,x)|,\end{aligned}$$

the first expectation in (2.15) converges to $L_t v(t,x)+v_t^{(0)}(t,x)$ as $\epsilon\to 0$. The second expectation in (2.15) converges to 0 as $\epsilon\to 0$. Recalling that $0\le \tau_\epsilon - t\le \epsilon^3$ and that $r\in[t,\tau_\epsilon]$ implies $|\xi_r-x|<\epsilon$, we immediately get the bound

$$\Big|I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} b(r)^* v_{xx}(t,x)(\xi_r-x)\,dr\Big| \le \frac{1}{\epsilon^3}\|v_{xx}(t,x)\|K\epsilon^4=\|v_{xx}(t,x)\|K\epsilon.$$

Hence dividing (2.13) by $\epsilon^3$ and letting $\epsilon\to 0$, we get $L_t v(t,x)+v_t^{(0)}(t,x)=0$. □

## 3. Fundamental solutions of the Kolmogorov equation - the analytic approach

Even the "analytic" proof that $v(t,x)=\mathbf{E}g(\xi_T(t,x))$ is a solution of the Kolmogorov equation relies on the well known probabilistic fact that since coefficients $\sigma(t)$, $b(t)$ are independent of $\omega$, the vector $\xi_T(t,x)$ is a Gaussian vector with parameters $\big(x+\int_t^T b(r)\,dr, \int_t^T a(r)\,dr\big)$. Hence, the distribution $P\xi_T(t,x)^{-1}$ has density function

$$p(T,t,y)=\frac{e^{-\frac12\langle C^{-1}(t)(y-x-\int_t^T b(r)dr),\,y-x-\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}},$$

where $C(t)=C_T(t)=\int_t^T a(r)\,dr$. From this, it follows that a solution to the problem

$$\begin{cases}\frac12 a^{ij}(t)v_{x^ix^j}(t,x)+b^i(t)v_{x^i}(t,x)+\frac{\partial v}{\partial t}(t,x)=0 & \text{a. e. } t\in[0,T)\\ v(T,x)=g(x) & x\in E_d\end{cases}\tag{3.1}$$

is given by

$$\begin{aligned} v(t,x)&=\mathbf{E}g(\xi_T(t,x))\\ &=\int_{E_d} g(y)\,P\,\xi_T^{-1}(t,x)\,(dy)\\ &=\int_{E_d} g(y)\frac{e^{-\frac12\langle C^{-1}(t)(y-x-\int_t^T b(r)dr),\,y-x-\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}}\,dy,\end{aligned}\tag{3.2}$$

where $a(r)=\sigma(r)\sigma^*(r)$ is non-degenerate. We prove this in Theorem 3.1 below, for slowly increasing $g\in C^0(E_d)$. Viewed analytically, since the function

$$p(T,t,x)=\frac{e^{-\frac12\langle C^{-1}(t)(x+\int_t^T b(r)dr),\,x+\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}}\tag{3.3}$$

is a fundamental solution (in $x$) of the equation $L_t p(t,x) + \frac{\partial p}{\partial t}(t,x) = 0$ a.e. $t \in [0,T)$, all $x \neq -\int_t^T b(r)dr$, where $L_t \equiv \frac{1}{2}a^{ij}(t)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t)\frac{\partial}{\partial x^i}$, a solution to (3.1) will be given by the convolution

$$\begin{aligned} v(t,x) &= [g * p(T,t,\cdot)](x) \\ &= \int_{E_d} g(y)\, p(T,t,x-y)\, dy \\ &= \int_{E_d} g(y) \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(x-y+\int_t^T b(r)dr),\, x-y+\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac{d}{2}}\sqrt{\det C(t)}}\, dy, \end{aligned}$$

providing, of course, we can differentiate under the integral sign. Regarding notation, by *fundamental solution*, we mean that for all $t \in [0,T)$, $p(T,t,x)$ is infinitely differentiable in $x$ and $\int_{E_d} p(T,t,x)\, dx = 1$.

By Lebesgue's differentiation theorem, $p(T,t,x)$ in (2) is differentiable with respect to $t$, only in the *almost everywhere* sense. This is in contrast to the case where $a(t) = I_d, b(t) = b\,(const.)$ and the Kolmogorov equation is simply $\frac{1}{2}\Delta u(t,x) + b \cdot u_x(t,x) + \frac{\partial u}{\partial t}(t,x) = 0$ for all $(t,x) \in H_T$. In this case, $p(T,t,x) = (2\pi(T-t))^{-\frac{d}{2}} e^{\frac{-|x+b(T-t)|^2}{2(T-t)}}$ is infinitely differentiable in both $t$ and $x$.

**Theorem 3.1.** *For $t \in [0,T]$ and $x \in E_d$ and $s \in [t,T]$, let $\xi_s(t,x) = x + \int_t^s \sigma(r)\, d\mathbf{w}_r + \int_t^s b(r)\, dr$, where $\sup_{t \le T}(\|\sigma(t)\| + |b(t)|) \le K$. Assume $\exists \delta > 0$ for which $\delta I_d \le a(t)$, for all $t \in [0,T]$, where $a(t) = \sigma(t)\sigma^*(t)$. Then for $p(T,t,x)$ as in (2) and $g$ continuous and slowly increasing, the function*

$$v(t,x) = \mathbf{E}g(\xi_T(t,x)) = \int_{E_d} g(y)\, p(T,t,x-y)\, dy$$

*satisfies the Kolmogorov equation $\frac{1}{2}a^{ij}(t)v_{x^i x^j}(t,x) + b^i(t)v_{x^i}(t,x) + \frac{\partial v}{\partial t}(t,x) = 0$ a.e. $t \in [0,T)$ and any $x \in E_d$.*

*Proof.* Direct calculation shows that for almost every $t \in [0,T)$ and any $x \neq -\int_t^T b(r)dr \in E_d$, $p(T,t,x)$ is a solution of the Kolmogorov equation. Thus we need only show that we can differentiate under the integral sign. Omitting the constant factor of $(2\pi)^{-d/2}$, direct calculation shows that for almost every $t \in [0,T)$, with $z = y - x$ and $\eta_t := \int_t^T b(r)\, dr$,

$$\begin{aligned} &\frac{\partial p}{\partial t}(T,t,x-y) \\ &= \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(z-\eta_t), z-\eta_t\rangle}}{2\sqrt{\det C(t)}}\Big\{ \operatorname{tr}[a(t)C^{-1}(t)] - \langle C^{-1}(t)\, a(t)\, C^{-1}(t)(z-\eta_t), z-\eta_t\rangle \\ &\quad + 2\langle C^{-1}(t)(y-x), b(t)\rangle + 2\langle C^{-1}(t)\, b(t), \eta_t\rangle \Big\} \end{aligned}$$

and hence

$$\begin{aligned} &\Big|\frac{\partial p}{\partial t}(T,t,x-y)\Big| \\ &\le \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(z-\eta_t), z-\eta_t\rangle}}{\sqrt{\det C(t)}}\Big\{ \|a(t)\|\|C^{-1}(t)\| \\ &\quad + \|C^{-1}(t)\, a(t)\, C^{-1}(t)\||z-\eta_t|^2 + \|C^{-1}(t)\||z||b(t)| + \|C^{-1}(t)\||b(t)||\eta_t| \Big\}. \end{aligned}$$

Since $\sup_{t\le T}(\|\sigma(t)\|+|b(t)|)\le K$ and $\|ab\|\le\|a\|\,\|b\|$, $\|a(t)\|=\|\sigma(t)\sigma^*(t)\|\le K^2$. From the estimate $\|C(t)\|\le\sqrt{T-t}\sqrt{\int_t^T\|a(r)\|^2\,dr}$, we have $\|C(t)\|\le K^2(T-t)$. Moreover, by the uniform non-degeneracy condition $\delta|\lambda|^2\le a^{ij}(t)\lambda^i\lambda^j$, which holds for all $t\in[0,T]$ and all $\lambda\in E_d$, we get $\|C^{-1}(t)\|\le\frac{\sqrt{d}}{\delta(T-t)}$. We also have $C^{ij}(t)\lambda^i\lambda^j\ge\delta|\lambda|^2(T-t)$, from which it immediately follows that $\det C(t)\ge[\delta(T-t)]^d$. Obviously, $|\eta_t|\le K(T-t)$. This gives

$$\begin{aligned}&\Big|\frac{\partial p}{\partial t}(T,t,x-y)\Big|\\&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\\&\times\Big\{\frac{K^2\sqrt{d}}{\delta(T-t)}+\frac{2dK^2}{\delta^2(T-t)^2}\left(|y-x|^2+K^2(T-t)^2\right)+\frac{K\sqrt{d}}{\delta(T-t)}|x-y|+\frac{K^2\sqrt{d}}{\delta}\Big\}.\end{aligned}$$

Similarly, the gradient and hessian of $p(T,t,x-y)$ satisfy

$$p_x(T,t,x-y)=\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{\sqrt{\det C(t)}}\;C^{-1}(t)\cdot(z-\eta_t)$$

$$\begin{aligned}&p_{xx}(T,t,x-y)\\&=\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{\sqrt{\det C(t)}}\left\{C^{-1}(t)(z-\eta_t)[C^{-1}(t)(z-\eta_t)]^*-C^{-1}(t)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}|p_x(T,t,x-y)|&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\|C^{-1}(t)\|\cdot|z-\eta_t|\\&\le\frac{\sqrt{d}\,e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2+1}}\Big\{|y-x|+K(T-t)\Big\},\end{aligned}$$

$$\begin{aligned}&\|p_{xx}(T,t,x-y)\|\\&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\left\{\|C^{-1}(t)\|^2\cdot|z-\eta_t|^2+\|C^{-1}(t)\|\right\}\\&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\Big\{\frac{2d}{\delta^2(T-t)^2}\left(|y-x|^2+K^2(T-t)^2\right)+\frac{\sqrt{d}}{\delta(T-t)}\Big\}.\end{aligned}$$

To estimate the exponential term in each derivative, we use the inequality $\frac{|z-\eta_t|^2}{K^2(T-t)}\le\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle$ and Young's inequality (twice): $|z-\eta_t|^2\ge\big||z|-|\eta_t|\big|^2\ge\frac12|z|^2-|\eta_t|^2\ge\frac12|y-x|^2-K^2(T-t)^2\ge\frac14|y|^2-\frac12|x|^2-K^2(T-t)^2$ to conclude

$$e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}\le e^{-\frac{|y|^2}{8K^2(T-t)}+\frac{|x|^2}{4K^2(T-t)}+\frac{T-t}{2}}.$$

Denoting any of the derivatives $p_t,p_x,p_{xx}$ by $p'(T,t,x-y)$, we see that

$$|p'(T,t,x-y)|\le\frac{N\cdot e^{-\frac{|y|^2}{8K^2(T-t)}+\frac{|x|^2}{4K^2(T-t)}+\frac{T-t}{2}}}{(T-t)^{\frac d2+2}}q(T-t,|y-x|),$$

where $N = N(\delta, d, K)$ and $q(a, b)$ is a paraboloid in $a$ and $b$. Hence if $(t, x) \in [0, t_0] \times B_R$, where $0 \le t_0 < T$, then

$$|p'(T, t, x - y)| \le \frac{N \cdot e^{-\frac{|y|^2}{8K^2 T} + \frac{R^2}{4K^2(T - t_0)} + T}}{(T - t_0)^{\frac{d}{2}+2}} q(T, |y| + R).$$

So if we require that $|g(x)| \le N e^{\frac{|x|^2}{16K^2 T}}$, we see that the integrals $\int_{E_d} g(y)p'(T, t, x - y)\,dy$ converge uniformly with respect to $(t, x) \in [0, t_0] \times B_R$. This implies $v(t, x)$ is twice differentiable with respect to $x$, once differentiable with respect to $t$ (almost everywhere) and its derivatives can be evaluated by differentiating under the integral sign. Since $p(T, t, x - y)$ satisfies the Kolmogorov equation for almost every $t \in [0, T)$, so does $v(t, x)$. $\square$

**Remark.** The above growth condition for $g$ is obviously satisfied when $g$ has polynomial growth, $|g(x)| \le K(1 + |x|^m)$. Furthermore, direct calculation shows that for any $d$-dimensional multi-index $\alpha$, any derivative of $p(T, t, x-y)$ with respect to $x$ satisfies

$$|D_x^\alpha p(T, t, x - y)| \le \frac{N \cdot e^{-\frac{|y-x|^2}{4K^2(T-t)} + \frac{T-t}{2}}}{(T - t)^{\frac{d}{2}+|\alpha|}} \cdot q_\alpha(T - t, |y - x|),$$

where $N = N(\delta, d, K, |\alpha|)$ and $q_\alpha(a, b)$ is a polynomial of degree less than or equal to $|\alpha|$ in $a$ and $b$, from which it follows, as above, that $v(t, x)$ is infinitely differentiable with respect to $x$.

More generally, if $c(t) \ge 0$ is bounded and measurable in $[0, T]$ and we define $\phi_s(t) = \int_t^s c(r)dr$, the function $\widetilde{p}(T, t, x) := p(T, t, x)e^{-\phi_T(t)}$ is an infinitely differentiable solution (in $x$) of the equation $L_t u(t, x) - c(t)u(t, x) + \frac{\partial u}{\partial t}(t, x) = 0$ a.e. $t \in [0, T)$. Since $[g * \widetilde{p}(T, t, \cdot)](x) = e^{-\phi_T(t)} [g * p(T, t, \cdot)](x)$, a solution to the problem

$$\begin{cases} \frac{1}{2}a^{ij}(t)v_{x^i x^j}(t, x) + b^i(t)v_{x^i}(t, x) - c(t)v(t, x) + \frac{\partial v}{\partial t}(t, x) = 0 \\ \qquad \text{a.e. } t \in [0, T), \text{ all } x \in E_d \\ \qquad v(T, x) = g(x) \quad x \in E_d \end{cases}$$

is given by

$$v(t, x) = e^{-\phi_T(t)}\mathbf{E}g(\xi_T(t, x)),$$

while if $\int_0^T |f(r)|e^{-\phi_r(t)}\,dr < \infty$, direct calculation shows that the function

$$v(t, x) = e^{-\phi_T(t)}\mathbf{E}g(\xi_T(t, x)) + \int_t^T f(r)e^{-\phi_r(t)}\,dr \tag{3.4}$$

satisfies

$$\begin{cases} \frac{1}{2}a^{ij}(t)v_{x^i x^j}(t, x) + b^i(t)v_{x^i}(t, x) - c(t)v(t, x) + f(t) + \frac{\partial v}{\partial t}(t, x) = 0 \\ \qquad \text{a.e. } t \in [0, T) \text{ all } x \in E_d \\ \qquad v(T, x) = g(x) \quad x \in E_d. \end{cases}$$

## 4. Paraboloid solutions of the Simplest Time-Measurable Bellman equations

In this section, we prove a result about the payoff function for the Bellman equation in the simple case where the equation depends only on second derivatives and $t$ and the coefficients are Borel measurable functions of $t$. Let $A$ be a separable metric space, where for $(\alpha, t) \in A \times [0,T]$, $\sigma(\alpha, t)$ is a $d \times d_1$ matrix and $f^\alpha(t)$ is a function, both continuous in $\alpha$ and Borel measurable in $t$. Now let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which $(\mathbf{w}_t, \mathcal{F}_t)$ is a $d_1$-dimensional Wiener process. We consider the controlled diffusion process $\xi_s(\alpha, t, x)$, defined for $s \in [0,T]$ by $\xi_s(\alpha, t, x) = x + \int_0^s \sigma(\alpha_r, t+r)\, d\mathbf{w}_r$, where $t \in [0,T], x \in E_d$ are fixed and $\alpha_t$ is a strategy in class $U$, that is, progressively measurable with values in $A$.

Suppose $g \in C^2(E_d)$ and satisfies $|g(x)|$, $|g_{(y)}(x)|$, $|g_{(y)(y)}(x)| \le K(1+|x|^m)$, $\forall\, x, y \in E_d$, where $K, m$ are nonnegative constants. It is known [4] that if for any $\alpha \in A$, $f^\alpha, \sigma(\alpha, \cdot)$ are differentiable with respect to $t$ with derivatives not exceeding $K$, then the payoff function

$$v(t,x) = \sup_{\alpha \in U} \mathbf{E}\Big[\int_0^{T-t} f^{\alpha_r}(r+t)\, dr + g(\xi^\alpha_{T-t}(t,x))\Big] \tag{4.1}$$

satisfies the Bellman equation

$$\sup_{\alpha \in A}\{a^{ij}(\alpha,t) v_{x^i x^j}(t,x) + f^\alpha(t)\} + \frac{\partial v}{\partial t}(t,x) = 0 \quad \text{a. e. } H_T, \quad v(T,x) = g(x).$$

In the special case where $g$ is a paraboloid, the payoff function takes a very convenient form and clearly satisfies the Bellman equation under the weak assumption that $\sup_{\alpha \in A} f^\alpha$, $\sup_{\alpha \in A} \sigma(\alpha, \cdot) \in L_1([0,T]), L_2([0,T])$, respectively.

**Theorem 4.1.** *Let $p(x)$ be any paraboloid defined on $E_d$, i.e. $p(x) = \frac{1}{2}x^* m x + l \cdot x + l_0$, where $m \in E_{d^2}, l \in E_d, l_0 \in E_1$. Then the probabilistic solution of the Bellman equation*

$$\begin{cases} \sup\limits_{\alpha \in A}\{a^{ij}(\alpha,t) v_{x^i x^j}(t,x) + f^\alpha(t)\} + \dfrac{\partial v}{\partial t}(t,x) = 0 & a.e.\ t \in [0,T) \\ v(T,x) = p(x) \quad x \in E_d. \end{cases}$$

*is given by*

$$v(t,x) = p(x) + \int_t^T \sup_{\alpha \in A}\{\operatorname{tr}[a(\alpha, r)m] + f^\alpha(r)\}\, dr. \tag{4.2}$$

*Proof.* From the theory of controlled diffusion processes [2], the probabilistic solution to this Bellman equation is the payoff function (4.1) with $g = p$ and $a(\alpha,t) = \frac{1}{2}\sigma(\alpha,t)\sigma(\alpha,t)^*$. It immediately follows from Itô's formula that $\forall \alpha \in U$, $t \in [0,T]$ and $x \in E_d$, we have

$$\mathbf{E} p(\xi^\alpha_{T-t}(t,x)) = p(x) + \mathbf{E}\int_0^{T-t} \operatorname{tr}[a(\alpha_r, t+r)m]\, dr. \tag{4.3}$$

We give a more direct proof of (4.3) using Wald's identity. Writing $\xi^\alpha_{T-t} = \xi^\alpha_{T-t}(t,x)$, we have $p(\xi^\alpha_{T-t}(t,x)) = \dfrac{\xi^{\alpha\,*}_{T-t} m \xi^\alpha_{T-t}}{2} + l \cdot \xi^\alpha_{T-t} + l_0$ and

$$\xi^{\alpha\,*}_{T-t} m \xi^\alpha_{T-t} = \langle m\xi^\alpha_{T-t}, \xi^\alpha_{T-t}\rangle = \langle mx, x\rangle + 2\langle mx, \eta^{\alpha,t}_{T-t}\rangle + \langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle,$$

where $\eta^{\alpha,t}_{T-t} := \int_0^{T-t} \sigma(\alpha_r, t+r)\, d\mathbf{w}_r$. Writing $m = ODO^*$, where $D = (\lambda^i \delta^{ij})$, we get

$$\langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle = \langle ODO^*\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle = \langle Dz^{\alpha,t}_{T-t}, z^{\alpha,t}_{T-t}\rangle = \sum_{i=1}^{d} \lambda^i \left(z^{\alpha,t,i}_{T-t}\right)^2$$

where

$$z^{\alpha,t}_{T-t} := O^* \cdot \eta^{\alpha,t}_{T-t} = \int_0^{T-t} O^* \cdot \sigma(\alpha_r, t+r)\, d\mathbf{w}_r := \int_0^{T-t} \widetilde{\sigma}(\alpha_r, t+r)\, d\mathbf{w}_r.$$

Orthogonality and the Wald identity yield

$$\mathbf{E}\big(z^{\alpha,t,i}_{T-t}\big)^2 = \mathbf{E}\sum_{k=1}^{d}\Big(\int_0^{T-t} \widetilde{\sigma}^{ik}(\alpha_r, t+r)\, dw^k_r\Big)^2 = \sum_{k=1}^{d} \mathbf{E}\int_0^{T-t} [\widetilde{\sigma}^{ik}(\alpha_r, t+r)]^2\, dr,$$

and hence

$$\begin{aligned}
\mathbf{E}\langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle &= \sum_{i=1}^{d} \lambda^i \sum_{k=1}^{d} \mathbf{E}\int_0^{T-t} [\widetilde{\sigma}^{ik}(\alpha_r, t+r)]^2\, dr \\
&= 2\mathbf{E}\int_0^{T-t} \operatorname{tr}[a(\alpha_r, t+r)m]\, dr.
\end{aligned}$$

By Wald's identity, we also have $\mathbf{E}\langle mx, \eta^{\alpha,t}_{T-t}\rangle = 0$ and $\mathbf{E}[l \cdot \xi^{\alpha}_{T-t} + l_0] = \mathbf{E}[l \cdot (x + \eta^{\alpha,t}_{T-t}) + l_0] = l \cdot x + l_0$. Thus

$$\begin{aligned}
\mathbf{E}p(\xi^{\alpha}_{T-t}(t,x)) &= \mathbf{E}\Big[\frac{\xi^{\alpha *}_{T-t} m \xi^{\alpha}_{T-t}}{2} + l\xi^{\alpha}_{T-t} + l_0\Big] \\
&= p(x) + \mathbf{E}\int_0^{T-t} \operatorname{tr}[a(\alpha_r, t+r)m]\, dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
v(t,x) &= p(x) + \sup_{\alpha \in U} \mathbf{E}\Big[\int_0^{T-t} \operatorname{tr}[a(\alpha_r, t+r)m] + f^{\alpha_r}(r+t)\, dr\Big] \\
&= p(x) + \int_t^T \sup_{\alpha \in \mathcal{A}} \{\operatorname{tr}[a(\alpha, r)m] + f^{\alpha}(r)\}\, dr.
\end{aligned}$$

□

This result is hardly a surprise since the second-order derivatives of any paraboloid are constant. Hence by Lebesgue's differentiation theorem, for any operator $F(b,t)$ for which $\int_0^T |F(b,t)|\, dt < \infty$, the function

$$u(t,x) = p(x) + \int_t^T F(p_{xx}(x), r)\, dr$$

satisfies

$$\begin{cases} F(u_{xx}(t,x), t) + \dfrac{\partial u}{\partial t}(t,x) = 0 & \text{a.e. } t \in [0,T) \\ u(T,x) = p(x) & x \in E_d\,. \end{cases}$$

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