CHARACTERISTICS OF THE BOSE GLASS PHASE
IN DISORDERED OPTICAL LATTICES

by

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"…It is our responsibility as scientists, knowing the great progress which comes from a satisfactory philosophy of ignorance, the great progress which is the fruit of freedom of thought, to proclaim the value of this freedom; to teach how doubt is not to be feared but welcomed and discussed; and to demand this freedom as our duty to all coming generations." – R. Feynman

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CHAPTER 1

1. INTRODUCTION

1.1 Motivation

From the seminal work done in the 1980's in regards to modeling bosonic systems, phase transitions at zero kelvin have been of immense interest [1]. The Bose Hubbard model, an extension of the Hubbard model, was first introduced in 1963 by Gersch and Knollman to quantify ground state phenomena of bosons [2]. This model can become very complicated very quickly and thus approximations were quickly in order. Quantum Monte Carlo simulations are the most detailed and are often considered the most accurate of approximations, but require immense computational power and thus are meant more for those with the necessary computational resources.[3,4,5,6.] Other approximations surfaced due to the need for less computationally heavy methods in modeling bosonic phase transitions. Local density approximations [7] allow for an accurate study of the insulating phase in bosons where a zero probability to tunnel between neighboring sites localizes particles into an insulating state. From local density approximations came mean field approximations which have been studied extensively in different dimensions with different system parameters as a focus [1,8,9,10]. With the minimized computation time offered by mean field approximations, this is used as a simplification to the Bose Hubbard model in this thesis. As much of the literature focuses on only specific system parameters and an expansive study in regards to the Bose glass phase in two dimensions is lacking, this thesis aims to extend the conclusions of past theory to encompass multiple variations of multiple different system parameters.

The Bose glass was first proposed as a phase in disordered bosonic systems at zero kelvin that prevents the direct transition from insulator to superfluid. The Bose glass is an emergent phase, which presents when disorder is added to a system of bosons trapped in a periodic potential (e.g. an optical lattice). Without disorder, either an insulating or a superfluid phase exists. If an external potential is introduced (through the use of a speckle field [11] or second commensurate lattice [12]), the system becomes disordered, and a third phase called the Bose glass presents [1].

Phases are defined in bosonic systems by order parameters, where an abrupt change in one marks a phase transition. Number density, superfluid density and compressibility are used in bosonic systems and are defined in **1.4**.

Bosons are of interest as they do not adhere to the Pauli Exclusion Principle, and thus, many may exist in the same quantum state at the same time. M.P.A Fisher and his group completed the first somewhat exhaustive study of bosons in periodic and/or external potentials, arguing that a Bose glass phase exists in the presence of uniformly random disorder [1]. Their paper states, however, that using mean field theory, the Bose glass phase is most often missed due to over estimations in the coherence (superfluidity) of the system. Fisher et al worked in one dimension and their ideas are extended to two dimensions.

A.E. Niederle and H. Rieger [10] studied mean field theory in two dimensions, comparing zero disorder and uniformly disordered systems. They postulate that a Bose glass phase is realized in fully disordered systems using the mean field approach, both in spatially averaged and local order parameters. The local phenomena of the Bose glass are defined using geometry called superfluid clusters which are regions in real space with non-zero superfluid density that appear as islands of superfluid in a background of Mott insulator. This definition of the Bose glass is extended to simple disorder configuration.

One of the most important visualization tools used to study the phase transitions of bosonic systems is a phase diagram. Buonsante et al [9] complete an in depth study of phase diagrams in one dimension, showing how increasing the complexity in the random disorder changes the behavior of phase space phenomena; that study is extended to two dimensions here.

Further, as computation time is usually a limiting factor in many scientific calculations, a method was devised to approximate phase diagrams; disordered system phase diagrams can often take many days to compute even in the mean field approximation. Zero disorder phase diagrams are "shifted" using the effective potential of the desired disordered system and the results are spatially averaged to give remarkably similar results to the fully calculated phase diagrams. From the study of comparing order parameters between zero disorder and disordered systems, phase transitions in mean field theory were determined to be a local phenomenon. This conclusion shows that real space plots are the important tools for studying phase phenomena using mean field approximations. Thus, the phase diagram approximation technique devised allows one to see approximate locations of phase transitions and points of interest without having to go through the laborious process of full phase diagram calculations. Only after full real space studies are the full phase spaces calculated for completeness.

1.2 Model

The model used to calculate energy in a two dimensional bosonic system is the Bose Hubbard Hamiltonian:

$$H_{BHH} = -t\sum_{<i,j>}\left(\hat{a}_i{}^\dagger\hat{a}_j + \hat{a}_i\hat{a}_j^\dagger\right) + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\sum_i \hat{n}_i + \sum_i \Delta_i. \tag{1}$$

Here, t is the hopping probability amplitude between neighboring sites i and j, $\hat{a}_i{}^\dagger(\hat{a}_i)$ is the creation (annihilation) operator for a boson in state i, U is the interaction potential between two bosons in the same state, $\hat{n}_i = \hat{a}_i{}^\dagger\hat{a}_i$ is the number operator, μ is the chemical potential, and Δ is the random on-site potential used to add disorder to the system. Throughout this thesis, all energy values in the Hamiltonian are in units of 1/U and are thus unit-less.

1.3 Mean Field Approximation

As computation time is often the most limiting factor in calculations, a mean field approximation technique is utilized. Mean field theory approximates a many body, many site Hamiltonian by reducing it to a many body single site Hamiltonian. This is achieved by approximating the hopping term in the Hamiltonian with average values for the operators at all neighboring sites,

$$\hat{a}_i^\dagger\hat{a}_j \approx \hat{a}_i\langle\hat{a}_j^\dagger\rangle + \langle\hat{a}_i\rangle\hat{a}_j^\dagger - \langle\hat{a}_j^\dagger\rangle\langle\hat{a}_i\rangle, \tag{2}$$

Where the total Hamiltonian becomes

$$H_{BHH(2)} = \sum_i h_i, \tag{3}$$

And the sub-Hamiltonian is of the form

$$\sum_j h_i = -t\left(a_i^\dagger\langle a_j\rangle + \langle a_j^\dagger\rangle a_i\right) + \frac{U}{2}n_i(n_i + 1) + (V_i - \mu)n_i. \tag{4}$$

1.4 Order Parameters

The system phases are defined both globally (in phase space) and locally (in real space) using number density (n), superfluid density (ρ) and compressibility (κ), either spatially averaged ($\bar{n}, \bar{\rho}, \bar{\kappa}$) or site-by-site ($n_i, \rho_i, \kappa_i$). Globally, the order parameters in real space are averaged over an n x n site, two dimensional lattice. Locally, each site is looked to individually. Number density is defined as the average value of the number operator,

$$\bar{n} = \frac{1}{N}\sum_i \langle \hat{n}_i \rangle = \sum_i \langle a_i^{\dagger} a_i \rangle, \qquad (5)$$

where N is the number of total lattice sites. Superfluid density is the measure of coherence (connectivity across a lattice) and is defined as the average value of the annihilation operator,

$$\bar{\rho} = \frac{1}{N}\sum_i \rho_i = \frac{1}{N}\sum_i \langle a_i \rangle \qquad (6)$$

Compressibility, however, is measured in two ways: (1) Globally, compressibility measures the change in particle density with respect to a change in chemical potential ($\bar{\kappa}$), and (2) Locally, compressibility measures the variance in number density (κ_i). The derivative definition of compressibility makes more sense globally as the chemical potential is constant at all sites in real space for local calculations. This then leads naturally to the fact that the variance in number density is more appropriate for local calculations as the variance only measures the change in number density across the lattice. Thus,

$$\bar{\kappa} = \frac{1}{N}\sum_i \frac{dn_i}{d\mu}, \tag{7}$$

$$\kappa_i = \frac{1}{N}\sum_i(\langle \hat{n}_i^2\rangle - \langle \hat{n}_i\rangle^2). \tag{8}$$

The phases of a bosonic system can thus be defined using the prescribed order parameters. The Mott insulating phase, characterized by low kinetic energy, is defined to have an integer number density. Due to the low kinetic energy (hopping amplitude) and thus low probability of particles hopping between sites, the Mott insulator has zero superfluid density and compressibility.

The superfluid phase is characterized by a large kinetic energy and a variable, non-integer number density. Although number density is non-integer in phase space, some sites in real space may contain integer number density; thus, number density is not necessarily an order parameter that can individually define a superfluid phase. However, since the hopping potential is large, particles have a greater propensity to move across a lattice and thus the superfluid density and compressibility are non-zero in a superfluid region.

In the Bose glass, theory states that number density and superfluid density behave like that of the Mott insulator, but that the glass is compressible like that of the superfluid. The addition of a phase definition for the Bose glass was given by reference 10 whereby certain real space geometry called “superfluid clusters” were defined to differentiate the phase. In this definition, the Bose glass presents with superfluid clusters in a background of Mott insulator in real space, such that there are islands of non-zero superfluid density that have not yet percolated (connected) completely across the lattice;

this definition holds more soundly in terms of reproducibility than the characterization of a Bose glass in spatially averaged quantities.

1.5 Calculation Method

Solutions to the Bose Hubbard model are calculated using nested functions in Matlab r2014a; functions work for all versions of Matlab from r2012a onward. Disorder is added using a random seed generator as a nested function, to allow for purely random configurations over real space lattices. The ladder operators and interaction term are a second nested function using switch and case statements to ensure the system is in the number basis. The two nested functions are called into the third, main function which solves for the energy eigenvalues using an iterative process for the mean field approximations in real space and simple sum functions for the spatially averaged values in phase space. All plots are created using pseudocolor, contour and histogram plots in Matlab.

1.6 Outline

In Chapter 2, the zero disorder system is studied as a starting point for the characterization of phases and transitions. The method for studying both phase space and real space is shown and is used as the foundation for the study of disordered system. As the clean system is the simplest system to understand, it is the most logical place to start.

Chapter 3 is the bulk of the thesis showing the effects of adding disorder slowly with simple disorder configurations. Binary, ternary and quaternary disorder are the simple disorder configurations studied and these are compared and contrasted with the uniformly disordered system. Conclusions are made about the Bose glass phase both in phase space and real space and these results are compared to the results of M.P.A. Fisher et al [1], Niederle and Rieger [10] and Buonsante [9] as described in section **1.1**.

Chapter 4 discusses the relationship between order parameters, specifically between number density and compressibility. It is shown that there is a distinct similarity between clean and disordered systems and the unique pair values of number density and compressibility. This relationship is used to show the local nature of phase transitions in disordered systems using mean field approximations.

The results of Chapter 4 are used in Chapter 5 to describe a method devised to quickly and accurately approximate phase diagrams.

The overall conclusions are made in Chapter 6 and the results of real and phase space characterizations are given for the Bose glass phase.

CHAPTER 2

2. CLEAN (ZERO DISORDER) SYSTEM

In the absence of disorder, a direct transition occurs from an insulating to a superfluid phase at a critical hopping potential value. To see this transition, one plots each order parameter as a function of hopping potential, while holding chemical potential constant.

2.1 Phase Space

Fig. 1 shows that each order parameter definitively indicates the phase transition at the same hopping potential value of $t/U = 0.03825$.

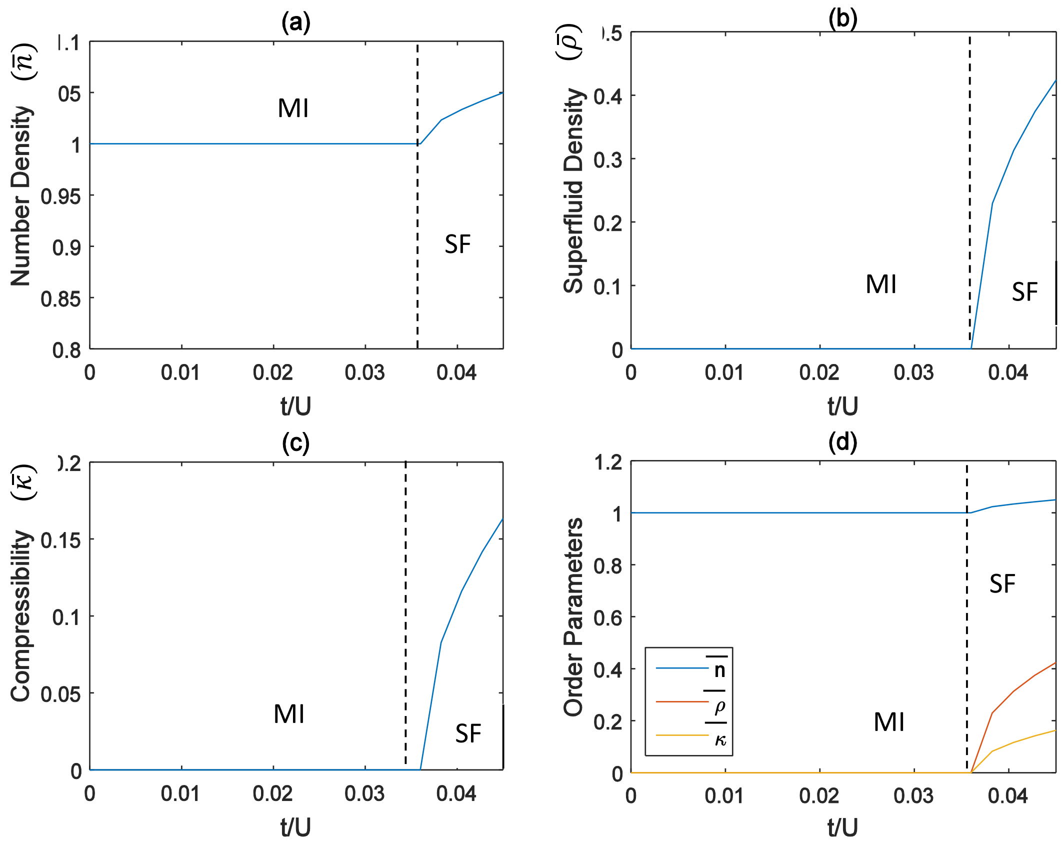

Fig. 1 Number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of hopping potential for zero disorder; chemical potential is held constant at μ = 0.5. All 3 order parameters **(d)** plotted together as a function of hopping potential. The dashed line marks the insulator to superfluid transition. Energy parameters are in terms of 1/U.

To see a more complete picture of the phase transitions, one looks to the phase diagrams of each order parameter as a function of both chemical and hopping potential over a range of each parameter. This allows one to see the typical lobe behavior of the Mott insulator for small hopping potential surrounded by a region of superfluid. Here chemical potential is ranged from μ = [0, 2] and hopping potential from t = [0, 0.045], while all energy parameters are in units of $\frac{1}{U}$.

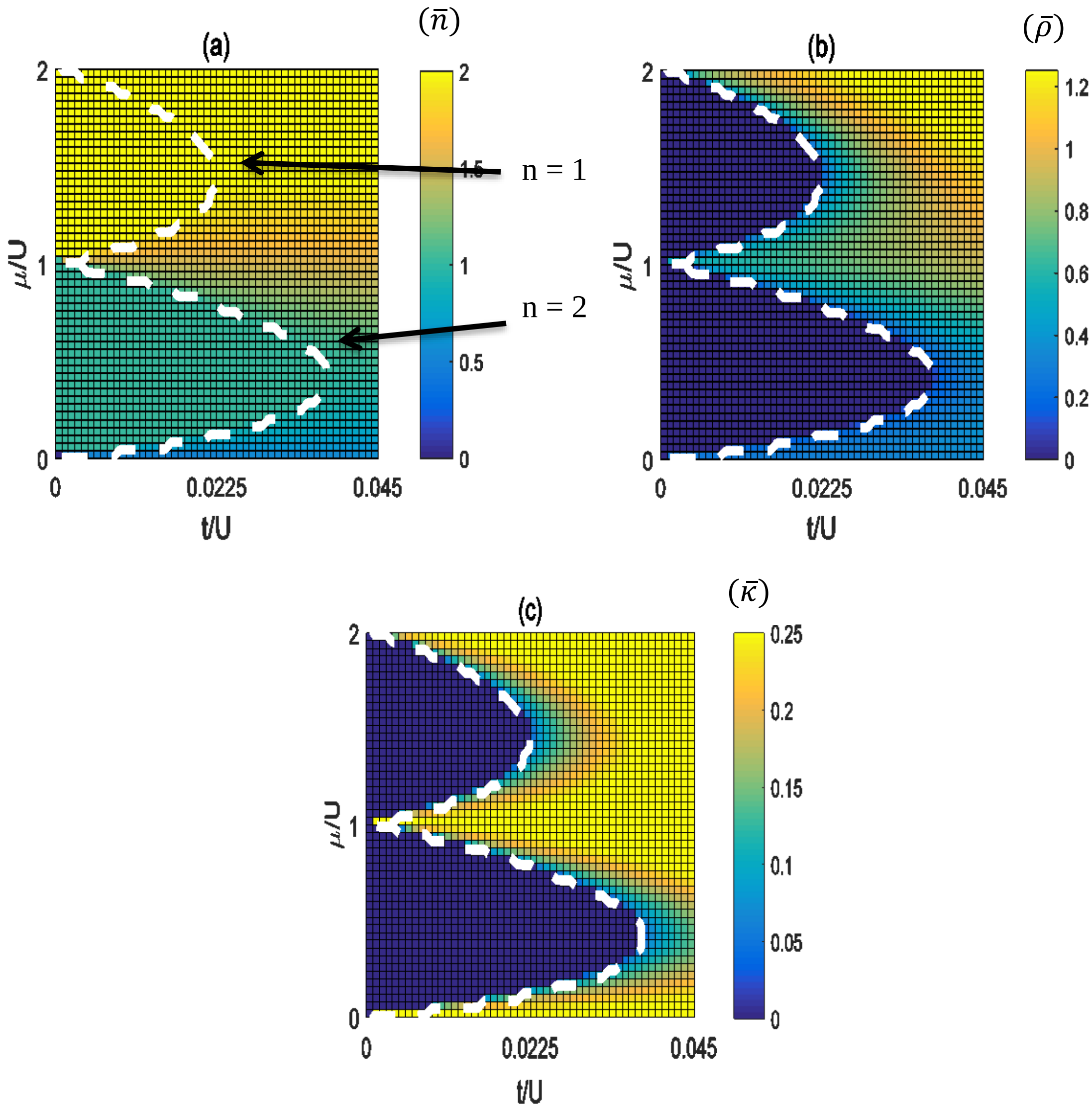

Fig. 2 Phase diagrams for number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of chemical and hopping potential for a zero disorder system. A white dashed line is superimposed to guide the eye along the phase boundaries

Number density in a clean system in phase space is an integer in the Mott insulator and is a non-integer as hopping potential is increased out of the Mott lobes into the superfluid region. Superfluid density is zero in the Mott insulator region and non-zero in the superfluid region. As with the superfluid density, compressibility is zero in the Mott insulating region and non-zero outside. The code used to compute these order parameters has a convergence of 10E-5, thus all values lower are effectively zero; on the logarithmic plots, these are values less than -5.

Using the tool of the phase diagram allows one to choose a chemical potential and hopping potential pair in phase space and map the order parameters over a real space lattice for a given phase space point; this allows one to study local phase behavior and compare and contrast average phase space phenomena and real space phenomena.

2.2 Real Space

Without an external disorder term, the clean system effective potential is a constant. This implies that each order parameter will be homogeneous in real space for a clean system, and one should expect a single value for each order parameter for each phase. This also implies that system size is negligible in a clean system. Thus, each order parameter in each phase is mapped over a 100 site, two dimensional lattice for the clean system. **Table 1** shows the chemical potential and hopping potential pairs chosen for the real space analysis of a clean system.

Table 1 Chemical potential and hopping potential value used to generate real space plots. Energy parameters are in units of 1/U.

Energy Parameter	MI	SF
μ/U	0.5	0.5
t/U	0.0275	0.1

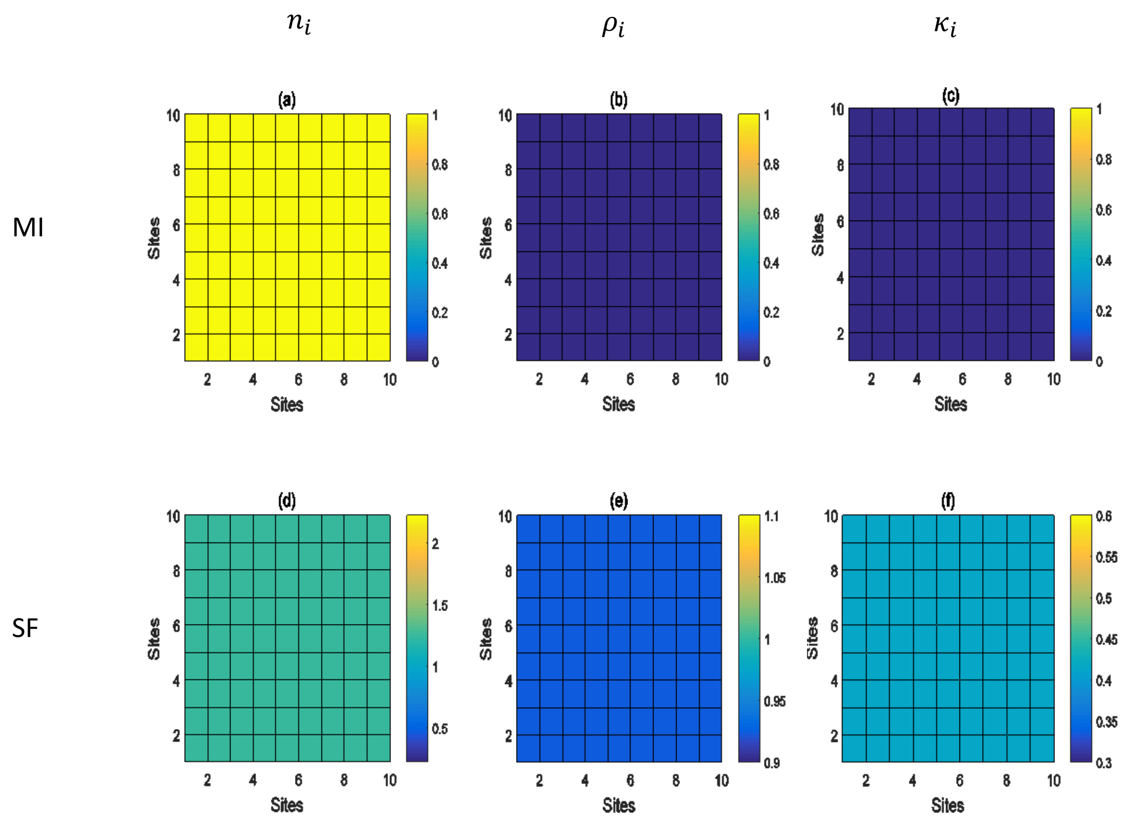

Fig. 3 Real space plots of each order parameter for zero disorder over a 100-site lattice (10 x 10).

As is predicted, **Fig. 3** shows that the order parameters are homogenous in real space for a zero disorder system in both the Mott insulator and the superfluid phases. In the Mott insulator, number density is an integer at all sites (n = 1) while both superfluid density and compressibility are both zero at all sites. In the superfluid, number density is a non-integer (n = 1.2246) at all sites and superfluid density and compressibility are both non-zero (ρ = 0.9223, κ = 0.4142) at all sites.

To ensure that every site in real space does in fact have the same value, one looks to population density plots; this may seem excessive for the clean system, but as disorder is added and becomes more complex, the distribution of order parameter values over the real space lattice become important in distinguishing phases. Histograms are used to show the percent of sites occupied by a given order parameter value.

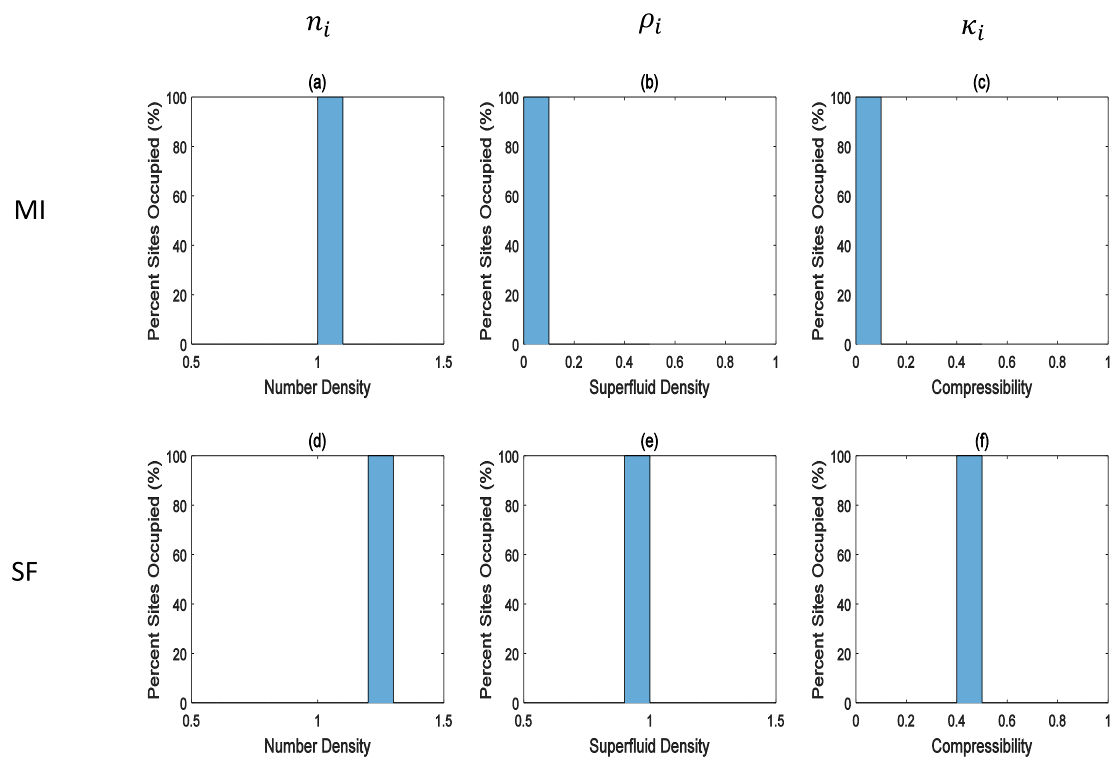

Fig. 4 Population density for zero disorder.

Fig. 4 confirms that every site in real space for a clean system contains the same value for a given order parameter.

The clean system is straightforward to understand. The potential energy term is the constant effective potential of the system and thus the energy outputs are homogenous in real space. In phase space, the Mott lobes each have an exact integer value for number density that increases with increasing chemical potential; the superfluid region surrounds the Mott lobes completely and has a non-integer number density value that increases both with hopping potential and chemical potential. Each Mott lobe is defined by a range of chemical potential and hopping potential values whereby the particles with these values are localized and thus superfluid density and compressibility is zero throughout. The opposite is true for the superfluid region, whereby the particles are able to tunnel or hop between sites given a large enough kinetic energy; this results in a non-zero superfluid density and compressibility due to the coherence across the system.

The next four chapters study the effects of adding disorder to the system in increasing complexity, ultimately looking to a uniformly disordered system in an attempt to mimic real world systems. It is shown that the true nature of the system is elusive in mean field theory and that the Bose glass only presents as a real space phenomenon.

CHAPTER 3

3. DISORDERED SYSTEMS

3.1 Binary Disorder

To study and characterize the Bose glass phase, disorder is introduced to the system by adding the random external potential term (Δ) to the effective potential of the Hamiltonian. The simplest disorder configuration is binary disorder, whereby the random on-site potential has a 50% chance of assuming either of two available values. Δ = 0.25 is the absolute disorder strength used for all disorder configurations in this thesis.

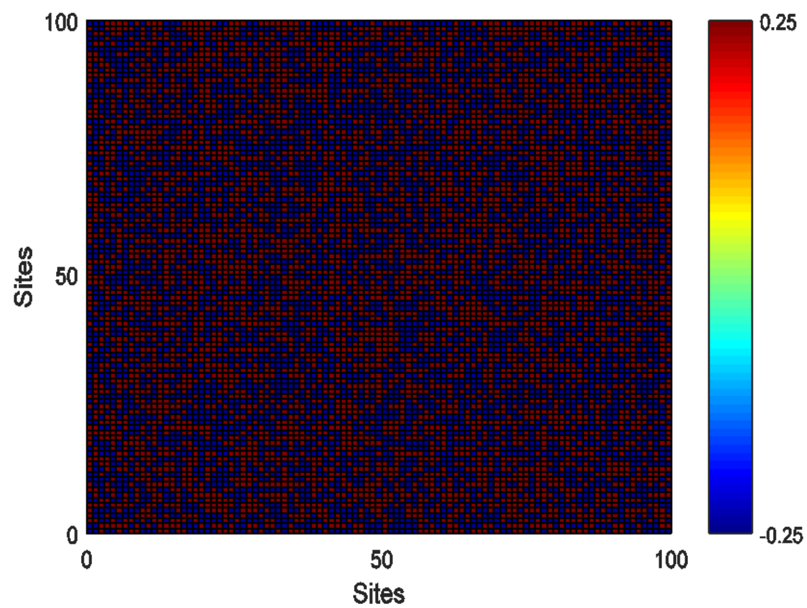

Fig. 5 Binary disorder map over a 10000 site lattice. Sites are randomly occupied by a disorder strength of -0.25 or +0.25.

The disorder potential is randomly distributed over an n x n site lattice (**Fig. 5**) as a term in the effective potential such that one of two sub-Hamiltonians determines site energy:

$$h_{+\Delta} = \frac{1}{U}\sum_i(\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^{\dagger}\hat{a}_j, \tag{9}$$

$$h_{-\Delta} = \frac{1}{U}\sum_i(-\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^{\dagger}\hat{a}_j. \tag{10}$$

Here, **Eqn. 9** is for the positive disorder potential and **eqn. 10** the negative. Looking at these two equations, it appears evident that in evolving from the clean system, the chemical potential values in the clean system will be shifted up or down by Δ/U, changing the shape and location of the Mott lobes.

3.1.1. Phase Space

Past theory states that in the presence of disorder, the Bose glass phase should present in phase space as an intermediate phase that prevents a direct transition from Mott insulator to superfluid [1]. As in the clean system, the first visualization method chosen is to plot each order parameter as a function of hopping potential. Initially, the major difference when disorder is added is the postulation of an emergent phase presenting which has compressibility behavior akin to the superfluid phase but superfluid behavior the same as an insulator. Since homogenity across real space is not expected in a disordered system, then one speaks of spatially averaged quantities in phase space and local, site-by-site quantities in real space. Thus, to look to the order parameters as a function of hopping potential, one must use spatially averaged quantites for the order parameters. Since the Bose glass is defined to have zero superfluid density but non-zero compressibility, one first looks to the spatially averaged order parameters to see if this definition holds.

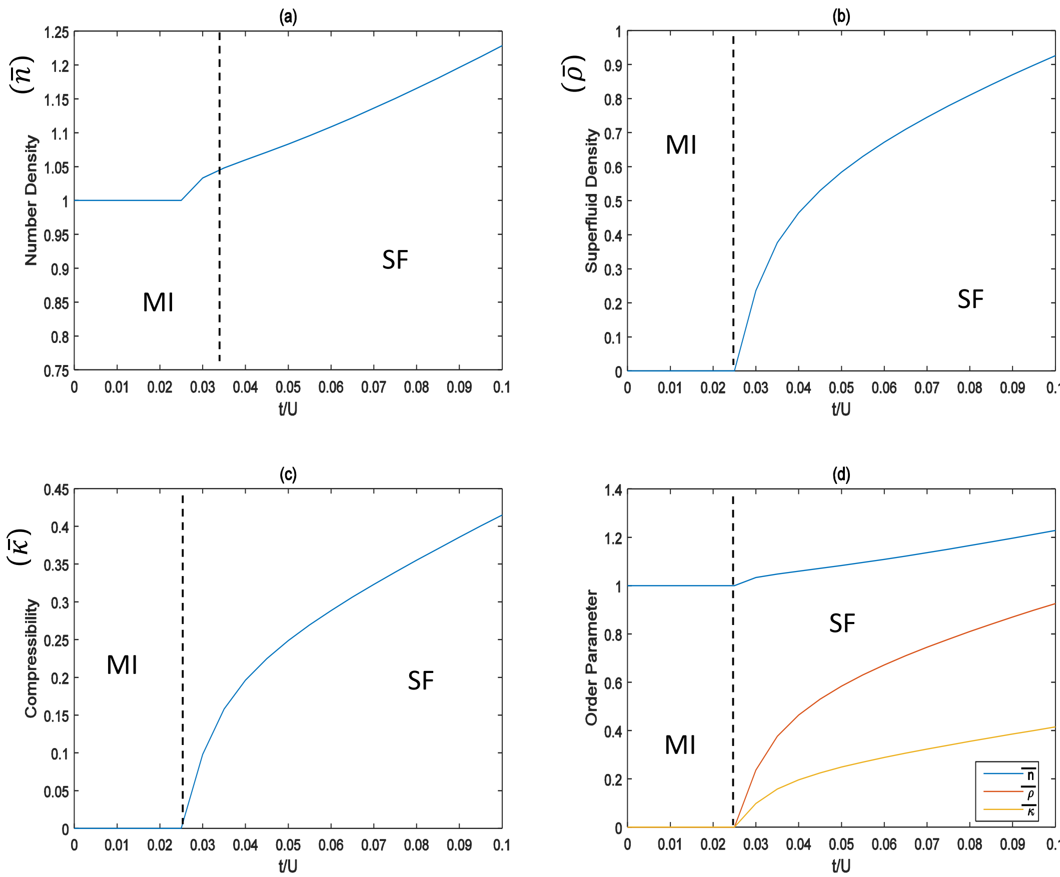

Fig. 6 Number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of hopping potential for binary disorder; chemical potential is held constant at μ = 0.5. All 3 order parameters **(d)** plotted together as a function of hopping potential. The dashed line marks the insulator to superfluid transition. Energy parameters are in terms of 1/U.

Plots (a-c) appear to show a phase transition from insulating to superfluid phase, with no intermediate phase containing a spatially averaged superfluid density of zero but non-zero compressibility. Plot (d) shows why it is important to plot each order parameter separately as both superfluid density and compressibility appear to become non-zero at the same hopping potential value. However, it does appear that both parameters do in fact become non-zero at the same hopping potential; therefore, a larger phase space picture is needed in an attempt to realize the Bose glass phase in binary disorder.

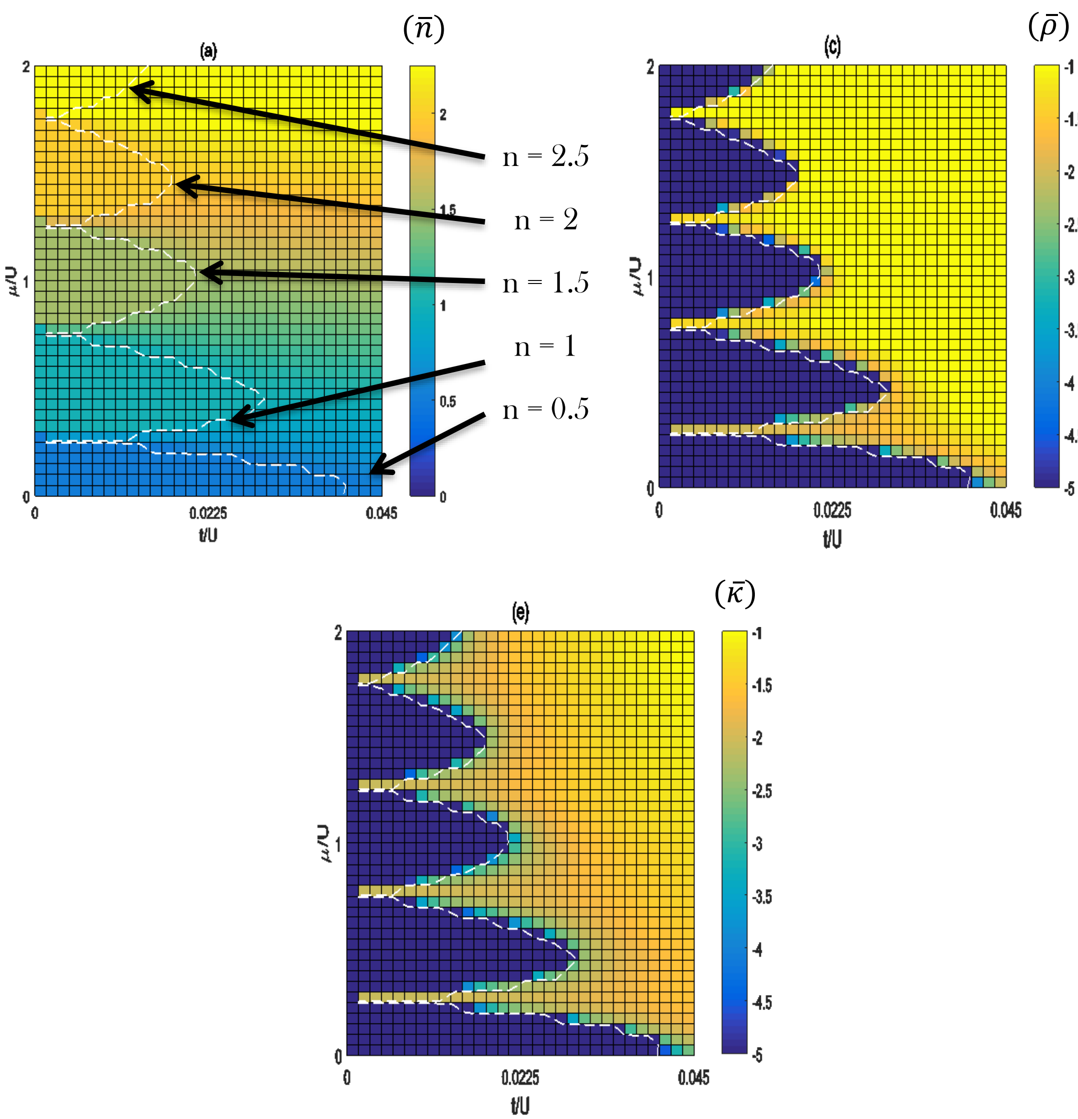

Fig. 7 Number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of chemical potential and hopping potential in a binary disordered system. A white dashed line is superimposed to guide the eye along the phase boundaries. Plots (**b-c**) are plotted logarithmically in the order parameter where values less than -5 are effectively zero.

Each Mott lobe has either an integer or half integer number density. The half integer number density arises from the addition of disorder and the disorder periodiocity; in this case f = 2 is defined as the periodicity of the disorder (binary disorder is said to have a peridiocity of 2, ternary 3 and so on, and f is used in these calculations to denote the division of the disorder or periodicity). As binary disorder is added, the integer filling Mott lobes shift up and down, and a half integer lobe appears between the integer lobes. **Fig. 7** is in direct agreement with the results obtained in the one dimensional case [9].

If a Bose glass is to present in phase space, there should be a region of zero superfluid density around the edge of each Mott lobe that relates to a region of non-zero compressibility. **Fig. 7** shows that the behavior in **Fig. 6 is** indicitive of the full phase space for binary disorder: no Bose glass phase presents in phase space for spatially averaged order parameters using mean field approximations. **Fig. 8** confirms this showing the phase boundary for both compressibility and superfluid density to be the same; thus no region in phase space contains a phase with zero superfluid density but finite compressibilty.

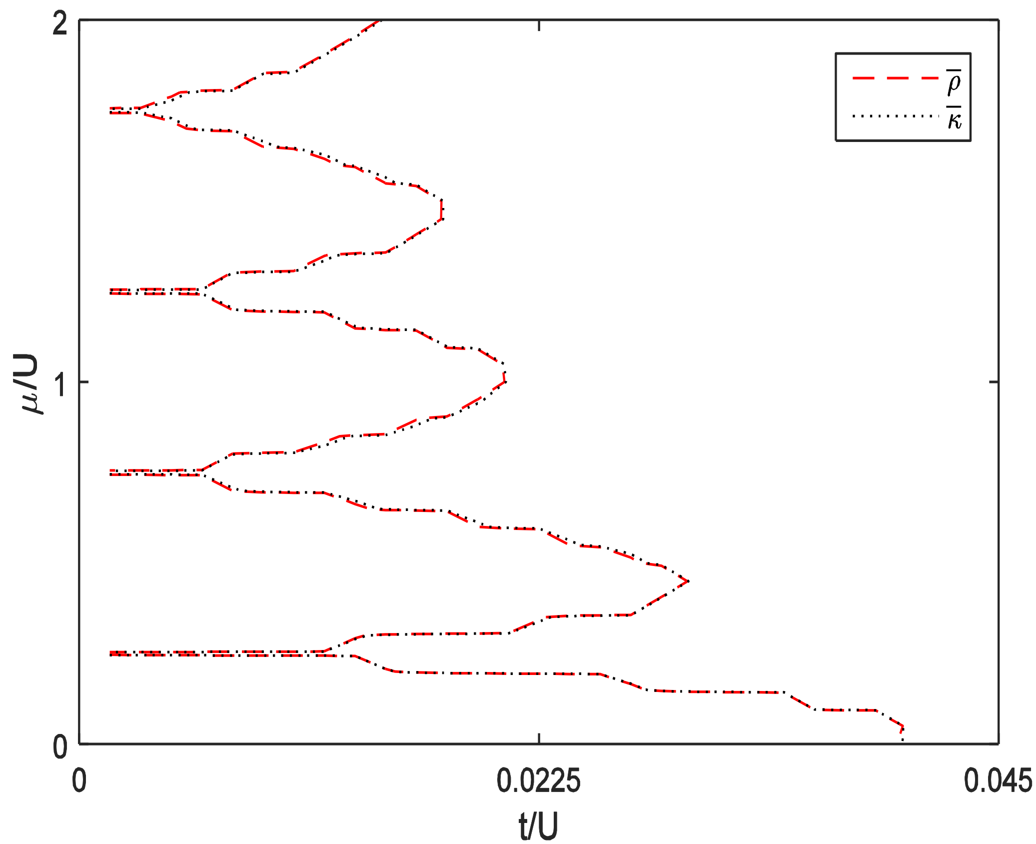

Fig. 8 Superfluid density ($\bar{\rho}$) and compressibility ($\bar{\kappa}$) phase diagram phase boundaries for binary disorder.

3.1.2. Real Space

Following the path of the clean system study, real space plots are looked to in an attempt to see if the Bose glass is a local phenomenon in mean field approximations. For completeness, points are chosen in phase space (**Fig. 9**) that on the Mott lobe boundaries in the so-called Bose Glass and in the superfluid region. **Table 2** gives the values tested in real space, while the red dots in **Fig. 9** show the location in phase space.

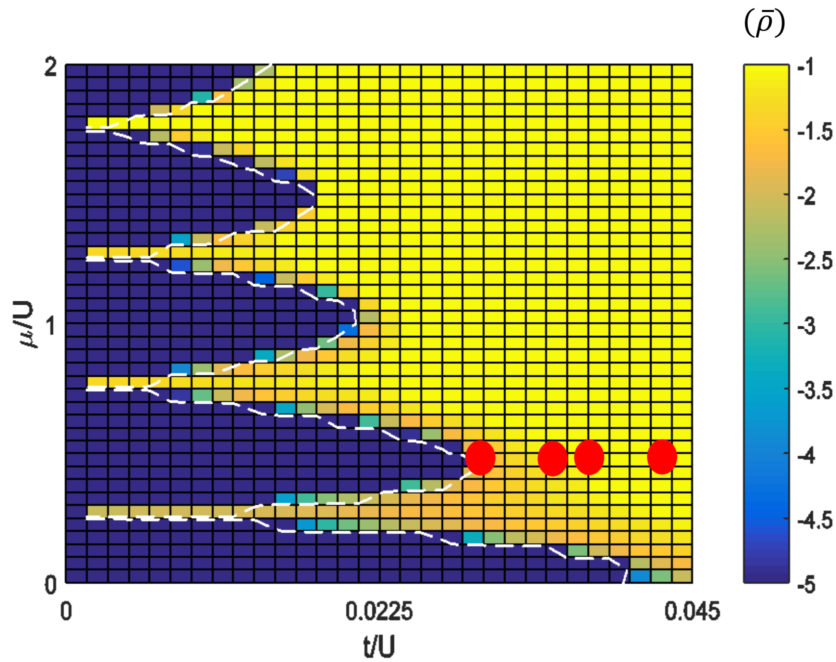

Fig. 9 Superfluid phase diagram for binary disorder used to determine μ, t pairs to be tested in real space. The red dots represent the four values tested: 1 in a Mott insulator (MI) region, 1 in a superfluid (SF) region, and 2 in a supposed Bose glass (BG) region. Superfluid density is plotted logarithmically where any value less than -5 is effectively zero.

Table 2 Phase space values tested in real space for binary disorder. Energy parameters are in units of 1/U.

Energy Parameter	MI	BG 1	BG 2	SF
μ/U	0.5	0.5	0.5	0.5
t/U	0.025	0.027	0.0275	0.029

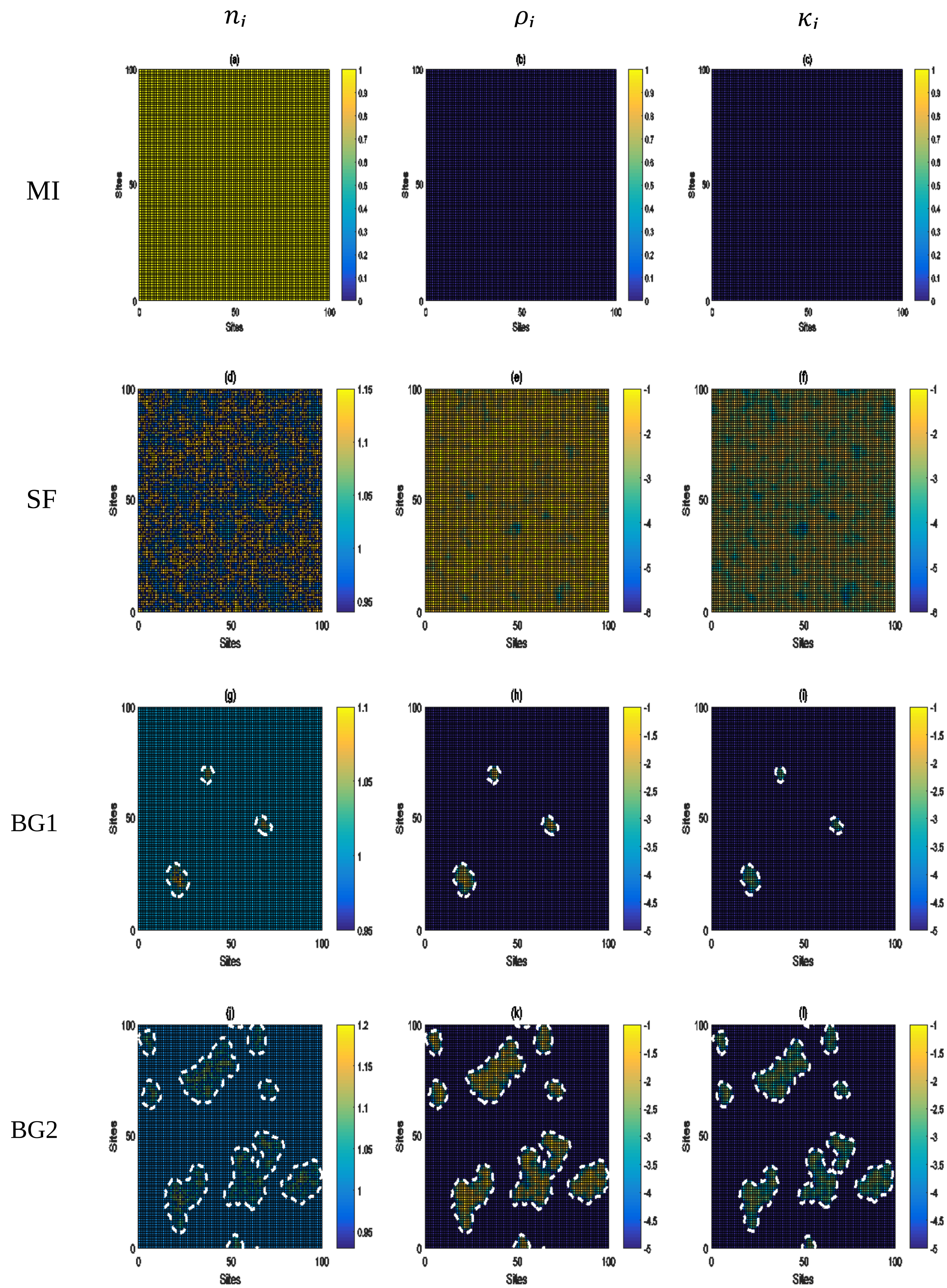

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Fig. 10 Real space plots of each order parameter for binary disorder over a 10000-site lattice (100 x 100).

As in the clean system, the definition of a Mott insulator is preserved in binary disorder; number density is an integer at all sites and superfluid density and compressibility are zero at all sites. The definition of a superfluid is also the same in binary disorder; however, as opposed to homogeneity in the clean system, there is a region of non-integer number density and non-zero superfluid density and compressibility values based on the random on site disorder potential. A mixture of insulating and superfluid sites coexists on the same real space lattice in the Bose glass region. This behavior, coined “superfluid clusters” is predicted and discussed in[10]. Here it is shown that the cluster phenomenon occurs in 2D as well as 1D for binary disorder.

Population density histograms will clarify the difference between the Bose glass and the insulating and superfluid phases. One expects similar behavior to the clean system for the insulating and superfluid phases while the Bose glass should show behavior indicative of a mixture of states.

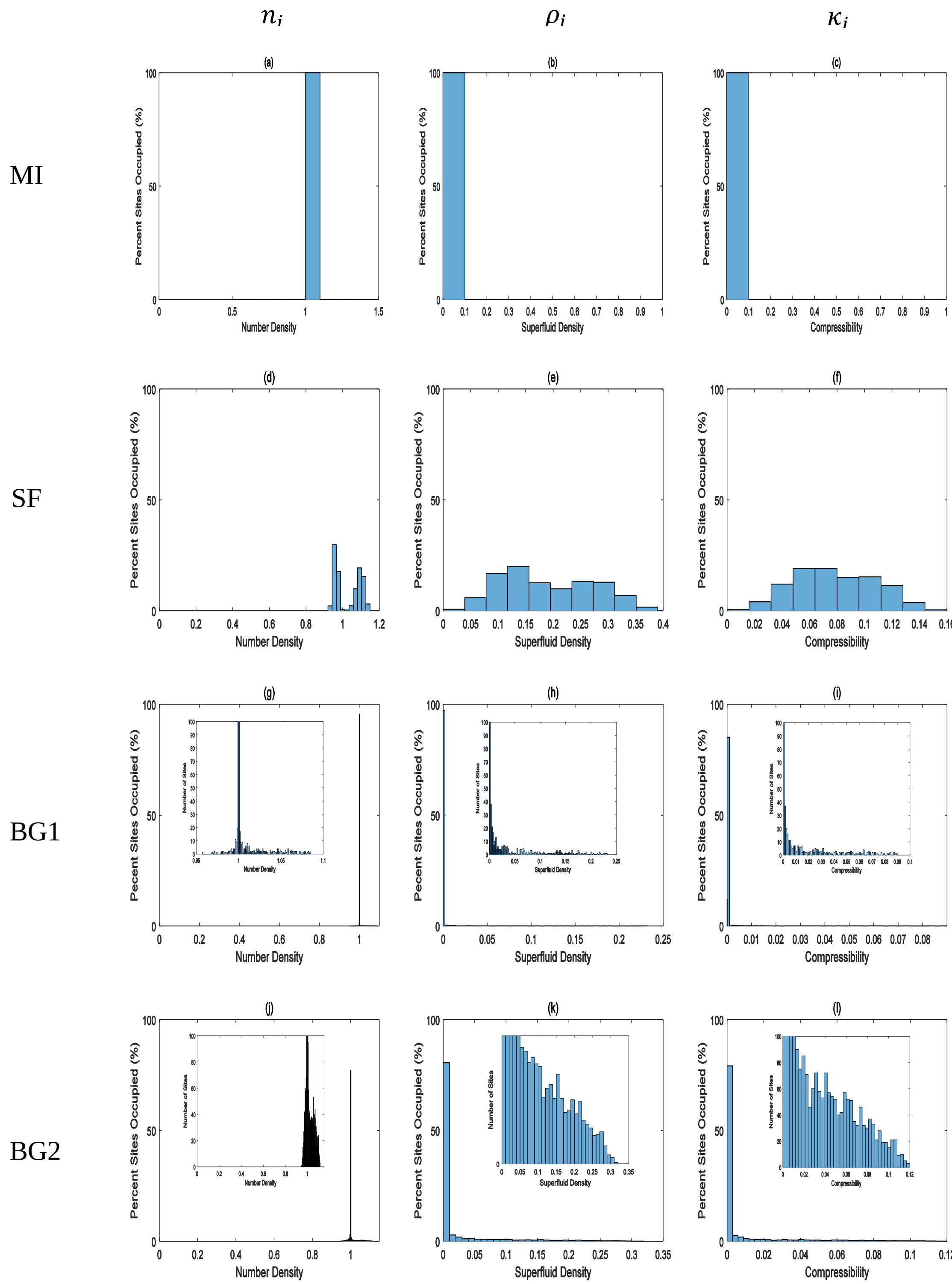

Fig. 11 Population density for binary disorder. The inserts in **(g-l)** show a zoomed in plot of number of sites as a function of each order parameter; each plot only shows up to 100 occupied sites to elucidate the tail behavior of the Bose glass region.

Fig. 11 (a-c) shows a 100% of sites contain an integer number density and zero superfluid density and compressibility confirming this point to be a Mott insulator. **Fig. 11 (d-f)** shows a range of non-integer number density and non-zero superfluid density and compressibility values confirming this point to be a superfluid. **Fig 11. (g-i)** shows the first Bose glass point has behavior similar to the Mott insulating phase. However, upon close inspection, the difference lies in the tail of values relating more to the superfluid phase. **Fig. 11 (j-l)** shows the second Bose glass point has similar 'tail' behavior to the previous Bose glass point. However, the tails in the second point (closer to the superfluid phase) are larger in the sense that more sites are populated with values in the tail, and the spike value around integer number density and zero superfluid density and compressibility is smaller. Thus, as hopping potential increases towards a superfluid, the Bose glass phase transitions from looking like a Mott insulator with a superfluid-like tail, to looking more and more like a superfluid.

When binary disorder is added to the system, changes occur in phase space as well as in real space. Although the Bose glass phase is not truly realized in phase space for mean field approximations, the behavior discussed by Buonsante and Vezzani [9] in regards to number density and the shifting Mott lobes observed in one dimension is confirmed here in two dimensions. Given the binary disorder configuration, the clean system Mott lobes containing one integer number density value shift up and down, and a lobe appears between these original lobes. This emergent lobe has chemical potential and hopping potential pair values that relate in real space to a lattice containing half the integer number density value of the lobe that shifted downward and half the integer number density value of the lobe that shifted upward. In other words, the n = 1 shifts upward and a lobe appears below it containing chemical potential and hopping potential pair values that result in a spatially averaged number density of n = 0.5. This is a result of having a real space lattice containing half sites with n = 0 and half sites with n =1; thus, the spatial average becomes n = 0.5. This is true for increasing chemical potential throughout phase space. In other words, every integer filling factor lobe shifts such that a single lobe is intermitant between two integer filling factor lobes with a number density that is the average of the upper and lower lobe's number density.

In regards to superfluid density and compressibilty, however, there is no region with spatially averaged zero superfluid density and non-zero compressibilty in phase space; the system transition directly from insulating to superfluid phases in phase space. Locally, the Bose glass presents as a global phenomenon over a real space lattice, but does not present at any single site; no new phases presents at any single site. Thus, the transition in a binary disorderd system occurs through a region that appears to be a superfluid in phase space but in real space appears as a mixture of states existing on the same lattice. The transition at any single site however, occurs directly from an insulating to a superfluid phase. This leads to the postulate of a number of sublattices equal to the number of divisions in the disorder (periodicity) e.g. two sublattices in binary disorder, that will be discussed in the phase diagram approximation technique chapter.

3.2 Ternary Disorder

Adding another division to the disorder potential from binary disorder results in ternary disorder, whereby the Hamiltonian may take one of three forms:

$$h_{+\Delta} = \frac{1}{U}\sum_i(\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j, \quad (11)$$

$$h_{-\Delta} = \frac{1}{U}\sum_i(-\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j, \quad (12)$$

$$h_0 = -\frac{1}{U}\sum_i \mu\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j. \quad (13)$$

The third Hamiltonian (**Eqn. 13**) is just that for a zero disorder Hamiltonian; therefore, ternary disorder may have one of three disorder strengths such that Δ = -Δ, 0 or +Δ. **Fig. 12** shows a 10000 site lattice populated randomly with ternary disorder of strength μ = 0.25.

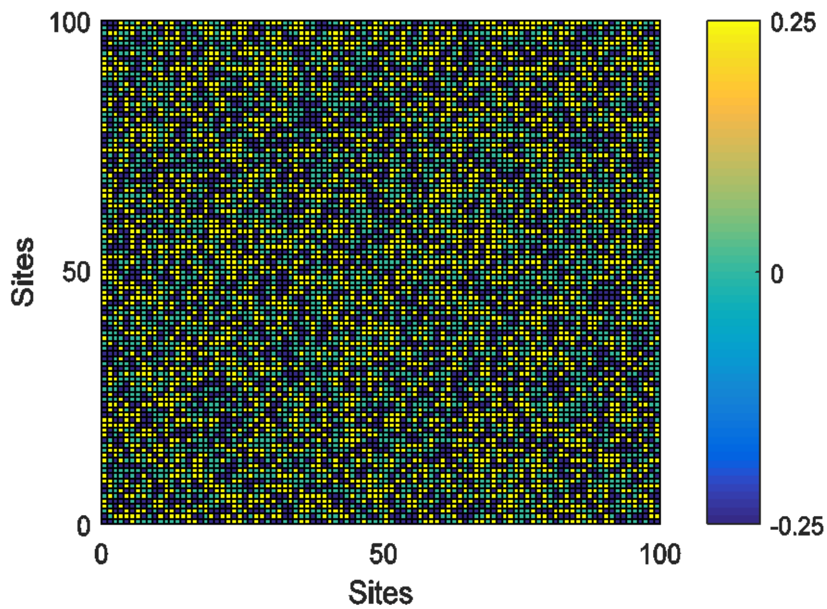

Fig. 12 Ternary disorder map over a 10000 site lattice; Δ = ± 0.25 or 0.

Making a prediction based on the evolution of the system when binary disorder is added from when the system contains no external random disorder, one should expect to see a direct transition in phase space from a Mott insulating phase to a superfluid phase; as no Bose glass was seen in phase space in binary disorder, it seems reasonable to assume that this would also be the case for ternary disorder.

3.2.1 Phase Space

One looks again to the plots of spatially averaged order parameters as a function of hopping potential (t/U).

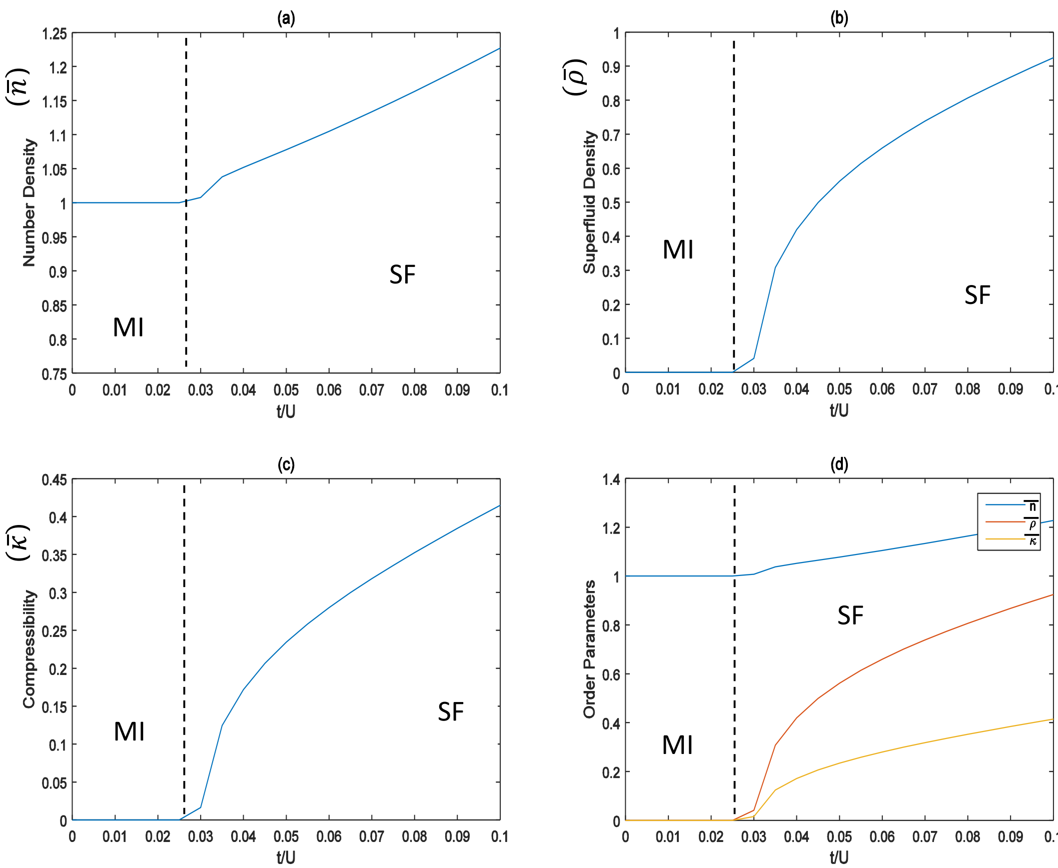

Fig. 13 Spatially averaged number density **(a)**, supefluid density **(b)** and compressibility **(c)** as a function of hopping potential for ternary disorder; chemical potential is held constant at μ = 0.5. All three are plotted on **(d)** to show the possible ambiguity between transition points for superfluid density and compressibility. The dashed line marks the insulator to superfluid transition. Energy parameters are in terms of 1/U.

The system does seem to transition directly from a Mott insulating phase to a superfluid phase with no region having zero superfluid density but finite compressibilty. However, as **Fig. 13(d)** shows, it is not completely apparent that both superfluid density and compressibilty become non-zero at the same hopping potential value. To see a more complete picture of the phase transition in phase space, phase diagrams of each spatially averaged order parameter are plotted as a function of chemical potential and hopping potential.

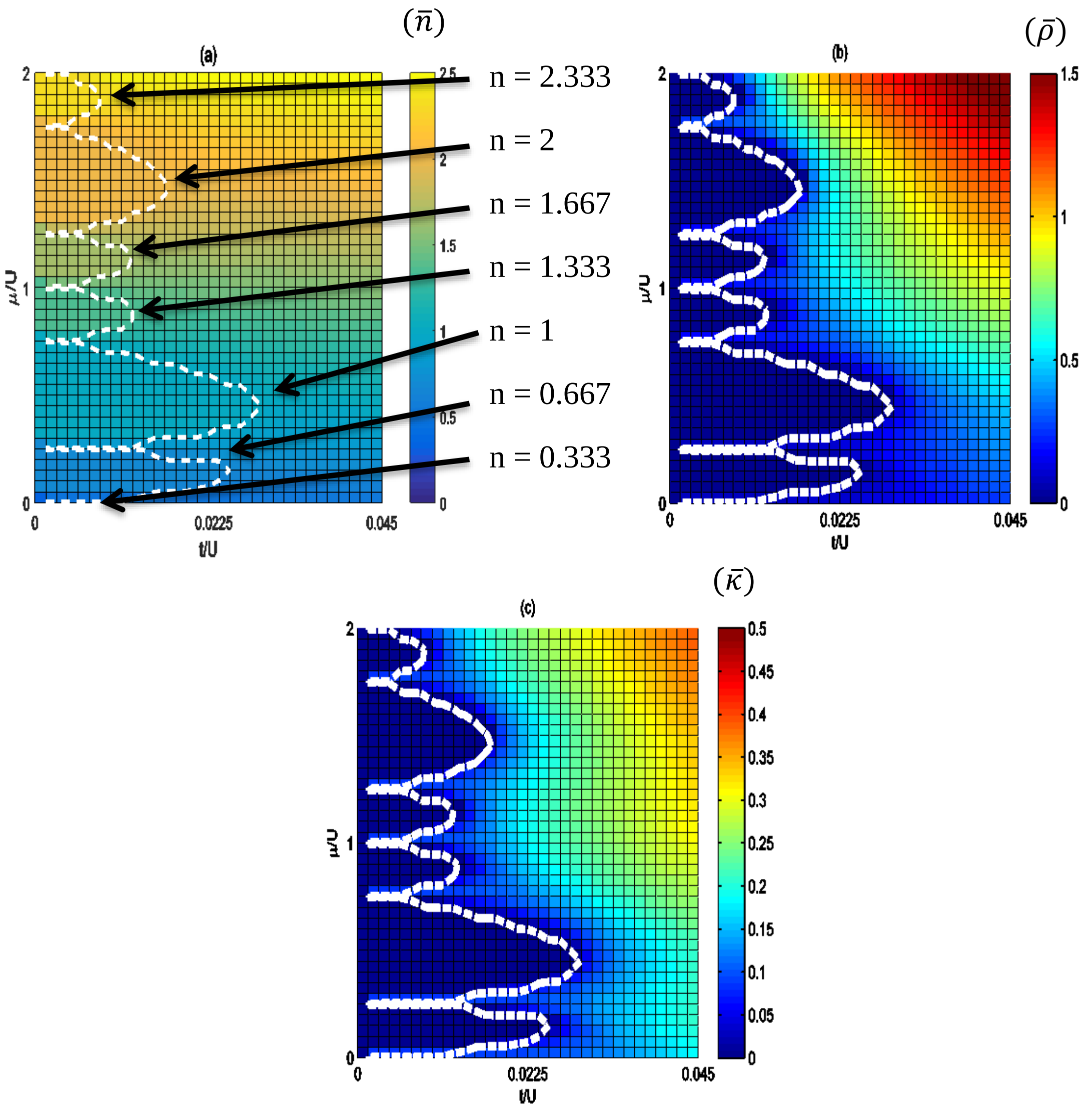

Fig. 14 Number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of chemical potential and hopping potential in a ternary disordered system. A white dashed line is superimposed to guide the eye along the phase boundaries.

Number density as a function of chemical potential and hopping potential shows that the behavior of the intermitent, partial integer filling Mott lobes increases with increasing disorder complexity; it appears that if the disorder division is n, then there are n – 1 intermitent, partial integer filling Mott lobes between the integer filling Mott lobes.

Again, this partial integer filling is due to the periodicity in the disorder (number of divisions) [9]. Since ternary disorder has three possible divisions, then the intermitent Mott lobes will have a third integer number density. This can be seen in **Fig. 14a** where the lowest Mott lobe has a number density of n = 1/3. This occurs because 1/3 of the sites in real space have a number density of n = 1 (the density of the nearest lobe above with integer filling) and 2/3 of the sites have a number density of n = 0 (the density of the nearest lobe below with integer filling). Moving one lobe upward (increasing chemical potential) results in an integer filling factor of 2/3, which occurs because 1/3 of the sites in real space have a number density of n = 0 and 2/3 have a number density of n = 1. This is a recurring pattern as chemical potential is increased. For ternary disorder, with a periodicity of three, two one-third integer filling lobes will appear between every integer filling lobe. The lower of the two lobes will have 1/3 of the sites in real space occupied with the number density of the nearest lobe above it with integer number density and 2/3 of the sites occupied with a number density of the neartes lobe below it with integer number density; the opposite is true for the upper third integer filling factor lobe. The boundaries for superfluid density and compressibility are shown to be the same and thus there is no apparent Bose glass region in phase space.

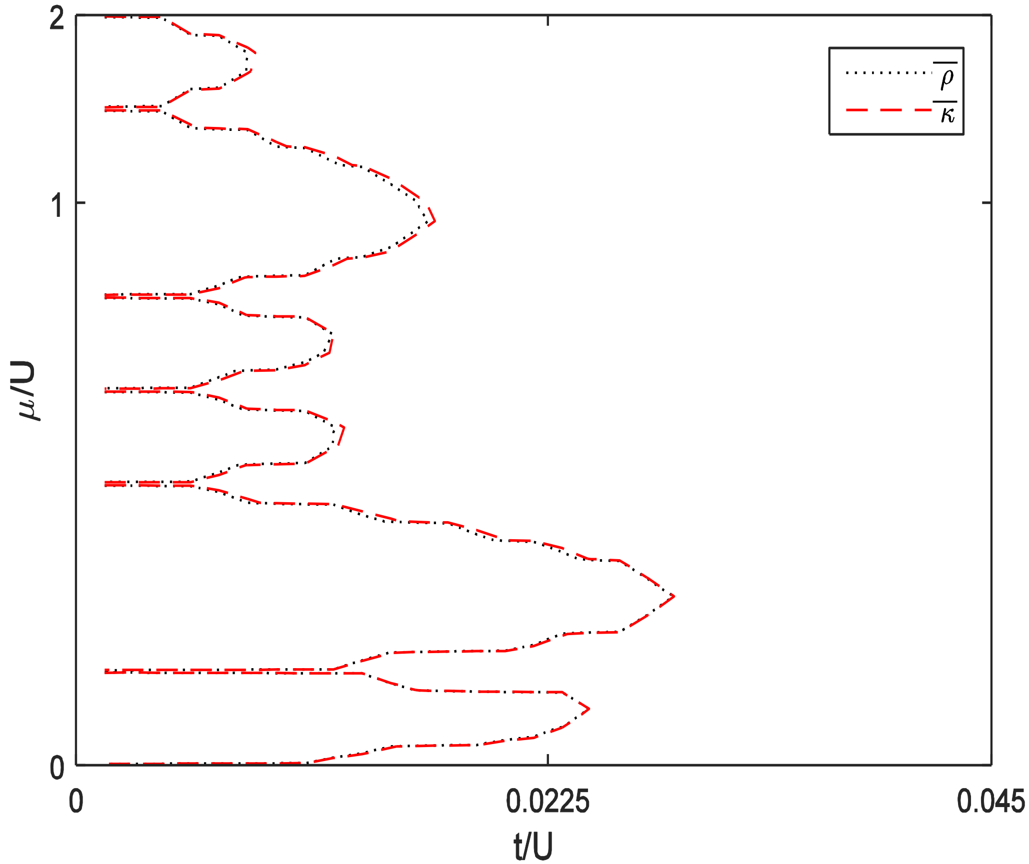

Fig. 15 Superfluid ($\bar{\rho}$) and compressibility ($\bar{\kappa}$) phase diagram boundaries for ternary disorder.

Fig. 15 shows that there is in fact no region in phase space for ternary disorder that has a zero superfluid density but finite compressibility. As in binary disorder, ternary disorder does not realize a Bose glass phase in phase space.

3.2.2. Real Space

Following the same methodollogy as before, one then looks to a real space analysis of ternary disorder in an attempt to access the Bose glass phase. **Fig. 16** shows the superfluid phase diagram again with the four points chosen to test in real space indicated by red dots.

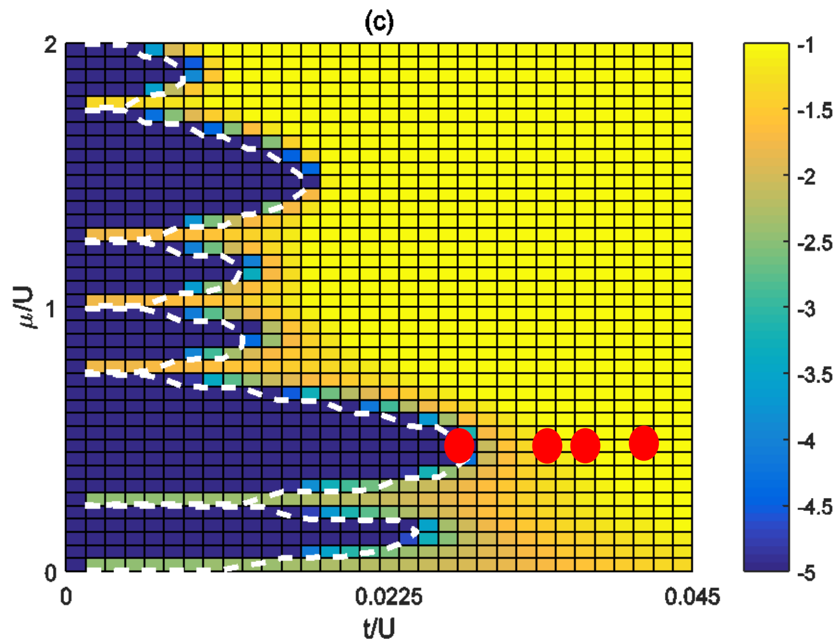

Fig 16. Superfluid phase diagram for ternary disorder used to determine μ, t pairs to be tested in real space. The red dots represent the four values tested: 1 in a Mott insulator region, 1 in a superfluid region, and 2 in a supposed Bose glass region. Superfluid density is plotted logarithmically where any value less than -5 is effectively zero.

The values of each point in phase space are given in **Table 3**.

Table 3 Phase space values tested in real space for ternary disorder. Energy parameters are in units of 1/U.

Order Parameter	MI	BG 1	BG 2	SF
μ/U	0.5	0.5	0.5	0.5
t/U	0.025	0.029	0.0295	0.035

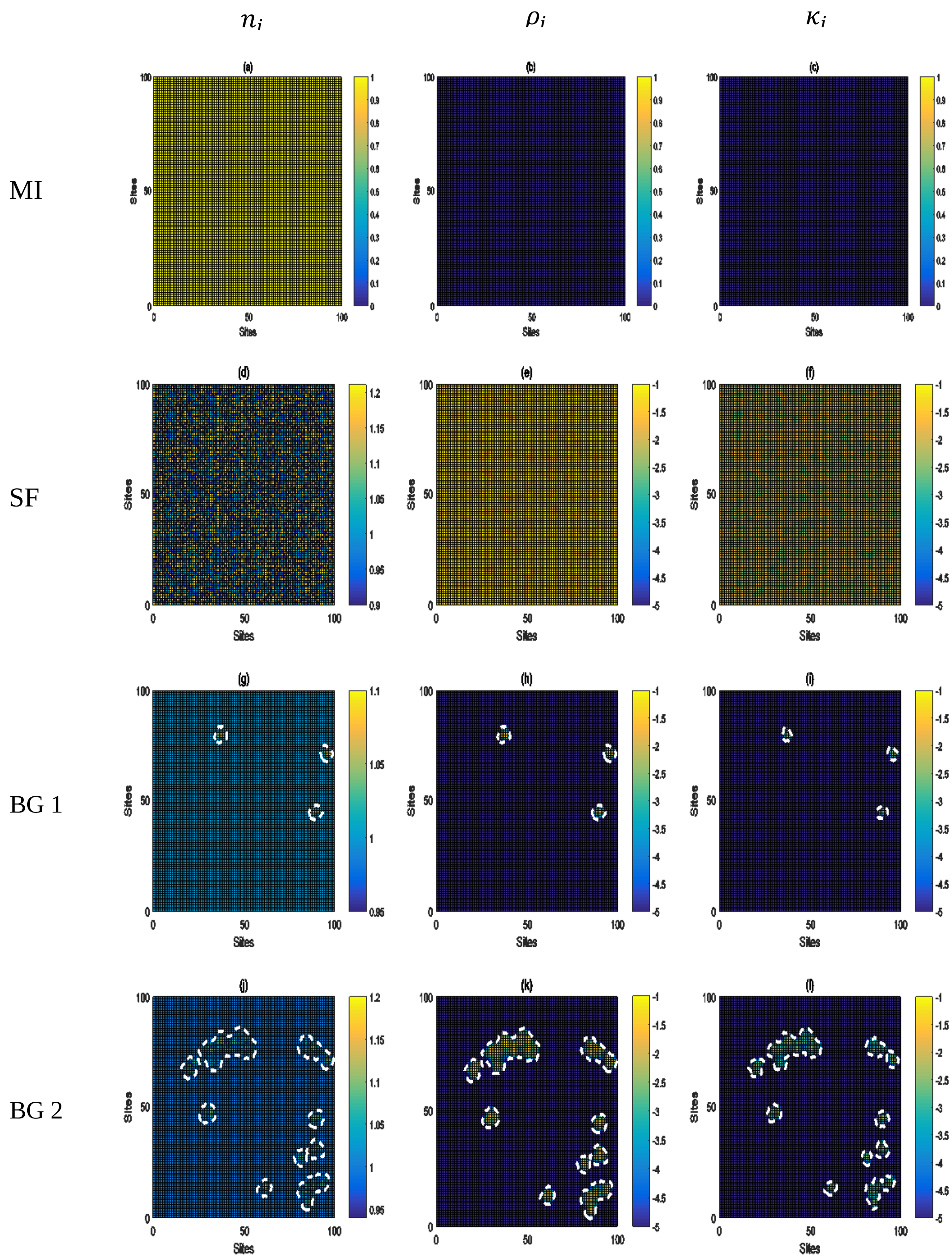

Fig. 17 Real space plots of each order parameters for ternary disorder over a 10000 site lattice (100 x 100).

The Mott insulator region in real space is the same as in binary, showing that as disorder is added, the definition of a Mott insulator does not change; local number density is an integer at every site and both superfluid density and compressibility are zero at every site. The superfluid region also has the same behavior in ternary as in binary disorder: sites of non-integer number density and non-zero superfluid density and compressibility percolate (connect) across the lattice.

The first point in the supposed Bose glass region has similar behavior to the Bose glass points in binary disorder. As **Fig. 17** shows, small superfluid clusters dot a mostly Mott insulating background. The relative size of the superfluid clusters in ternary disorder are slightly smaller than that of binary disorder, but this is most likely a coincidence of the randomness of the disorder distribution in the calculation. There are, however, no sites with zero superfluid density and finite compressibility, and thus no Bose glass at any single site. In **Fig. 17(i – k)**, hopping potential is increased infinitesimally to see the evolution of the Bose glass region in real space. As with binary disorder, as hopping potential is increased through the Bose glass region, superfluid clusters increase in both quantity and size without yet percolating entirely across the lattice; and as with the first point in the Bose glass phase in **Fig. 17(f – h)**, no single site has zero superfluid density and finite compressibility. In real space for ternary disorder, the Bose glass presents as a global phenomenon over a full real space lattice; the Bose glass can be characterized in real space with the use of Niederle and Rieger's superfluid cluster geometry. However, when looking to any one site, one sees that the phase transition is directly from an insulator to a superfluid.

Population density histograms (**Fig. 18**) are again used to confirm real space phenomena and phase definitions.

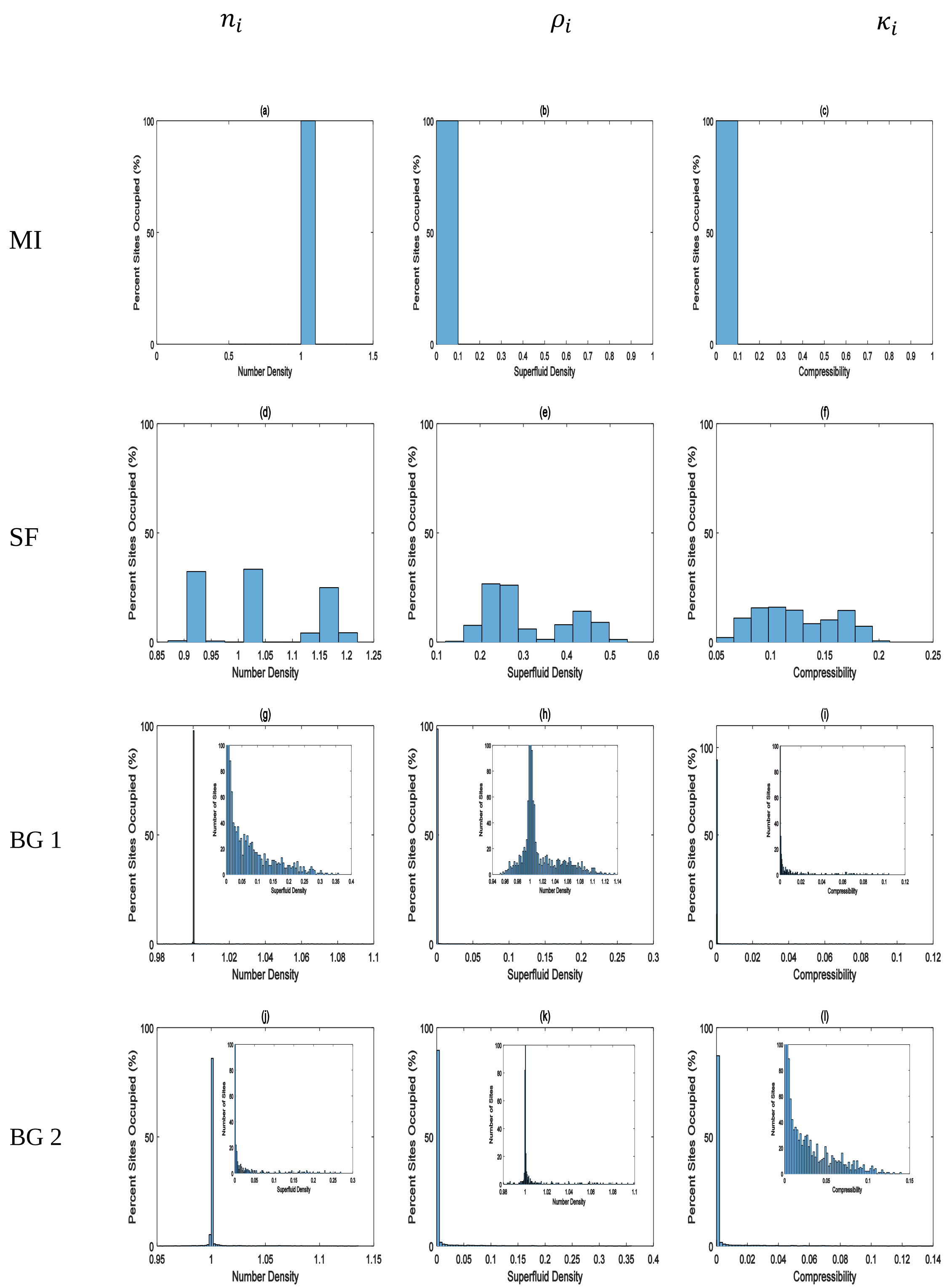

Fig. 18 Population density for ternary disorder. The inserts in **(g-l)** show a zoomed in plot of number of sites as a function of each order parameter; each plot only shows up to 100 occupied sites to elucidate the tail behavior of the Bose glass region.

As in binary disorder, all sites in a Mott insulator for ternary disorder have integer number density and both zero superfluid density and compressibility. Comparing binary and ternary superfluid population density plots, one sees that number density is distributed around an equal number of values as the number of possible disorder values. In other words, binary disorder number density is distributed around two values while ternary is distributed around three values. As in binary disorder, the Bose glass in real space has a spike at values in each order parameter relating to a Mott insulator with a tail of values that relate to superfluid characteristics. The second Bose glass point also has a large spike of values at a Mott insulating value with a tail of values characteristic of a superfluid. However, and similar to that of the binary case, as hopping potential is increased away from the Mott lobe, the superfluid tail increases and the plot becomes more similar to a superfluid.

The Bose glass in ternary disorder presents many similarities as in binary disorder, with the simple extension that any phenomena inferred from the periodicity (number of divisions) in the disorder increases with the disorder. Although not mathematically rigorous, this idea simply means that if the result in binary disorder is two fold (number density in the Mott lobes in phase space), then in ternary disorder it becomes three fold (third integer number density Mott lobes as opposed to half integer in binary disorder). As in binary disorder, the Bose glass presents in ternary disorder as a global phenomenon in real space over a full real space lattice. The most noticable difference between binary and ternary disorder configurations is the distribution of number density values in real space and how that affects the spatially averaged quantities in phase space. However, all real space phenomena as well as population density characteristics in the Bose glass are similar between binary and ternary disorder. This leads to the prediction that quaternary disorder will present an extension of the phase space phenomena with one added possibility for disorder strength, but will present mostly similar real space phenomena to that of binary and ternary disorder.

3.3 Quaternary Disorder

To complete the study of simple disorder configurations before looking to uniformly distributed disorder, the periodicity of the random external disorder is increased again by one to quaternary disorder. The Hamiltonian now has four possible forms, two positive and two negative such that:

$$h_{+\Delta} = \frac{1}{U}\sum_i(\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j, \tag{14}$$

$$h_{-\Delta} = \frac{1}{U}\sum_i(-\Delta_i - \mu)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j, \tag{15}$$

$$h_{+\Delta/3} = \frac{1}{U}\sum_i\left(\frac{\Delta_i}{3} - \mu\right)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j, \tag{16}$$

$$h_{-\Delta/3} = \frac{1}{U}\sum_i\left(-\frac{\Delta_i}{3} - \mu\right)\hat{n}_i + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1) - \frac{t}{U}\sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j. \tag{17}$$

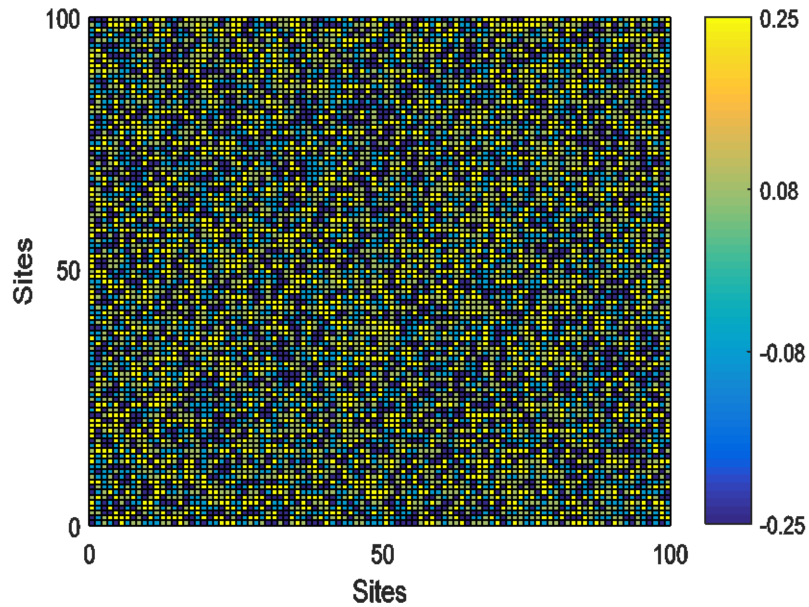

Fig. 19 Quaternary disorder mapped in real space over 10000 sites; $\Delta = \pm\, 0.25, \pm\, 0.08333$.

Using the prediction from ternary disorder, one can again predict that no Bose glass will be seen in phase space. Although past theory states that a Bose glass is realized in truly randomly disordered systems [10], quaternary disorder does not have a large enough periodicity to be considered uniformly random nor predict that a Bose glass will be realized in phase space.

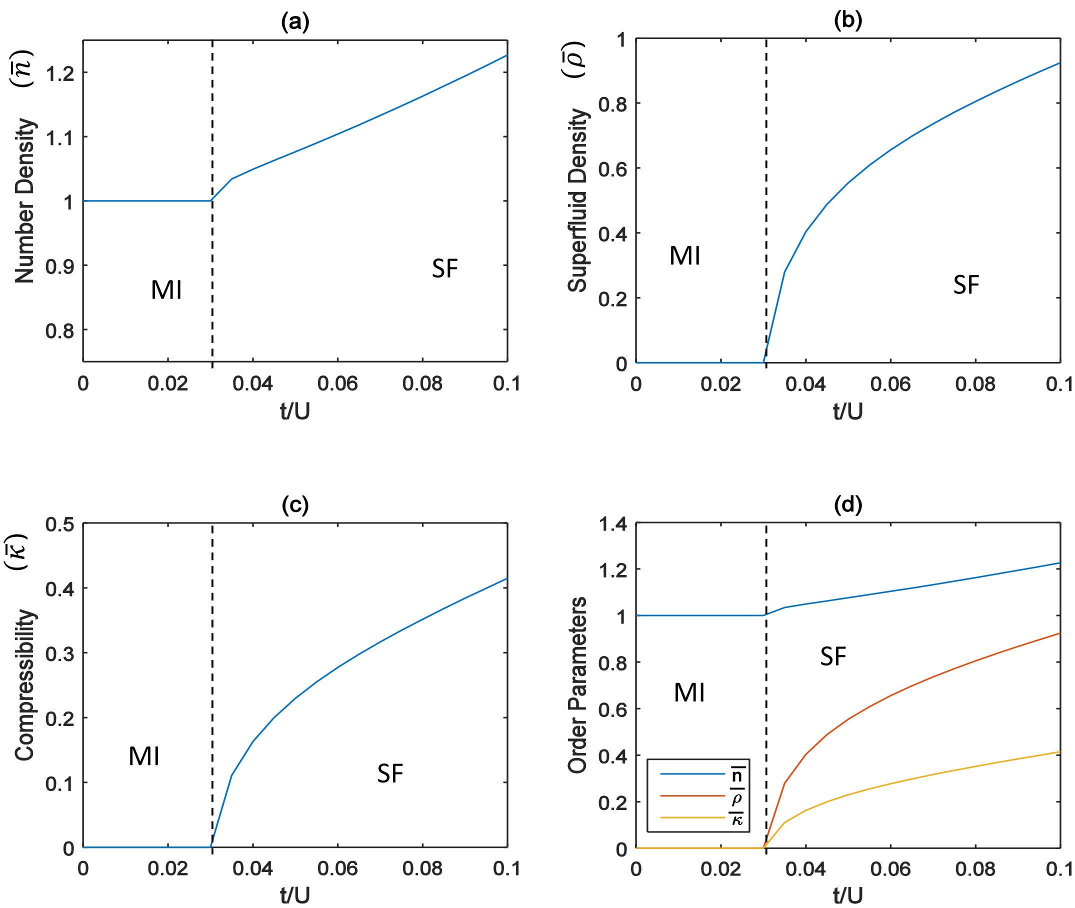

Fig. 20 Spatially averaged number density **(a)**, supefluid density **(b)** and compressibility **(c)** as a function of hopping potential for quaternary disorder; chemical potential is held constant at μ = 0.5. All three are plotted on **(d)** to show the possible ambiguity between transition points for superfluid density and compressibility. The dashed line marks the insulator to superfluid transition. Energy parameters are in terms of 1/U.

As in ternary and binary disorder, quaternary disorder appears to transition directly from insulator to superfluid all at the same hopping potential value; the transition occurs at t = 0.035. Unlike ternary disorder, however, quaternary disorder shows far less ambiguity in the critical hopping potential value where superfulid density and compressibilty both become non-zero. Again, for completeness, full phase diagrams help to confirm that this notion in accurate.

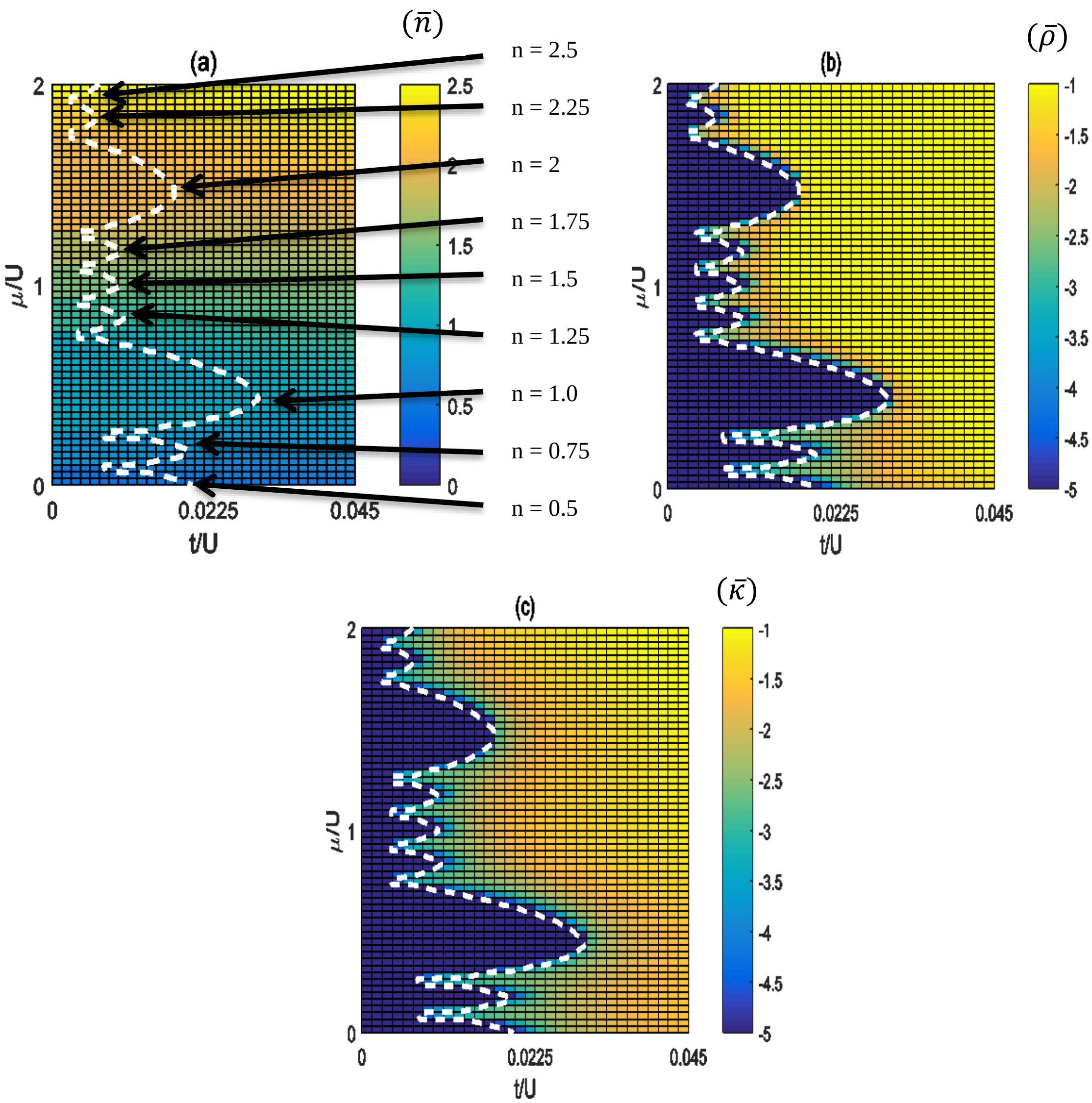

Fig. 21 Number density **(a)**, superfluid density **(b)** and compressibility **(c)** each as a function of chemical potential and hopping potential in a quaternary disordered system. Superfluid density and compressibiltiy are each plotted logarithmically. A white dashed line is superimposed to guide the eye along the phase boundaries.

Number density in phase space has the same behavior as in binary and ternary disorder with f – 1 intermittent, partial integer occupancy Mott lobes appearing between the integer occupancy lobes of a zero disorder system (f is the periodicity of the disorder). Thus, quaternary disorder with a periodicity of four has three intermitent Mott lobes each with a quarter integer number density. The filling factors for the quarter integer lobes is as follows: Looking to the three lobes between the $n = 1$ and $n = 2$ lobes, one finds that the lowest of the three lobes contains, in real space, 75% of sites with $n = 1$ number density and 25% of sites with $n = 2$ number density resulting in a spatially averaged value of $n = 1.5$. The middle intermittent lobe has 50% of sites with $n = 1$ and 50% of sites with $n = 2$ resulting in a spatially averaged number density of $n = 1.5$. The upper intermittent lobe has 25% of sites with $n = 1$ and 75% of sites with $n = 2$ resulting in a spatially averaged number density of $n = 1.75$. A general method for determining the number density of an intermittent lobe for any disorder periodicity is shown in **Chapter 6**.

The superfluid and compressibility boundaries appear to be the same and, again, looking to the phase boundaries plotted together in **Fig. 22**, one sees that the boundaries are in fact the same. As was predicted, no Bose glass presents in phase space for quaternary disorder.

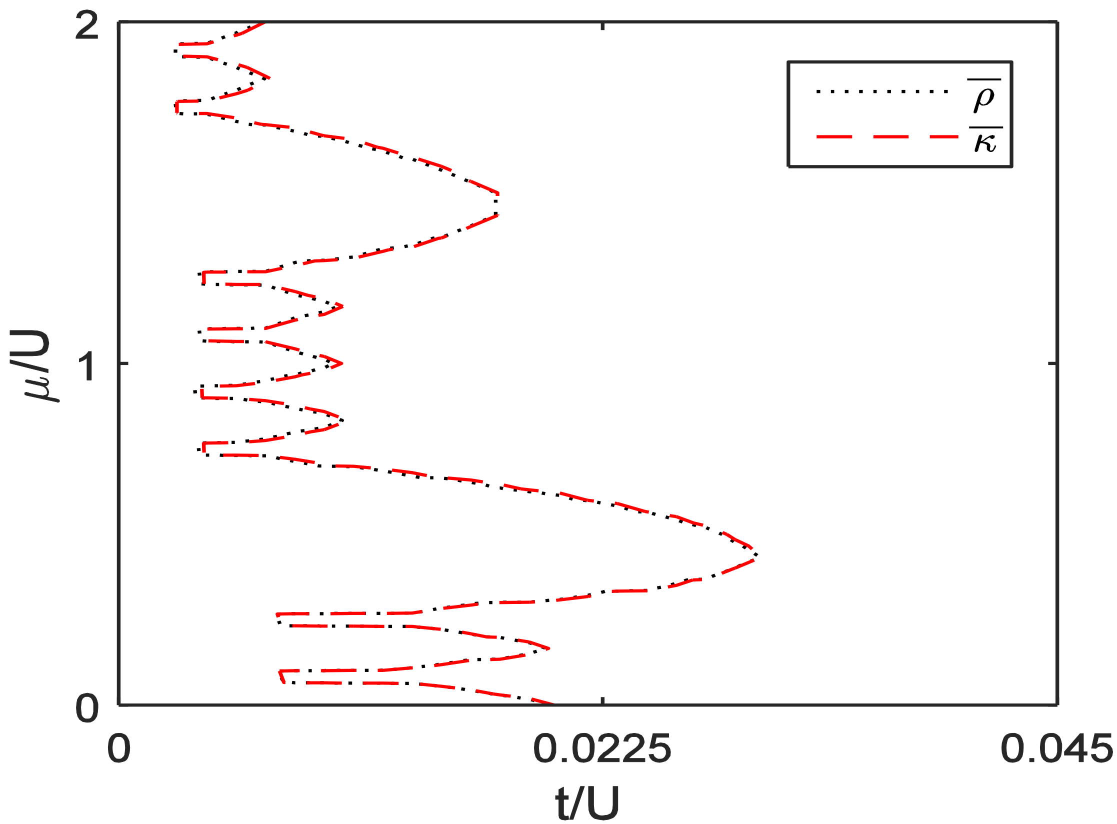

Fig. 22 Superfluid ($\bar{\rho}$) and compressibility ($\bar{\kappa}$) phase diagram boundaries for quaternary disorder.

Concluding the simple disorder configuration study of phase spaces shows that no Bose glass presents in phase space using mean field approximations. Either a Mott insulator or a superfluid presents in phase space with a direct transition occuring at some critical energy value. Although phase space may not show a region with the definition of a Bose glass, one may still acertain information from phase diagrams e.g. phase boundaries. As has been shown, the phase boundaries allow one to determine values of chemical and hopping potential to study real space phenomena. Thus, it is prudent to start a study of phase transitions in bosonic systems by looking to phase diagrams. **Chapter 5** will show a method to accurately approximate phase diagrams to avoid the extensive computation time required to accurately calculate full, disordered phase diagrams. This will then justify the claim that phase space is a qualitatively appropriate place to start when studying these types of phase phenomena, even though the full calculation will eventually be time consuming. It is, however imperative that the full phase space be calculated eventually to ensure that the approximation technique has been properly implemented. The approximation, ultimately, is a way to determine which general phase regions and points of interest are to be tested in real space.

3.3.2. Real Space

The critical points chosen to study in real space are shown in **Fig. 23** as red dots and are tabulated in **Table 4.**

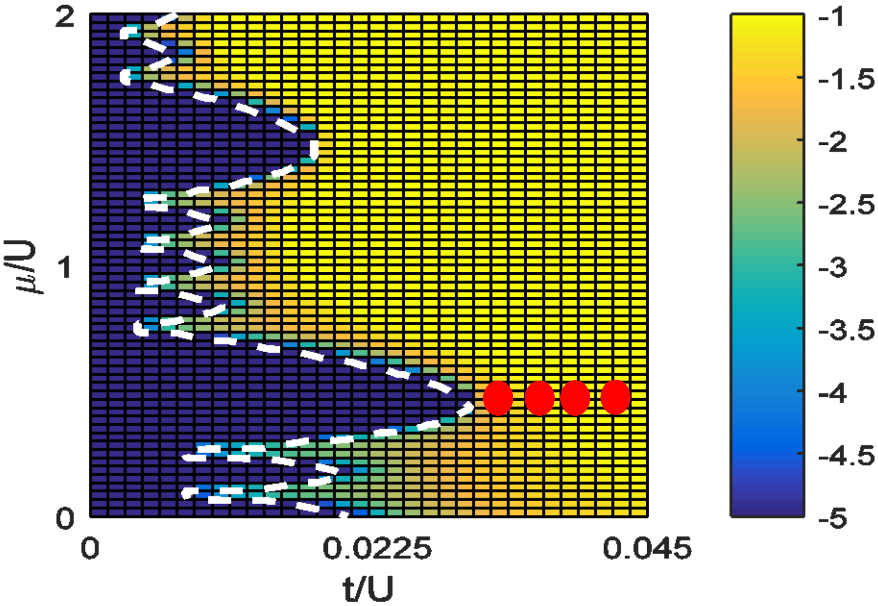

Fig. 23 Superfluid phase diagram for quaternary disorder used to determine μ, t pairs to be tested in real space. The red dots represent the four values tested: 1 in a Mott insulator region, 1 in a superfluid region, and 2 in a supposed Bose glass region. Superfluid density is plotted logarithmically where any value less than -5 is effectively zero.

Table 4 Phase space values tested in real space for quaternary disorder. All energy parameters are in units of 1/U.

Order Parameter	MI	BG 1	BG 2	SF
μ/U	0.5	0.5	0.5	0.5
t/U	0.03	0.0305	0.031	0.032

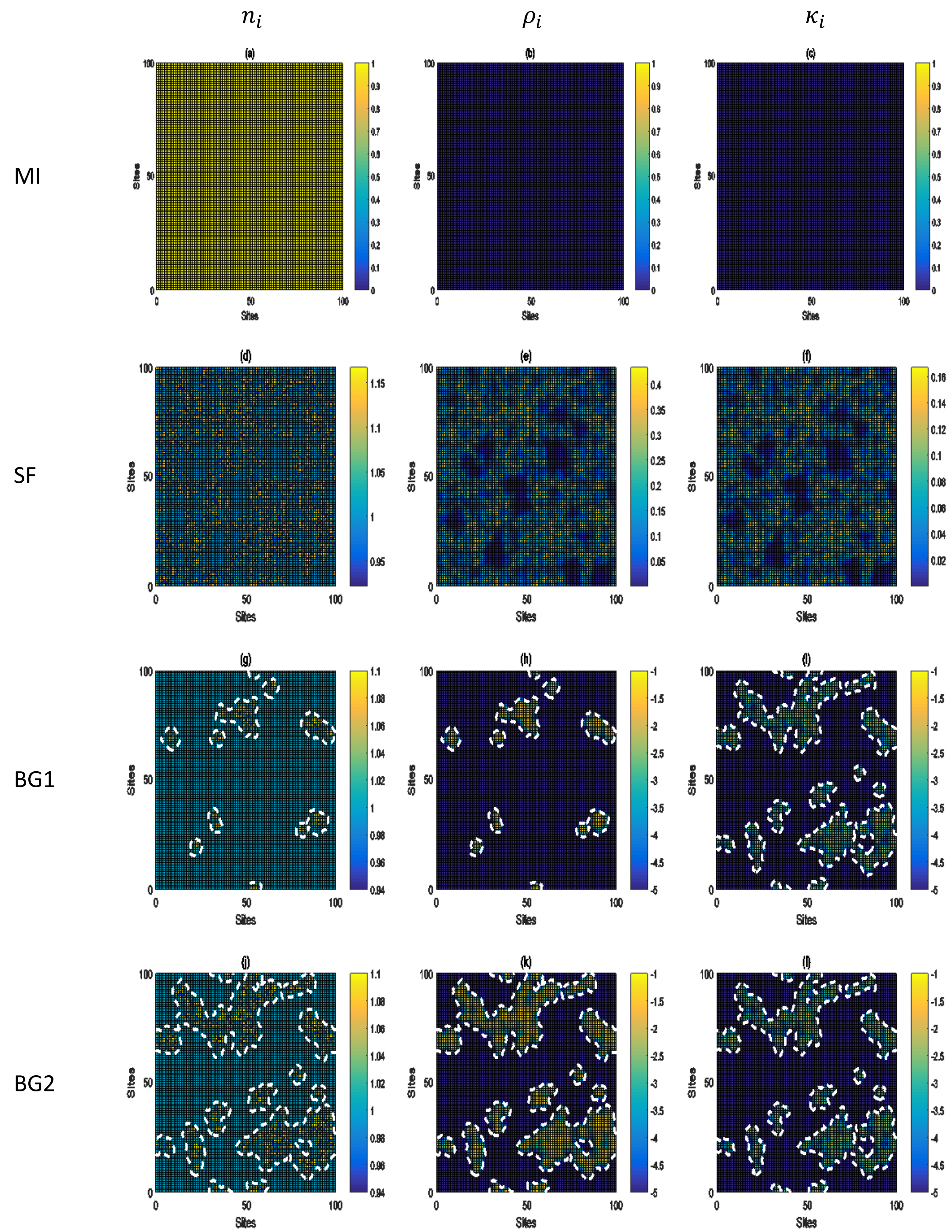

Fig. 24 Real space plots of each order parameters for quaternary disorder over a 10000 site lattice (100 x 100).

The real space plots for a Mott insulator and superfluid region show the same behavior as in binary and ternary disorder. Number density is an integer at every site and superfluid density and compressibility are both zero in the Mott insulator, while in the superfluid region, number density is a non-integer at every site and superfliud density and compressibility are both non-zero. The Bose glass region also shows the same behavior as in binary and ternary disorder with islands of superfluid in a background of Mott insulator. Also, as in binary and ternary disorder, as hopping potential is increased away from the Mott insulator region, the Bose glass "superfluid clusters" increase in size, until the superfluid region is reached in phase space and the superfluid clusters percolate (connect) completely across the real space lattice.

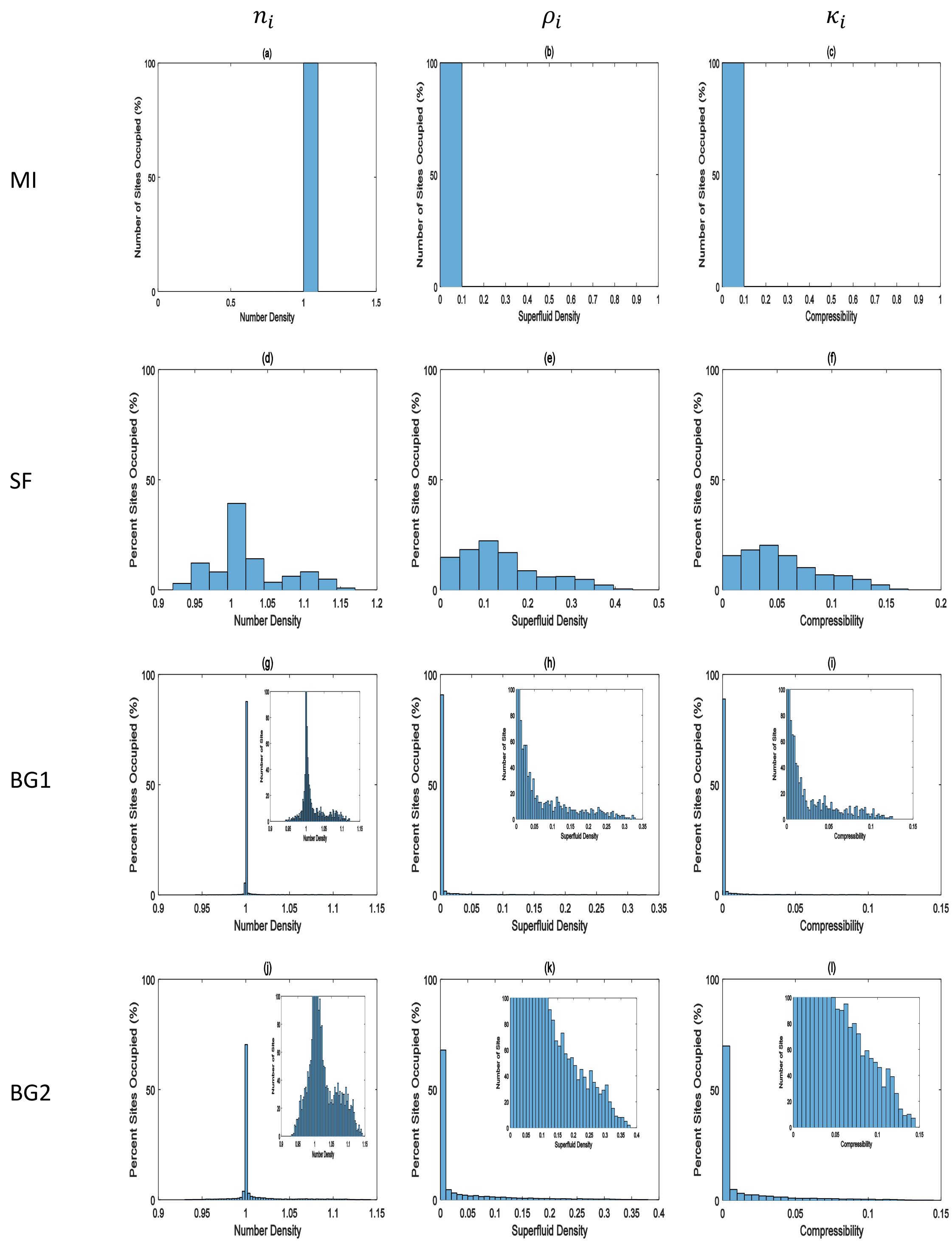

Fig. 25 Population density for quaternary disorder. The inserts in **(g-l)** show a zoomed in plot of number of sites as a function of each order parameter; each plot only shows up to 100 occupied sites to elucidate the tail behavior of the Bose glass region.

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Population density plots confirm that the real space behavior is simliar between all simple disorder configurations. The major difference is in the tail of values that make the Bose glass a unique phase; input parameters determine the shape and size of the tail.

A Mott insulator is generally the same, except with the increase in periodicity of the disorder changing the number density such that multiple integer values exist on the same real space lattice. In other words, the spatially averaged number density over a large real space lattice will change from being an integer to being a partial integer; this arising from the periodicity of the disorder configuration. However, in real space, a Mott insulator has an integer number density at every site. Superfluid density and compressibility are always zero at all sites in real space for all simple disorder configurations.

A superfluid has the same behavior in all simple disorder configurations such that number density is a non-integer at every site. The defining characteristic for a superfluid in simple disorder configurations, however is coherence and thus percolation of non-zero superfluid density and compressibility across a real space lattice. It must be stated here that a true superfluid can only be realized in infinite systems [1], but mean field theory allows the testing of smaller systems. The superfluid density and compressibility are not required to be non-zero at every site for the chemical potential and hopping potential pair value to be considered a superfluid in phase space. To fully determine if a point in phase space is a superfluid, however it is required to test coherence in real space.

The Bose glass has both integer and non-integer values of number density populating a real space lattice and thus may appear like a superfluid. However, when looking to the superlfluid density and compressibilty one sees that there are the "islands" of superfluid clusters as predicted by Niederle and Rieger [10]. What makes the point in phase space a Bose glass is the fact that any site with integer number density and zero superfluid density also has zero compressibility. In other words, the Bose glass is a real space phenomenon that presents across a real space lattice, but not at any site. Thus, in the mean field approximation using simple disorder configurations, all transitions in both phase space and at any single site are from Mott insulator to superfluid; no Bose glass exists in the mean field approximation at any sites in phase or real space.

As the periodicity is increased in the disorder, the system will have more potential values of disorder randomly distributed over a finite lattice. Thus, increasing the periodicity to ten creates a system with randomly distributed disorder across a 10,000 site lattice where the behavior is more similar to experiment. M.P.A. Fisher et al stated that the Bose glass would only be truly realized in the mean field approximation using uniformly random disorder [1]. However, Niederle and Rieger showed that a real space study is necessary to realize the Bose glass in the 1-D mean field approximation [10], and this has been confirmed in this thesis in 2-D. Thus, the random uniformly disordered system is studied to compare and contrast the results with simple disorder configurations and the postulates of Fisher et al.

3.4 Uniform Disorder

To see a more realistic picture, the disorder configuration must be more uniformly random. Thus, f = 10 is studied to see if the results agree with the initial postulates of [1] in that a Bose glass may only be realized in mean field theory in a uniformly random disordered system, and if the Bose glass phase found in [10] can be repeated for a uniformly disordered system in two dimensions.

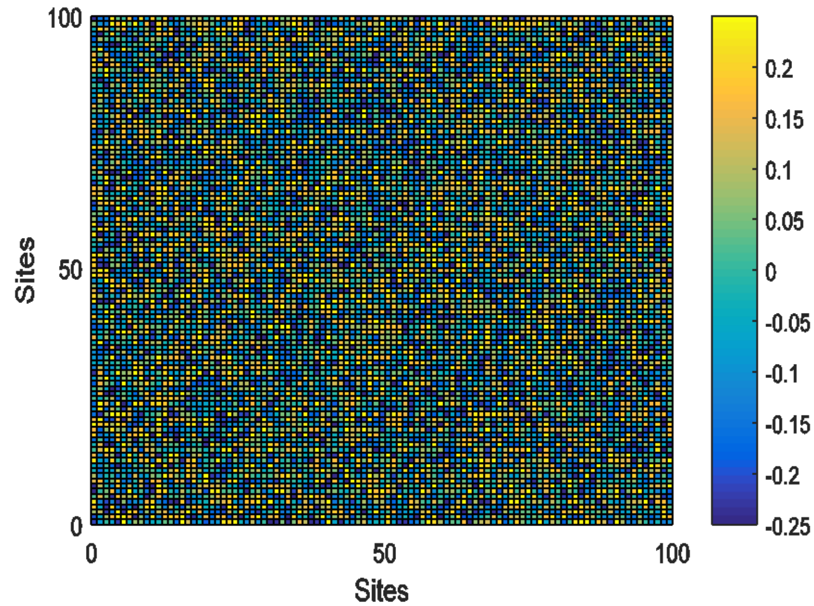

Fig. 26 Uniform disorder map over a 10000 site lattice; Δ has a periodicity of 10.

Ten possible disorder values are randomly distributed over a 10,000 site lattice in **Fig. 26** and one can see that this will result in a more realistic system study; any behavior in real space will be truly unique to the chemical potential and hopping potential pair in phase space.

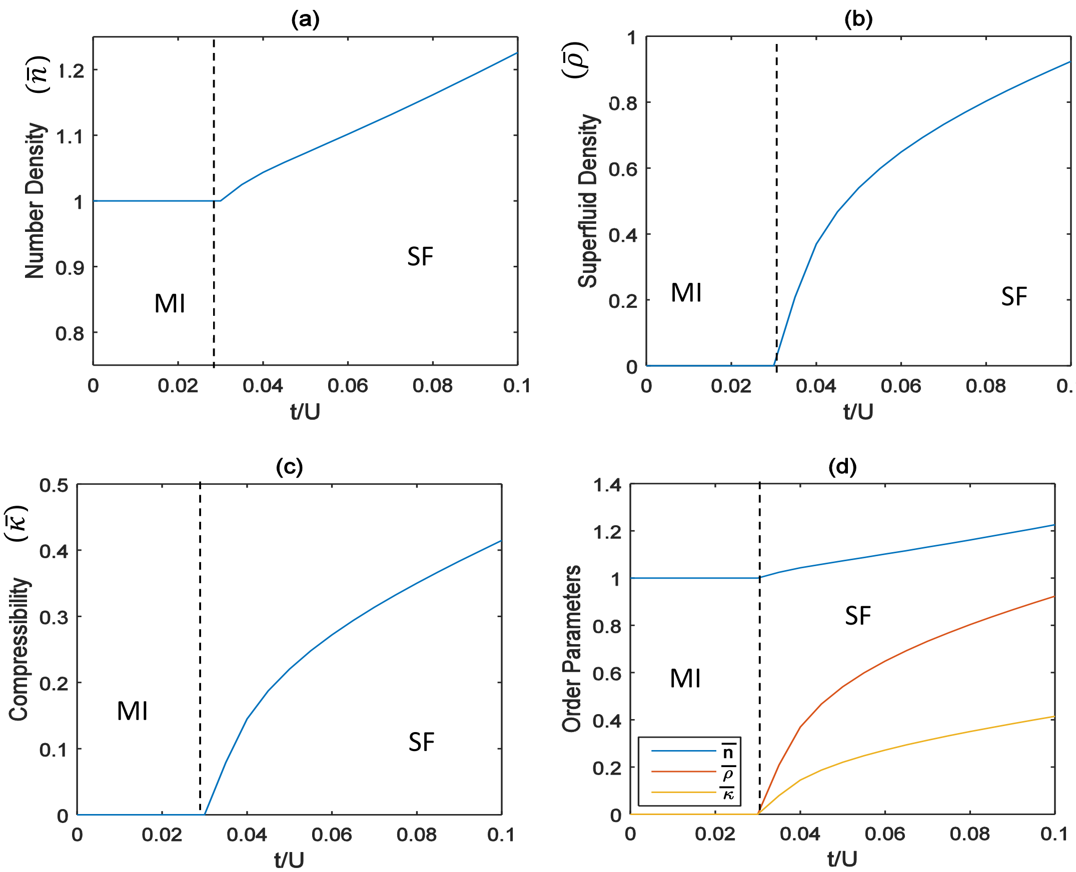

Fig. 27 Spatially averaged number density **(a)**, supefluid density **(b)** and compressibility **(c)** as a function of hopping potential for uniform disorder; chemical potential is held constant at μ = 0.5. All three are plotted on **(d)** to show the possible ambiguity between transition points for superfluid density and compressibility. The dashed line marks the insulator to superfluid transition. Energy parameters are in terms of 1/U.

Plot **(d)** in **Fig. 27** shows that the mean field approximation does not agree among different studies; this uniformly disordered system has a direct insulator to superfluid transition. The conclusion is that certain system parameters may be slightly different between different codes and coding software used to solve for energy outputs. As the Bose glass is purported to be a very small region in phase space for a uniformly disordered system, any deviation in methodolgy may show quite different results using the same general approach to modeling the system. The full phase diagram is necessary to determine if there is any small compressibile region of zero superfluid density for very small values of hopping potential.

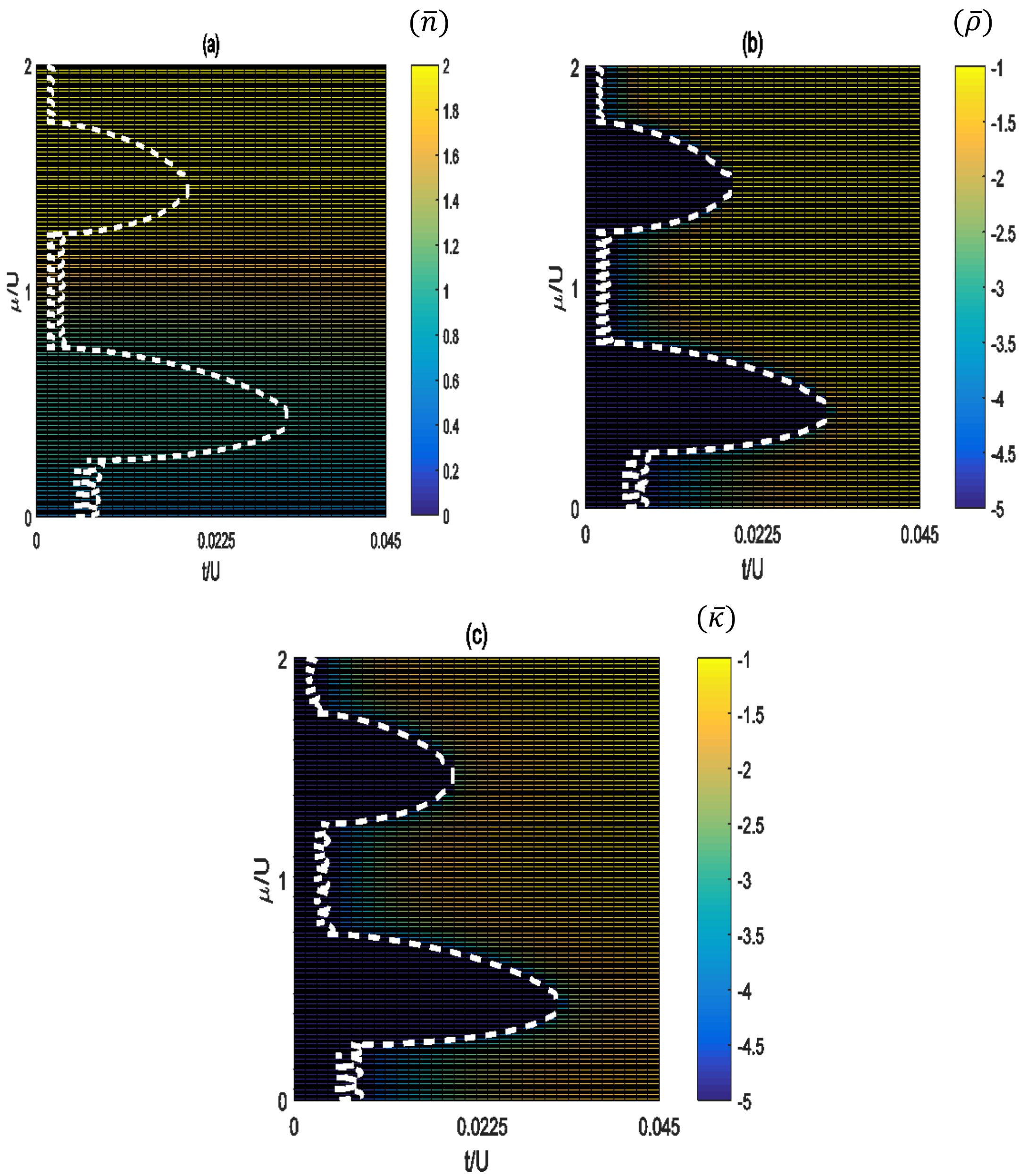

Fig. 28 Spatially averaged number density **(a)**, supefluid density **(b)** and compressibility **(c)** as a function of hopping potential for uniform disorder; chemical potential is held constant at μ = 0.5. All three are plotted on **(d)** to show the possible ambiguity between transition points for superfluid density and compressibility. Plots (**b-c**) order parameters are plotted logarithmically where any value less than -5 is effectively zero.

Number density now has a periodicity of ten, where there are nine intermitent, partial integer Mott integer lobes between the two lobes of spatially averaged integer density. The superfluid density and compressibility phase diagrams seem to have the same boundaries; at least no compressible region of zero superfluid density exists. This is illustrated in **Fig. 29**. All inputs being the same, it appears that a real space study is in fact required to determine the phase space region of Bose glass using the mean field approximation.

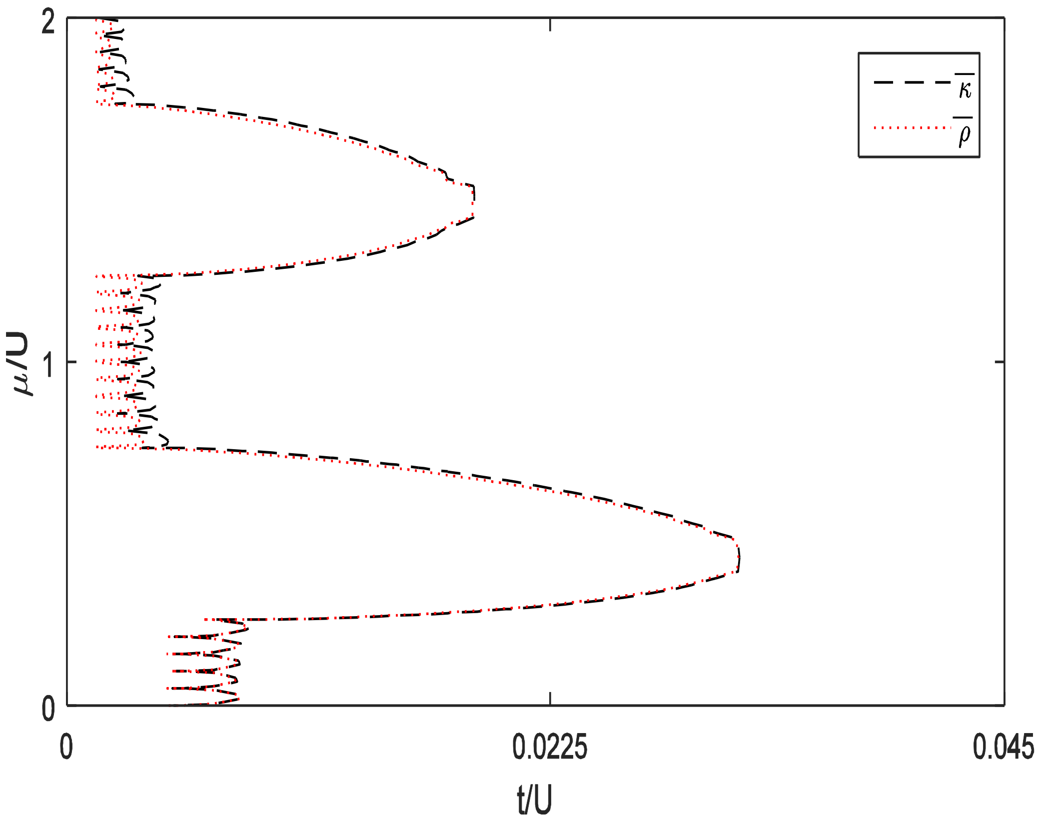

Fig. 29 Superfluid ($\overline{\rho}$) and compressibility ($\overline{\kappa}$) phase diagram boundaries for uniform disorder.

3.4.2 Real Space

The phase space points tested in real space are shown as red dots on **Fig. 30** and are tabulated in **Table 5**.

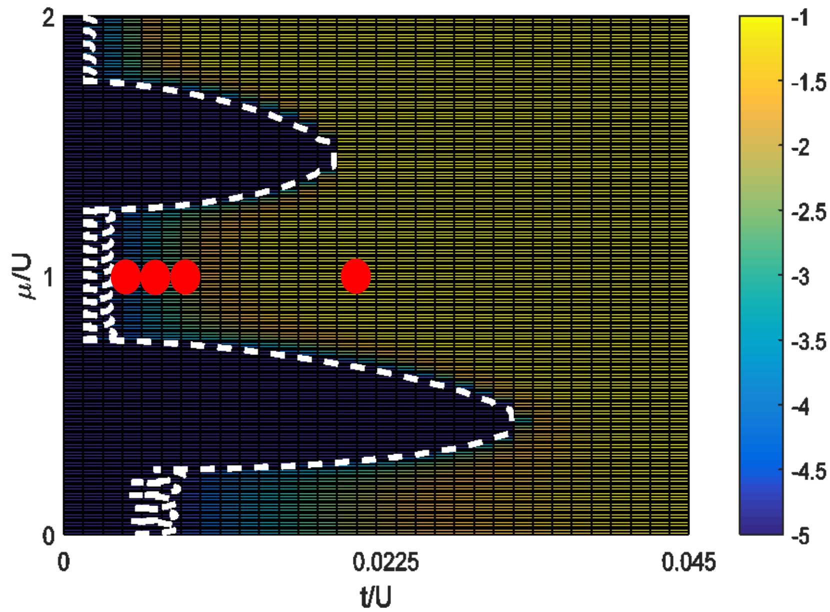

Fig. 30 Superfluid phase diagram for uniform disorder used to determine μ, t pairs to be tested in real space. The red dots represent the four values tested: 1 in a Mott insulator region, 1 in a superfluid region, and 2 in a supposed Bose glass region. Superfluid density is plotted logarithmically where any value less than -5 is effectively zero.

Table 5 Phase space values tested in real space for quaternary disorder. All energy parameters are in units of 1/U.

Order Parameter	MI	BG 1	BG 2	SF
μ/U	1	1	1	1
t/U	0	0.005	0.0105	0.02

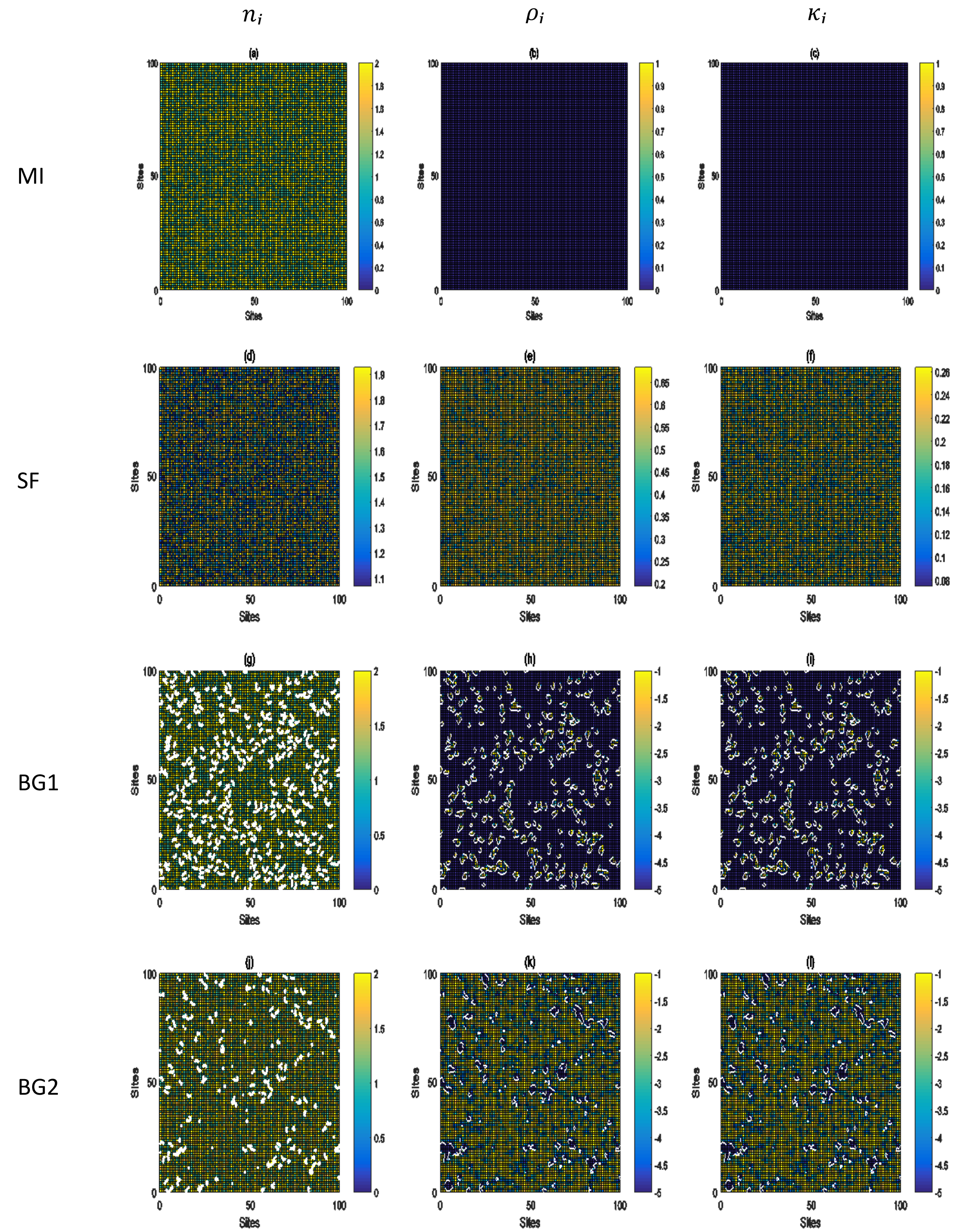

Fig. 31 Real space plots of each order parameters for ternary disorder over a 10000 site lattice (100 x 100).

In a uniformly disordered system, the Mott insulator has many values of number density across a real space lattice; it is crucial to look to the full numerical output and ensure all sites are integer to define the Mott insulator. Superfluid density and compressibility are zero at all sites in a Mott insulator.

Not all sites must be non-integer in a superfluid, as coherence across a system does not necessarily mean all sites have percolated. Thus, in a superfluid, the spatially averaged number density is a non-integer. The same is true of superfluid density and compressibility, in that not all sites have to be non-zero for the phase space point to be considered a superfluid. However, the sites that do have non-zero superfluid density also have non-zero superfluid density.

In the Bose glass, number density is similar to the superfluid region in that some sites have integer number density and some sites have non-integer number density. The difference is that regions of non-zero superfluid density and compressibility present as islands in a background of zero superfluid density and compressibility for a Bose glass; percolation has not occurred thus these islands are separated from one another and the region is not yet a superfluid. The Bose glass is in fact a global phenomenon over a real space lattice presented as islands of superfluid in a background of Mott insulator that are completely separated (no percolation).

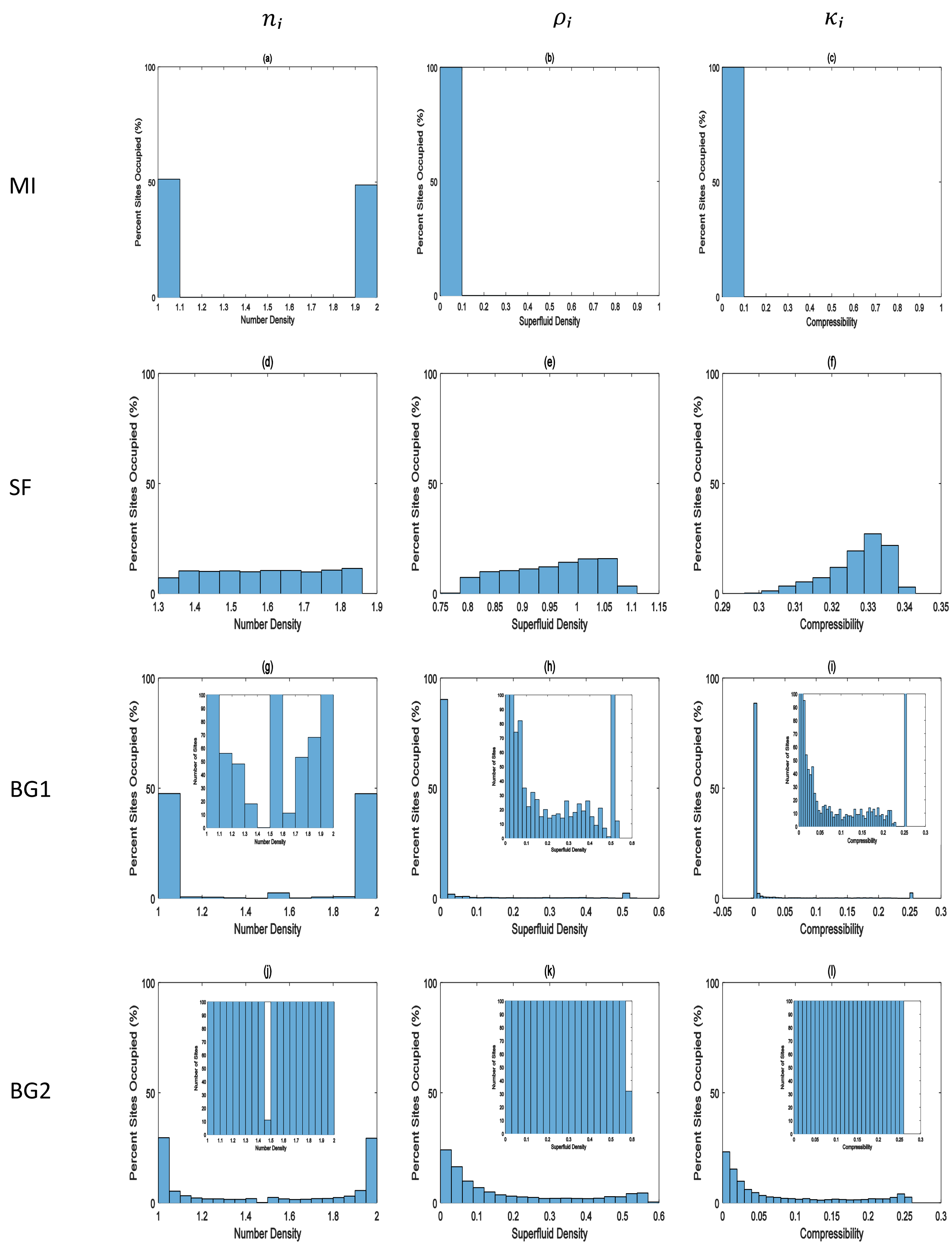

Fig. 32 Population density for uniform disorder. The inserts in **(g-l)** show a zoomed in plot of number of sites as a function of each order parameter; each plot only shows up to 100 occupied sites to elucidate the tail behavior of the Bose glass region.

Finally, the population density plots (**Fig. 32**) show similar behavior between a uniformly disordered system and simple disorder configurations.

In the Mott insulator region, all sites in real space contain an integer number density (does not have to be the same integer, but every site has an integer number density) and both zero compressibility and superfluid density. The Mott insulator maintains the same definition in real space across all disorder configurations. However, in phase space, number density can be a partial integer based on the periodicity of the disorder. Thus it is important to look to real space and population density plots of a point in a Mott insulator in phase space to ensure all sites contain an integer number density and both zero superfluid density and compressibility.

In the superfluid region, sites may contain either integer or non-integer number density. However, all sites contain non-zero superfluid density and compressibility. The superfluid region maintains the same definition in real and phase space across all disorder configurations. The major difference between phase and real space is that the spatially averaged number density in phase space is always a non-integer. As with the Mott insulator it is important to look to real space; for a slightly different reason than the Mott region, however. Real space allows one to differentiate between the Bose glass and superfluid region by determining if the superfluid clusters have connected (percolated), thus indicating a superfluid region.

The distinct difference in uniform disorder lies in the Bose glass. This region, for a very small hopping potential has behavior much more like a superfluid; little to no tail exists for an order parameter. In simple disorder configurations, the Bose glass is a region with a population density spike at a value relating to a Mott insulator (integer number density and both zero superfluid density and compressibility) and tail of values relating to a superfluid (non-zero superfluid density and compressibilty). However, with uniform disorder, there is no spike at an integer number density or zero superfluid density or compressibility; the values are more spread out like a superfluid.

A Bose glass was indeed found for very small hopping potential in a uniformly disordered system as predicted by Fisher et al [1], but was not found in phase space as predicted by Niederle and Rieger [10].

CHAPTER 4

4. ORDER PARAMETER RELATIONSHIPS

4.1 Number Density and Compressibility

After a thorough comparison of order parameters in disordered systems to those of the clean system, it was discovered that pairs of compressibility and number density in the disordered system were very similar to those in a clean system. A full phase space study was done on a clean system over a range of chemical and hopping potential values and this was compared to a single chemical potential and hopping potential pair in a uniformly disordered system. As there is only one value of each order parameter for a given chemical potential and hopping potential pair in a clean system, it was required that many chemical potential and hopping potential pairs in a clean system be compared to one pair in a disordered system. **Fig. 33** shows a range of chemical potential and hopping potential pairs from a large phase space study, and one can see that not all pair values exist. This motivated the comparison to a disordered system.

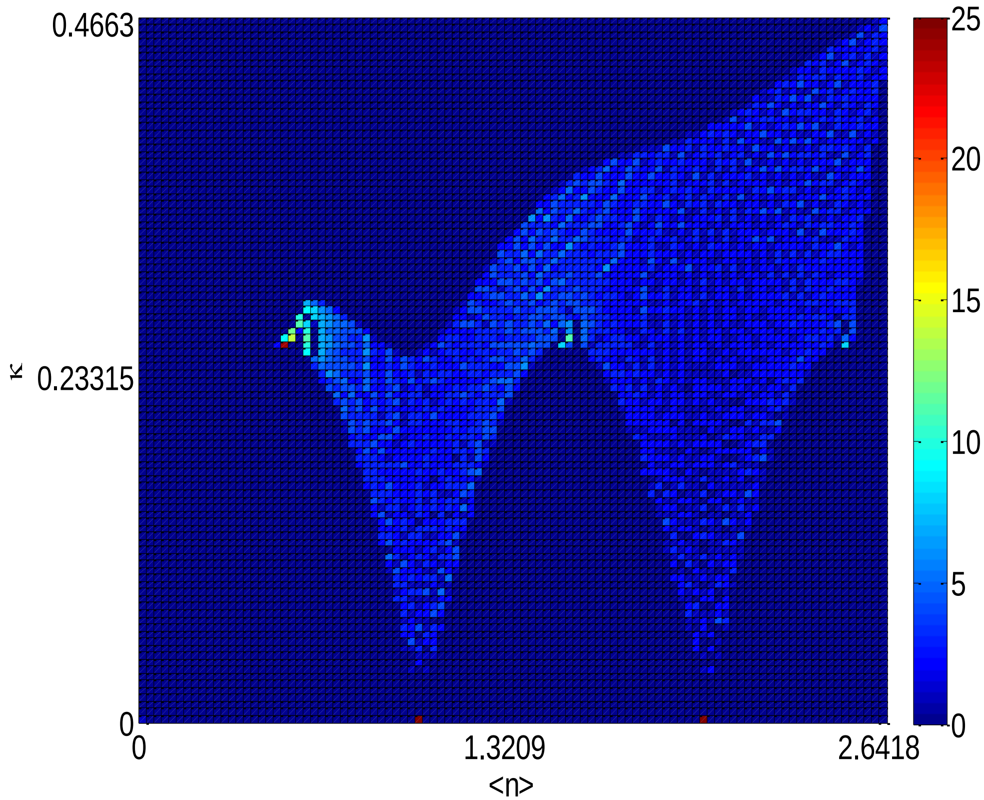

Fig. 33 Compressibility and number density for many different μ,t pairs in phase space for a clean system. Color here represents the quantity of sites with the given compressibility and number density values.

Only a few certain pair values of compressibility and number density exist at a large number of sites in the clean system, so this region is compared to the same range of order parameter values in **Fig. 34**.

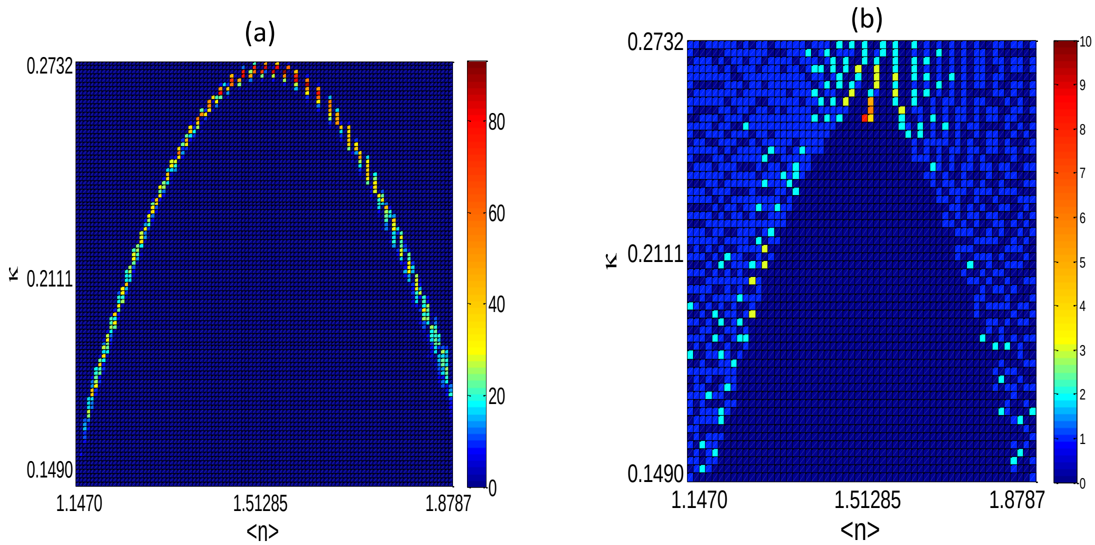

Fig. 34 Density maps of lattice sites that correspond with a certain number density and compressibility value. **(a)** is over a single disordered lattice with a system chemical potential of $\mu = 1.0$ and $t = 0.0255$. **(b) i**s over a large phase space of many chemical potential and hopping potential values in a clean system ranging over $\mu = [0,2]$ and $t = [0,0.045]$. All energy parameters are in units of 1/U.

Looking at **Fig. 34**, one can see that the pair values of compressibility and number density occupying a large number of sites is remarkably similar in both the clean and disordered system. This leads to two conclusions: (1) phase behavior and phase transitions are a local phenomenon and (2) the system may be looked at as a number of sublattices equal to the periodicity of the disorder, where a certain number density value relates to only a given compressibility. The second conclusion is what led to the hypothesis that the superfluid clusters of Niederle and Rieger [10] are actually sublattices of only Mott insulator or only superfluid; no mixture of states exists on any sublattice. This comes from the extensive study of simple and uniformly random disordered systems and there never being any single site with the definition of a Bose glass. Consequently, these two conclusions also led to the technique described in **chapter 5**, whereby the energy inputs required to describe a disordered system are used to "shift" and spatially average values in phase space of a clean system which results in phase diagrams remarkably similar to fully calculated disordered phase diagrams.

CHAPTER 5

5. PHASE DIAGRAM APPROXIMATION TECHNIQUE

The conclusion of phase transitions being a local phenomenon allows one to see that phase diagrams may be approximated. The approximation technique uses the inputs from the disorder periodicity being studied to shift and spatially average values in a clean system. Since the phase diagram is the most important tool in determining energy inputs to test in real space, being able to accurately approximate a disordered phase diagram alleviates the need to fully calculate disordered phase diagrams until after the real space study. Full calculations must be done, though, to ensure the approximation technique is accurate enough for the individual's need.

5.1. Method

As the clean (zero disorder) system is relatively easy to calculate (pencil and paper), a method that uses the clean phase diagram will help to reduce computation time significantly. Thus, the simple clean system phase diagram parameters are used as a base to shift chemical potential values up and down by the disorder strength and periodicty of the system being studied. After this "shift," order parameter values are spatially averaged over the respective real space lattice and a new phase diagram is plotted. The results for three different disorder configurations are shown in **5.2**.

5.2. Plots

The techinque was created after full phase diagrams had been studied, so the original calculations were used to validate the accuracy of the approximation technique. As **Fig. 34** shows, the results are remarkably simliar to the full calculations. However, as stated before, the full calculation must eventually be done for comparisson. **Fig. 34** shows phase diagrams for f = ½ (binary), 1/5 and 1/41.

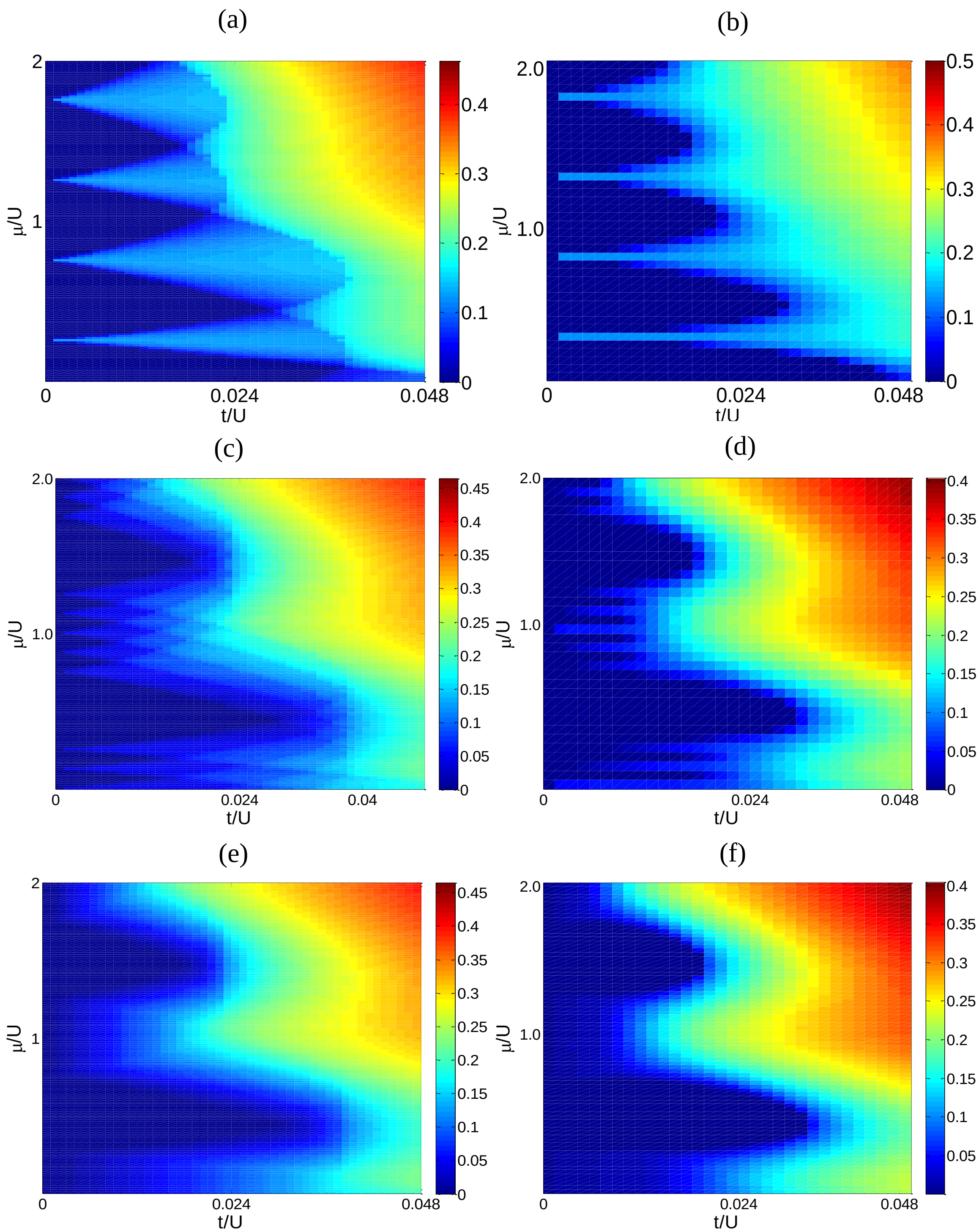

Fig. 35 Phase diagram comparison for approximation technique. The first column is the approximated phase diagram and the second is the fully calculated phase diagram. The rows are different disorder configurations such that row one f = ½, row two f = 1/5 and row three f = 1/41.

The most astonishing realization after studying the results of the approximation technique are that the accuracy appears to increase with the increase in disorder complexity. Thus, as theory stated that a fully uniform disorder configuraion was required to realize a Bose glass, this technique is powerful in the mean field approximation and allows one to “compute” (approximate) a fully disorded phase diagram in a fraction of the time using Matlab programing. Periodidicity of f = 1/10 takes roughly four days to compute having broken the phase space down into subphase spaces and computing each subphase space on a separate computer. It allows one to look to phase space for general phase behavior, and determine points of interest to test in real space. As this is the general methodolgy for studying phase transitons in bosonic system, it is highly appropriate to use the approximation technique early in a study to look to real space phenomena in the mean field approximation.

CHAPTER 6

6. CONCLUSION

The Bose glass phase has been shown to be a global phenomenon in real space using mean field approximations.

The postulates of M.P.A. Fisher et al [1] about the Bose glass in the one dimensional mean field theory are confirmed in two dimensions; no Bose glass presents in phase space using mean field approximations. Although their conclusions were made for a uniformly disordered system, it has been shown that the same conclusions hold true in simple disorder configurations.

The superfluid clusters of Niederle and Rieger's [10] work are confirmed in real space in simple disorder configurations. However, the claim that a Bose glass may be found in phase space for a uniformly disordered system is not able to be corroborated; it is shown that the system transitions directly from an insulator to a superfluid in phase space. As stated by M.P.A. Fisher et al, this is likely due to the overestimate of coherence in the system in using mean field approximations. [1] Thus, any small deviation between the two individual approaches to mean field theory could result in different results using the same general method.

Phase space behavior in simple disorder configurations, first studied at length in one dimension by Buonsante [9] are confirmed in two dimensions. The increase in periodicity of simple disorder configurations creates a quantifiable change in phase space in regards to the order parameter number density. If the periodicity of disorder is f, then f – 1 emergent Mott lobes form in phase space between the clean system's, single valued integer filling lobes. Number density in phase space, thus will be a partial integer with the same periodicity as the disorder. The value in each lobe in phase space will depend on where it is positioned in regards to the single valued integer filling lobes. If the emergent lobe is immediately next to the lower full integer lobe, in real space, the majority of sites will be populated with the same number density as the lower full integer lobe. One must determine the percent of sites occupied by the lower full integer lobe to determine how the percentage of sites changes for each lobe as chemical potential is increased. This is a straightforward task and the result allows one to determine how each emergent lobe is populated in real space.

Although the Bose glass presents as a phenomenon in real space, it must be emphasized that the phase transition at any one site in real space is still directly from insulator to superfluid; no new phase is found at any one site. Thus, the Bose glass is a phase characterized by superfluid like behavior in phase space and superfluid cluster behavior over a full real space. To realize a Bose glass in the mean field approximation it is necessary to look to both phase space and real space as well as the full real space data (population density plots) to see the robust nature of the emergent phase. Though the Bose glass may be elusive in phase space, this space is still of importance to determine the values to be tested over a real space lattice where the Bose glass may be fully studied.

Therefore, to fully study the Bose glass phase, more advanced and powerful calculation methods must be studied. The mean field approximation is computationally efficient and allows for qualitatively appropriate studies of the Bose glass; however, the limitations of mean field approximations are apparent. However, mean field theory does give a quantitatively accurate measure of real space phenomena and thus may be used to understand the important qualities discussed within.

REFERENCES

[1] M.P.A. Fisher, P.B. Weichman, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).

[2] G. Gersch, H.A. and Knollman, G. C. (School of Physics, Gerogia Institute of Technology, Atlanta, Phys. Rev. Lett. **129**, (1963).

[3] J. Pablo, Á. Zúñiga, D.J. Luitz, G. Lemarié, and N. Laflorencie, Phys. Rev. Lett. **155301**, 1 (2015).

[4] J. Kisker and H. Rieger, Phys. Rev. B **55**, R11981 (1997).

[5] B.M. Kemburi and V.W. Scarola, Phys. Rev. B - Condens. Matter Mater. Phys. **85**, (2012).

[6] Ş.G. Söyler, M. Kiselev, N. V. Prokof'Ev, and B. V. Svistunov, Phys. Rev. Lett. **107**, 1 (2011).

[7] D. Delande and J. Zakrzewski, Phys. Rev. Lett. **102**, 1 (2009).

[8] P. Buonsante, V. Penna, a. Vezzani, and P.B. Blakie, Phys. Rev. A - At. Mol. Opt. Phys. **76**, 1 (2007).

[9] P. Buonsante and a. Vezzani, Phys. Rev. A - At. Mol. Opt. Phys. **70**, 1 (2004).

[10] A.E. Niederle and H. Rieger, New J. Phys. **15**, 075029 (2013).

[11] M. White, M. Pasienski, D. McKay, S.Q. Zhou, D. Ceperley, and B. Demarco, Phys. Rev. Lett. **102**, 1 (2009).

[12] L. Fallani, J.E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Phys. Rev. Lett. **98**, 1 (2007).