Electronic Journal of Differential Equations, Vol. **1998**(1998), No. 01, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

Existence of continuous and singular ground states for semilinear elliptic systems *

Cecilia S. Yarur

Abstract

We study existence results of a curve of continuous and singular ground states for the system

$$-\Delta u = \alpha(|x|)f(v)$$
$$-\Delta v = \beta(|x|)g(u)\,,$$

where $x \in \mathbb{R}^N \setminus \{0\}$, the functions f and g are increasing Lipschitz continuous functions in $\mathbb{R}$, and α and β are nonnegative continuous functions in $\mathbb{R}^+$. We also study general systems of the form

$$\Delta u(x) + V(|x|)u + a(|x|)v^p = 0$$
$$\Delta v(x) + V(|x|)v + b(|x|)u^q = 0\,.$$

1 Introduction

The purpose of this paper is to prove existence of a curve of positive radially symmetric continuous ground states, and a curve of singular ground states with the singularity at zero, for the system

$$\begin{aligned} -\Delta u &= \alpha(|x|)f(v) \\ -\Delta v &= \beta(|x|)g(u) \end{aligned} \qquad x \in \mathbb{R}^N \setminus \{0\}\,. \tag{1.1}$$

The existence of these curves depends on conditions on the functions f, g, α and β. For some functions there exists a curve of continuous ground states, and not a curve of singular ground states and vice versa.

We will assume throughout this paper that $f \in C(\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$, $\alpha \in C^1(\mathbb{R}^+,\mathbb{R}^+)$, and $\beta \in C^1(\mathbb{R}^+,\mathbb{R}^+)$. Recall that no smoothness at zero for the function (α,β) is required. This leads to obtain existence of ground states for more general systems. Moreover, we assume that f and g are increasing functions with $f(0) = 0$ and $g(0) = 0$. Serrin and Zou [13], proved the existence

* *1991 Mathematics Subject Classifications:* 35J60, 31A35.
Key words and phrases: Semilinear elliptic systems, ground states.
©1998 Southwest Texas State University and University of North Texas.
Submitted September 3, 1997. Published January 16, 1998.
Partially supported by FONDECYT grant 1961235 and by DICYT.

of a curve of ground states for system (1.1) when $\alpha = \beta \equiv 1$, $(f(v), g(u)) = (v^p, u^q)$ and (p, q) above the critical hyperbola (see (1.6) below with $(\alpha_o, \beta_o) = (0, 0)$). P.L Lions [9] proved existence of ground states on the hyperbola by studying the scalar equation

$$-\Delta((-\Delta u)^{1/p}) = u^q\,.$$

In section 2, we give some preliminary results, such as existence results for the Cauchy problem, continuous dependence theorem and some properties for the function (α, β). Section 3 is devoted to prove the existence of a curve of regular (no classical) ground states. We assume that (f, g) are Lipschitz continuous functions such that for all $u \geq 0$ and $v \geq 0$,

$$vf(v) \geq (p+1)F(v), \quad ug(u) \geq (q+1)G(u)\,, \tag{1.2}$$

and that the functions α and β satisfy

$$\int^{\infty} r\alpha(r)dr = \infty, \quad \int^{\infty} r\beta(r)dr = \infty\,. \tag{1.3}$$

As shown in section 1, condition (1.3) implies that positive radially symmetric solutions to (1.1) are ground states. The existence of a solution, near zero, to the Cauchy problem needs

$$\int_0 s\alpha(s)ds < \infty, \quad \text{and} \quad \int_0 s\beta(s)ds < \infty. \tag{1.4}$$

We also assume the existence of α_o and β_o such that

$$r^{\alpha_o}\alpha(r), \quad \text{and} \quad r^{\beta_o}\beta(r) \quad \text{are non-increasing functions,} \tag{1.5}$$

and that

$$\frac{N-\alpha_o}{p+1} + \frac{N-\beta_o}{q+1} \leq N-2. \tag{1.6}$$

In section 4, we prove two kinds of results of existence of singular ground states. The first, will be given as a limit of regular ground states constructed in section 3, and the second type will be given by those results of section 3 and the Kelvin transform. This last type, gives existence of a curve of singular ground states under the critical hyperbola (1.6) and above the “first critical hyperbola”. In section 5 we applied results of the previous sections to general systems of the form

$$\begin{aligned}
\Delta u(x) + V(|x|)u + a(|x|)v^p &= 0\\
\Delta v(x) + V(|x|)v + b(|x|)u^q &= 0
\end{aligned}$$

with x in $\mathbb{R}^N \setminus \{0\}$, and V not necessarily negative. In particular, if $V(r) = -dr^{-2}$, $d > -(N-2)^2/4$, and $\theta_{1,0} = (2 - N \pm ((N-2)^2 + 4d)^{1/2})/2$ we get the following results.

Corollary 5.1 *Assume that there exists (a_o, b_o) such that $r^{a_o}a(r)$ and $r^{b_o}b(r)$ are non-increasing functions for some (a_o, b_o) satisfying*

$$\frac{N-a_o}{p+1} + \frac{N-b_o}{q+1} \le N-2\,.$$

Also assume that a and b satisfy (5.5) and (5.6). Then, there exists $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ strictly increasing and such that for any $c > 0$ there exists a radially symmetric solution (u, v) to (5.8) such that

$$\lim_{r\to 0} \frac{u(r)}{r^{\theta_1}} = g(c), \quad \text{and} \quad \lim_{r\to 0} \frac{v(r)}{r^{\theta_1}} = c\,.$$

Corollary 5.2 *Assume that there exists (a_o, b_o) such that $r^{a_o}a(r)$ and $r^{b_o}b(r)$ are nondecreasing functions for some (a_o, b_o) satisfying*

$$\frac{N-a_o}{p+1} + \frac{N-b_o}{q+1} \ge N-2.$$

Also assume that a and b satisfy (5.10) and (5.11). Then, there exists $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ strictly increasing and such that for any $c > 0$ there exists a radially symmetric solution (u, v) to (5.8) such that

$$\lim_{r\to \infty} \frac{u(r)}{r^{\theta_0}} = \delta(c), \quad \lim_{r\to \infty} \frac{v(r)}{r^{\theta_0}} = c.$$

After this work was completed, we learned of a paper by Serrin and Zou, [14], in which they prove existence of classical ground states for a general elliptic Hamiltonian system.

2 Preliminary results

This section is devoted to prove existence, continuous dependence results and some previous properties of nonnegative solutions to the problem

$$\begin{aligned} -u''(r) - \tfrac{N-1}{r}u'(r) &= \alpha(r)f(v) \\ -v''(r) - \tfrac{N-1}{r}v'(r) &= \beta(r)g(u), \\ u(0) = c_1, \quad v(0) &= c_2, \end{aligned} \tag{2.1}$$

where c_1 and c_2 are positive constants. In this section, the functions α and β are C^1 functions defined in $(0, \infty)$, satisfying

$$\int^\infty r\alpha(r)dr = \infty, \quad \int^\infty r\beta(r)dr = \infty. \tag{2.2}$$

Condition (2.2) implies that positive radial solutions to (1.1) are ground states, as is shown below. Moreover,

$$\int_0 s\alpha(s)ds < \infty, \quad \text{and} \quad \int_0 s\beta(s)ds < \infty \tag{2.3}$$

ensures the existence of nonnegative solutions to (2.1). Condition (2.3) appears usually on existence problems ([1], [11]), and is the Kato class for radially symmetric functions.

The solutions of (2.1) defined in $[0,\infty)$ are radially symmetric continuous solutions to (1.1), in the sense of distributions in $\mathbb{R}^N$. We distinguish the radially symmetric continuous solutions to (1.1) from those which satisfy

$$\begin{aligned} -u''(r) - \frac{N-1}{r}u'(r) &= \alpha(r)f(v), \\ -v''(r) - \frac{N-1}{r}v'(r) &= \beta(r)g(u), \\ u(0)=c_1, u'(0)=0, \qquad & v(0)=c_2, v'(0)=0, \end{aligned} \tag{2.4}$$

by calling these last type **classical solutions**. Our interest in the study of continuous solutions and not necessary classical solutions, comes from the existence of singular ground states treated in Section 4. The classical nature of a solution to (2.1) with $c_1>0$ and $c_2>0$ only depends on the functions α and β as is shown in the following.

Proposition 2.1 *Assume that* (α,β) *satisfies* (2.3). *Let* (u,v) *be a solution to* (2.1) *with* $c_1>0$ *and* $c_2>0$. *Then*

$$\lim_{r\to 0} ru'(r)=0, \quad \textit{and} \quad \lim_{r\to 0} rv'(r)=0. \tag{2.5}$$

Moreover, if

$$\lim_{r\to 0} r^{1-N}\int_0^r s^{N-1}\alpha(s)ds=0, \quad \textit{and} \quad \lim_{r\to 0} r^{1-N}\int_0^r s^{N-1}\beta(s)ds=0, \tag{2.6}$$

then, $u'(0)=0$, *and* $v'(0)=0$.

Proof. Since u is continuous at zero, $\lim_{r\to 0} r^{N-1}u'(r)=0$, and thus, since from the system (2.1) $-r^{N-1}u'(r)$ is increasing for r small, we have that u is decreasing for r small enough. Similarly, v is decreasing for all r small. Integrating from 0 to r the first equation of (2.1) and if $C=\max_{v\in[c_2/2,c_2]} f(v)$, we get

$$\begin{aligned} -r^{N-1}u'(r) &= \int_0^r s^{N-1}\alpha(s)f(v(s))\,ds \\ &\le C\int_0^r s^{N-1}\alpha(s)ds \le Cr^{N-2}\int_0^r s\alpha(s)\,ds\,, \end{aligned}$$

and thus (2.5) follows for u, and similarly for v. From the above estimate we also get $u'(0)=0$ assuming (2.6). $\diamondsuit$

The following is a nonexistence result of solutions to (2.1), without condition (2.3). For a similar result in this direction, see [3].

Proposition 2.2 *Assume that* (u,v) *is a nonnegative solution to* (2.1) *with* $c_1>0$ *and* $c_2>0$, *then* (α,β) *satisfies* (2.3). *Moreover, if* (u,v) *is a classical solution to* (2.1), *then* (α,β) *satisfies* (2.6).

Proof. Integrating twice (2.1) from 0 to r_0 with r_0 small enough such that $c_1 \geq u(r) \geq c_1/2$ and $c_2 \geq v(r) > c_2/2$, for all $r \in [0, r_0]$, we obtain

$$c_1 - \int_0^{r_0} s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v(t)) dt \right] ds \geq c_1/2,$$

and thus,

$$m \int_0^{r_0} s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) dt \right] ds \leq \int_0^{r_0} s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v(t)) dt \right] ds \leq \frac{c_1}{2} ,$$

where $m = \min_{r \in [c_2/2, c_2]} \{f(v)\}$. Therefore,

$$\int_0^{r_0} s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) dt \right] ds < \infty,$$

which is equivalent to $\int_0 s\alpha(s)ds < \infty$. Moreover,

$$-u'(r) = r^{1-N} \int_0^r s^{N-1} \alpha(s) f(v(s)) ds \geq m r^{1-N} \int_0^r s^{N-1} \alpha(s) ds$$

and thus if $u'(0) = 0$,

$$0 = -u'(0) \geq m \lim_{r \to 0} r^{1-N} \int_0^r s^{N-1} \alpha(s) ds.$$

Similarly, it can be proven that

$$\int_0 s\beta(s)ds < \infty, \quad \text{and} \quad \lim_{r \to 0} r^{1-N} \int_0^r s^{N-1} \beta(s) ds = 0 \, .$$

$\diamondsuit$

Condition (2.2) implies that nonnegative solutions to (2.1) defined in $(0, \infty)$ are ground states, as is proved in the following proposition.

Proposition 2.3 *Assume that* (α, β) *satisfies* (2.2) *and let* (u, v) *be a nonnegative solution to* (2.1) *in* $(0, \infty)$. *Then,*

$$\lim_{r \to \infty} u(r) = 0, \quad \lim_{r \to \infty} v(r) = 0.$$

Proof. From [4]), we deduce the following:

Let $0 \leq h \in L^1_{\rm loc}(B_1(0) \setminus \{0\})$, and $w \in L^1_{\rm loc}(B_1(0) \setminus \{0\})$ be a nonnegative function, such that $-\Delta w \in L^1_{\rm loc}(B_1(0) \setminus \{0\})$ in the sense of distributions in $B_1(0) \setminus \{0\}$, and $-\Delta w = h$. Then, $h \in L^1_{\rm loc}(B_1(0))$. Moreover, for a radially symmetric w; $\lim_{r \to 0} r^{N-2} w(r)$ exists and it is finite.

Next, we prove that v tends to zero as r tends to infinity. Consider w, be the Kelvin transform of the function u (see [6]), that is

$$w(r) = r^{2-N} u(1/r).$$

We easily get that

$$-\Delta w = h, \quad \text{where} \quad h(x) = |x|^{-(N+2)}\alpha(1/|x|)f(v(1/|x|)),$$

and thus $h \in L^1_{\rm loc}(B_1(0))$, and $\lim_{r\to\infty} u(r) = \lim_{r\to 0} r^{N-2}w(r)$ exists and is finite. Similarly, there exists $\lim_{r\to\infty} v(r) = c$. Now, $h \in L^1_{\rm loc}(B_1(0))$, is equivalent to

$$\int^{\infty} s\alpha(s)f(v(s)) < \infty.$$

Therefore, from (2.2), we obtain $c = 0$. ♢

The next results will be needed in the following sections.

Proposition 2.4 *Let α be any nonnegative function, and $\alpha_o \in \mathbb{R}$, be such that $\int_0 s\alpha(s)ds < \infty$ and $r^{\alpha_o}\alpha(r)$ is non-increasing. Then, $\alpha_o < 2$, and $\lim_{r\to 0} r^2\alpha(r) = 0$.*

Proof. Let $0 < r < r_0$, with r_0 small, then

$$\int_0^r s\alpha(s)ds \ge \int_{r/2}^r s^{\alpha_o}\alpha(s)s^{1-\alpha_o}ds \ge Cr^2\alpha(r) \ge Kr^{2-\alpha_o},$$

and the conclusion follows. ♢

For $(\alpha_o, \beta_o) \in \mathbb{R}^2$ and $pq > 1$ we define,

$$\gamma_1(\alpha_o,\beta_o) = \frac{\alpha_o - 2 + (\beta_o - 2)p}{pq-1}, \quad \gamma_2(\alpha_o,\beta_o) = \frac{\beta_o - 2 + (\alpha_o - 2)p}{pq-1}. \tag{2.7}$$

We have the following

Proposition 2.5 *(i) If $pq > 1$, then conditions (1.6) and $\gamma_1(\alpha_o,\beta_o) + \gamma_2(\alpha_o,\beta_o) + N - 2 \ge 0$ are equivalent.*

(ii) Assume that $pq > 1$ and assume that there exist (α_o, β_o) such that $r^{\alpha_o}\alpha(r)$, $r^{\beta_o}\beta(r)$ are non-increasing functions, and condition (2.3) is satisfied. Then, $\gamma_i(\alpha_o,\beta_o) < 0$, for $i = 1, 2$.

(iii) Assume that (α, β, p, q) satisfy the hypothesis given on (ii) and (1.6). Then $2 - \alpha_o - (N-2)p < 0$ and $2 - \beta_o - (N-2)q < 0$.

Proof. The proof of (i) is a consequence of the definition of γ_i for $i = 1, 2$. Item (ii) follows from Proposition 2.4. From that proposition $2 - \alpha_o > 0$ and $2 - \beta_o > 0$, and thus $N - \alpha_o > 0$ and $N - \beta_o > 0$. Therefore, if we are in the assumption of (iii), we get

$$\frac{N-\alpha_o}{p+1} \le N - 2 - (\frac{N-\beta_o}{q+1}) < N - 2,$$

and the conclusion follows. ♢

Theorem 2.1 *Assume that* (α, β) *satisfies* (2.3) *and* f *and* g *are continuous nonnegative functions defined in* $\mathbb{R}^+ \cup \{0\}$. *Then, for any* $(c_1, c_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, *there exists a solution* (u, v) *to* (2.1) *defined in some interval* $[0, k)$. *Moreover, the solution* (u, v) *is a classical solution if* (α, β) *satisfies* (2.6).

Proof. Let

$$S := \{(u, v) \in C([0, \epsilon]) \times C([0, \epsilon])| \quad 0 \le u \le c_1, 0 \le v \le c_2\},$$

where $\epsilon > 0$ is small enough. In S we define $T(u, v) = (T_1(u, v), T_2(u, v))$ as follows

$$\begin{aligned} T_1(u,v)(r) &= c_1 - \int_0^r s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v(t)) dt \right] ds \\ T_2(u,v)(r) &= c_2 - \int_0^r s^{1-N} \left[\int_0^s t^{N-1} \beta(t) g(u(t)) dt \right] ds. \end{aligned}$$

Next, we will show that for ϵ small $T(S) \subset S$. Since, $v \ge 0$, from the definition of T_1 it follows that $T_1(u, v) \le c_1$. We will prove that $T_1(u, v) \ge 0$. Let $M \ge \max\{f(v) : v \in [0, c_2]\}$, then if $(u, v) \in S$, we have

$$\begin{aligned} 0 \le \int_0^\epsilon s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v(t)) dt \right] ds &\le M \int_0^\epsilon s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) dt \right] ds \\ &\le \frac{M}{N-2} \int_0^\epsilon s\alpha(s) ds, \end{aligned}$$

and thus, if $\frac{M}{N-2} \int_0^\epsilon s\alpha(s)ds \le c_1$, we obtain that $T_1(u, v) \ge 0$. A similar argument can be used to prove $0 \le T_2(u, v) \le c_2$. It is not difficult to prove that T is a continuous and compact operator in S, with respect to the uniform convergence, and thus it has a fixed point, which corresponds to a nonnegative (u, v) to (2.1). $\diamondsuit$

Lemma 2.1 *Let* α *and* β *be two nonnegative functions defined in* $(0, \infty)$, *satisfying* (2.3). *Let* f *and* g *be locally Lipschitz continuous functions, and nonnegative near zero. Let* (u_1, v_1), (u_2, v_2) *be nonnegative solutions to* (2.1), *with* $(u_1(0), v_1(0)) = (c_1, c_2)$, $(u_2(0), v_2(0)) = (d_1, d_2)$, *and* c_1, c_2, d_1, d_2 *positive numbers. Consider* $I_1 := [0, r_1]$ *to be any interval where the functions* u_i, *and* v_i, *for* $i = 1, 2$, *are defined. Then, there exists a positive constant* C *such that for all* $r \in I_1$

$$\begin{aligned} |u_1(r) - u_2(r)| &\le C \max\{|c_1 - d_1|, |c_2 - d_2|\}, \\ |v_1(r) - v_2(r)| &\le C \max\{|c_1 - d_1|, |c_2 - d_2|\}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} r|u_1'(r) - u_2'(r)| &\le C \max\{|c_1 - d_1|, |c_2 - d_2|\}, \\ r|v_1'(r) - v_2'(r)| &\le C \max\{|c_1 - d_1|.|c_2 - d_2|\}. \end{aligned} \tag{2.9}$$

Proof. Consider I_1 as in the hypothesis of this lemma. Define $b := \max\{|c_1 - d_1|, |c_2 - d_2|\}$, then integrating twice (2.1), and since from proposition 2.1, $\lim_{r\to 0} ru'(r) = 0$, we get

$$|u_1'(r) - u_2'(r)| \le r^{1-N} \int_0^r s^{N-1}\alpha(s)|f(v_1(s)) - f(v_2(s))|ds, \tag{2.10}$$

and

$$|u_1(r) - u_2(r)| \le b + H, \tag{2.11}$$

where

$$H := \int_0^r t^{1-N} \left[\int_0^t s^{N-1}\alpha(s)|f(v_1(s)) - f(v_2(s))|ds\right] dt.$$

Since

$$|f(v_1(r)) - f(v_2(r))| \le m|v_1(r) - v_2(r)| \quad \text{for any} \quad r \in [0, r_1],$$

where we take m as a Lipschitz constant for f and g, from (2.11) we get

$$|u_1(r) - u_2(r)| \le b + m \int_0^r t^{1-N} \left[\int_0^t s^{N-1}\alpha(s)|v_1(s) - v_2(s)|ds\right] dt. \tag{2.12}$$

Arguing as above, but now with the second equation of (2.1), we obtain

$$|v_1(r) - v_2(r)| \le b + m \int_0^r t^{1-N} \left[\int_0^t s^{N-1}\beta(s)|u_1(s) - u_2(s)|ds\right] dt. \tag{2.13}$$

Define

$$X(r) := b + m \int_0^r t^{1-N} \left[\int_0^t s^{N-1}\alpha(s)|v_1(s) - v_2(s)|ds\right] dt\,, \tag{2.14}$$

$$Y(r) := b + m \int_0^r t^{1-N} \left[\int_0^t s^{N-1}\beta(s)|u_1(s) - u_2(s)|ds\right] dt\,. \tag{2.15}$$

The functions X and Y are nondecreasing functions such that $X(0) = Y(0) = b$, and

$$\begin{aligned} (r^{N-1}Y')'(r) &= mr^{N-1}\beta(r)|u_1(r) - u_2(r)| \le mr^{N-1}\beta(r)X(r), \\ (r^{N-1}X')'(r) &= mr^{N-1}\alpha(r)|v_1(r) - v_2(r)| \le mr^{N-1}\alpha(r)Y(r), \end{aligned} \tag{2.16}$$

and thus using that Y is increasing in the second inequality of (2.16), we get

$$X(r) \le b + mC_\alpha Y(r), \tag{2.17}$$

where $C_\alpha := (1/(N-2) \int_0^{r_1} s\alpha(s)ds$. Using (2.17) in the first inequality of (2.16), we have that

$$(r^{N-1}Y')'(r) \le mr^{N-1}\beta(r)\left(b + mC_\alpha Y(r)\right). \tag{2.18}$$

Integrating twice (2.18) from 0 to r, we easily get

$$Y(r) \leq b(1+mC_\beta) + \frac{m^2 C_\alpha}{N-2} \int_0^r t\beta(t) Y(t) dt, \tag{2.19}$$

where C_β is define by replacing β for α in the definition of C_α. From Gronwall's inequality and (2.19), we deduce

$$Y(r) \leq b(1+mC_\beta) \exp\left\{ \frac{m^2 C_\alpha}{N-2} \int_0^r t\beta(t) dt \right\} \leq b(1+mC_\beta) \exp\{m^2 C_\alpha C_\beta\},$$

and similarly for X, and thus the conclusion follows from the above inequality, (2.12) and (2.13). For the bound of u', we use (2.10), and the bound for v. $\diamondsuit$

3 Existence of grounds states

In this section we prove the existence of a curve of ground states for the system (1.1), under the following conditions: the functions $f \in C(\mathbb{R})$ and $g \in C(\mathbb{R})$ are increasing functions, Lipschitz continuous such that $f(0) = 0$, $g(0) = 0$. Moreover, we assume the existence of two positive constants p and q such that for $u \geq 0$ and $v \geq 0$

$$vf(v) \geq (p+1)F(v), \quad ug(u) \geq (q+1)G(u), \tag{3.1}$$

where

$$F(v) = \int_0^v f(t) dt, \quad G(u) = \int_0^u g(t)\, dt\,.$$

For the functions α and β we assume that they are nonnegative C^1 functions defined in $(0,\infty)$ and such that there exist α_o and β_o such that

$$r^{\alpha_o}\alpha(r), \quad \text{and} \quad r^{\beta_o}\beta(r) \quad \text{are non-increasing functions,} \tag{3.2}$$

and

$$\frac{N-\alpha_o}{p+1} + \frac{N-\beta_o}{q+1} \leq N-2. \tag{3.3}$$

Moreover, we assume that (2.2) and (2.3) are satisfied.

Nonexistence results can be found in [10], [12] and the references therein.

Consider the system

$$\begin{aligned} -u''(r) - \frac{N-1}{r} u'(r) &= \alpha f(v), \\ -v''(r) - \frac{N-1}{r} v'(r) &= \beta g(u), \end{aligned} \tag{3.4}$$

and let $= (c_1, c_2)$ be such that $c_1 > 0$ and $c_2 > 0$, and (u, v) be the solution to (3.4) such that $(u(0), v(0)) = (c_1, c_2)$. We define the subsets

$$\begin{aligned} \mathcal{U}_0 &:= \{(c_1,c_2) \in \mathbb{R}^+ \times \mathbb{R}^+ | \quad \text{such that} \quad u(\tau) = 0 \quad \text{for some} \quad \tau > 0\}, \\ \mathcal{V}_0 &:= \{(c_1,c_2) \in \mathbb{R}^+ \times \mathbb{R}^+ | \quad \text{such that} \quad v(\tau) = 0 \quad \text{for some} \quad \tau > 0\}, \\ \mathcal{G} &:= \{(c_1,c_2) \in \mathbb{R}^+ \times \mathbb{R}^+ | \quad (u,v) \quad \text{is a ground state}\}. \end{aligned}$$

Our main result of this section is the following.

Theorem 3.1 *Assume that conditions* (2.2), (2.3), (3.1), (3.2) *and* (3.3) *are satisfied. Then, there exists a function* $h \in C(\mathbb{R}^+, \mathbb{R}^+)$, *strictly increasing such that*

1. *For any* $c_2 > 0$, *the solution to* (2.1) *with* $c_1 = h(c_2)$, *is a ground state.*
2. $\mathcal{U}_0 = \{(c_1, c_2) | \quad c_1 < h(c_2)\}$.
3. $\mathcal{V}_0 = \{(c_1, c_2) | \quad h(c_2) < c_1\}$.

Before proving this theorem, we need some preliminary results. The following is a Pohozaev-Pucci-Serrin identity. The proof can be obtained by a direct computation.

Proposition 3.1 *Let* (u, v) *be a solution to* (3.4) *in some interval* I, *and let* a *and* b *be two constants. For any* $r \in I$ *define*

$$\begin{aligned} E_{ab}(r) &= r^N \left[u'(r)v'(r) + ar^{-1}u'(r)v(r)\right] \\ &\quad + r^N \left[br^{-1}v'(r)u(r) + \alpha F(v) + \beta G(u)\right]. \end{aligned} \tag{3.5}$$

Then, the derivative with respect to r *of* E_{ab} *satisfies*

$$\begin{aligned} E'_{ab}(r) &= (a + b + 2 - N)r^{N-1}u'(r)v'(r) + (r^N\alpha)'F(v) \\ &\quad - ar^{N-1}\alpha v f(v) + (r^N\beta)'G(u) - br^{N-1}\beta u g(u). \end{aligned} \tag{3.6}$$

Next, we study the sets $\mathcal{U}_0$ and $\mathcal{V}_0$.

Theorem 3.2 *Assume that conditions in theorem 3.1 are satisfied. We have the following:*

(i) If $(c_1, c_2) \in \mathcal{U}_0$ *and* (u, v) *is the solution to* (2.1) *defined on his maximal right hand side interval* $I = [0, k)$, *then there exists* $\tau \in I$ *such that*

$$\begin{array}{rl} u(\tau) = 0, & u(r)(\tau - r) > 0 \quad r \in I \setminus \{\tau\} \\ v(r) > 0 \quad r \in I, & v'(r) < 0 \in (0, \tau) \\ \lim_{r\to k} u(r) = -\infty, & \lim_{r\to k} v(r) = \infty. \end{array} \tag{3.7}$$

(ii) If $(c_1, c_2) \in \mathcal{V}_0$ *and* (u, v) *is the solution to* (2.1) *defined on his maximal right hand side interval* $I = [0, k)$, *then there exists* $\tau \in I$ *such that*

$$\begin{array}{rl} v(\tau) = 0, & v(r)(\tau - r) > 0 \quad r \in I \setminus \{\tau\} \\ u(r) > 0 \quad r \in I, & u'(r) < 0 \in (0, \tau) \\ \lim_{r\to k} u(r) = \infty, & \lim_{r\to k} v(r) = -\infty. \end{array}$$

(iii) $\mathbb{R}^+ \times \mathbb{R}^+ = \mathcal{G} \cup \mathcal{U}_0 \cup \mathcal{V}_0$.

Proof. If (u, v) is not a ground state, from proposition 2.3 either u or v has a zero at some point $\tau < \infty$, and thus (iii) is proved.

Let (u, v) be a solution to (2.1) defined on his maximal right hand side interval $I = (0, k)$, (positive or not) and such that $u(\tau) = 0$ with u and v positive functions in $I' = (0, \tau)$.

Assertion. $r^{N-2}v$ is increasing on I, and $u(r) < 0$ for all $r \in (\tau, k)$.

We prove the assertion in two steps.

Step 1. $r^{N-2}v$ is increasing on I'.

The functions u and v are non-increasing functions on I'. For a and b such that $b \geq (N - \beta_o)/(q + 1)$, $a \geq (N - \alpha_o)/(p + 1)$, and $a + b \leq N - 2$; we obtain that $E'_{ab}(r)$ is non-positive for all $r \in I'$. In that case the energy is a decreasing function of r and thus, $E_{ab}(r) \leq E_{ab}(0)$. Next, we prove that $E_{ab}(0) = 0$. From Proposition 2.1, the three first terms on $E'_{ab}(r)$, goes to zero as r does. The convergence to zero of the other two terms, follows from Proposition 2.4; Thus for all $r \in (0, \tau)$

$$E_{ab}(r) \leq 0. \tag{3.8}$$

For $r \in (0, \tau)$, let

$$d(r) := -\frac{ru'(r)}{u(r)} \quad \text{and} \quad e(r) := -\frac{rv'(r)}{v(r)}. \tag{3.9}$$

From (3.8) in particular we obtain

$$e(r)d(r) - ad(r) - be(r) < 0. \tag{3.10}$$

As a consequence of $u(\tau) = 0$ and the concave nature of $w_1(s) := su(r)$, $s = r^{N-2}$, there exists $r_1 < \tau$ satisfying $d(r_1) = N - 2$, and for all $r \in (r_1, \tau)$, $d(r) > N - 2$. Therefore,

$$\frac{N - \alpha_o}{p + 1} + \frac{N - \beta_o}{q + 1} \leq d(r_1) = N - 2. \tag{3.11}$$

For this $d(r_1)$ we can choose a, b such that $a + b = d(r_1)$ and $\frac{N-\alpha_o}{p+1} \leq a$, $\frac{N-\beta_o}{q+1} \leq b$. Returning to (3.10) we get $e(r_1) < d(r_1) = N-2$. On the other hand, the function $w_2(s) := sv(r)$, $\quad s = r^{N-2}$ is a concave function with $w_2(0) = 0$ and thus, $e(r) < N - 2$ for all $r \in [0, r_1]$. Moreover, if for some $r_2 > r_1$ we have that $e(r_2) = N - 2$, then arguing as above we get that $d(r_2) < N - 2$, which is a contradiction. Thus we conclude that $e(r) < N - 2$ for all $r \in (0, \tau)$. Therefore, $r^{N-2}v(r)$ is an increasing function in $(0, \tau)$. In particular $v(\tau) > 0$.

Step 2. Next, we will prove that $u(r) < 0$ for all $r \in (\tau, k)$, and $v(r) > 0$ for all $r \in I$. For it, let $w_2(s) := r^{N-2}v(r)$ with $s = r^{N-2}$ and $s \in (\tau^{N-2}, k^{N-2})$. Assume that $u(r) < 0$ for $r \in (\tau, x)$. Since $g(u) \leq 0$, for $u \leq 0$, it can be easily verified that w_2 is a convex function for all $s \in (\tau^{N-2}, x^{N-2})$, and thus in such interval

$$\frac{dw_2}{ds}(s) \geq \frac{dw_2}{ds}(\tau^{N-2}) \geq 0,$$

which in turn implies that $w_2(s) \geq w_2(\tau^{N-2}) > 0$, for all $s \in (\tau^{N-2}, x^{N-2})$. Now, consider the function $w_1(s) := r^{N-2}u(r)$. Using now, that $w_2(s) > 0$ and thus $v(r) > 0$ for all $r \in (0, x)$, it can be verified that the function w_1 is concave in $(0, x)$. We have used that $f(v) \geq 0$, for $v \geq 0$. Therefore, in particular

$$\frac{dw_1}{ds}(s) \leq \frac{dw_1}{ds}(\tau^{N-2}) < 0,$$

for all $s \in (\tau^{N-2}, x^{N-2})$, and thus $w_1(s)$ is decreasing for $s > \tau^{N-2}$ and it cannot be zero for $s > \tau^{N-2}$. Also in the above argument we can take $x = k$ to conclude that v is positive and $r^{N-2}v(r)$ is increasing on I. Moreover, from (2.1) and since v is positive we obtain that u is decreasing on I. Thus the assertion follows.

By changing the role between u and v in the above assertion, we obtain that $\mathcal{U}_0 \cap \mathcal{V}_0 = \emptyset$. Next, we will show that

$$\lim_{r\to k} u(r) = -\infty, \quad \lim_{r\to k} v(r) = \infty.$$

Assume first that $k < \infty$. Since $r^{N-2}v(r)$ is increasing, we obtain the existence of $\lim_{r\to k} v(r) = l$. Assume, by contradiction that $l < \infty$, and thus v is bounded. Integrating the first equation of (2.1) and using the facts that v bounded and $k < \infty$, it can be easily verified that u is also bounded. Therefore, (u, v) can be extended to the right of k, contradicting the definition of k. The proof that $\lim_{r\to k} u(r) = -\infty$ is analogous to the above.

Now, for $k = \infty$, we prove first that $l = \lim_{r\to\infty} v(r) = \infty$. Integrating twice the first equation of (2.1) we obtain for any $r > 2\tau$,

$$\begin{aligned} v(r) - v(2\tau) &= -\int_{2\tau}^{r} s^{1-N}\left[\int_0^s t^{N-1}\beta(t)g(u(t))dt\right]ds \\ &= \frac{1}{N-2}[r^{2-N} - (2\tau)^{2-N}]\left[\int_0^{2\tau} t^{N-1}\beta(t)g(u(t))dt\right] \\ &\quad -\int_{2\tau}^{r} s^{1-N}\left[\int_{2\tau}^s t^{N-1}\beta(t)g(u(t))dt\right]ds. \end{aligned} \tag{3.12}$$

From (3.12) and since $|g(u(s))| = -g(u(s)) \geq |g(u(2\tau))|$, for all $s \geq 2\tau$, $l = \infty$ is implied by

$$\lim_{r\to\infty}\int_{2\tau}^{r} s^{1-N}\left[\int_{2\tau}^s t^{N-1}\beta(t)dt\right]ds = \infty.$$

If $r > 4\tau$, we get

$$\begin{aligned} \int_{2\tau}^{r} s^{1-N}\left[\int_{2\tau}^s t^{N-1}\beta(t)dt\right]ds &\geq \int_{4\tau}^{r} s^{1-N}\left[\int_{2\tau}^s t^{N-1}\beta(t)dt\right]ds \\ &\geq \int_{4\tau}^{r} s^{1-N}\left[\int_{s/2}^s t^{N-1-\beta_o}t^{\beta_o}\beta(t)dt\right]ds \\ &\geq C\int_{4\tau}^{r} s\beta(s)\,ds\,, \end{aligned} \tag{3.13}$$

in the last inequality of (3.13) we have used that $t^{\beta_o}\beta(t)$ is non-increasing. Thus, from (3.13) and (2.2) we get that $l = \infty$. A similar argument with

$$\begin{aligned} -u(r) &= \int_\tau^r s^{1-N}\left[\int_0^s t^{N-1}\alpha(t)f(v(t))dt\right]ds \\ &\geq f(c)\int_\tau^r s^{1-N}\left[\int_0^s t^{N-1}\alpha(t)dt\right]ds, \end{aligned}$$

shows that $\lim_{r\to\infty} u(r) = \infty$. $\diamondsuit$

As a consequence of the above result we get the following for the particular system

$$\begin{aligned} -\Delta u &= \alpha f(v), \\ -\Delta v &= \alpha f(u). \end{aligned} \tag{3.14}$$

Corollary 3.1 *Assume that $f = g$ and $\alpha = \beta$ satisfy conditions in theorem 3.1. Then,*

$$\mathcal{G} = \{(c,c)| \quad c > 0\}.$$

Proof. Let $c > 0$, and let u be a nonnegative solution to

$$-\Delta u = \alpha f(u),$$

defined near zero and such that $u(0) = c$. Thus, since (u,u) is a solution to (3.14) with $u = v$, we get that $(c,c) \notin \mathcal{U}_0 \cup \mathcal{V}_0$ (solutions with initial data in $\mathcal{U}_0 \cup \mathcal{V}_0$ satisfy that only one between u and v have a zero). Therefore, $(c,c) \in \mathcal{G}$. $\diamondsuit$

With some extra conditions, it can be proven that the ground states are the unique solutions to (2.1) defined in $[0,\infty)$.

Theorem 3.3 *Assume that conditions in theorem 3.2 are satisfied. Moreover, assume that f and g are odd functions. If there exist constants α_1, β_1 and a positive constant C such that*

$$\alpha(r) \geq Cr^{-\alpha_1}, \quad \beta(r) \geq Cr^{-\beta_1}, \quad \textit{for all } r \textit{ large}$$

with

$$\min\{\gamma_1(\alpha_1,\beta_1), \gamma_2(\alpha_1,\ \beta_1)\} \leq 0, \tag{3.15}$$

then for any (c_1,c_2) with $c_1 > 0$ and $c_2 > 0$, we have that the corresponding solution (u,v) is either a ground state or the maximal right hand side interval is $(0,k)$ with $k < \infty$.

Proof. Let (u,v) be a solution which is not a ground state, and thus from theorem 3.2, we can assume the existence of τ such that $u(\tau) = 0$, $u(r) < 0$ for all $r \in (\tau,k)$, and $v(r) > 0$ for all $r \in (0,\tau)$. If $k = \infty$, let $u_1(r) = -u(r)$, thus (u_1,v) is a positive solution in an exterior domain to

$$\begin{aligned} \Delta u_1 &\geq C|x|^{-\alpha_1}f(v) \\ \Delta v &\geq C|x|^{-\beta_1}g(u_1). \end{aligned} \tag{3.16}$$

Now, since $u_1(r)$ and $v(r)$ tend to infinity as r does, we have $f(v) \geq Cv^p$ and $g(u_1) \geq Cu_1^q$, and thus (u_1, v) is a solution to

$$\begin{aligned} \Delta u_1 &\geq C|x|^{-\alpha_1} v^p \\ \Delta v &\geq C|x|^{-\beta_1} u_1^q. \end{aligned}$$

From the hypothesis (3.15) and Theorem 3.1 or Theorem 3.2 in [16], we get that either u_1 or v are bounded, and the contradiction follows. Therefore, $k < \infty$. ♢

The following lemmas allow us to define the function h given in theorem 3.1.

Lemma 3.1 *Assume that conditions on theorem 3.1 are satisfied. Then, for any $c_1 > 0$ (respectively $c_2 > 0$) there exists at most one $c_2 > 0$ (respectively $c_1 > 0$) such that $(c_1, c_2) \in \mathcal{G}$.*

Proof. Let $c_1 > 0$. Assume that there exists c_2 such that (u_1, v_1) is a ground state with $u_1(0) = c_1$ and $v_1(0) = c_2$. Let us prove that solutions (u, v) of (2.1) corresponding to $u(0) = c_1$ and $v(0) = c_2 - \delta$, with $0 < \delta < c_2$, are not ground states. Assume by contradiction that some of the above (u, v) is also a ground state. Consider $[0, r_1)$ be the maximal right hand side interval where $v_1(r) > v(r) > 0$, for all $r \in [0, r_1)$. Integrating twice (2.1) we get for all $r \in [0, r_1)$

$$u(r) - u_1(r) = \int_0^r s^{1-N} \left[\int_0^s t^{N-1} \alpha(t)(f(v_1(t)) - f(v(t)))dt \right] ds > 0,$$

and thus, $u(r) > u_1(r)$, for all $r \in [0, r_1)$. Moreover, in such interval,

$$-r^{N-1}(v'(r) - v_1'(r)) = \int_0^r t^{N-1} \beta(t)(g(u(t)) - g(u_1(t)))dt]ds > 0,$$

implying that $v_1(r) - v(r)$ is increasing in $[0, r_1)$, and thus

$$v_1(r) - v(r) > \delta. \tag{3.17}$$

If $r_1 < \infty$, then from the definition of r_1 and (3.17) we get $v(r_1) = 0$, and thus (u, v) is not a ground state. If $r_1 = \infty$, the contradiction follows from (3.17) and proposition 2.3. Therefore, if (c_1, c_2) corresponds to a ground state, then $(c_1, c_2 - \delta)$ does not correspond to a ground state, and thus $(c_1, c_2 + \delta)$ also cannot correspond to a ground state (if it does, $(c_1, c_2) = (c_1, c_2 + \delta - \delta)$ does not). ♢

Using similar arguments to those used in the proof of the above lemma, we have.

Lemma 3.2 *Assume that conditions on theorem 3.1 are satisfied. Then,*

(*i*) *If $(c_1, c_2) \in \mathcal{V}_0$, then $[c_1, \infty) \times (0, c_2) \subset \mathcal{V}_0$.*

(*ii*) *If* $(c_1, c_2) \in \mathcal{U}_0$, *then* $(0, c_1) \times [c_2, \infty) \in \mathcal{U}_0$.

Lemma 3.3 *Assume that conditions on theorem 3.1 are satisfied. Then,*

(*i*) *For any* $c_1 > 0$ *there exists a* $\bar{c}_2 > 0$ *such that* $\{c_1\} \times (0, \bar{c}_2) \subset \mathcal{V}_0$.

(*ii*) *For any* $c_2 > 0$ *there exists a* $\bar{c}_1 > 0$ *such that* $(0, \bar{c}_1) \times \{c_2\} \subset \mathcal{U}_0$.

Proof. We prove (i). Assume by contradiction that for some $c_1 > 0$ there exists no $\bar{c}_2$ such that $\{c_1\} \times (0, \bar{c}_2) \subset \mathcal{V}_0$. In that case, we choose a positive sequence $\{c_{2n}\}$ decreasing to zero such that $(c_1, c_{2n}) \in \mathcal{U}_0$. We call (u_n, v_n), the solution to (2.1) corresponding to $u_n(0) = c_1$ and $v_n(0) = c_{2n}$. If τ_n is the point where $u_n(\tau_n) = 0$, we obtain $0 \leq v_n(r) \leq c_{2n}$ for all $r \in (0, \tau_n)$ and

$$c_1 = \int_0^{\tau_n} s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v(t)) dt \right] ds \leq \frac{f(c_{2n})}{N-2} \int_0^{\tau_n} s\alpha(s) ds,$$

and thus τ_n tends to infinity as n does.

Next, we will prove that for any $a > 0$, $(\{u_n\}_n, \{v_n\}_n)$ has a subsequence which converges uniformly in $[0, a]$ to a solution (u, v) of (2.1). Let n be large enough such that $\tau_n > a$, and thus $0 \leq v_n(r) \leq c_{2n}$, for all $r \in (0, a)$. Since $\{c_{2n}\}_n$ tends to zero as n tends to infinity, we get that that $v \equiv 0$, which is a contradiction to $u(0) = c_1$. We prove the existence of such a converging subsequence by showing that $(\{u_n\}_n, \{v_n\}_n)$ is equicontinuous and bounded in $[0, a]$. We have

$$0 \leq v_n(r) \leq c_{2n}, \quad 0 \leq u_n(r) \leq c_1 \quad r \in (0, a), \tag{3.18}$$

and thus integrating twice (2.1) we get

$$|u_n(r) - u_n(x)| \leq \int_r^x s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) f(v_n(t)) dt \right] ds, \tag{3.19}$$

Therefore from (3.19) and (3.18)

$$|u_n(r) - u_n(x)| \leq f(c_{2n}) \int_r^x s^{1-N} \left[\int_0^s t^{N-1} \alpha(t) dt \right] ds, \tag{3.20}$$

and thus $\{u_n\}$ is equicontinuous. Similarly, we obtain

$$|v_n(r) - v_n(x)| \leq g(c_1) \int_r^x s^{1-N} \left[\int_0^s t^{N-1} \beta(t) dt \right] ds.$$

The proof of (ii) follows the same ideas given on (i). ♢

Lemma 3.4 *Assume that conditions on theorem 3.1 are satisfied. Then,*

(*i*) *For any* $c_2 > 0$ *there exists* c_1^+, *such that* $(c_1^+, \infty) \times \{c_2\} \subset \mathcal{V}_0$.

(*ii*) *For any* $c_1 > 0$ *there exists* c_2^+, *such that* $\{c_1\} \times (c_2^+, \infty) \subset \mathcal{U}_0$.

Proof. Assume by contradiction the existence of $c_2 > 0$ and a sequence $\{c_{1n}\}$ increasing to infinity such that $(c_{1n}, c_2) \in \mathcal{U}_0$. We consider r_n to be such that $u_n(r_n) = c_{1n}/2$, where (u_n, v_n) is the solution to (2.1) with $(u_n(0), v_n(0)) = (c_{1n}, c_2)$. Therefore, since v_n is decreasing before the zero of u_n, we get

$$\begin{aligned} \tfrac{c_{1n}}{2} &= u_n(0) - u_n(r_n) = \int_0^{r_n} s^{1-N}[\int_0^s t^{N-1}\alpha(t)f(v_n(t))dt]ds \\ &\leq f(c_2)\int_0^{r_n} s^{1-N}[\int_0^s t^{N-1}\alpha(t)dt]ds, \end{aligned} \tag{3.21}$$

and thus r_n tends to infinity as n does. On the other hand, $v_n(r_n) > 0$ implies

$$\begin{aligned} c_2 &> \int_0^{r_n} s^{1-N}\left[\int_0^s t^{N-1}\beta(t)g(u_n(t))dt\right]ds \\ &\geq g(c_{1n}/2)\int_0^{r_n} s^{1-N}\left[\int_0^s t^{N-1}\beta(t)dt\right]ds, \end{aligned}$$

which is a contradiction to (3.21). ♢

Lemma 3.5 *Assume that conditions on theorem 3.1 are satisfied. Then, $\mathcal{U}_0$ and $\mathcal{V}_0$ are open subsets of $\mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. Let $(d_1, d_2) \in \mathcal{U}_0$, then $u(\tau_d) = 0$, for some $\tau_d > 0$, and thus

$$\begin{aligned} d_1 &= \int_0^{\tau_d} s^{1-N}[\int_0^s t^{N-1}\alpha(t)f(v(t))dt]ds \\ &\leq f(d_2)\int_0^{\tau_d} s^{1-N}[\int_0^s t^{N-1}\alpha(t)dt]ds \\ &\leq \frac{f(d_2)}{N-2}\int_0^{\tau_d} s\alpha(s)ds. \end{aligned} \tag{3.22}$$

Similarly if $(d_1, d_2) \in \mathcal{V}_0$, we get

$$d_2 \leq \frac{g(d_1)}{N-2}\int_0^{\tau_d} s\beta(s)ds. \tag{3.23}$$

Now, let $\mathbf{c} := (c_1, c_2) \in \mathcal{U}_0$, and let $\mathbf{d} := (d_1, d_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, with $|c_i - d_i| < \epsilon$, for $i = 1, 2$. Consider (u_1, v_1) to be the solution to (2.1) associated to $\mathbf{c}$ and (u, v) the corresponding $\mathbf{d}$. Then, if $\epsilon < \max\{c_1/2, c_2/2\}$, from (3.22) and (3.23)we get that the solution (u, v) is at least defined on $I_1 = [0, \tau_1]$, where τ_1 satisfies

$$\frac{c_1(N-2)}{2f(3c_2/2)} \leq \int_0^{\tau_1} s\alpha(s)ds, \quad \text{and} \quad \frac{c_2(N-2)}{2g(3c_1/2)} \leq \int_0^{\tau_1} s\beta(s)ds.$$

Therefore, from Lemma 2.1 for the above interval I_1 we get that $|u_1(\tau_1) - u(\tau_1)| < \epsilon_0$, $|u_1'(\tau_1) - u'(\tau_1)| < \epsilon_0$, $|v_1(\tau_1) - v(\tau_1)| < \epsilon_0$, and $|v_1'(\tau_1) - v'(\tau_1)| < \epsilon_0$, where ϵ_0 is small. Thus, from classical continuous dependence results for (u_1, v_1) defined in $[\tau_1, k_0]$, for any $k_0 < k$, we get that the solution (u, v) is also defined in $[\tau_1, k_0]$, and it is near to (u_1, v_1). Since, u_1 is negative at some point, we conclude that u must be negative at that point. ♢

Proof of theorem 3.1. We first construct the function h. Let $c_2 > 0$, thus from lemma 3.3 and lemma 3.4 we obtain

$$0 < h(c_2) := sup\{c_1 > 0 \quad | \quad (0, c_1) \times \{c_2\} \subset \mathcal{U}_0\} < \infty.$$

Next, we will prove that $(h(c_2), c_2)$ corresponds to a ground state. Since by lemma 3.5 $\mathcal{U}_0$ is open and $h(c_2)$ is a supremum, we get that $(h(c_2), c_2) \notin \mathcal{U}_0$. If $(h(c_2), c_2) \in \mathcal{V}_0$, and since $\mathcal{V}_0$ is open $(h(c_2)-\epsilon, c_2) \in \mathcal{V}_0$, for some ϵ small, which is a contradiction to the definition of $h(c_2)$, and thus $(h(c_2), c_2)$ correspond to a ground state.

The proof that h is increasing is as follows. Let $c_2^* > c_2$. From the definition of $h(c_2)$ we get that $(0, h(c_2)) \times \{c_2\} \subset \mathcal{U}_0$, and thus from lemma 3.2 $(0, h(c_2)) \times [c_2, \infty) \subset \mathcal{U}_0$. Therefore from the definition of $h(c_2^*)$ we get $h(c_2) \leq h(c_2^*)$. Now, since $(h(c_2), c_2)$ and $(h(c_2^*), c_2^*)$ are ground states, from lemma 3.1 $h(c_2) < h(c_2^*)$.

We end by showing item 1 and 2 of the theorem. Let (c_1, c_2) such that $c_1 < h(c_2)$, from the definition of $h(c_2)$, we have that $(c_1, c_2) \in \mathcal{U}_0$. Assume now that $c_1 > h(c_2)$. If $(c_1, c_2) \in \mathcal{U}_0$, from lemma 3.2, $(h(c_2), c_2)$ also belongs to $\mathcal{U}_0$, which is a contradiction, and thus $(c_1, c_2) \in \mathcal{V}_0$, for any $c_1 > h(c_2)$. $\diamondsuit$

The following result has to do with the particular system

$$\begin{aligned} u''(r) + \frac{N-1}{r}u'(r) &= -r^{-\alpha_o}v^p, \\ v''(r) + \frac{N-1}{r}v'(r) &= -r^{-\beta_o}u^q, \\ u(0) = c_1 \qquad & v(0) = c_2. \end{aligned} \tag{3.24}$$

Corollary 3.2 *Let $(f(v), g(u)) = (v^p, u^q)$, with $p \geq 1$, $q \geq 1$. Assume that $(\alpha(r), \beta(r)) = (r^{-\alpha_o}, r^{-\beta_o})$ with $\alpha_o < 2$, $\beta_o < 2$, and $(\alpha_o, \beta_o, p, q)$ satisfies* (3.3)*. Then, there exists a positive constant C_0 such that h given on theorem 3.1 is*

$$h(x) = C_0 x^{(\alpha_o - 2 + (\beta_o - 2)p)/(\beta_o - 2 + (\alpha_o - 2)q)}.$$

Remark. The above result was proven by Serrin and Zou [13], when $\alpha_o = \beta_o = 0$. The behavior at infinity of the ground states when (3.3) is an equality an $\alpha_o = \beta_o = 0$ is given in [8].

Proof. Let x positive, then $(h(x), x)$ is an initial value for a ground state. If (u, v) is such ground state, then for any positive constants a, b and k, we define (w, y) by

$$w(r) = au(kr), \quad y(r) = bv(kr).$$

Thus, (w, y) is a ground state to

$$\begin{aligned} \Delta w + ab^{-p}k^{2-\alpha_o}r^{-\alpha_o}y^p = 0, \\ \Delta y + ba^{-q}k^{2-\beta_o}r^{-\beta_o}w^q = 0. \end{aligned}$$

If $ab^{-p}k^{2-\alpha_o} = 1$, and $ba^{-q}k^{2-\beta_o} = 1$, which implies that

$$a = b^\nu \quad \text{where} \quad \nu = \frac{\alpha_o - 2 + (\beta_o - 2)p}{\beta_o - 2 + (\alpha_o - 2)q}.$$

In that case, (w, y) is a ground state to (3.24) and $(w(0), y(0)) = (b^\nu h(c), bc)$. Let x be positive and choose b such that $x = bc$. Since the ground state corresponding to x is unique we get that

$$h(x) = C_0 x^\nu,$$

where $C_0 = h(c)c^{-\nu}$. $\diamondsuit$

4 Existence of singular ground states

In this section we study the problem of existence of singular ground states for (1.1). A singular ground state means a radially symmetric nonnegative solution (u, v) to (1.1) such that either $\lim_{r\to 0} u(r) = \infty$, or $\lim_{r\to 0} v(r) = \infty$.

Next, we will prove an existence result of singular radially symmetric ground state as a limit of ground states constructed in Section 3.

Theorem 4.1 *Assume that condition on Theorem 3.1 are satisfied and $pq > 1$. Moreover, assume that*

$$f(v) \geq Cv^p, \quad v \geq 0, \quad g(u) \geq Cu^q, \quad u \geq 0,$$

and for any k positive

$$\int_0 t^{N-1}\alpha(t) f(kt^{\gamma_2})dt < \infty, \quad \int_0 t^{N-1}\beta(t) g(kt^{\gamma_1})dt < \infty, \tag{4.1}$$

where γ_1 and γ_2 are defined by (2.7)*. Then, there exists a nonnegative singular ground state to* (1.1)*. Moreover,*

$$\lim_{r\to 0} u(r) = \infty, \quad \textit{and} \quad \lim_{r\to 0} v(r) = \infty.$$

Remark. If in the above theorem $f(v) \leq Dv^p$, and $g(u) \leq Du^q$, and $\alpha(r) \leq Dr^{\alpha_o}$, $\beta(r) \leq Dr^{\beta_o}$, for some positive constant D, then condition (4.1) is satisfy, since

$$\int_0 t^{N-1}\alpha(s) f(kt^{\gamma_2})dt \leq C \int_0 t^{N-1-\alpha_o+p\gamma_2}dt < \infty.$$

The last integral is finite since $N - \alpha_o + p\gamma_2 = N - 2 + \gamma_1 > 0$, and we are in the region where $N - 2 + \gamma_1 + \gamma_2 \geq 0$ and $\gamma_1 < 0$ and $\gamma_2 < 0$.

Proof. As usual it can be proved that for any (u, v) nonnegative radially symmetric solution to (1.1) we have

$$u(r) \geq Cv^p(r) r^2 \alpha(r), \tag{4.2}$$

In the proof of (4.2) we have used that $r^{N-2}u(r)$ is nondecreasing and $r^{\alpha_o}\alpha(r)$ is non-increasing. Similarly, we get

$$v(r) \geq Cu^q(r)) r^2 \beta(r), \tag{4.3}$$

and thus from (4.2) and (4.3) we obtain

$$\begin{aligned} u^{pq-1}(r) &\leq \frac{C}{r^{2(p+1)}\beta^p(r)\alpha(r)}, &(4.4)\\ v^{pq-1}(r) &\leq \frac{C}{r^{2(q+1)}\alpha^q(r)\beta(r)}, &(4.5)\end{aligned}$$

Moreover, for any interval $[0,a]$, there exist a constant $A>0$, depending on a such that $r^{\alpha_o}\alpha(r)\geq A$, and $r^{\beta_o}\beta(r)\geq A$, for all $r\in[0,a]$. Thus, we obtain from (4.4) that for all $r\in[0,a]$

$$\begin{aligned} u(r) &\leq Kr^{\gamma_1}, &(4.6)\\ v(r) &\leq Kr^{\gamma_2}, &(4.7)\end{aligned}$$

where K is some positive constant depending on a. Let $(h(c_n),c_n)$ with $c_n>0$ converging to ∞, and (u_n,v_n) the associated ground state. From (4.4) the sequence $\{u_n,v_n)\}$ is uniformly bounded in any compact subset of $(0,\infty)$. We will show that the sequence $\{u_n,v_n)\}$ is equicontinuous away from zero. Let $x>0$ be fixed, we prove equicontinuity on x. From the definition of (u_n,v_n), we have

$$\begin{aligned} |u_n(r)-u_n(x)| &\leq \int_r^x s^{1-N}\left[\int_0^s t^{N-1}\alpha(s)f(v_n(t))dt\right]ds\\ &\leq \int_r^x s^{1-N}\left[\int_0^s t^{N-1}\alpha(s)f(Kt^{\gamma_2})dt\right]ds,\end{aligned}$$

and thus from (4.1) we get the equicontinuity of the sequence $\{(u_n,v_n)\}$, in x. Therefore, there exists a subsequence of $\{(u_n,v_n)\}$, which converges uniformly on any compact subset of $(0,\infty)$ to a continuous function (u,v). Moreover, (u,v) is a solution of (1.1) in the sense of distribution in $\mathbb{R}^N\setminus\{0\}$. $\diamondsuit$

Next, by using Kelvin transform on the results of Section 3, we prove existence of singular ground states for the system

$$\begin{aligned} -\Delta u &= \alpha^*(|x|)v^p\\ -\Delta v &= \beta^*(|x|)u^q \end{aligned}\qquad x\in\mathbb{R}^N\setminus\{0\}, \tag{4.8}$$

where, $p\geq 1$, $q\geq 1$ and $pq>1$, under the following conditions

$$\int_0 s^{N-1-(N-2)p}\alpha^*(s)ds=\infty,\quad \int_0 s^{N-1-(N-2)q}\beta^*(s)ds=\infty, \tag{4.9}$$

$$\int^\infty s^{N-1-(N-2)p}\alpha^*(s)ds<\infty,\quad \int^\infty s^{N-1-(N-2)q}\beta^*(s)ds<\infty. \tag{4.10}$$

Moreover, we will assume the existence of α_0' and β_0' such that

$$r^{\alpha_o'}\alpha^*(r)\quad\text{and}\quad r^{\beta_o'}\beta^*(r)\quad\text{are nondecreasing and} \tag{4.11}$$

$$\frac{N-\alpha_o'}{p+1}+\frac{N-\beta_o'}{q+1}\geq N-2. \tag{4.12}$$

Theorem 4.2 *Assume that conditions* (4.9), (4.10), (4.11) *and* (4.12) *are satisfied. Then, there exists a function* $h^* \in C(\mathbb{R}^+, \mathbb{R}^+)$, *strictly increasing such that for any* $c_2 > 0$ *there exists a positive ground state* (u, v) *such that*

$$\lim_{r\to\infty} r^{N-2}u(r) = h^*(c_2), \quad \textit{and} \quad \lim_{r\to\infty} r^{N-2}v(r) = c_2. \tag{4.13}$$

Moreover, assume that either $r^{\alpha'_o}\alpha^*(r)$ *or* $r^{\beta'_o}\beta^*(r)$ *are strictly increasing at some point or* (4.12) *is not an equality, then*

$$\lim_{r\to 0} u(r) = \infty, \quad \textit{and} \quad \lim_{r\to 0} v(r) = \infty. \tag{4.14}$$

Proof. The proof of this result is fundamentally based on Kelvin transform and on theorems of the previous section. If (u, v) is a radially symmetric nonnegative solution to (4.8), we define

$$u_1(r) := r^{2-N}u(r^{-1}), \quad v_1(r) := r^{2-N}v(r^{-1}),$$

the Kelvin transform of the function u and v. The pair (u_1, v_1) is thus a solution to

$$\begin{array}{rcl} -\Delta u_1 &=& \alpha(|x|)v_1^p \\ -\Delta v_1 &=& \beta(|x|)u_1^q \end{array} \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{4.15}$$

where the functions α and β are given by

$$\begin{array}{rcl} \alpha(r): &=& r^{(N-2)p-(N+2)}\alpha^*(r^{-1}) \\ \beta(r): &=& r^{(N-2)q-(N+2)}\beta^*(r^{-1}). \end{array} \tag{4.16}$$

Define

$$\alpha_o = N + 2 - (N-2)p - \alpha'_o, \quad \beta_o = N + 2 - (N-2)q - \beta'_o,$$

and thus we easily get that (α, β, p, q) satisfies the hypothesis on Theorem 3.1 and the existence of h^* follows.

Next, we will show that those ground states satisfy (4.14) if $r^{\alpha'_o}\alpha^*(r)$ or $r^{\beta'_o}\beta^*(r)$ are strictly increasing at some point or (4.12) is not an equality.

Let (u, v) be a radially symmetric ground state constructed as above. The functions u and v are decreasing functions such that $r^{N-2}u(r)$ and $r^{N-2}v(r)$ are increasing. Moreover,

$$\lim_{r\to 0} r^{N-2}u(r) = 0, \quad \text{and} \quad \lim_{r\to 0} r^{N-2}v(r) = 0, \tag{4.17}$$

as it follows from the behavior at infinity of (u_1, v_1). The energy function given by (see Proposition 3.1)

$$\begin{array}{rcl} E^*_{ab}(r) &=& r^N \left[u'(r)v'(r) + ar^{-1}u'(r)v(r) + br^{-1}v'(r)u(r)\right] \\ && +r^N \left[\dfrac{\alpha^*}{p+1}v^{p+1} + \dfrac{\beta^*}{q+1}u^{q+1}\right], \end{array} \tag{4.18}$$

satisfies

$$
\begin{aligned}
(E^*_{ab})'(r) &= (a+b+2-N)r^{N-1}u'(r)v'(r) + (\frac{r^N\alpha^*}{p+1})'v^{p+1} \qquad (4.19)\\
&\quad -ar^{N-1}\alpha^* v^{p+1} + (\frac{r^N\beta^*}{q+1})'u^{q+1} - br^{N-1}\beta^* u^{q+1}.
\end{aligned}
$$

Moreover, for $a = (N-\alpha'_o)/(p+1)$ and $b = (N-\beta'_o)/(q+1)$, we get that E^*_{ab} is a nondecreasing function. First, we will prove that $\lim_{r\to\infty} E^*_{ab}(r) = 0$. From (4.13) we get that the first three terms in (4.18) tend to zero as r tends to infinity. Moreover, from Proposition 2.4, we get that $\lim_{r\to 0} r^2\alpha(r) = 0$ and $\lim_{r\to 0} r^2\beta(r) = 0$, where (α,β) is given by (4.16). Therefore $\lim_{r\to\infty} r^{N-(N-2)p}\alpha^*(r) = 0$ and $\lim_{r\to\infty} r^{N-(N-2)q}\beta^*(r) = 0$, and thus the last two terms in (4.18) tend to zero as r tends to infinity.

Assume by contradiction that $\lim_{r\to 0} u(r) = l < \infty$.

Assertion: If $l < \infty$, then $\lim_{r\to 0} E^*_{ab}(r) = 0$, and thus $(E^*_{ab})'(r) = 0$, for all r.

Assume that the assertion is true. If either $r^{\alpha'_o}\alpha^*(r)$ or $r^{\beta'_o}\beta^*(r)$ are increasing functions at some point x, or the inequality in (4.12) is strictly, then $(E^*_{ab})'(x) = 0$, implies that either $u(x) = 0$, or $v(x) = 0$, or $u'(x) = 0$, or $v'(x) = 0$, and the contradiction follows.

Now, we prove the assertion. If u is bounded, the three first term in $E^*_{ab}(r)$, goes to zero as a consequence of (4.17). For the fourth term, we argue as follows. Since $r^{N-2}u(r)$ is nondecreasing, v is non-increasing and $r^{\alpha'_0}\alpha^*(r)$ is nondecreasing, we get

$$
\begin{aligned}
r^{N-2}u(r) &\geq C\int_{r/2}^{r} s^{N-1}\alpha^*(s)v^p(s)\\
&\geq Cr^N\alpha^*(r/2)v^p(r), \qquad (4.20)
\end{aligned}
$$

and thus, using that u is non-increasing and $r^{N-2}v(r)$ is nondecreasing, we can change in (4.20) $u(r)$ by $Cu(r/2)$ and $v^p(r)$ by $Cv^p(r/2)$ to get

$$
u(r) \geq Cr^2\alpha^*(r)v^p(r). \qquad (4.21)
$$

Therefore, from (4.21) and since $l < \infty$ we get

$$
Cr^{N-2}v(r) \geq C'r^{N-2}u(r)v(r) \geq r^N\alpha^*(r)v^{p+1}(r),
$$

and thus $\lim_{r\to 0} r^N\alpha^*(r)v^{p+1}(r) = 0$. Now, from Proposition 2.5 we conclude that $r^N\beta^*(r) = r^{(N-2)q-2}\beta(r^{-1})$ goes to zero as r does, and thus the last term in $E^*_{ab}(r)$ goes to zero as r does, and the assertion follows. $\diamondsuit$

Remark. Assume that conditions on theorem 4.2 are satisfied with $p = q = 1$. In this case (4.12) is equivalent to

$$
2-\alpha'_o + 2 - \beta'_o \geq 0. \qquad (4.22)
$$

The conclusion of the above theorem is in this case also valid. If either $r^{\alpha'_o}\alpha^*(r)$ or $r^{\beta'_o}\beta^*(r)$ is non constant, no changes on the proof of the above result. If $\alpha^*(r) = C_1 r^{-\alpha'_o}$ and $\beta^*(r) = C_2 r^{-\beta'_o}$, then from (4.10) we get that (4.22) is not an equality, and the proof follows that of the above theorem.

On the other hand, it is known that the existence of radially symmetric positive solutions to

$$-\Delta u = c|x|^{-2}u,$$

depends on the constant c. For $4c > (N-2)^2$, those solutions do not exist.

For system (3.24) we have the following

Corollary 4.1 *Assume that $\alpha'_o > N - (N-2)p$, $\beta'_o > N - (N-2)q$, and $(\alpha'_o, \beta'_o, p, q)$ satisfies (4.12). Then, there exists a positive constant C^* such that*

$$h^*(x) = C^* x^{(2-\alpha'_o+(2-\beta'_o)p)/(2-\beta'_o+(2-\alpha'_o)q)}.$$

Moreover, if

$$(\alpha'_o, \beta'_o) = (0,0), \qquad \frac{N}{p+1} + \frac{N}{q+1} > N-2,$$

and if (u,v) is a ground state and $pq > 1$, then there exist two positive constants c and d such that for all r near zero we have

$$\begin{aligned} dr^{(-2(p+1))/(pq-1)} &\le u(r) \le cr^{(-2(p+1))/(pq-1)}\,, \\ dr^{(-2(p+1))/(pq-1)} &\le v(r) \le cr^{(-2(p+1))/(pq-1)}\,, \end{aligned}$$

Proof. The proof of the behavior at zero of the ground states follows from Theorem 3.2 in [5]. ♢

From theorem 4.1 and Kelvin transform, we obtain the following.

Theorem 4.3 *Assume that conditions on theorem 4.2 are satisfied. Moreover, assume that*

$$\int^{\infty} s^{1+p\gamma_2}\alpha^*(s)ds < \infty\,, \qquad \int^{\infty} s^{1+q\gamma_1}\beta^*(s)ds < \infty\,, \tag{4.23}$$

where $\gamma_{1,2} = \gamma_{1,2}(\alpha'_o, \beta'_o)$ are given by (2.7). Then, there exists (u,v) a nonnegative ground state to (4.8), such that

$$\lim_{r\to\infty} r^{N-2}u(r) = \infty\,, \quad \textit{and} \quad \lim_{r\to\infty} r^{N-2}v(r) = \infty\,. \tag{4.24}$$

Moreover, if (4.12) is not an equality, then those ground states satisfy

$$\lim_{r\to 0} u(r) = \infty, \quad \textit{and} \quad \lim_{r\to 0} v(r) = \infty.$$

Proof. The existence of a positive solution (u,v) of (4.8) satisfying (4.24) follows from Theorem 4.1 and Kelvin transform. Next, we will prove that $(u(r), v(r))$ tends to zero as r tends to infinity. Similarly to (4.21) we get

$$v(r) \ge Cr^2\beta^*(r)u^q(r), \tag{4.25}$$

and thus from (4.21), (4.25) and since $r^{\alpha'_o}\alpha(r)$ and $r^{\beta'_o}\beta(r)$ are nondecreasing functions we obtain the existence of r_0 and a positive constant C such that for all $r \geq r_0$

$$\begin{aligned} u(r) &\leq Cr^{\gamma_1(\alpha'_o,\beta'_o)}, \\ v(r) &\leq Cr^{\gamma_2(\alpha'_o,\beta'_o)}. \end{aligned} \tag{4.26}$$

Moreover, from (4.23) we obtain that $2-\alpha'_o+\gamma_2(\alpha'_o,\beta'_o)p<0$, but it can be easily verified that $\gamma_1(\alpha'_o,\beta'_o)=2-\alpha'_o+\gamma_2(\alpha'_o,\beta'_o)p$ and thus $\gamma_1(\alpha'_o,\beta'_o)<0$. Similarly $\gamma_2(\alpha'_o,\beta'_o)<0$. Therefore, the solution (u,v) is a ground state.

Assume now that (4.12) is a strict inequality. To prove that u and v are singular at zero we use the same argument used on the proof of theorem 4.2. The unique difference with that proof will be the proof of $\lim_{r\to\infty} E^*_{ab}(r)=0$, which is as follows: From (4.21), (4.25) and (4.26) we obtain that

$$\begin{aligned} r^{N-2+\gamma_1+\gamma_2} &\geq Cr^{N-2}u(r)v(r) \geq r^N\beta^*(r)u^{q+1}(r), \\ r^{N-2+\gamma_1+\gamma_2} &\geq Cr^{N-2}u(r)v(r) \geq r^N\alpha^*(r)v^{p+1}(r). \end{aligned} \tag{4.27}$$

Now since $r^{N-2}u$ and $r^{N-2}v$ are nondecreasing we also get that

$$\begin{aligned} r^{\gamma_1-1} &\geq C|u'(r)|, \\ r^{\gamma_2-1} &\geq C|v'(r)|. \end{aligned} \tag{4.28}$$

On the other hand, since condition (4.12) (without equality) is equivalent to $\gamma_1+\gamma_2+N-2<0$, from (4.27) and (4.28) the conclusion follows. ♢

5 Applications

In this section we will apply the results of the above sections to the more general system

$$\begin{aligned} \Delta u(x)+V(|x|)u+a(|x|)v^p &= 0, \\ \Delta v(x)+V(|x|)v+b(|x|)u^q &= 0, \quad \text{in} \quad \mathbb{R}^N\setminus\{0\}. \end{aligned} \tag{5.1}$$

where a and b are nonnegative functions, $p\geq 1$, $q\geq 1$. The "potential" V is not necessarily non-positive.

Let $V\in L^\infty_{\mathrm{loc}}(0,\infty)$, such that the equation

$$h''(r)+\frac{N-1}{r}h'(r)+V(r)h(r)=0, \quad r\in(0,\infty) \tag{5.2}$$

is disconjugate in $(0,\infty)$; i.e., there exists a positive solution h_0 of (5.2) such that

$$\int^\infty r^{1-N}h_0^{-2}(r)dr=\infty.$$

(See, e.g.,[7], [15] for the definition and properties of disconjugacy.) Note that for $V = 0$, (5.2) is disconjugate, and in this case $h_0 = r^{2-N}$. We define as in [2], $h_1(r) = h_0(r) \int_0^r t^{1-N} h_0^{-2}(t)dt$ if

$$D \equiv \int_0 t^{1-N} h_0^{-2}(t)dt < \infty,$$

and $h_1(r) = h_0(r) \int_R^r t^{1-N} h_0^{-2}(t)dt$ if $D = \infty$, where $R > 0$ is fixed. In any case h_0, h_1 are two linearly independent solutions to (5.2). In the sequel we assume that $D < \infty$. If (u, v) is a radially symmetric solution to (5.1), then (u_1, v_1) given by

$$u_1(s) = \frac{u(r)}{h_1(r)}, \quad v_1(s) = \frac{v(r)}{h_1(r)}\,, \tag{5.3}$$

where $s = \frac{h_1}{h_0}$, and v is a solution for all $s > 0$ to

$$\begin{aligned} u_1''(s) + \frac{2}{s}u_1'(s) + \alpha(s)v_1^p &= 0, \\ v_1''(s) + \frac{2}{s}v_1'(s) + \beta(s)u_1^q &= 0, \end{aligned} \tag{5.4}$$

where

$$\alpha(s) = a(r)h_1^{p-1}(r)h_0^4(r)r^{2(N-1)}, \quad \beta(s) = b(r)h_1^{q-1}(r)h_0^4(r)r^{2(N-1)},$$

and thus with the appropriate conditions on α and β, the results of the above sections give existence of positive solutions to (5.1). The function $(\alpha(s), \beta(s))$ satisfies (2.3) if and only if $(a(r), b(r))$ satisfies

$$\int_0 r^{N-1} h_0 h_1^p a(r)dr < \infty, \quad \int_0 r^{N-1} h_0 h_1^q b(r)dr < \infty, \tag{5.5}$$

and $(\alpha(s), \beta(s))$ satisfies (2.2) if and only if $(a(r), b(r))$ satisfies

$$\int^\infty r^{N-1} h_0 h_1^p a(r)dr = \infty, \quad \int^\infty r^{N-1} h_0 h_1^q b(r)dr = \infty. \tag{5.6}$$

We have the following

Theorem 5.1 *Assume that a and b satisfy (5.5) and (5.6). Assume that there exists (α_o, β_o) such that*

$$\frac{3-\alpha_o}{p+1} + \frac{3-\beta_o}{q+1} \le 1, \tag{5.7}$$

and

$$r^{2(N-1)} h_1^{p-1+\alpha_o} h_0^{4-\alpha_o} a(r), \quad r^{2(N-1)} h_1^{q-1+\beta_o} h_0^{4-\beta_o} b(r)$$

are non-increasing functions. Then, there exists $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ strictly increasing and such that for any $c > 0$ there exists a radially symmetric solution (u, v) to (5.1) such that

$$\lim_{r\to 0} \frac{u(r)}{h_1(r)} = g(c), \quad \textit{and} \quad \lim_{r\to 0} \frac{v(r)}{h_1(r)} = c.$$

For instance, if $V(r) = -dr^{-2}$, with $d > -(N-2)^2/4$, the equation

$$h''(r) + \frac{N-1}{r}h'(r) - \frac{d}{r^2}h(r) = 0, \quad r \in (0,\infty)$$

has the solutions

$$\begin{aligned} &h_0(r) = r^{\theta_0} \quad \text{and} \quad h_1(r) = r^{\theta_1}, \\ &\theta_{1,0} = \frac{2 - N \pm ((N-2)^2 + 4d)^{1/2}}{2}, \end{aligned}$$

and for the system

$$\begin{aligned} \Delta u(x) - \frac{d}{|x|^2}u + a(|x|)v^p &= 0, \\ \Delta v(x) - \frac{d}{|x|^2}v + b(|x|)u^q &= 0, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \end{aligned} \tag{5.8}$$

we have

Corollary 5.1 *Assume that there exists (a_o, b_o) such that $r^{a_o}a(r)$ and $r^{b_o}b(r)$ are non-increasing functions for some (a_o, b_o) satisfying*

$$\frac{N - a_o}{p+1} + \frac{N - b_o}{q+1} \le N - 2. \tag{5.9}$$

Moreover, assume that a and b satisfy (5.5) and (5.6). Then, there exists $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ strictly increasing and such that for any $c > 0$ there exists a radially symmetric solution (u, v) to (5.8) such that

$$\lim_{r\to 0} \frac{u(r)}{r^{\theta_1}} = g(c), \quad \text{and} \quad \lim_{r\to 0} \frac{v(r)}{r^{\theta_1}} = c.$$

Remark. If in the above corollary $d > 0$, then for any $c > 0$, the solution (u, v) corresponding to $(g(c), c)$ satisfies that $u(0) = 0$ and $v(0) = 0$. On the other hand, if $d < 0$, the solutions are singular at zero. Moreover, if $(a(r), b(r)) = (r^{-a_o}, r^{-b_o})$ with (a_o, b_o) satisfying (5.9) and

$$a_o < 2 + (p-1)\theta_1, \quad b_o < 2 + (q-1)\theta_1$$

then there exists a positive constant D such that $g(x) = Dx^k$, where

$$k = \frac{a_o - 2 + (b_o - 2)p - \theta_1(pq-1)}{b_o - 2 + (a_o - 2)q - \theta_1(pq-1)}.$$

Now, we will obtain existence of a curve of ground states of (5.1) which behaves at infinity as h_0. To this end, we will apply results of section 4 to the system (5.4). Condition (4.9) for $(\alpha(s), \beta(s))$ in terms of $(a(r), b(r))$ is the following

$$\int_0 r^{N-1}h_1h_0^p a(r)dr = \infty, \quad \int_0 r^{N-1}h_1h_0^q b(r)dr = \infty, \tag{5.10}$$

and condition (4.10) for $(\alpha(s),\beta(s))$ reads for $(a(r),b(r))$

$$\int^{\infty} r^{N-1}h_1h_0^p a(r)dr<\infty,\quad \int^{\infty} r^{N-1}h_1h_0^q b(r)dr<\infty. \tag{5.11}$$

Theorem 5.2 *Assume that a and b satisfy* (5.10) *and* (5.11). *Assume that there exists (α_o',β_o') such that*

$$\frac{3-\alpha_o'}{p+1}+\frac{3-\beta_o'}{q+1}\geq 1\,. \tag{5.12}$$

and

$$r^{2(N-1)}h_1^{p-1+\alpha_o'}h_0^{4-\alpha_o'}a(r),\quad r^{2(N-1)}h_1^{p-1+\beta_o'}h_0^{4-\beta_o'}b(r),$$

are nondecreasing functions. Then, there exists $\delta\in C(\mathbb{R}^+,\mathbb{R}^+)$ strictly increasing and such that for any $c>0$ there exists a radially symmetric solution (u,v) to (5.1) *such that*

$$\lim_{r\to\infty}\frac{u(r)}{h_0(r)}=\delta(c),\quad \textit{and}\quad \lim_{r\to\infty}\frac{v(r)}{h_0(r)}=c\,.$$

In particular for system (5.8) we have

Corollary 5.2 *Assume that there exists (a_o,b_o) such that $r^{a_o}a(r)$ and $r^{b_o}b(r)$ are nondecreasing functions for some (a_o,b_o) satisfying*

$$\frac{N-a_o}{p+1}+\frac{N-b_o}{q+1}\geq N-2\,.$$

Moreover, assume that a and b satisfy (5.10) *and* (5.11). *Then, there exists $\delta\in C(\mathbb{R}^+,\mathbb{R}^+)$ strictly increasing and such that for any $c>0$ there exists a radially symmetric solution (u,v) to* (5.8) *such that*

$$\lim_{r\to\infty}\frac{u(r)}{r^{\theta_0}}=\delta(c),\quad \textit{and}\quad \lim_{r\to\infty}\frac{v(r)}{r^{\theta_0}}=c\,.$$

References

[1] M. Aizenman, B. Simon, Brownian motion and Harnack inequality for Schroedinger operators, *Comm. Pure Appl. Math.* **35** (1982), 209-273.

[2] R. Benguria, S. Lorca and C. Yarur, Nonexistence results for solutions of semilinear elliptic equations, *Duke Mathematical Journal* **74** (1994), 615-634.

[3] M.F. Bidaut-Véron, Local behaviour of the solutions of a class of nonlinear elliptic systems, *Advances in Differential Equations*, to appear.

[4] H. Brezis and P.L. Lions, A note on isolated singularities for linear elliptic equations, *J. Math. Anal. and Appl.* **7A** (1981), 263-266.

[5] M. García-Huidobro and C. Yarur, Classification of positive singular solutions for a class of semilinear elliptic systems, *Advances in Differential Equations* **2** (1997), 383-402.

[6] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1977.

[7] P. Hartman, Ordinary Differential Equations, John Wiley, New York, 1964.

[8] J. Hulshof and R. van der Vorst, Asymptotic Behaviour of ground states, preprint.

[9] P.L. Lions, The concentration-compactness principle in the calculus of variations, part 1, *Rev. Iberoam.* **1** (1985), 145-201.

[10] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential and Integral Equations*, **9** (1996), 465-479.

[11] W.M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations, and applications in geometry, *Indiana Univ. Math. J.* **31** (1982), 493-529.

[12] J. Serrin and H. Zou, Nonexistence of positive solutions of the Lane-Emden system, *Differential Integral Eqns.* **9** (1996), 635-653.

[13] J. Serrin and H. Zou, Existence of positive solutions of the Lane-Emden system, to appear.

[14] J. Serrin and H. Zou, Existence of positive solutions of elliptic Hamiltonian systems, *Comm. Partial Diff. Eqns.,* to appear.

[15] D. Willett, Classification of second order linear differential equations with respect to oscillation, *Adv. Math.* **3** (1969), 594-623.

[16] C. Yarur, Nonexistence of positive singular solutions for a class of semilinear elliptic systems, *Electr. J. Diff. Eqns. Vol* **1996** (1996), No 8, 1-22.

Cecilia S. Yarur
Departamento de Matemática y C. C.
Universidad de Santiago de Chile
Casilla 307, Correo 2, Santiago, Chile
E-mail address: cyarur@fermat.usach.cl