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STURMIAN COMPARISON RESULTS FOR QUASILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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Abstract. We obtain Sturmian comparison results for the nonnegative solutions to Dirichlet problems associated with p-Laplacian operators. From Picone-type identities [4, 9], we obtain results comparing solutions of two types of equations. We also present results related to those operators using Picone-type identities.

1. Introduction

In this work Ω denotes an open and bounded subset of $\mathbb{R}^n$, $n \geq 2$ with $\partial\Omega \in C^\ell$, $\ell \geq 1$. Also $a \in C^1(\overline{\Omega}; 0, \infty))$, $c \in C(\overline{\Omega}; \mathbb{R})$ and functions $f, g \in C^1(\overline{\Omega}; \mathbb{R})$. Define in Ω the operators

$$\begin{aligned} pu &:= \nabla.\{a(x)\Phi(\nabla u)\} \\ Pu &:= \nabla.\{a(x)\Phi(\nabla u)\} + c(x)\phi(u). \end{aligned} \tag{1.1}$$

Associated with the functions f and g define

$$Fu := Pu + f(x, u), Gu := Pu + g(x, u) \tag{1.2}$$

where for $(\zeta, t) \in \mathbb{R}^n \times \mathbb{R}$, $\Phi(\zeta) = |\zeta|^{\alpha-1}\zeta, \phi(t) = |t|^{\alpha-1}t$ and $\alpha > 0$. Solutions of (1.1) or (1.2) with regular boundary data
(e.g. $u|_{\partial\Omega} = g \in C(\overline{\partial\Omega})$) will be supposed to belong to the space

$$D_p(\Omega) := \left\{w \in C^1(\overline{\Omega}; \mathbb{R}) : a(x)\Phi(\nabla w) \in C^1(\Omega; \mathbb{R}) \cap C(\overline{\Omega}; \mathbb{R})\right\}. \tag{1.3}$$

For any other similar domain E, $D_P(E)$ is defined similarly.

1.1. **Picone-type formulae.** Similar to [3, Theorem 1.1], let E be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a regular boundary (e.g. $\partial E \in C^\ell$, $\ell \geq 1$), and define for $\alpha > 0$ and $f, g \in C(\overline{E} \times \mathbb{R}; \mathbb{R})$ the operators

$$\begin{aligned} Fu &:= \nabla.\{a\Phi(\nabla u)\} + c\phi(u) + f(x, u) \\ Gv &:= \nabla.\{A\Phi(\nabla v)\} + C\phi(v) + g(x, v) \end{aligned} \tag{1.4}$$

where $a, A \in C^1(\overline{E}; \mathbb{R}_+)$, $c, C \in C(\overline{E}; \mathbb{R})$.

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Lemma 1.1. *If $u, v \in D_P(E)$ with $v \neq 0$ in E, then from*

$$\nabla.\big\{\frac{u}{\phi(v)}[\phi(v)a\Phi(\nabla u)]\big\} = a|\nabla u|^{\alpha+1} + uFu - c|u|^{\alpha+1} - uf(x,u),$$

and

$$\begin{aligned}\nabla.\big\{u\phi(u)\frac{A\Phi(\nabla v)}{\phi(v)}\big\} &= (\alpha+1)A\phi(u/v)\nabla u.\Phi(\nabla v) - \alpha A|\frac{u}{v}\nabla v|^{\alpha+1}\\ &\quad + \frac{u}{\phi(v)}\phi(u)Gv - C|u|^{\alpha+1} - \frac{u}{\phi(v)}\phi(u)g(x,v),\end{aligned}$$

we obtain

$$\begin{aligned}&\nabla.\big\{\frac{u}{\phi(v)}[\phi(v)a\Phi(\nabla u) - \phi(u)A\Phi(\nabla v)]\big\}\\ &= (a-A)|\nabla u|^{\alpha+1} + (C-c)|u|^{\alpha+1}\\ &\quad + A\big\{|\nabla u|^{\alpha+1} - (\alpha+1)|\frac{u}{v}\nabla v|^{\alpha-1}\nabla u.(\frac{u}{v}\nabla v) + \alpha|\frac{u}{v}\nabla v|^{\alpha+1}\big\}\\ &\quad + \frac{u}{\phi(v)}\big\{[\phi(v)Fu - \phi(u)Gv] + [\phi(u)g(x,v) - \phi(v)f(x,u)]\big\}.\end{aligned} \tag{1.5}$$

The following important inequality is also from [3, Lemma 2.1]: For all $\alpha > 0$ and all $\xi, \eta \in \mathbb{R}^n$,

$$Y(\xi,\eta) := |\xi|^{\alpha+1} + \alpha|\eta|^{\alpha+1} - (\alpha+1)|\eta|^{\alpha-1}\xi.\eta \geq 0\,. \tag{1.6}$$

The equality holds if and only if $\xi = \eta$. For $u, v \in C^1$ define

$$Z(u,v) := Y(\nabla u, \nabla v).$$

Some identities. If $a = A$, $c = C$, $Fu = Gv = 0$ in E then (1.5) becomes

$$\begin{aligned}&\nabla.\big\{\frac{u}{\phi(v)}a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\big\}\\ &= a\{|\nabla u|^{\alpha+1} - (\alpha+1)|\frac{u}{v}\nabla v|^{\alpha-1}\nabla u.(\frac{u}{v}\nabla v) + \alpha|\frac{u}{v}\nabla v|^{\alpha+1}\}\\ &\quad + u\phi(u)\Big[\frac{g(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)}\Big]\\ &:= aZ(u,v) + u\phi(u)\Big[\frac{g(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)}\Big].\end{aligned} \tag{1.7}$$

Define

$$\chi(x,t) := \frac{f(x,t)}{\phi(t)}.$$

For the functions u and v above, if $\Omega \subset E$ is open , non empty and $f(x,t) \equiv g(x,t)$, then after integrating (1.6) over Ω we get for positive u and v

$$\begin{aligned}&\int_{\partial\Omega} au\big\{|\nabla u|^{\alpha-1}\frac{\partial u}{\partial\nu_\Omega} - \phi(\frac{v}{u})|\nabla v|^{\alpha-1}\frac{\partial v}{\partial\nu_\Omega}\big\}ds\\ &= \int_\Omega \big[aZ(u,v) + |u|^{\alpha+1}\{\chi(x,v) - \chi(x,u)\}\big]\,dx\,.\end{aligned} \tag{1.8}$$

After interchanging u and v,

$$\begin{aligned}&\int_{\partial\Omega} av\big\{|\nabla v|^{\alpha-1}\frac{\partial v}{\partial\nu_\Omega} - \phi(\frac{u}{v})|\nabla u|^{\alpha-1}\frac{\partial u}{\partial\nu_\Omega}\big\}ds\\ &= \int_\Omega [aZ(v,u) + |v|^{\alpha+1}\{\chi(x,u) - \chi(x,v)\}]dx\end{aligned} \tag{1.9}$$

where ν_Ω denotes the outward normal unit vector to $\partial\Omega$.

For the operators F and G in (1.1)-(1.2), if u and v satisfy respectively $Fu = Gv = 0$ in Ω, Equation (1.6) leads to

$$\begin{aligned}
&\nabla.\{\frac{u}{\phi(v)}a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\} \\
&:= aZ(u,v) + u\phi(u)\chi(x,v) \quad \text{if } v > 0 \text{ in } \Omega, \\
&\nabla.\{\frac{v}{\phi(u)}a[\phi(u)\Phi(\nabla v) - \phi(v)\Phi(\nabla u)]\} \\
&:= aZ(v,u) - v\phi(v)\chi(x,v) \quad \text{if } u > 0 \text{ in } \Omega.
\end{aligned} \tag{1.10}$$

Remark 1.2. It is a classical result that if u and v are continuous and piecewise-C^1 in $\overline{\Omega}$ and for $pw := \nabla.\{a(x)\Phi(\nabla w)\}$ satisfies weakly

$$\begin{gathered}
G_1 u := pu + g(x,u) \geq 0 \geq pv + g(x,v) \quad \text{in } \Omega; \\
u \leq v \quad \text{in } \overline{\Omega}\,,
\end{gathered}$$

then if $g \in C(\Omega \times \mathbb{R})$ is non decreasing in its second argument , the existence of such u and v leads to the existence of a solution $w \in D_P(\Omega)$ of $pw + g(x,w) = 0$ in $\Omega; w|_{\partial\Omega} = w_0$ for any continuous w_0 satisfying $u \leq w_0 \leq v$ on $\partial\Omega$.

Remark 1.3. Let Ω be bounded, Ω' be an open subset of $\Omega, c \in C(\overline{\Omega})$ and $h \in C(\overline{\Omega} \times \mathbb{R})$. It is known (e.g. [1, 7]) that if $u, v \in D_p(\Omega)$ satisfy (weakly) for $H(w) := \nabla.\{a(x)\Phi(\nabla w)\} + c(x)\phi(w) + h(x,w)$,

$$Hu \geq Hv \quad \text{in } \Omega; \quad (u-v)|_{\partial\Omega'} \leq 0 \tag{1.11}$$

then $(u - v) \leq 0$ in Ω' provided that $\forall x \in \Omega$, $c(x)\phi(w) + h(x,w)$ is non increasing in w for $|w| \leq \max\{|u|_{L^\infty(\Omega)}, |v|_{L^\infty(\Omega)}\}$.

2. Main Results

Let $a, c, \dots$ be as defined in the Introduction. Define in Ω the equations:

$$Pu := \nabla.\{a(x)\Phi(\nabla u)\} + c(x)\phi(u) = 0, \tag{2.1}$$

$$Fv := \nabla.\{a(x)\Phi(\nabla v)\} + c(x)\phi(v) + f(x,v) = 0, \tag{2.2}$$

$$G_1 w := \nabla.\{a(x)\Phi(\nabla w)\} + g(x,w) = 0. \tag{2.3}$$

Following the Remarks 1.2-1.3, we have the following result for the problem

$$G_1 w := pw + g(x,w) = 0 \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0 \tag{2.4}$$

Theorem 2.1. *(1) Assume that for all x in Ω, g is increasing in the second argument and that $a(x) > 0$ is constant in Ω. Then if there is a strictly positive $v \in D_P(\Omega)$ which satisfies $G_1 v \leq 0$ in Ω and $v|_{\partial\Omega} \geq 0$, then* (2.4) *has a solution $u \in D_P(\Omega)$ which satisfies $0 \leq u \leq v$ in Ω.*
(2) If for all x in Ω, g is non increasing in the second argument then (2.4) *has at most one solution in $D_P(\Omega)$.*

Theorem 2.2. *Assume that Ω is bounded and connected and $c \in C(\overline{\Omega})$ is non positive.*

(1) *Let $u \in D_p(\Omega)$ be a solution of*

$$\begin{gathered}
Pu := \nabla.\{a(x)\Phi(\nabla u)\} + c(x)\phi(u) = 0 \quad \text{in } \Omega \\
u|_{\partial\Omega} = 0.
\end{gathered}$$

Then $u > 0$ *in* Ω *if* $\operatorname{meas}\{x \in \Omega : u(x) > 0\} > 0$.

(2) *For the solutions* $w \in D_p(\Omega)$ *of*

$$Fu := \nabla.\{a(x)\Phi(\nabla w)\} + c(x)\phi(w) + f(x, w) = 0 \quad \text{in } \Omega$$
$$w|_{\partial\Omega} = 0$$

the same conclusion holds provided that in $\overline{\Omega}$, $f(x, t) \le 0$ *for* $t \ge 0$.

Theorem 2.3.

(1) *Assume that for all* $x \in \Omega$, $f(x, t) \ge 0$ *for* $t \ge 0$. *Then if* (2.1) *has a strictly positive solution* u *which satisfies* $u|_{\partial\Omega} = 0$, (2.2) *cannot have a solution strictly positive in* Ω . *Consequently if* (2.1) *has a positive solution* u *with the boundary condition* $u|_{\partial\Omega} = 0$ *then any non negative solution* v *of* (2.2) *has a zero inside* Ω.

(2) *If* (2.1) *has a solution strictly positive in* Ω *then if for all* $x \in \Omega$, $f(x, t) \le 0$ *for* $t \ge 0$, (2.2) *has no nontrivial and nonnegative solution* v *satisfying* $v|_{\partial\Omega} = 0$.

Theorem 2.4. *Let* $f \in C(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ *and let* $u, v \in D_p(\Omega)$ *be two solutions of*

$$Fw := \nabla.\{a\Phi(\nabla w)\} + c\phi(w) + f(x, w) = 0; \quad w > 0 \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0.$$

(1) If for all x *in* Ω, $t \mapsto \chi(x, t) = f(x, t)/\phi(t)$ *is strictly increasing and positive in* $t > 0$ *then*

(i) *the two solutions intersect in* Ω ;

(ii) *if for some open* $D \subset \Omega$, $v \ge u$ *in* D *then*

$$\int_{\partial D} au\big\{|\nabla u|^{\alpha-1}\frac{\partial u}{\partial \nu_D} - \phi(\frac{u}{v})|\nabla v|^{\alpha-1}\frac{\partial v}{\partial \nu_D}\big\}ds \ge 0 \tag{2.5}$$

and if in addition $u = v$ *on* ∂D, *then*

$$\begin{gathered}\int_D \big\{aZ(v, u) + |v|^{\alpha+1}X(x, u : v)\big\}dx \le 0 \quad \text{and} \\ \int_D \big\{aZ(u, v) + |u|^{\alpha+1}X(x, v : u)\big\}dx \ge 0,\end{gathered} \tag{2.6}$$

where $X(x, w : z) := \chi(x, w) - \chi(x, z)$.

(2) If for all x *in* Ω

(i) $t \mapsto \chi(x, t) = f(x, t)/\phi(t)$ *is positive and strictly decreasing in* $t > 0$ *or*

(ii) *if* f *is positive and decreasing in* $t > 0$ *then the two solutions coincide.*

(3) For connected Ω, *the problem*

$$Pw = \nabla.\{a\Phi(\nabla w)\} + c\phi(w) = 0 \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0$$

has at most one non negative solution in $D_P(\Omega)$.
This problem has at most one strictly positive solution even if Ω *is not connected.*

3. Proofs of the main results

Proof of Theorem 2.1. (1) Taking in account remark 1.2, we just need to build a subsolution $w \in D_P(\Omega)$, such that

$$G_1 w \ge 0 \ge G_1 v \quad \text{and} \quad 0 \le w \le v \quad \text{in } \Omega.$$

Because $v > 0$ in Ω we consider any nonnegative $U \in C(\overline{\Omega})$ which is piecewise afine; i.e., there exists $\mathcal{N} := \{\eta_i; i = 1, 2, \dots, M\}$ and some finite number (pairwise disjoint) of subsets B_i, $1 \le i \le N$ of Ω such that with $x = (x_1, x_2, \dots, x_n) \in \Omega$

(i) $B := \bigcup_{i=1}^N B_i \subset \Omega$;
(ii) $\forall i, U(x) = \sum_{i=1}^n \eta_i x_i < v(x)$ for $x \in B_i$;
(iii) $U|_{\partial B} = 0$ and is extended by 0 outside B in Ω.

Thus as $a(x)$ is positive and constant in Ω,

$$GU = g(x, U) \ge 0 \ge Gv \quad \text{and} \quad 0 \le U \le v \quad \text{in } \Omega.$$

The solution u of $pu + g(x, U) = 0$ in $\Omega; u|_{\partial\Omega} = 0$ is in $D_P(\Omega)$ and satisfies $G_1 u = pu + g(x, u) \ge 0 \ge G_1 v$ and $0 \le u \le v$ in Ω . Thus from Remark 1.2, this leads to the existence of such a required solution.
(2) Let g be decreasing in the second argument. Suppose that there are two distinct solutions u and $v \in D_P(\Omega)$ such that for some subset B of Ω whose measure is strictly positive $v > u$ in B and $(u - v)|_{\partial B} = 0$. In that case, as g is decreasing,

$$pu - pv = g(x, v) - g(x, u) \le 0 \quad \text{in } B \quad \text{and} \quad (u - v)|_{\partial B} \ge 0.$$

This leads to $u \ge v$ in B, conflicting with the assumption. Therefore any such two solutions have to coincide in Ω. □

The proof of Theorem 2.2, follows from the lemma below.

Lemma 3.1. *(1) Let $u \in D_p(\Omega)$ be a solution of*

$$\begin{gathered} pu := \nabla.\{a(x)\Phi(\nabla u)\} = 0 \quad \text{in } \Omega; \\ u|_{\partial\Omega} = 0; \quad \text{meas}\{\Omega^+\} > 0 \end{gathered} \tag{3.1}$$

where $\Omega^+ := \{x \in \Omega : u(x) > 0\}$ and $\Omega^- := \{x \in \Omega : u(x) > 0\}$. Then $u \ge 0$ a.e. in Ω. Moreover if in addition Ω is connected then $u > 0$ in Ω.
(2) The same conclusions hold for the problems

$$\begin{gathered} Pu := \nabla.\{a(x)\Phi(\nabla u)\} + c(x)\phi(u) = 0 \quad \text{in } \Omega; \\ u|_{\partial\Omega} = 0; \quad \text{meas}\{\Omega^+\} > 0 \end{gathered} \tag{3.2}$$

where $c \in C(\overline{\Omega}; \mathbb{R})$ remains non positive in Ω.

The same conclusion holds for the operator F if in $\overline{\Omega} \times \mathbb{R}_+$ the function f is non positive.

Proof. (1) Let $k := \max_{\Omega^-} |u(x)|$ and the function $v(x) := u(x)_+ + k$.
As $(\nabla u - \nabla v)|_{\Omega^+} \equiv 0$, $Z(u, v) = 0$ and weakly in Ω^+,

$$\nabla.\{\frac{u}{\phi(v)} a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\} = \frac{u}{\phi(v)}\{\phi(v) - \phi(u)\}\nabla[a(x)\Phi(\nabla u)] = 0$$

by (1.5) and (3.1). So, as v is constant in Ω^-,

$$\nabla.\{\frac{u}{\phi(v)} a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\} = \begin{cases} aZ(u, k) & \text{in } \Omega^-, \\ 0 & \text{otherwise.} \end{cases}$$

This implies after integration over Ω that

$$0 = \int_{\Omega^-} a(x)Z(u, k)dx = \int_{\Omega^-} a(x)|\nabla u|^{\alpha+1}dx > 0$$

which is absurd unless meas$\{\Omega^-\} = 0$. The fact that $a \in C^1(\overline{\Omega}; (0,\infty))$ makes the operator p here satisfy the conditions required for the case of the following maximum principle.

> [1, Theorem 2.2] If the bounded domain Ω is connected , $p \in (1,\infty)$ and $u \in W^{1,p}_{\rm loc}(\Omega) \bigcap C^0(\Omega)$ satisfies $-\operatorname{div} A(x, Du) + \Lambda |u|^{p-2}u \geq 0$, $u \geq 0$ in Ω for a constant $\Lambda \in \mathbb{R}$ then either $u \equiv 0$ or $u > 0$ in Ω.

(2) If $c \leq 0$ in Ω and meas$\{\Omega^-\} > 0$ proceeding as above with v defined as before,

$$\begin{aligned}
&\nabla.\big\{\frac{u}{\phi(v)} a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\big\} \\
&= \begin{cases} u\{pu + c\phi(u)\} - u\phi(\frac{u}{v})\{pu + c\phi(v)\} & \text{in } \Omega^+ \\ aZ(u,k) + u\{pu + c\phi(u)\} - u\phi(\frac{u}{v})c\phi(v) & \text{in } \Omega^-. \end{cases} \\
&= \begin{cases} upu\{1 - \phi(\frac{u}{v})\} & \text{in } \Omega^+ \\ aZ(u,k) + upu & \text{in } \Omega^-. \end{cases}
\end{aligned} \tag{3.3}$$

From (3.2), $upu = -c\phi(u) \geq 0$ in Ω provided that c is non positive there.

For the operator F , (3.3) reads

$$\begin{aligned}
&\nabla.\big\{\frac{u}{\phi(v)} a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)]\big\} \\
&= \begin{cases} -c(x)u\phi(\frac{u}{v})\phi(v)\{\phi(v) - \phi(u)\} - u\phi(\frac{u}{v})\{f(x,v) - f(x,u)\} & \\ +u\phi(\frac{u}{v})f(x,v) - uf(x,u) & \text{in } \Omega^+ \\ aZ(u,k) + \frac{u}{\phi(k)}\{-\phi(u)[c(x)\phi(k) + f(x,k)] & \\ +\phi(u)f(x,k) - \phi(k)f(x,u)\} & \text{in } \Omega^- \end{cases} \\
&= \begin{cases} -c(x)u\phi(u)\{1 - \phi(\frac{u}{v})\} + uf(x,u)\{\phi(\frac{u}{v}) - 1\} & \text{in } \Omega^+ \\ aZ(u,k) - c(x)u\phi(u) - uf(x,u) & \text{in } \Omega^-. \end{cases}
\end{aligned} \tag{3.4}$$

Integrating of both sides of (3.3) and (3.4) over Ω provides an absurdity as the left would be zero while the right would be strictly positive, unless Ω^- has measure zero. This completes the proof. □

Proof of Theorem 2.3. (1) If v and u are respectively solutions of

$$\begin{gathered} Fv = 0; \quad v > 0 \quad \text{in } \Omega \quad \text{and} \\ Pu = 0; \quad u \geq 0 \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0 \end{gathered} \tag{3.5}$$

with $f \in C(\overline{\Omega} \times \mathbb{R}; [0,\infty))$. As in (1.5) we have

$$\nabla.\big\{\frac{u}{\phi(v)}[\phi(v)a\Phi(\nabla u) - \phi(u)a\Phi(\nabla v)]\big\} = aZ(u,v) + u\phi(\frac{u}{v})f(x,v) > 0\,.$$

Then integrating both sides of the equation leads to a contradiction.
(2) Similarly if in (3.5), $u > 0$ in Ω and $v|_{\partial\Omega} = 0$ after interchanging u and v in (1.5) we get to

$$\nabla.\big\{\frac{v}{\phi(u)} a[\phi(u)\Phi(\nabla v) - \phi(v)\Phi(\nabla u)]\big\} = aZ(v,u) - vf(x,v) > 0.$$

Then we complete as above. □

Proof of Theorem 2.4. The statement (2.5) follows from (1.8). Adding (1.8) and (1.9), we get

$$\begin{aligned}
&\int_{\partial D} a(u-v)\{\Phi(\nabla u)-\Phi(\nabla v)\}.\nu_D ds \\
&= \int_D \big\{aZ(u,v)+aZ(v,u)+[|u|^{\alpha+1}-|v|^{\alpha+1}](\chi(x,v)-\chi(x,u))\big\}dx
\end{aligned}$$

leading to (2.6). For the two solutions, (1.6) (and interchanging u and v) leads (after integration over Ω) to

$$\begin{aligned}
0 &\le \int_\Omega aZ(u,v)dx \\
&= -\int_\Omega u\phi(u)\Big\{\frac{f(x,v)}{\phi(v)}-\frac{f(x,u)}{\phi(u)}\Big\}dx \\
&= -\int_\Omega |u|^{\alpha+1}\{\chi(x,v)-\chi(x,u)\}dx.
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
0 &\le \int_\Omega aZ(v,u)dx \\
&= -\int_\Omega v\phi(v)\Big\{\frac{f(x,u)}{\phi(u)}-\frac{f(x,v)}{\phi(v)}\Big\}dx \\
&= \int_\Omega |v|^{\alpha+1}\{\chi(x,v)-\chi(x,u)\}dx.
\end{aligned} \tag{3.7}$$

Assume that $\chi(x,t)$ is increasing: If we suppose that $v>u$ in Ω then (3.6) provides a contradiction and if we suppose that $u>v$, (3.7) would lead to a contradiction. Assume that $\chi(x,t)$ is decreasing and define $\Omega_+ := \{x\in\Omega : X(x) := \chi(x,v)-\chi(x,u)>0\}$ and $\Omega_- := \{x\in\Omega : X(x) := \chi(x,v)-\chi(x,u)<0\}$. Then (without loss of generality) $0<v<u$ in Ω_+ and $v>u>0$ in Ω_- whence

$$\begin{aligned}
\int_{\Omega_+}|v|^{\alpha+1}X(x)dx &\le \int_{\Omega_+}|u|^{\alpha+1}X(x)dx, \\
\int_{\Omega_-}|v|^{\alpha+1}X(x)dx &\le \int_{\Omega_-}|u|^{\alpha+1}X(x)dx.
\end{aligned} \tag{3.8}$$

This implies from (3.6) and (3.7) that

$$0 \le \int_\Omega |v|^{\alpha+1}X(x)dx \le \int_\Omega |u|^{\alpha+1}X(x)dx \le 0$$

whence $\int_\Omega Z(u,v)dx = 0$, leading to $v\equiv u$ in Ω by (1.6). If f is nonnegative and decreasing in t, χ is decreasing in t and the same conclusion is reached.
(3) The statement follows immediately from (1.8) or (1.9) as we would get for any such two solutions $0=\int_\Omega a(x)Z(u,v)dx$ the right hand side being strictly positive unless $u\equiv v$ in Ω. □

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