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AN EQUATION FOR THE LIMIT STATE OF A SUPERCONDUCTOR WITH PINNING SITES

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Abstract. We study the limit state of the inhomogeneous Ginzburg-Landau model as the Ginzburg-Landau parameter $\kappa = 1/\epsilon \to \infty$, and derive an equation to describe the limit state. We analyze the properties of solutions of the limit equation, and investigate the convergence of (local) minimizers of the Ginzburg-Landau energy with large κ. Our results verify the pinning effect of an inhomogeneous superconductor with large κ.

1. Introduction

Since the presence of vortices is inevitable for high temperature superconductors in high magnetic fields, it is desirable to pin the vortices to some specific locations, so that the supercurrent pattern around the vortices will be stable under the influence of the applied magnetic field and thermal fluctuation, which are important in applications (see [15, 18, 13]). One of the pinning mechanisms is to add normal impurities to the superconductors to attract the vortices, however, this procedure destroys the homogeneity of the superconductors, introduces an inhomogeneous structure inside the superconductor. The analysis of the behavior of inhomogeneous superconductors provides a good help for the understanding of such pinning mechanism.

Inhomogeneous models of superconductor under Ginzburg-Landau frame work have been discussed in both physics and mathematical literatures (see [2, 4, 11, 12] [17] etc.). We consider a Ginzburg-Landau system describing an inhomogeneous superconducting material used in [4], through the study of the limit case of such system,we derive an equation to describe the limit system, which is useful to understand the pinning effect. The following is the energy of the inhomogeneous superconductor with the parameter ϵ:

$$J_\epsilon(\psi, A) = \int_\Omega (|(\nabla - i\,A)\psi|^2 + \frac{1}{2\epsilon^2}(a - |\psi|^2)^2 + |\operatorname{curl} A - H_e|^2)dx, \tag{1.1}$$

where the parameter $\epsilon = 1/\kappa$ is a nonnegative number, and κ is the Ginzburg-Landau parameter of the superconductor material; $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with a smooth ($C^{2,\beta}$) boundary, represents the cross-section of an infinite cylindrical body with $\mathbf{e_3}$ as its generator; $H_e = h_e\mathbf{e_3}$ is the applied magnetic

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field with h_e being a constant; $A \in H^1(\Omega;\mathbb{R}^2)$ is the magnetic potential and $\operatorname{curl} A = \nabla \times (A_1, A_2, 0)$ is the induced magnetic field in the cylinder; $\psi \in H^1(\Omega;\mathbb{C})$ is complex-valued, with $|\psi|^2 = \psi^*\psi$ represents the density of superconducting electron pairs and $j = \dfrac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2 A$ denotes the superconducting current density circulating in Ω; $a : \Omega \to [0,1]$ is a bounded continuous function, describing the inhomogeneities of the material, the zero set of $a(x)$ corresponds to normal regions in the material.

In order to analyze the limit problem as $\epsilon \to 0$, we define the energy

$$J_0(\psi, A) = \int_\Omega |(\nabla - i\,A)\psi|^2 + |\operatorname{curl} A - H_e|^2\,dx, \tag{1.2}$$

where $(\psi, A) \in H_a^1 \times H^1(\Omega;\mathbb{R}^2)$, $a(x)$ is the same as in (1.1),

$$H_a^1 \equiv \{\psi \in H^1(\Omega;\mathbb{C}) \text{ such that } |\psi|^2 = a \text{ almost everywhere}\}. \tag{1.3}$$

In Lemma 2.1, we show that for each $u \in H_a^1$, there is a unique well-defined degree $D \equiv (d_1, \dots, d_n) \in \mathbb{Z}^n$ around $\overline{\Omega}_H$, denote the homotopy class in H_a^1 corresponding to D as $H_{a,D}^1$, then

$$H_a^1 = \bigcup_{D \in \mathbb{Z}^n} H_{a,D}^1\,.$$

Since $H_{a,D}^1$ is a nonempty open and (sequentially weakly) closed subspace of H_a^1 (Theorem 2.3), we can find the minimizer of J_0 in $H_{a,D}^1 \times H^1(\Omega;\mathbb{R}^2)$, and call it the local minimizer of J_0 in $H_a^1 \times H^1(\Omega;\mathbb{R}^2)$.

In [4] Andre, Bauman and Phillips have considered the case $a(x)$ vanishes at a finite number of points $\{x_1, \dots, x_n\}$, and showed that for sufficiently large $\kappa = 1/\epsilon$ the local minimizers of J_ϵ in (1.1) have nontrivial vortex structures, which are pinned near the zero points of $a(x)$ with any prescribed vortex pattern. In this paper we consider the case where $a(x)$ vanishes in subdomains (holes), which is more realistic in the presence of normal inclusions.

Our situation is different from the cases studied in [19] or [20], where they have considered the energy J_ϵ with $a \equiv 1$ in a multiply connected domain without applied magnetic field, they have shown the existence of the local minimizers of J_ϵ with prescribed vortex structures within certain homotopy class. In a recent paper [3] by Alama and Bronsard, they studied the energy J_ϵ with $a \equiv 1$ in a multiply connected domain with applied magnetic field, and achieved deeply results related to the pinning phenomena. They proved the interior vortex will not be shown until the applied magnetic field exceeds H_{c_1} of order $|\ln \epsilon|$, when the applied magnetic filed exceeds H_{c_1}, the vortices are nucleated strictly inside the multiply connected domain. Their techniques and results are similar to those from [1], [2], [4], [21] and [22].

We analyze the limit state through the investigation of the structure of local minimizers of J_0, and derive an equation to describe the limit state. Our methods and results are similar to those of [4]. While, we concentrate more on the analysis of the properties of the solutions of the limit equation, especially various nontrivial properties of the base functions of the solutions, which is a consequence of the setting of $a(x)$.

In detail, $a(x)$ satisfies the following conditions:
$a \in C^1(\overline{\Omega} \setminus \Omega_H)$, $\sqrt{a} \in H^1(\Omega)$, $a(x) \geq 0$ for all x in $\overline{\Omega}$, and $a(x) = 0$ iff $x \in \overline{\Omega}_H \subset \Omega$, where $\Omega_H = \cup_{j=1}^n \Omega_j$ corresponds to the inhomogeneities of the superconductor,

$n \in \mathbb{N}$, and $\Omega_j, j = 1, \dots, n$, are simply connected Lipschitz subdomains with $\overline{\Omega}_j \subset \Omega$. There also exists a constant $0 < r_1 < 1$ such that

$$\operatorname{dist}\{\Omega_i, \Omega_j\} > r_1, i \neq j, 1 \leq i, j \leq n, \text{ and } \operatorname{dist}\{\Omega_H, \partial\Omega\} > r_1.$$

In addition, for $x \in \Omega \setminus \overline{\Omega}_j$, with $d_j(x) = \operatorname{dist}\{x, \Omega_j\} < r_1$, there are positive constants C_0, C_1, α_j, such that

$$C_0 d_j^{\alpha_j}(x) \leq a(x) \leq C_1 d_j^{\alpha_j}(x), \quad \big|d_j(x)\frac{\nabla a(x)}{a(x)}\big| \leq C_1 \quad j = 1, 2, \dots, n. \tag{1.4}$$

Choose one $x_j \in \Omega_j$, and fix it, for any $x \in \Omega \setminus \Omega_j$, write

$$\mathbf{n}_j(x) = \frac{x - x_j}{|x - x_j|}\,, \tag{1.5}$$

then $\mathbf{n}_j \in C^\infty(\Omega \setminus \Omega_j, \mathbb{R}^2)$ with $|\mathbf{n}_j| = 1$. Moreover, we can rewrite $\mathbf{n}_j(x)$ in term of its azimuthal angle $\theta_j(x)$, so that

$$\mathbf{n}_j(x) = (\cos\theta_j(x), \sin\theta_j(x)), 1 \leq j \leq n. \tag{1.6}$$

Note that $e^{i\theta_j(x)}$ and $\nabla\theta_j(x)$ are single-valued with $e^{i\theta_j(x)} \in H^1(\Omega \setminus \overline{\Omega}_j, \mathbb{C})$.

Set

$$\mathcal{M} \equiv H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2), \quad \mathcal{M}_0 \equiv H_a^1 \times H^1(\Omega; \mathbb{R}^2),$$

then $\mathcal{M}$ and $\mathcal{M}_0$ are the domains of the functional J_ϵ and J_0 respectively.

If $(\psi, A) \in \mathcal{M}(\mathcal{M}_0)$ and $\phi \in H^2(\Omega)$, the gauge transformation of (ψ, A) under ϕ is defined by

$$(\psi', A') = G_\phi(\psi, A) \equiv (\psi e^{i\phi}, A + \nabla\phi) \in \mathcal{M}(\mathcal{M}_0)\,. \tag{1.7}$$

(ψ', A') is **gauge equivalent** to (ψ, A) whenever (1.7) has been satisfied for some $\phi \in H^2(\Omega)$. As is well-known that J_ϵ, $\epsilon \geq 0$, are gauge invariant, i.e. $J_\epsilon(\psi, A) = J_\epsilon(\psi', A')$. Hence if (ψ, A) is a (local) minimizer of J_ϵ in $\mathcal{M}_0$, so is (ψ', A').

In this paper, we fix a gauge by requiring that A satisfy

$$\begin{aligned} \operatorname{div} A = 0 \quad &\text{in } \Omega, \\ A \cdot \mathbf{n} = 0 \quad &\text{on } \partial\Omega\,. \end{aligned} \tag{1.8}$$

This can be done by choosing a gauge ϕ such that

$$\begin{aligned} \triangle\phi = -\operatorname{div} A \quad &\text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = -A \cdot \mathbf{n} \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.9}$$

Apply (1.7) to (ψ, A), we get $(\psi', A') = G_\phi(\psi, A)$ satisfies (1.8).

Since $J_\epsilon(\sqrt{a}, 0) = J_0(\sqrt{a}, 0) = \int_\Omega |\nabla\sqrt{a}| + h_e^2|\Omega| < \infty$, it makes sense to talk about the minimizers and local minimizers of J_0 and J_ϵ in $\mathcal{M}_0$ and $\mathcal{M}$ respectively. In Section II, we derive a few preliminary results. In Section 3, we analyze the local minimizers of J_0 in $\mathcal{M}_0$, and establish the following equation to describe them.

Theorem 1.1 (see Theorem 3.3)**.** *Fix h_e. Let (ψ_D, A_D) be a minimizer of J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ under gauge (1.8), define h_D by* $\operatorname{curl} A_D = h_D\mathbf{e_3}$*, then $h_D \in V$ is the unique solution of*

$$\begin{gathered} \int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla h \cdot \nabla f dx + \int_\Omega h f dx = \sum_{j=1}^n 2\pi d_j f_j\,, \\ \forall f(x) \in V \cap H_0^1(\Omega), \quad \textit{and} \quad h - h_e \in V \cap H_0^1(\Omega)\,, \end{gathered} \tag{1.10}$$

where the space

$$V \equiv \{f \in H^1(\Omega) \quad | \; f|_{\Omega_j} = f_j = \mathit{constant}, 1 \le j \le n, \\ \int_{\Omega\setminus\overline{\Omega}_H} a^{-1}(x)\,|\nabla f(x)|^2\,dx < \infty\}, \tag{1.11}$$

and f_j is the constant for f on $\Omega_j, j = 1, 2, \dots, n$.

Note that V is nontrivial, since $a \in V$ by $\sqrt{a} \in H^1(\Omega)$. We further reveal the relation between the local minimizers and critical points of J_0 in $\mathcal{M}_0$, namely critical points are the same as local minimizers, as below.

Theorem 1.2 (see Theorem 3.5). *Fix h_e. For each $D = (d_1, \dots, d_n) \in \mathbb{Z}^n$, J_0 has a unique minimizer in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2) \subset H^1_a \times H^1(\Omega;\mathbb{R}^2)$ in sense of gauge equivalence; moreover for any two such minimizers, say (ψ, A) and (ψ', A'), under gauge (1.8), then $A = A'$ and $\psi = \psi' e^{ic}$ for some $c \in \mathbb{R}$.*

Combine Theorem 1.1 and Theorem 1.2, we can see there is a one to one relation between the solutions of (1.10) and the gauge equivalent minimizers of J_0 in $\mathcal{M}_0$.

In Section IV, we study the properties of solutions of (1.10), where we show its solution can be represented by a linear combination of $n+1$ independent functions in $C(\overline{\Omega}) \cap V$ (see Theorem 4.1), we derive more detailed properties of the independent functions. Under a slightly stronger assumption on $a(x)$, we also achieve higher regularity of the solution.

In Section V, we discuss the motivation of our analysis of the limit problem. We show (see Theorem 5.2) the minimizers of J_ϵ converge to the minimizer of J_0 in $\mathcal{M}$. Moreover, for ϵ sufficiently small, all vortices of minimizers of J_ϵ are pinned near $\overline{\Omega}_H$, the zero set of $a(x)$. Since the zero set of $a(x)$ corresponds to the normal regions, the result confirms the effectiveness of the pinning mechanism by adding normal impurities to a superconductor to attract vortices.

Consider the local minimizers of J_ϵ in the neighborhood of a local minimizer of J_0, similar to the above result, we have the following theorem.

Theorem 1.3. *Fix h_e and $D \in \mathbb{Z}^n$. Let (ψ_D, A_D) be a minimizer for J_0 in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ under gauge (1.8). Choose $r > 0$ such that $\mathcal{B}_r \cap \mathcal{M}_0 = \mathcal{B}_r \cap [H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)]$. Then for all $\epsilon > 0$ sufficiently small, $\mathcal{B}_r(\psi_D, A_D)$ contains a local minimizer, $(\psi_\epsilon, A_\epsilon)$, of J_ϵ in $\mathcal{M}$, such that, $|\psi_\epsilon| \to \sqrt{a}$ in $C(\overline{\Omega})$, and $(\psi_\epsilon, A_\epsilon) \to (\psi_D, A_D)$ in $\mathcal{M}$ as $\epsilon \to 0$. In addition, for each $0 < \sigma < r_1$ and all ϵ sufficiently small, $|\psi_\epsilon|$ is uniformly positive outside $\bigcup_{j=1}^n \overline{\Omega}_j^\sigma$ and the degree of ψ_ϵ around $\overline{\Omega_j^\sigma}$ is d_j, $j = 1, 2, \dots, n$.*

Where $\mathcal{B}_r \equiv \mathcal{B}_r(\psi_D, A_D) = \{(\psi, A) \in \mathcal{M} | \|(\psi, A) - (\psi_D, A_D)\|_{H^1(\Omega)} \le r\}$.

2. Preliminaries

In this section, we describe some properties of two Sobolev spaces to be used for our later analysis. Section 2.1 is about the properties of H^1_a defined in (1.3), Section 2.2 is about the properties of the weighted Sobolev space V in (1.11), where we generalize the space ideas from [4].

2.1. Space H^1_a.

Recall that we have defined

$$H^1_a \equiv \{\psi \in H^1(\Omega;\mathbb{C}), \text{ such that } |\psi|^2 = a \text{ almost everywhere}\}.$$

By the assumption on $a(x)$, $\sqrt{a} \in H^1_a$, H^1_a is nonempty. The following lemma justifies the existence of the degree for any $u \in H^1_a$.

Lemma 2.1. *For every $u \in H_a^1$, there is a unique $D \equiv (d_1, \dots, d_n) \in \mathbb{Z}^n$, depending only on u, such that for any subdomain G_j and any function f_{G_j}, $1 \le j \le n$, satisfying*

$$G_j \subset \Omega, \text{ be a simply connected smooth subdomain with } G_j \cap \overline{\Omega}_H = \overline{\Omega}_j, \tag{2.1}$$

$$f_{G_j} \in C^\infty(\overline{\Omega}), 0 \le f_{G_j} \le 1, f_{G_j} = 1 \text{ on } \overline{\Omega} \setminus G_j, \text{ and } supp\{f_{G_j}\} \subset \overline{\Omega} \setminus \overline{\Omega}_j, \tag{2.2}$$

then we have the representation

$$d_j = \deg(u/\sqrt{a}, \partial G_j) = \frac{1}{\pi} \int_{G_j \setminus \overline{\Omega}_j} J(\frac{u f_{G_j}}{\sqrt{a}}) dx\,. \tag{2.3}$$

Where $J(\mathbf{w})$ is the Jacobian of the map $\mathbf{w} : G_j \to \mathbb{C}$. Write $x = (x_1, x_2) \in G_j$, $\mathbf{w} = w_1 + i w_2$, then

$$J(\mathbf{w}) = \frac{\partial(w_1, w_2)}{\partial(x_1, x_2)} = \det \begin{bmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{bmatrix}.$$

Proof. Fix $u \in H_a^1$, set $v(x) = u(x)/\sqrt{a(x)} = u(x)/|u(x)|$ in $\Omega \setminus \overline{\Omega}_H$, and $v(x) = 0$ for other case. By the assumption on $a(x)$, $v \in H^1_{\rm loc}(\Omega \setminus \overline{\Omega}_H; S^1)$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Let G_j be as in (2.1) and f_{G_j} as in (2.2), we have $v f_{G_j}$ is well defined on G_j, in addition, $supp\{v f_{G_j}\} \cap \overline{G}_j \subset \overline{G}_j \setminus \overline{\Omega}_j$, $v f_{G_j} \in H^1(G_j)$, and $\left|v f_{G_j}\right| = |v| = 1$ a.e. on ∂G_j. From [8] (Property 5 at page 220 and lemma 11 at page 337),

$$\deg(v, \partial G_j) = \deg(v f_{G_j}, \partial G_j) = \frac{1}{\pi} \int_{G_j} J(v f_{G_j}) dx = \frac{1}{\pi} \int_{G_j \setminus \overline{\Omega}_j} J(\frac{u f_{G_j}}{\sqrt{a}}) dx\,, \tag{2.4}$$

$\deg(v, \partial G_j)$ is well-defined, integer-valued and independent of f_{G_j},

Now to show $\deg(v, \partial G_j)$ is independent of the choice of G_j.

Claim: If two subdomains G_j^1, G_j^2 satisfy (2.1) with $G_j^2 \subset G_j^1 \subset \Omega$, then $\deg(v, \partial G_j^1) = \deg(v, \partial G_j^2)$.

Proof of the Claim: By $v \in H^1(\overline{G}_j^1 \setminus G_j^2)$, there is a constant $\delta = \delta(G_j^1, G_j^2, v)$, such that for any set $A \subset G_j^1 \setminus G_j^2$ and $\text{meas}\{A\} < \delta$, $\|v\|^2_{H^1(A)} < 1$. Then for any two simply connected smooth subdomains B^1, B^2 with $G_j^2 \subset B^2 \subset B^1 \subset G_j^1$ and $\text{meas}\{B^1 \setminus B^2\} < \delta$, from (2.4), we have

$$\begin{aligned} \left|\deg(v, \partial B^1) - \deg(v, \partial B^2)\right| &= \Big|\frac{1}{\pi} \int_{B^1} J(v f_{G_j^2}) dx - \frac{1}{\pi} \int_{B^2} J(v f_{G_j^2}) dx\Big| \\ &= \Big|\frac{1}{\pi} \int_{B^1 \setminus B^2} J(v) dx\Big| \\ &\le 2\|v\|^2_{H^1(B^1 \setminus B^2)}/\pi < 1\,. \end{aligned}$$

Since the left-hand side is integer-valued, $\deg(v, \partial B^1) = \deg(v, \partial B^2)$.

Choose a finite number of nested simply connected smooth subdomains, say $G_j^1 = A^1 \supset\supset A^2 \supset\supset A^2 \supset\supset \cdots \supset\supset A^k = G_j^2$, such that $\text{meas}\{A^\ell \setminus A^{\ell+1}\} < \delta$, $\ell = 1, \dots, k-1$, then $\deg(v, \partial G_j^1) = \deg(v, \partial A^1) = \deg(v, \partial A^2) = \cdots = \deg(v, \partial G_j^2)$. From the above claim, we know for any two subdomains G_j^1, G_j^2 satisfy (2.1),

$$\deg(v, \partial G_j^1) = \deg(v, \partial(G_j^1 \cap G_j^2)) = \deg(v, \partial G_j^2).$$

Hence $\deg(v,\partial G_j)$ is constant for any G_j satisfying (2.1), i.e., (2.3) is well-defined, and $d_j\equiv\deg(v,\partial G_j)=\deg(u/\sqrt{a},\partial G_j)$ depends on u only. $\square$

Lemma 2.2. *Each $u\in H_a^1$ can be written in the form of*

$$u(x)=\sqrt{a(x)}e^{i\Theta(x)},\ x\in\Omega\setminus\overline{\Omega}_H,$$

where $\Theta(x)=\phi(x)+\sum_{j=1}^n d_j\theta_j$, $\theta_j(x)$ is from (1.6) defined on $\Omega\setminus\overline{\Omega}_j$, $D\in\mathbb{Z}^n$ is from (2.3), uniquely decided by $u\in H_a^1$, and $\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$ is unique up to an additive constant $2\pi k$ for $k\in\mathbb{Z}$, satisfying $\int_{\Omega\setminus\overline{\Omega}_j}a|\nabla\phi|^2\le C(\Omega_H,a,D)+\int_\Omega|\nabla u|^2$.

We follow the same idea as in [4, Theorem 1.4] to prove the lemma, please see the proof in the appendix.

For each $D=(d_1,\dots,d_n)\in\mathbb{Z}^n$, we define the homotopy class

$$H^1_{a,D}=\{u\in H_a^1|\text{ degree for }u\text{ around }\overline{\Omega}_j\text{ is }d_j, j=1,2,\dots,n\}.$$

By Lemma 2.2, $u\in H^1_{a,D}$, if and only if $u=\sqrt{a}e^{i[\phi(x)+\sum_{j=1}^n d_j\theta_j]}$, where $\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_j)$ and $\int_{\Omega\setminus\overline{\Omega}_j}a|\nabla\phi|^2\le C(\Omega_H,\Omega,a,D)+\int_\Omega|\nabla u|^2$; Lemma 2.1 implies that $H_a^1=\bigcup_{D\in\mathbb{Z}^n}H^1_{a,D}$ and $H^1_{a,D}\cap H^1_{a,D'}=\emptyset$ for $D\neq D'$ in $\mathbb{Z}^n$; the following theorem further reveals the topology of H_a^1.

Theorem 2.3. *For each $D\in\mathbb{Z}^n$, $H^1_{a,D}$ is a nonempty, open and closed subset of H_a^1. In addition, $H^1_{a,D}$ is sequentially weakly closed in $H^1(\Omega;\mathbb{C})$, i.e., if $\{u_k\}_{k=1}^\infty\subset H^1_{a,D}$ and $u_k\rightharpoonup u$ in $H^1(\Omega;\mathbb{C})$ as $k\to\infty$, then $u\in H^1_{a,D}$.*

Proof. Since $\sqrt{a}\in H^1(\Omega)$ and $\mathbf{n}_j\in C^\infty(\overline{\Omega}\setminus\Omega_j;\mathbb{R}^2)$, $1\le j\le n$, according to Lemma 2.2, $\sqrt{a}e^{i\sum_{j=1}^n d_j\theta_j}\in H^1_{a,D}$, so that $H^1_{a,D}\neq\emptyset$.

Assume $u_0\in H^1_{a,D}$, let $B_r(u_0)=\{u\in H_a^1:\|u-u_0\|_{H^1(\Omega;\mathbb{C})}<r\}$, where $r>0$ to be chosen later. Pick any $u\in B_r(u_0)$, set $v_0=u_0/|u_0|=a^{-1/2}u_0$, $v=u/|u|=a^{-1/2}u$. Fix G_j as in (2.1) and f_{G_j} as in (2.2), $1\le j\le n$, by (2.3),

$$d_j=\frac{1}{\pi}\int_{G_j\setminus\Omega_j}J(v_0f_{G_j})dx\quad\text{and}\quad\tilde d_j=\frac{1}{\pi}\int_{G_j\setminus\Omega_j}J(vf_{G_j})dx,$$

then

$$\|J(v_0f_{G_j})-J(vf_{G_j})\|_{L^1(G_j)}\le C\cdot(1+\|u-u_0\|_{H^1(G_j)})\cdot\|u-u_0\|_{H^1(G_j)}\le Cr(1+r),$$

where $C=C(a,v_0,G_j)$. It follows that for r small (say $r=\frac{1}{2C+1}$) $d_j=\tilde d_j$ and $u\in H^1_{a,D}$. Thus $B_r(u_0)\subset H^1_{a,D}$ for r small, $H^1_{a,D}$ is an open subset of H^1_a.

Since $H_a^1=\bigcup_{D\in\mathbb{Z}^n}H^1_{a,D}$ and $H_{a,D}\cap H^1_{a,D'}=\emptyset$ for $D\neq D'$ in $\mathbb{Z}^n$, from the closeness of H_a^1, we obtain that $H^1_{a,D}$ is a closed subset of H_a^1.

Now prove $H^1_{a,D}$ is weakly sequentially closed in H_a^1. Assume that $\{u_k\}_{k=1}^\infty\subset H^1_{a,D}$ and $u_k\rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{C})$ as $k\to\infty$. By compactness, a subsequence (which we relabel as $\{u_k\}_{k=1}^\infty$) satisfies $u_k\to u$ in $L^2(\Omega)$ as $k\to\infty$, so $|u|=a^{1/2}$ a.e. in Ω, and $u\in H_a^1$, according to Lemma 2.1, $u\in H^1_{a,\tilde D}$ for some $\tilde D\in\mathbb{Z}^n$. We show $D=\tilde D$.

Set $v_k(x)=u_k(x)/|u_k(x)|, v(x)=u(x)/|u(x)|$ in $\Omega\setminus\Omega_H$, then $v_k\rightharpoonup v$ in $H^1_{\rm loc}(\Omega\setminus\Omega_H)$ and $v_k\to v$ in $L^2_{\rm loc}(\Omega\setminus\Omega_H)$, as $k\to\infty$.

Choose $\Omega_j^1 \supset \Omega_j^2$ satisfying (2.1), assume $\|v_k\|_{H^1(\Omega_j^1\setminus\Omega_j^2)} < M, M \in \mathbb{Z}, k = 1,2,3,\dots$. Partition $\Omega_j^1 \setminus \Omega_j^2$ into $4M^2+1$ subdomains enclosing Ω_j^2, say, they are $G^{(l)} \setminus G^{(l+1)}$, $l = 1,2\dots 4M^2+1$, where

$$\Omega_j^1 = G^{(1)} \supset\supset G^{(2)} \supset\supset \dots \supset\supset G^{(4M^2+2)} = \Omega_j^2 .$$

For any v_k, at least on one of the $G^{(l)} \setminus G^{(l+1)}$, $\|v_k\|_{H^1(G^{(l)}\setminus G^{(l+1)})} < \frac{1}{2}$. Choose $G^{(l)} \setminus G^{(l+1)}$ with infinitely many v_k such that $\|v_k\|_{H^1(G^{(l)}\setminus G^{(l+1)})} < \frac{1}{2}$. Let $G_j = G^{(l)}, \tilde{G}_j = G^{(l+1)}$, and take the corresponding subsequence on $G^{(l)} \setminus G^{(l+1)}$ (still labelled as $\{v_k\}$), then $\Omega_j \subset\subset \tilde{G}_j \subset\subset G_j$ and $\|v_k\|_{H^1(G_j\setminus\tilde{G}_j)} < \frac{1}{2}$, for all $k \geq 1$. By the weak convergence, $\|v\|_{H^1(G_j\setminus\tilde{G}_j)} < \frac{1}{2}$, $\int_{G_j\setminus\tilde{G}_j} |J(v_k)| \leq \|v_k\|^2_{H^1(G_j\setminus\tilde{G}_j)} < \frac{1}{4}$, and $\int_{G_j\setminus\tilde{G}_j} |J(v)| \leq \|v\|^2_{H^1(G_j\setminus\tilde{G}_j)} < \frac{1}{4}$. Pick f_{G_j} satisfying (2.2). By (2.3),

$$d_j = \frac{1}{\pi}\int_{G_j\setminus\tilde{G}_j} J(v_k f_{G_j})dx, \quad \tilde{d}_j = \frac{1}{\pi}\int_{G_j\setminus\tilde{G}_j} J(v f_{G_j})dx.$$

$$\begin{aligned}
&\int_{G_j\setminus\tilde{G}_j} (J(v_k f_{G_j}) - J(v f_{G_j}))dx \\
&= \int_{G_j\setminus\tilde{G}_j} f_{G_j}^2 (J(v_k) - J(v))dx \\
&\quad + \int_{G_j\setminus\tilde{G}_j} f_{G_j}\frac{\partial f_{G_j}}{\partial x_1}\operatorname{Re}\Big(iv_k(\frac{\partial v_k}{\partial x_2})^* - iv(\frac{\partial v}{\partial x_2})^*\Big)\,dx \\
&\quad + \int_{G_j\setminus\tilde{G}_j} f_{G_j}\frac{\partial f_{G_j}}{\partial x_2}\operatorname{Re}\Big(iv_k(\frac{\partial v_k}{\partial x_1})^* - iv(\frac{\partial v}{\partial x_1})^*\Big)\,dx\,.
\end{aligned}$$

Since $f_{G_j}\frac{\partial f_{G_j}}{\partial x_1} \in C^\infty(\overline{G}_j)$, $v_k \to v$ in L^2, and $\frac{\partial v_k}{\partial x_2} \rightharpoonup \frac{\partial v}{\partial x_2}$ weakly in L^2, it follows that $\int_{G_j\setminus\tilde{G}_j} f_{G_j}\frac{\partial f_{G_j}}{\partial x_1}\operatorname{Re}\Big(iv_k(\frac{\partial v_k}{\partial x_2})^* - iv(\frac{\partial v}{\partial x_2})^*\Big)\,dx \to 0$, as $k \to \infty$. Similarly $\int_{G_j\setminus\tilde{G}_j} f_{G_j}\frac{\partial f_{G_j}}{\partial x_2}\operatorname{Re}\Big(iv_k(\frac{\partial v_k}{\partial x_1})^* - iv(\frac{\partial v}{\partial x_1})^*\Big)\,dx \to 0$, as $k \to \infty$.

By $0 \leq f_{G_j} \leq 1$, $\int_{G_j\setminus\tilde{G}_j} f_{G_j}^2 |J(v_k) - J(v)|\,dx \leq \frac{1}{2}$, for any k, $\Big|d_j - \tilde{d}_j\Big| < \frac{1}{2}$. Since $d_j, \tilde{d}_j \in \mathbb{Z}$, $d_j = \tilde{d}_j, j = 1,2,\dots n$. Thus $D = \tilde{D}$ and $u \in H^1_{a,D}$. □

2.2. **Space V.** By (1.11), $V \equiv \{f \in H^1(\Omega) : f|_{\Omega_j} = f_j =$ constant, $1 \leq j \leq n$, $\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}(x)\,|\nabla f(x)|^2\,dx < \infty\}$ is a weighted Sobolev space. Define the norm of V as

$$\|f\|_V = \Big(\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}(x)\,|\nabla f(x)|^2\,d\,x + \int_\Omega f^2\Big)^{1/2}. \tag{2.5}$$

Lemma 2.4. *V is a Hilbert space with norm (2.5).*

Proof. Assume $\{f_k\}_{k=1}^\infty \subset V$ is a Cauchy sequence under norm (2.5). By $1/a(x) > c > 0$ in $\overline{\Omega} \setminus \Omega_H$ for some constant $c \in \mathbb{R}$, we know $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $H^1(\Omega)$. Hence there is a $f \in H^1(\Omega)$, such that $f_k \to f$ in $H^1(\Omega)$. Also $f_k \in V$ implies that f is constant on $\Omega_j, 1 \leq j \leq n$. By $\{\nabla f_k/\sqrt{a}\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Omega \setminus \Omega_H)$, there are g_1, g_2 in $L^2(\Omega \setminus \Omega_H)$, such that $\nabla f_k/\sqrt{a} \to (g_1, g_2)$ in $L^2(\Omega \setminus \Omega_H)$, from $1/\sqrt{a}$ is bounded away from 0, we get $\nabla f_k \to (\sqrt{a}g_1, \sqrt{a}g_2)$ in $L^2(\Omega \setminus \Omega_H)$. Therefore $(\sqrt{a}g_1, \sqrt{a}g_2) = \nabla f$ by the uniqueness of the convergence in $L^2(\Omega \setminus \Omega_H)$, i.e. $\nabla f/\sqrt{a} \in L^2(\Omega \setminus \Omega_H)$, and we get $f \in V$, $f_k \to f$ in V. □

Using the same idea as above, we can obtain that V is weakly closed under the norm (2.5). In addition, we have the following lemma proved in the appendix.

Lemma 2.5. *$C^1(\Omega)\cap V$ is dense in V.*

To go forward, let us first investigate properties of the Lipschitz domain $\Omega_j \subset \mathbb{R}^d, 1\le j\le n$, d is the dimension. By saying Ω_j is Lipschitz, we means that for every point $p\in\partial\Omega_j$, there is a neighborhood $\mathcal{U}_p$ of p, and a function $\phi_p:\mathbb{R}^{d-1}\to\mathbb{R}$, such that there is a Cartesian coordinate system in $\mathcal{U}_p$ with p as the origin, satisfying:

- (i) $|\phi_p(\tilde{x})-\phi_p(\tilde{y})|\le A\,|\tilde{x}-\tilde{y}|$, where $A=A(\Omega_j)$, $\tilde{x},\tilde{y}\in\mathbb{R}^{d-1}$.
- (ii) $\Omega_j\cap\mathcal{U}_p=\{(\tilde{x},x_d)|x_d<\phi_p(\tilde{x})\}\cap\mathcal{U}_p$, and $\mathcal{U}_p\setminus\Omega_j=\{(\tilde{x},x_d)|x_d>\phi_p(\tilde{x})\}\cap\mathcal{U}_p$, where $\tilde{x}\in\mathbb{R}^{d-1}$.
- (iii) For all $x\in\mathcal{U}_p$, $d(x)=\operatorname{dist}\{x,\,\partial\Omega_j\}>|x_d-\phi_p(\tilde{x})|/g_p$, for some constant $g_p>1$.

Since $\partial\Omega_j$ is compact, we can choose $\{\mathcal{U}_k^j\}_{k=1}^{n_j}$ to cover it, $j=1,2,\dots,n$, where $\mathcal{U}_k^j=\{x=(\tilde{x},x_d)\in\mathbb{R}^d|\,|\tilde{x}|\le\lambda_k^j$, and $\left|x_d-\phi_k^j(\tilde{x})\right|<\lambda_k^j\}$, λ_k^j is constant, ϕ_k^j is as in (i) and (ii).

Apply (iii), for any $x=(\tilde{x},x_d)\in\mathcal{U}_k^j$, there is a constant $g=g(\Omega_H)>1$,

$$\left|x_d-\phi_k^j(\tilde{x})\right|/g\le d(x)\le\left|x_d-\phi_k^j(\tilde{x})\right|\qquad k=1,2,\dots,n_j,\ j=1,\dots,n. \tag{2.6}$$

Since $\partial\Omega_j\subset\cup_{k=1}^{n_j}\mathcal{U}_k^j$ and $\mathcal{U}_k^j$ is open, there is a constant r_1, such that for $\sigma<r_1$, $\overline{\Omega}_j^\sigma\setminus\Omega_j\subset\cup_{k=1}^{n_j}\mathcal{U}_k^j$, where $\Omega_j^\sigma=\{x\in\Omega|\operatorname{dist}\{x,\,\Omega_j\}<\sigma\}$, $j=1,2,\dots,n$. Choose a partition of unity for $\overline{\Omega}_j^{r_1}\setminus\Omega_j$ subordinate to $\{\mathcal{U}_k^j\}_{k=1}^{n_j}$, say, $\{\beta_k^j\}_{k=1}^{n_j}$, such that,

$$\beta_k^j\in C_0^\infty(\mathcal{U}_k^j),0\le\beta_k^j\le1,\ \text{and}\ \sum_{k=1}^{n_j}\beta_k^j(x)=1\quad x\in\overline{\Omega}_j^{r_1}\setminus\Omega_j,\ 1\le j\le n. \tag{2.7}$$

Lemma 2.6. *Assume $f\in C^1(\Omega)\cap V$, and $f|_{\Omega_j}=f_j$, $1\le j\le n$, pick the constant g satisfying* (2.6)*, then for any $\sigma_0<r_1/g$,*

$$\int_{\partial\Omega_j^{\sigma_0}}a^{-1}(x)\,|f-f_j|^2\,ds\le c(\Omega_j)\sigma_0\int_{\Omega_j^{g\sigma_0}\setminus\Omega_j}a^{-1}(x)\,|\nabla f|^2\,dx\,.$$

Proof. By $\partial\Omega_j^{\sigma_0}\subset\Omega_j^{r_1}\setminus\Omega_j\subset\cup_{k=1}^{n_j}\mathcal{U}_k^j$ and the partition of unity,

$$\begin{aligned}\int_{\partial\Omega_j^{\sigma_0}}a^{-1}(x)\,|f-f_j|^2\,ds&=\int_{\partial\Omega_j^{\sigma_0}}\sum_{k=1}^{n_j}\beta_k^j(x)a^{-1}(x)\,|f_j-f|^2\,ds\\&=\sum_{k=1}^{n_j}\int_{\partial\Omega_j^{\sigma_0}\cap\mathcal{U}_k^j}\beta_k^j(x)a^{-1}(x)\,|f_j-f|^2\,ds\,.\end{aligned}$$

Then apply the local coordinate system on $\mathcal{U}_k^j$, we obtain

$$\begin{aligned}
&\int_{\partial\Omega_j^{\sigma_0}\cap\mathcal{U}_k^j} \beta_k^j(x)a^{-1}(x)\left|f-f_j\right|^2 ds\\
&=\int_{\partial\Omega_j^{\sigma_0}\cap\mathcal{U}_k^j} \beta_k^j(x)a^{-1}(x)\left|\int_{\phi_k^j(\tilde{x})}^{x_d}\nabla f\cdot\mathbf{e_d}dt\right|^2 ds\\
&\le\int_{\partial\Omega_j^{\sigma_0}\cap\mathcal{U}_k^j} a^{-1}(x)\left(\int_{\phi_k^j(\tilde{x})}^{x_d}\left|\nabla f\right|^2 dt\int_{\phi_k^j(\tilde{x})}^{x_d} dt\right) ds\\
&\le\int_{\partial\Omega_j^{\sigma_0}\cap\mathcal{U}_k^j}\left|\phi_k^j(\tilde{x})-x_d\right|\int_{\phi_k^j(\tilde{x})}^{g\sigma_0} a^{-1}(x(s))\left|\nabla f\right|^2 dtds\\
&\le c(\Omega_j)\sigma_0\int_{\Omega_j^{g\sigma_0}\setminus\Omega_j} a^{-1}(x)\left|\nabla f\right|^2 dx\,.
\end{aligned}$$

Where $\mathbf{e_d}$ is the $d-$th unit vector in the local coordinate system,. In the proof, (1.4), (2.6), $0\le\beta_k^j(x)\le 1$ and the Lipschitz property (i) are used. Hence $\int_{\partial\Omega_j^{\sigma_0}} a^{-1}(x)\left|f-f_j\right|^2 ds\le c(\Omega_j)\sigma_0\int_{\Omega_j^{g\sigma_0}\setminus\Omega_j} a^{-1}(x)\left|\nabla f\right|^2 dx$, $1\le j\le n$. □

3. Limit Equation

In this section, we prove Theorem 1.1 and Theorem 1.2 stated in the introduction. First let us give a result concerning the existence of the minimizers of J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$.

Lemma 3.1. *For fixed h_e and $D\in\mathbb{Z}^n$, there is a minimizer of J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ under gauge (1.8), which is a local minimizer of J_0 in $\mathcal{M}_0$.*

Proof. By the gauge equivalence in (1.7) and (1.9), we need only to consider the situation under the fixed gauge (1.8), i.e., in the space

$$\{(\psi,A)\in H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2):\ \operatorname{div}A=0\text{ in }\Omega\text{ and }A\cdot\mathbf{n}=0\text{ on }\partial\Omega\}.$$

According to Theorem 2.3, $H^1_{a,D}$ is sequentially weakly closed in $H^1(\Omega;\mathbb{C})$, we can apply direct method in the calculus of variations to find the minimizer of J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$. Since $H^1_{a,D}$ is both open and closed in H^1_a, the minimizer in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ is also a local minimize of J_0 in $\mathcal{M}_0$. □

From (1.2), we get the Euler-Lagrange equations of the minimizer of J_0,

$$\begin{gathered}
\operatorname{div}\Big[-\frac{i}{2}(\psi^*\nabla\psi-\psi\nabla\psi^*)-|\psi|^2A\Big]=0\quad\text{in }\Omega\,,\\
\Big[-\frac{i}{2}(\psi^*\nabla\psi-\psi\nabla\psi^*)-|\psi|^2A\Big]\cdot\mathbf{n}=0\quad\text{on }\partial\Omega\,,
\end{gathered}\tag{3.1}$$

and

$$\begin{gathered}
\operatorname{curl}\operatorname{curl}A=-\frac{i}{2}(\psi^*\nabla\psi-\psi\nabla\psi^*)-|\psi|^2A\equiv j_0\quad\text{in }\Omega\,,\\
\operatorname{curl}A=h_e\mathbf{e_3}\quad\text{on }\partial\Omega.
\end{gathered}\tag{3.2}$$

Where $A=(A_1,A_2)$, $\operatorname{curl}\operatorname{curl}A=(\partial_{x_2x_1}A_2-\partial_{x_2x_2}A_1,-\partial_{x_1x_1}A_2+\partial_{x_2x_1}A_1)$.

Note: Taking divergence on both sides of the second equation (3.2) in above, we could get the first equation of (3.1) in distribution sense.

Assume (ψ, A) is under gauge (1.8), from (1.9), we see that (3.2) becomes

$$\begin{gathered}\triangle A = -\frac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2 A \quad \text{in} \quad \Omega.\\ \operatorname{curl} A = h_e \mathbf{e_3} \quad \text{on} \quad \partial\Omega.\\ A \cdot \mathbf{n} = 0 \quad \text{in} \quad \partial\Omega.\end{gathered}$$

Since $\operatorname{div} A = 0$ in Ω and $A \cdot \mathbf{n} = 0$ on $\partial\Omega$, according to Poincaré's lemma, rewrite $A = (A_1, A_2)$ with $(A_2, -A_1) = \nabla\zeta$ for some $\zeta \in H_0^1(\Omega)$, from the above equation, $\zeta \in W^{3,2}(\Omega)$, so that we obtain the following regularity result on A:

Lemma 3.2. *If (ψ_D, A_D) under gauge (1.8) is a minimizer in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$, then $A_D \in W^{2,2}(\Omega)$.*

Now we prove Theorem 1.1 in the introduction, for the convenience to read, let us restate it.

Theorem 3.3. *Fix h_e. Let (ψ_D, A_D) be a minimizer of J_0 in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ under gauge (1.8), define h_D by $\operatorname{curl} A_D = h_D\mathbf{e_3}$, then $h_D \in V$ is the unique solution of*

$$\begin{gathered}\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla h \cdot \nabla f dx + \int_\Omega hf dx = \sum_{j=1}^n 2\pi d_j f_j\,. \\ \forall f(x) \in V \cap H_0^1(\Omega), \quad \textit{and} \quad h - h_e \in V \cap H_0^1(\Omega).\end{gathered} \tag{3.3}$$

Proof. First we show $h_D \in V$. From the boundary condition, $h = h_e$ on $\partial\Omega$. By $\psi_D \in H^1_{a,D}$ and $\psi|_{\Omega_H} = 0$, (3.2) implies,

$$\operatorname{curl}(h_D\mathbf{e_3}) = 0 \quad \text{in } \Omega_H. \tag{3.4}$$

Hence in Ω_j, $\nabla h_D = 0$, i.e., $h_D = h_{D,j}$ a.e., where $h_{D,j}$ is a constant depending on Ω_j, $1 \le j \le n$. On $\Omega \setminus \overline{\Omega}_H$, $|\psi_D| = \sqrt{a} \ne 0$, we can write $\psi_D = \sqrt{a}e^{i\theta_D}$, so that

$$\operatorname{curl}(h_D\mathbf{e_3}) = j_D = a(\nabla\theta_D - A_D) \quad \text{in } \Omega \setminus \overline{\Omega}_H\,. \tag{3.5}$$

Since $|(\nabla - iA_D)\psi_D|^2 = |\nabla\sqrt{a}|^2 + |\sqrt{a}(\nabla\theta_D - A_D)|^2$ and $J_0(\psi_D, A_D)$ is bounded, $a^{-1/2}|\nabla h_D| = \sqrt{a}|\nabla\theta_D - A_D| \in L^2(\Omega)$, so that $h_D \in H^1(\Omega)$, and $h_D \in V$.

Now we prove h_D satisfies (3.3). Divide on both sides of (3.5) by $a(x)$, then take curl to annihilate $\nabla\theta_D$, then $\operatorname{curl}\frac{1}{a(x)}\operatorname{curl}(h_D\mathbf{e_3}) = -\operatorname{curl} A_D = (0, 0, -h_D)$, rewriting the equation, we obtain

$$\nabla \cdot \frac{1}{a(x)}\nabla h_D = h_D \quad \text{in } \Omega \setminus \overline{\Omega}_H \tag{3.6}$$

in the sense of distributions. Set

$$\Omega_j^\sigma = \{x \in \Omega | \operatorname{dist}\{x, \Omega_j\} < \sigma\}, \quad \Omega^\sigma = \bigcup_{j=1}^n \Omega_j^\sigma.$$

Since $a \in C^1(\overline{\Omega} \setminus \Omega_H)$ and $a > 0$ in $\overline{\Omega} \setminus \Omega_H$, $h_D \in H^2_{\rm loc}(\Omega \setminus \overline{\Omega}_H)$, and $\nabla\theta_D \in H^1_{\rm loc}(\Omega \setminus \overline{\Omega}_H)$. Take the test function $f(x) \in C^1(\Omega) \cap V \cap H_0^1(\Omega)$ for (3.6), and integrate by parts,

$$\begin{aligned}
&\int_{\Omega\setminus\overline{\Omega}^\sigma}(a^{-1}(x)(\nabla h_D\cdot\nabla f)+h_D(x)f(x))dx\\
&=\int_{\partial(\Omega\setminus\overline{\Omega}^\sigma)}\frac{\nu\cdot\nabla h_D f}{a(x)}ds\\
&=-\int_{\partial\Omega^\sigma}\frac{\mathbf{n}\cdot\nabla h_D f}{a(x)}ds\\
&=-\int_{\partial\Omega^\sigma}\frac{\mathbf{n}\cdot\nabla h_D f_j}{a(x)}ds-\int_{\partial\Omega^\sigma}\frac{\mathbf{n}\cdot\nabla h_D(f-f_j)}{a(x)}ds\,.
\end{aligned}$$

Here $\mathbf{n}$ is the outward normal of $\partial\Omega^\sigma$, $\nu=-\mathbf{n}$ is the inward normal. Assume τ is the counterclockwise tangent vector field of $\partial\Omega^\sigma$, use (3.2) on $\partial\Omega^\sigma$,

$$\begin{aligned}
-\int_{\partial\Omega^\sigma}\frac{\mathbf{n}\cdot\nabla h_D f_j}{a(x)}ds&=f_j\int_{\partial\Omega^\sigma}\frac{\tau\cdot\operatorname{curl}h_D}{a(x)}ds=f_j\int_{\partial\Omega^\sigma}(\tau\cdot\nabla\theta_D-\tau\cdot A_D)ds\\
&=2\pi d_jf_j-f_j\int_{\Omega^\sigma}\operatorname{curl}A_Ddx=2\pi d_jf_j-f_j\int_{\Omega^\sigma}h_Ddx\\
&=2\pi d_jf_j-\int_{\Omega^\sigma}h_Dfdx-\int_{\Omega^\sigma}h_D(f_j-f)dx\\
&=2\pi d_jf_j-\int_{\Omega^\sigma}h_Dfdx-o(1),\quad\text{as }\sigma\to0.
\end{aligned}$$

Using the Cauchy inequality,

$$\Big|\int_{\partial\Omega^\sigma}\frac{\mathbf{n}\cdot\nabla h_D(f-f_j)}{a(x)}ds\Big|\le\Big(\int_{\partial\Omega^\sigma}\frac{|f-f_j|^2}{a(x)}ds\Big)^{1/2}\Big(\int_{\partial\Omega^\sigma}\frac{|\nabla h_D|^2}{a(x)}ds\Big)^{1/2}.$$

By Lemma 2.6, for $d(x)\le r_1$,

$$\int_{\partial\Omega^\sigma}\frac{|f-f_j|^2}{a(x)}ds\le c(r_1)\sigma\int_{\Omega^{g\sigma}\setminus\overline{\Omega}_H}\frac{|\nabla f|^2}{a(x)}dx\,.$$

Because $\int_{\Omega^\sigma\setminus\overline{\Omega}_H}\frac{|\nabla h_D|^2}{a(x)}dx\to0$, as $\sigma\to0$, there is a sequence $\{\sigma_m\}_{m=1}^\infty$, $\sigma_m\to0$, $\sigma_{m+1}<\sigma_m$, and $\int_{\partial\Omega^{\sigma_m}}\frac{|\nabla h_D|^2}{a(x)}ds\le\frac{c(a,\Omega)}{\sigma_m}$, as $m\to\infty$, then

$$\Big|\int_{\partial\Omega^{\sigma_m}}\frac{\mathbf{n}\cdot\nabla h_D(f-f_j)}{a(x)}ds\Big|\le c(a,\Omega)\Big(\int_{\Omega^{g\sigma_m}\setminus\overline{\Omega}_H}\frac{|\nabla f|^2}{a(x)}dx\Big)^{1/2}\to0,$$

as $\sigma_m\to0$. Now we have

$$\int_{\Omega\setminus\overline{\Omega}^{\sigma_m}}a^{-1}(\nabla h_D\nabla f+h_Df)dx=\sum_{j=1}^n2\pi d_jf_j-\int_{\Omega^{\sigma_m}}h_Dfdx-o(1)\,,$$

i.e.

$$\int_{\Omega\setminus\overline{\Omega}^{\sigma_m}}a^{-1}\nabla h_D\cdot\nabla fdx+\int_\Omega h_Dfdx=\sum_{j=1}^n2\pi d_jf_j-o(1)\,.$$

If $\sigma \in (\sigma_{m+1}, \sigma_m)$,

$$\begin{aligned}&\int_{\Omega\setminus\overline{\Omega}^{\sigma}} a^{-1}\nabla h_D\cdot\nabla f dx+\int_{\Omega} h_D f dx\\&=\int_{\Omega\setminus\overline{\Omega}^{\sigma_m}} a^{-1}\nabla h_D\cdot\nabla f dx+\int_{\Omega} h_D f dx+\int_{\Omega^{\sigma_m}\setminus\overline{\Omega}^{\sigma}} a^{-1}\nabla h_D\cdot\nabla f dx\,.\end{aligned}$$

As $\sigma\to 0$, $\mathrm{meas}\{\Omega^{\sigma_m}\setminus\overline{\Omega}^{\sigma}\}\to 0$, $\int_{\Omega^{\sigma_m}\setminus\overline{\Omega}^{\sigma}} a^{-1}\nabla h_D\cdot\nabla f dx\to 0$, then

$$\int_{\Omega\setminus\overline{\Omega}^{\sigma}} a^{-1}\nabla h_D\cdot\nabla f dx+\int_{\Omega} h_D f dx\to\sum_{j=1}^{n}2\pi d_j f_j,\quad \text{as } \sigma\to 0\,.$$

Hence the weak form of h_D becomes

$$\begin{gathered}\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla h\cdot\nabla f dx+\int_{\Omega} h f dx=\sum_{j=1}^{n}2\pi d_j f_j\,,\\ \forall f(x)\in C^1(\Omega)\cap V\cap H_0^1(\Omega)\quad\text{and}\quad h-h_e\in V\cap H_0^1(\Omega).\end{gathered}$$

By Lemma 2.5, $C^1(\Omega)\cap V\cap H_0^1(\Omega)$ is dense in $V\cap H_0^1(\Omega)$, thus the above equation is true for $\forall f(x)\in V\cap H_0^1(\Omega)$, i.e. we get (3.3).

Now to prove the solution of (3.3) is unique. Assume that h_1 and h_2 are solutions, then $h=h_1-h_2\in V\cap H_0^1(\Omega)$. Apply h as a test function to the corresponding equations about h_1 and h_2 respectively, then take their difference,

$$\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}|\nabla h|^2 dx+\int_{\Omega} h^2 dx=0\,,$$

whence $h_1-h_2=0$ in V. □

Lemma 3.4. *For fixed $h_e\in\mathbb{R}$ and $D\in\mathbb{Z}^n$, there is a unique solution for* (3.3).

Proof. Existence: For the given h_e and $D\in\mathbb{Z}^n$, by Lemma 3.1, we can find (ψ_D, A_D) as the minimizer(i.e. a minimizer) of J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ under gauge (1.8), apply Theorem 3.3, we know $h_D\mathbf{e_3}=\operatorname{curl}A_D$ satisfying eq (3.3). Uniqueness is exactly the last part of Theorem 3.3. □

Note that for any $h_e\in H^1(\Omega)$, Lemma 3.4 holds.

Theorem 3.5. *For any fixed h_e and $D=(d_1,\dots,d_n)\in\mathbb{Z}^n$, J_0 has a unique minimizer in the space $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)\subset H^1_a\times H^1(\Omega;\mathbb{R}^2)$ in the sense of gauge equivalence; moreover, for any two such minimizers, say (ψ,A) and (ψ',A'), under gauge (1.8), then $A=A'$ and $\psi=\psi' e^{ic}$ for some $c\in\mathbb{R}$.*

Proof. The existence follows from Lemma 3.1. Uniqueness: By (1.7) and (1.9), we only need to consider the situation under gauge (1.8). Without loss of generality, we assume the two minimizers (ψ,A) and (ψ',A') in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ are under gauge (1.8), so that $\operatorname{div}A=\operatorname{div}A'=0$ and $A\cdot\mathbf{n}=A'\cdot\mathbf{n}=0$, according to Poincaré's lemma, we have $A-A'=(-\frac{\partial\zeta}{\partial y},\frac{\partial\zeta}{\partial x})$ for some $\zeta\in H^1(\Omega)$, where (x,y) are the coordinates in 2−dim. we can also derive that ζ is constant on $\partial\Omega$ from $(A-A')\cdot\mathbf{n}=0$. Through Theorem 3.3, we get $\operatorname{curl}A=\operatorname{curl}A'$, which implies $\triangle\zeta=0$ in Ω, thus ζ is constant on Ω, i.e. $A=A'$. Then by (3.2), we have $j_0=j_0'$, so $\nabla\theta=\nabla\theta'$ in $\Omega\setminus\overline{\Omega}_H$, i.e. $e^{i\theta-i\theta'}=e^{ic}$, for some $c\in\mathbb{R}$, hence $\psi=\psi' e^{ic}$, and We have proved the later part of the theorem. If take $\phi=c=$ constant, we then have $(\psi',A')=G_\phi(\psi,A)$, i.e. they are gauge equivalent. □

We need to mention that Theorem 3.5 is a generalization of the Theorem 3.2 in [4] under our setting.

Now suppose (ψ_D, A_D) is a critical point of J_0, i.e. a solution of (3.1) and (3.2), then from the first part of Theorem 3.3, h_D in $\operatorname{curl} A_D = h_D\mathbf{e_3}$ is the unique solution of (3.3), hence (ψ_D, A_D) is a local minimizer of J_0 in $\mathcal{M}_0$. On the other hand, by Lemma 2.2, every local minimizer (ψ, A) belongs to $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ for some $D \in \mathbb{Z}^n$, hence it satisfies (3.1) and (3.2), and by Theorem 3.5, it is gauge equivalent to the minimizer in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$. Thus we have the following statement.

Corollary 3.6. *All critical points of J_0 are local minimizers in $H^1_a \times H^1(\Omega;\mathbb{R}^2)$.*

As is easy to see that if we obtain the solution h_D of (3.3), then we can recover A_D with the condition $\operatorname{div} A_D = 0$ in Ω, and recover ψ_D from (3.2), so that (3.3) describes the limit system completely.

4. Properties Of The Solutions Of The Limit Equation

Consider the $n+1$ functions in $V \cap H^1_0(\Omega)$, $\{\eta_0, \eta_1, \dots \eta_n\}$ satisfying

$$\begin{gathered}\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla\eta_0 \cdot \nabla f dx + \int_\Omega \eta_0\, f dx = 0\,,\\ \forall f(x) \in V \cap H^1_0(\Omega) \quad \text{and} \quad \eta_0 = 1 \text{ on } \partial\Omega\end{gathered} \tag{4.1}$$

and

$$\begin{gathered}\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla\eta_j \cdot \nabla f dx + \int_\Omega \eta_j f dx = 2\pi f_j\,,\\ \forall f(x) \in V \cap H^1_0(\Omega) \quad \text{and} \quad \eta_j = 0 \text{ on } \partial\Omega\,,\ j = 1, \dots, n\,.\end{gathered} \tag{4.2}$$

The existence and uniqueness of solutions in V for both (4.1) and (4.2) follows the result in Lemma 3.4. We can use them to represent the solution of (3.3).

Theorem 4.1. *Fix $h_e \in \mathbb{R}$, $D \in \mathbb{Z}^n$. If h_D solves (3.3), then $h_D \in C(\overline{\Omega})$ and*

$$h_D = \sum_{j=1}^n d_j\eta_j + h_e\eta_0\,. \tag{4.3}$$

Moreover, if $\alpha_k > 1, k = 1, 2, \dots, n$, then $h_D \in C^1(\overline{\Omega})$, where α_k is from (1.4).

The proof of Theorem 4.1 is a consequence of properties of η_0 and $\eta_j, j = 1, 2, \dots, n$. We will postpone it to the end of this section. We first discuss some properties of $\{\eta_1, \dots \eta_n\}$.

Property(i) $\eta_j \geq 0$ in Ω, $1 \leq j \leq n$.

Property(ii) $\eta_1, \dots, \eta_n$ are linear independent in $V \cap H^1_0(\Omega)$, i.e., $\sum_{j=1}^n w_j\eta_j \equiv 0$ for $w_j \in \mathbb{R}, 1 \leq j \leq n$, if and only if $w_j = 0, 1 \leq j \leq n$.

Proof. To prove this property (i), we use the test function $f = \min\{\eta_j, 0\}$ in (4.2) and obtain

$$\int_{\Omega\setminus\overline{\Omega}_H} a^{-1} |\nabla f|^2 dx + \int_\Omega |f|^2 dx \leq 0.$$

So that $f \equiv 0$, i.e., $\eta_j \geq 0$ in Ω for $1 \leq j \leq n$, and (i) is proved.

Assume $g = \sum_{j=1}^{n} w_j \eta_j \equiv 0$ for some $w_j \in \mathbb{R}, 1 \le j \le n$. From (4.2), g satisfies the equation

$$\begin{gathered} \int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla g \cdot \nabla f dx + \int_{\Omega} g f dx = 2\pi \sum_{j=1}^{n} w_j f_j \equiv 0\,, \\ \forall f(x) \in V \cap H_0^1(\Omega), \quad \text{and} \quad g = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{4.4}$$

Fix $k \in \{1, \dots, n\}$. Choose σ, m such that $\Omega_k^\sigma \cap \Omega_H = \Omega_k$ and $m > 2/\sigma$. Set χ_k^σ as the characteristic function of Ω_k^σ, $\Omega_k^\sigma = \{x \in \Omega | \operatorname{dist}\{x, \Omega_k\} < \sigma\}$. Let $f^k \equiv \chi_k^\sigma * \rho_m(x)$, where $\rho_m(x) = m^2\rho(mx)$, $\rho(x)$ is defined as in (6.4). Then $f^k \in V \cap H_0^1(\Omega)$, $f_k^k = 1$ and $f_j^k = 0$ if $k \neq j$, for $1 \le j \le n$. Apply f^k as a test function for (4.4), we have $w_k = 0$, for $k = 1, 2, \dots, n$. Thus $\eta_1, \dots, \eta_n$ are linear independent in $V \cap H_0^1(\Omega)$, we have (ii). □

Property (iii) Assume η_j^k is the value of η_j on Ω_k, then $\eta_j^j = \operatorname{ess\,sup}_\Omega \eta_j < 2\pi/\operatorname{meas}\{\Omega_j\}$ and $\eta_j^j > \eta_j^k$ for $k \neq j$, $1 \le k, j \le n$.

Proof. Using $f = \eta_j$ as a test function for (4.2), then

$$2\pi\eta_j^j = \int_{\Omega\setminus\overline{\Omega}_H} a^{-1} |\nabla\eta_j|^2 dx + \int_\Omega |\eta_j|^2 dx > 0.$$

On the other hand, Using $f = (\eta_j - \eta_j^j)_+$ as a test function for (4.2), then

$$\int_{\Omega\setminus\overline{\Omega}_H} a^{-1} \left|\nabla(\eta_j - \eta_j^j)_+\right|^2 dx + \int_\Omega \eta_j(\eta_j - \eta_j^j)_+ dx = 0,$$

so that $(\eta_j - \eta_j^j)_+ = 0$ a.e., i.e. $\eta_j^j = \operatorname{ess\,sup}_\Omega \eta_j$. By using test functions from $C_0^1(\Omega \setminus \overline{\Omega}_H)$ in (4.2), we see that

$$\begin{gathered} -\nabla \cdot a^{-1}\nabla\eta_j + \eta_j = 0 \quad \text{in } \Omega \setminus \overline{\Omega}_H, \\ \eta_j = 0 \quad \text{on } \partial\Omega, \text{ for } 1 \le j \le n. \end{gathered} \tag{4.5}$$

Since η_j is nonconstant, by the maximum principle in (4.5), there is no local maxima for η_j in $\Omega \setminus \overline{\Omega}_H$. If $\eta_j^k = \eta_j^j$ for some $k \neq j$, fix σ such that $\Omega_k^\sigma \cap \Omega_H = \Omega_k$. Let $c_k = \operatorname{ess\,sup}_{\Omega_k^\sigma \setminus \Omega_k^{\sigma/2}} \eta_j$, then $c_k < \eta_j^j = \text{ess}, \sup_\Omega \eta_j$.

Using $f = \chi_k^\sigma(\eta_j - c_k)_+ \not\equiv 0$ as a test function in (4.2), we have

$$0 = \int_{\Omega_k^\sigma\setminus\overline{\Omega}_k} a^{-1} |\nabla f|^2 dx + \int_{\Omega_k^\sigma} f\eta_j dx > 0,$$

a contradiction, so that $\eta_j^j > \eta_j^k$ for $k \neq j$, $1 \le k, j \le n$. Using η_j as a test function in (4.2), we have $(\eta_j^j)^2 \operatorname{meas}\{\Omega_j\} < 2\pi\eta_j^j$, so that $\eta_j^j < 2\pi/\operatorname{meas}\{\Omega_j\}$. Therefore, (iii) is proved. □

Property (iv) $\eta_j \in C^0(\overline{\Omega}) \cap V$, $j = 1, 2, \dots, n$.

Proof. By (4.5) and $a^{-1} \in C^1(\overline{\Omega} \setminus \Omega_H)$, we apply the standard estimate for the weak solution of an elliptic equation (say [16] theorem 8.8 at page 183 and theorem 8.12 at page 186), $\eta_j(x)$ in $H^2(\Omega')$, for any $\Omega' \subset\subset \overline{\Omega} \setminus \overline{\Omega}_H$, by Sobolev embedding, $\eta_j(x) \in C^0(\overline{\Omega} \setminus \overline{\Omega}_H)$. Since $\eta_j(x)$ is a bounded constant in $\overline{\Omega}_k, 1 \le k \le n$, with $\eta_j \in H^1(\overline{\Omega})$, i.e.,

$$\eta_j(x) \in H^1(\overline{\Omega}) \cap C^0(\overline{\Omega} \setminus \overline{\Omega}_H). \tag{4.6}$$

We show $\eta_j(x)$ is C^0 on the boundary of Ω_H. Since Ω_k is Lipschitz, $k=1,2,\dots,n$, there is a constant $\sigma_k < r_1$, for any $x_0 \in \Omega\backslash\Omega_H$ with $d = \operatorname{dist}\{x_0, \overline{\Omega}_k\} < \sigma_k$, we can find a rectangle with sides parallel to the local coordinate axes, its top $\overline{x_0x_1}$, above the boundary graph and its bottom $\overline{y_0y_1}$, below the graph, $\operatorname{dist}\{x_0, x_1\} = d^{1+\alpha_k/3}$, $\operatorname{dist}\{x_0, y_1\} = c_k d$, where c_k is a constant depending on Ω_k, $\overline{x_0x_1}$ represents the line segment starting at x_0 and ending at x_1, $\overline{y_0y_1}$ is the line segment starting at y_0 and ending at y_1. Note the rectangle is not unique, but it does not matter.
Claim: For all $f \in C^0(\overline{\Omega} \setminus \overline{\Omega}_H) \cap V$,

$$d^{-1-\alpha_k/3} \int_{x_0}^{x_1} |f(x_s) - f_k| \, dH^1 \le b_k d^{\alpha_k/3}(x) \|f\|_V ,$$

where the integral is in $\overline{x_0x_1}$, and $x_s = (1-s)x_0 + sx_1$, and $s \in [0,1]$.
Proof of the Claim: Assume $f \in C^1(\Omega) \cap V$, then $f(y) = f_k$ in $\overline{y_0y_1}$ and

$$\int_{x_0}^{x_1} |f(x_s) - f_k| \, dH^1 = \int_{x_0}^{x_1} |f(x_s) - f(y_s)| \, dH^1 = \int_{x_0}^{x_1} \Big| \int_{x_s}^{y_s} \nabla f \cdot \mathbf{n}_{x_s y_s} dH^1 \Big| dH^1 ,$$

where $\mathbf{n}_{x_s y_s}$ is the unit vector from x_s to y_s. So that

$$\begin{aligned} \int_{x_0}^{x_1} |f(x_s) - f_k| \, dH^1 &\le \int_{x_0}^{x_1} \int_{x_s}^{y_s} |\nabla f| \, dx \\ &\le \Big(\int_{x_0}^{x_1} \int_{x_s}^{y_s} a^{-1} |\nabla f|^2 \, dx \Big)^{1/2} \cdot \Big(\int_{x_0}^{x_1} \int_{x_s}^{y_s} a(x) dx \Big)^{1/2} \\ &\le \|f\|_V \Big(\int_{x_0}^{x_1} \int_{x_s}^{y_s} C_1 d^{\alpha_k} dx \Big)^{1/2} \\ &\le b_k d^{1+2\alpha_k/3} \|f\|_V , \end{aligned}$$

here $b_k = C_1 c_k$. From the above inequality,

$$d^{-1-\alpha_k/3} \int_{x_0}^{x_1} |f(x_s) - f_k| \, dH^1 \le b_k d^{\alpha_k/3} \|f\|_V .$$

Because $C^1(\Omega) \cap V$ is dense in $C^0(\Omega) \cap V$, we have proved the claim.

Now we continue the proof of Property (iv). Since $\eta_j(x) \in C^0(\Omega_k^{\sigma_k} \setminus \overline{\Omega}_k)$, for every $x_0 \in \Omega_k^{\sigma_k} \setminus \overline{\Omega}_k$, we have

$$\begin{aligned} &\big|\eta_j(x_0) - \eta_j^k\big| \\ &=\le d^{-1-\alpha_k/3} \int_{x_0}^{x_1} \big|\eta_j(x_s) - \eta_j^k\big| \, dH^1 + d^{-1-\alpha_k/3} \int_{x_0}^{x_1} |\eta_j(x_s) - \eta_j(x_0)| \, dH^1 \\ &\le b_k d^{\alpha_k/3} \|\eta_j\|_V + d^{-1-\alpha_k/3} \int_{x_0}^{x_1} |\eta_j(x_s) - \eta_j(x_0)| \, dH^1, \end{aligned}$$

To estimate $d^{-1-\alpha_k/3} \int_{x_0}^{x_1} |\eta_j(x_s) - \eta_j(x_0)| \, dH^1$, consider (4.5) in the ball $B_{d/2}(x_0)$. Scaling $B_{d/2}(x_0)$ to a unit ball, (4.5) becomes

$$-4d^{-2} \nabla_y \cdot a^{-1}(yd/2 + x_0) \nabla_y \eta_j = -\eta_j \quad \text{in } B_1(0) .$$

Apply Hölder estimate [16, theorem 8.22 page 200] in the ball $B_{2d^{\alpha_k/3}}(0)$ for the dilated equation, we have $\operatorname{osc} \eta_j \le C d^{\beta \alpha_k/3} \|\eta_j\|_{L^\infty}$, where C depends on

β, Ω_k and α_k, $\beta \in (0,1)$. By Property (iii), $\|\eta_j\|_{L^\infty} \le 1/\operatorname{meas}\{\Omega_j\}$, so that $|\eta_j(x_s) - \eta_j(x_0)| \le C(\Omega_k, \alpha_k) d^{\beta\alpha_k/3}$, i.e.,

$$\left|\eta_j(x_0) - \eta_j^k\right| \le b_k d^{\alpha_k/3} \|\eta_j\|_V + C(\Omega_k, \alpha_k) d^{\beta\alpha_k/3} \le \tilde{b}_k d^{\beta\alpha_k/3}(1 + \|\eta_j\|_V), \quad (4.7)$$

for all $x_0 \in \Omega_k^{\sigma_k} \setminus \overline{\Omega}_k$. Since $d(x) \to 0$ as $x \to \partial\Omega_k$, $\eta_j(x) \to \eta_j^k$ at $\partial\Omega_k$, i.e., $\eta_j(x)$ is continuous at $\partial\Omega_k$. Combining this with (4.6), we have $\eta_j \in C^0(\overline{\Omega})$, and Property (iv) is proved. □

Property (v) $\eta_j > 0$ a.e. in Ω, $j = 1, 2, \dots, n$. Hence $\eta_j^k > 0$, $1 \le k, j \le n$.

Proof. We prove that $\operatorname{meas}\{x \in \overline{\Omega} | \eta_j(x) = 0\} = 0$ for $j = 1, 2, \dots, n$. By (iv), it makes sense to talk about the level set of $\eta_j(x)$. Let $\Gamma_0^j = \{x \in \overline{\Omega} | \eta_j(x) = 0\}$, $\Gamma_\delta^j = \{x \in \overline{\Omega} | \eta_j(x) \le \delta\}$, then $\Gamma_0^j \subset \Gamma_\delta^j$, $\operatorname{meas}\{\Gamma_\delta^j \setminus \Gamma_0^j\} \to 0$ as $\delta \to 0$. By $\eta_j^j > 0$, $\overline{\Omega}_j \cap \Gamma_\delta^j = \emptyset$ for $\delta < \eta_j^j$. In the proof, we always assume $\delta < \eta_j^j$, i.e., $\overline{\Omega}_j \cap \Gamma_\delta^j = \emptyset$.

Denote $\chi_{\Gamma_\delta^j}$ as the characteristic function of the set Γ_δ^j, $\delta \ge 0$. Set

$$g_\delta(v) = \begin{cases} v, & v \le \delta/2 \\ (\delta - v)_+, & v > \delta/2 \end{cases} \quad \text{and} \quad h_\delta(v) = \begin{cases} v/2, & v \le 2\delta/3 \\ (\delta - v)_+, & v > 2\delta/3\,. \end{cases}$$

Using $f_\delta = \chi_{\Gamma_\delta^j}\, h_\delta(\eta_j) \in H_0^1(\Omega) \cap V$ as a test function in (4.2), we have

$$\int_{\Gamma_\delta^j \setminus \overline{\Omega}_H} a^{-1} h_\delta'(\eta_j) \left|\nabla \eta_j\right|^2 dx + \int_{\Gamma_\delta^j} \eta_j f_\delta dx = 0.$$

By the sign of h_δ' in Γ_δ^j,

$$\begin{aligned} \int_{\Gamma_\delta^j \setminus (\overline{\Omega}_H \cup \Gamma_{2\delta/3}^j)} a^{-1} \left|\nabla \eta_j\right|^2 dx &= 1/2 \int_{\Gamma_{2\delta/3}^j \setminus \overline{\Omega}_H} a^{-1} \left|\nabla \eta_j\right|^2 dx + \int_{\Gamma_\delta^j} \eta_j f_\delta dx \\ &\le 1/2 \int_{\Gamma_{2\delta/3}^j \setminus \overline{\Omega}_H} a^{-1} \left|\nabla \eta_j\right|^2 dx + \int_{\Gamma_\delta^j} \left|\eta_j\right|^2 dx\,. \end{aligned}$$

Do the same thing in $\Gamma_{2/3\delta}^j, \Gamma_{2^2/3^2\delta}^j, \Gamma_{2^3/3^3\delta}^j, \dots$, to get

$$\begin{aligned} \int_{\Gamma_\delta^j \setminus (\overline{\Omega}_H \cup \Gamma_{2\delta/3}^j)} a^{-1} \left|\nabla \eta_j\right|^2 dx &\le \sum_{k=0}^\infty 2^{-k} \int_{\Gamma_{2^k\delta/3^k}^j} \left|\eta_j\right|^2 dx \\ &< \sum_{k=0}^\infty 2^{-k} \int_{\Gamma_\delta^j} \left|\eta_j\right|^2 dx = 2 \int_{\Gamma_\delta^j} \left|\eta_j\right|^2 dx\,. \end{aligned}$$

Similarly,

$$\int_{\Gamma_{2\delta/3}^j \setminus (\overline{\Omega}_H \cup \Gamma_{4\delta/9}^j)} a^{-1} \left|\nabla \eta_j\right|^2 dx < 2 \int_{\Gamma_{2\delta/3}^j} \left|\eta_j\right|^2 dx\,.$$

Summing the above two equations, we have

$$\int_{\Gamma_\delta^j \setminus (\overline{\Omega}_H \cup \Gamma_{4\delta/9}^j)} a^{-1} \left|\nabla \eta_j\right|^2 dx < 4 \int_{\Gamma_\delta^j} \left|\eta_j\right|^2 dx\,.$$

Use the test function $\chi_{\Gamma_\delta^j}\, g_\delta(\eta_j)$ in (4.2), get

$$\int_{\Gamma_\delta^j \setminus (\overline{\Omega}_H \cup \Gamma_{\delta/2}^j)} a^{-1} \left|\nabla \eta_j\right|^2 dx = \int_{\Gamma_{\delta/2}^j \setminus \overline{\Omega}_H} a^{-1} \left|\nabla \eta_j\right|^2 dx + \int_{\Gamma_\delta^j} \eta_j g_\delta(\eta_j) dx\,,$$

i.e.,

$$\int_{\Gamma_\delta^j\setminus(\overline{\Omega}_H\cup\Gamma_{\delta/2}^j)} a^{-1}\left|\nabla\eta_j\right|^2 dx > \int_{\Gamma_{\delta/2}^j\setminus\overline{\Omega}_H} a^{-1}\left|\nabla\eta_j\right|^2 dx\,.$$

So that

$$\begin{aligned}\int_{\Gamma_\delta^j\setminus\overline{\Omega}_H} a^{-1}\left|\nabla\eta_j\right|^2 dx &\le 2\int_{\Gamma_\delta^j\setminus(\overline{\Omega}_H\cup\Gamma_{\delta/2}^j)} a^{-1}\left|\nabla\eta_j\right|^2 dx\\ &\le 2\int_{\Gamma_\delta^j\setminus(\overline{\Omega}_H\cup\Gamma_{4\delta/9}^j)} a^{-1}\left|\nabla\eta_j\right|^2 dx \le 8\int_{\Gamma_\delta^j}\left|\eta_j\right|^2 dx\,.\end{aligned}$$

Consider the function $g(x)=\chi_{\Gamma_\delta^j}(\delta-\eta_j)_+\in H^1(\Omega)$, then $\nabla g=\nabla\eta_j$ a.e. in Γ_δ^j. By $g\equiv 0$ on Ω_j, and $\operatorname{meas}\{\Omega_j\}>0$, we can apply Sobolev inequality to $g(x)$, then

$$\begin{aligned}\int_{\Gamma_\delta^j}|g|^2 dx &\le c(\Omega)\int_{\Gamma_\delta^j}|\nabla g|^2 dx = c(\Omega)\int_{\Gamma_\delta^j}\left|\nabla\eta_j\right|^2 dx\\ &\le c(a,\Omega)\int_{\Gamma_\delta^j\setminus\overline{\Omega}_H} a^{-1}\left|\nabla\eta_j\right|^2 dx \le c(a,\Omega)\int_{\Gamma_\delta^j}\left|\eta_j\right|^2 dx\,.\end{aligned}$$

Therefore,

$$\delta^2\operatorname{meas}\{\Gamma_0^j\}\le\int_{\Gamma_\delta^j}|g|^2 dx\le c(a,\Omega)\int_{\Gamma_\delta^j}\left|\eta_j\right|^2 dx\le c(a,\Omega)\delta^2\operatorname{meas}\{\Gamma_\delta^j\setminus\Gamma_0^j\}\,,$$

i.e., $\operatorname{meas}\{\Gamma_0^j\}\le c(a,\Omega)\operatorname{meas}\{\Gamma_\delta^j\setminus\Gamma_0^j\}\to 0$, as $\delta\to 0$, thus $\operatorname{meas}\{\Gamma_0^j\}=0$, $\eta_j>0$ a.e.. Since $\operatorname{meas}\{\Omega_k\}>0$, for $1\le k\le n$, and $\eta_j>0$ a.e., $\eta_j^k=\frac{\int_{\Omega_k}\eta_j(x)dx}{\operatorname{meas}\{\Omega_k\}}>0$. So that Property (v) is proved. □

Property (vi) For every domain G_k with $G_k\cap\overline{\Omega}_H=\overline{\Omega}_k$, $\eta_j^k<\sup_{G_k}\eta_j$, where $k\ne j$, $k,j=1,2,\dots,n$.

Proof. If η_j is constant on any subdomain of $\Omega\setminus\overline{\Omega}_H$, then $\nabla\eta_j$ is zero, from (4.5), η_j is also zero, which contradict with (v). So that, η_j is not a constant on any subdomain of $\Omega\setminus\overline{\Omega}_H$. We use contradiction to prove Property (vi). Assume $\eta_j^k=\sup_{G_k}\eta_j$, for some G_k as in (vi). Set

$$\sigma<\operatorname{dist}\{\partial G_k,\overline{\Omega}_k\},\quad c_k=\operatorname*{ess\,sup}_{\Omega_k^\sigma\setminus\overline{\Omega}_k^{\sigma/2}}\eta_j,$$

use $f=\chi_{\Omega_k^\sigma}(\eta_j-c_k)_+$ as a test function in (4.2), for $k\ne j$,

$$\int_{\Omega_k^\sigma\setminus\overline{\Omega}_k} a^{-1}\left|\nabla f\right|^2 dx+\int_{\Omega_k^\sigma} f\eta_j dx=0$$

i.e., $f\equiv 0,\eta_j^k=c_k>0$. So that η_j achieves its nonzero local maximum in $\Omega_k^\sigma\setminus\overline{\Omega}_k^{\sigma/2}$, which contradicts with the maximum principle applicable to (4.5), hence Property (vi) holds. □

Property (vii) If $\alpha_k>1$, $k=1,2,\dots,n$, then $\eta_j\in C^1(\overline{\Omega})$, $j=1,2,\dots,n$, where α_k is from (1.4).

Proof. Without loss of generality, we write η_j as η in the proof. The C^1 continuity of η in $\overline{\Omega}\setminus\overline{\Omega}_H$ is from the standard elliptic argument (See [16], theorem 8.33 on page 210). We focus on the proof of the C^1 continuity of η close to Ω_H, and show that $|\nabla\eta|$ is forced to 0 as x close $\partial\Omega_H$. For $T\le\eta_j^j$, denote $\Sigma_T=\{x|\eta(x)\ge T\}$, use the test function $(\eta-T)_+$ in (4.2), then

$$\int_{\Sigma_T\setminus\overline{\Omega}_H} a^{-1}|\nabla\eta|^2\,dx+\int_{\Sigma_T}\eta(\eta-T)dx=2\pi(\eta_j^j-T).$$

Apply the co-area formula (see [14]),

$$\int_{\{\eta=T\}\setminus\overline{\Omega}_H} a^{-1}|\nabla\eta|\,dH^1(x)+\int_{\Sigma_T}\eta dx=2\pi. \tag{4.8}$$

For any point x_0 close to Ω_k with $d=\operatorname{dist}\{x_0,\Omega_k\}\le r_1$, (4.2) becomes (4.5) in $B_{d/2}(x_0)$, with $a(x)$ of order $d_k^{\alpha_k}(x)$, where $d_k(x)=\operatorname{dist}\{x,\Omega_k\}$. Scaling $B_{d/2}(x_0)$ to the unit ball $B_1(0)$, write $\tilde\eta(y)=\eta(dy/2+x_0)$, then (4.5) can be simplified as

$$-\triangle_y(\tilde\eta-\eta(x_0))+\frac{d}{2}\frac{\nabla_x a}{a}\cdot\nabla_y(\tilde\eta-\eta(x_0))=-a\big(\frac{d}{2}\big)^2\tilde\eta,\qquad \text{in } B_1(0).$$

Apply Hölder estimate [16, Theorem 8.32 page 210] in the ball $B_{1/2}(0)$ for the above dilated equation, then $\tilde\eta\in C^{1,\beta}(B_{1/2}(0))$, $\forall\beta\in(0,1)$, and $\exists\tilde C$ depending only on α_k, C_1, such that

$$|\nabla_y\tilde\eta|_{C^{0,\beta}(B_{1/2}(0))}\le\tilde C\Big(|\tilde\eta-\eta(x_0)|_{C^0(B_1(0))}+\Big|a(x_0)(\frac{d}{2})^2\tilde\eta\Big|_{C^0(B_1(0))}\Big).$$

Fix β. From (1.4), $a(x_0)$ is bounded by d^{α_k}. Pull back to $B_{d/2}(x_0)$, then

$$\Big|\big(\frac{d}{2}\big)^{1+\beta}\nabla_x\eta\Big|_{C^{0,\beta}(B_{d/4}(x_0))}\le\tilde C\left(|\eta-\eta(x_0)|_{C^0(B_{d/2}(x_0))}+d^{\alpha_k+2}\right). \tag{4.9}$$

Let $T=\eta(x_0), M=|\nabla\eta(x_0)|$,we iterate to obtain the bound on $\nabla\eta(x_0)$. First by the uniform bound of η and (4.9), $|\nabla\eta|_{C^{0,\beta}(B_{d/4}(x_0))}\le\tilde C_1 d^{-1-\beta}$, so that $|\nabla\eta(x)|>\frac{M}{2}$ for $x\in B_{\tilde r_1}(x_0)$, where $\tilde r_1=\big(\frac{M}{2\tilde C_1}\big)^{1/\beta}d^{1+1/\beta}$.

By (4.8), $\int_{\{\eta=T\}\cap B_{\tilde r_1}(x_0)}a^{-1}|\nabla\eta|\,dH^1(x)\le 2\pi$, then $\frac{M}{2d^{\alpha_k}}\operatorname{meas}\{\{\eta=T\}\cap B_{\tilde r_1}(x_0)\}\le 2\pi$. If $\operatorname{meas}\{\{\eta=T\}\cap B_{\tilde r_1}(x_0)\}\ge\tilde r_1$, then $\frac{M^{1+1/\beta}d^{1+1/\beta}}{2d^{\alpha_k}(2\tilde C_1)^{1/\beta}}\le 2\pi$, i.e. $M\le\tilde C_2 d^{\beta\alpha_k/(1+\beta)-1}$.

If $\operatorname{meas}\{\{\eta=T\}\cap B_{\tilde r_1}(x_0)\}<\tilde r_1$, by the continuity of η and the intermediate value theorem, $\{\eta=T\}\cap B_{\tilde r_1}(x_0)$ will be a closed curve inside $B_{\tilde r_1}(x_0)$, we use $(T-\eta)_+\chi_{B_{\tilde r_1}(x_0)}$ as the test function in (4.2), then

$$\int_{\{\eta\le T\}\cap B_{\tilde r_1}(x_0)}\Big(-a^{-1}|\nabla\eta|^2+\eta(T-\eta)\Big)\,dx=0\,,$$

so that

$$\begin{aligned}\frac{M}{2C_1d^{\alpha_k}}\operatorname{meas}\{\{\eta\le T\}\cap B_{\tilde r_1}(x_0)\}&\le\int_{\{\eta\le T\}\cap B_{\tilde r_1}(x_0)}\eta(T-\eta)dx\\&\le C\operatorname{meas}\{\{\eta\le T\}\cap B_{\tilde r_1}(x_0)\},\end{aligned}$$

we have $M\le\tilde C_2 d^{\alpha_k}$. Hence $|\nabla\eta(x)|\le\tilde C_2 d^{-1+\beta\alpha_k/(1+\beta)}$ for all $x\in\Omega_k^{r_1}$, which yields

$$|\tilde\eta-\eta(x_0)|_{C^0(B_{d/2}(x_0))}\le\tilde C_2 d^{\beta\alpha_k/(1+\beta)}\,.$$

Back to (4.9), then $|\nabla\eta|_{C^{0,\beta}(B_{d/4}(x_0))} \le \tilde{C}_3 d^{-1-\beta+\beta\alpha_k/(1+\beta)}$. Consider in $B_{\tilde{r}_2}(x_0)$, where $\tilde{r}_2 = \big(\frac{M}{2\tilde{C}_3}\big)^{1/\beta} d^{1+1/\beta-\alpha_k/(1+\beta)}$. Using the same way as above, we obtain

$$\frac{M^{1+1/\beta} d^{1+1/\beta-\alpha_k/(1+\beta)}}{2d^{\alpha_k}(2\tilde{C}_3)^{1/\beta}} \le 2\pi,$$

i.e., $M \le \tilde{C}_4 d^{-1+\beta\alpha_k/(1+\beta)+\beta\alpha_k/(1+\beta)^2}$. Iterate N times,

$$M \le \tilde{C}_{2N} d^{-1+\beta\alpha_k((1+\beta)^{-1}+(1+\beta)^{-2}+\dots+(1+\beta)^{-N})} = \tilde{C}_{2N} d^{-1+\alpha_k(1-(1+\beta)^{-N-1})}.$$

Take $N = 1 + \lfloor \log_{1+\beta} \frac{\alpha_k}{\alpha_k-1} \rfloor$, then $\gamma_k = (1-(1+\beta)^{-N-1})\alpha_k - 1 > 0$, and $|\nabla\eta(x)| \le \tilde{C}_{2N} d^{\gamma_k}$ for any x with $\operatorname{dist}\{x, \Omega_k\} \le r_1/2$. Thus as x approaches Ω_k, $|\nabla\eta(x)|$ approaches 0 with the order of $d_k^{\gamma_k}(x)$. The above argument is held for all x close to $\Omega_k, k = 1, 2, \dots, n$, hence as x approaches Ω_H, $|\nabla\eta(x)|$ approaches 0. $\square$

Similar to Properties (i)–(vii) of η_k, $k = 1, 2, \dots, n$, we have the following results.

Lemma 4.2. *η_0 has the following properties:*

- (i) $0 \le \eta_0 \le 1$ *in* Ω
- (ii) $\eta_0 \in C^0(\overline{\Omega}) \cap V$
- (iii) *Assume* η_0^k *is the value of* η_0 *on* Ω_k, *then* $\eta_0^j \ne 1$, *for* $1 \le k \le n$.
- (iv) $\eta_0 \ne 0$ *a.e..* $\eta_0 \ne 1$ *a.e.; i.e.* $\eta_0^j \ne 0$ *for* $1 \le k \le n$.
- (v) *For any subdomain* G_k *with* $G_k \cap \overline{\Omega}_H = \overline{\Omega}_k$, $\eta_0^k < \sup_{G_k} \eta_0$, *for* $1 \le k \le n$.
- (vi) *If* $\alpha_k > 1$, $k = 1, 2, \dots, n$, *then* $\eta_0 \in C^1(\overline{\Omega})$.

Proof. (i) can be proved by using test functions $(\eta_0 - 1)_+ = \max\{\eta_0 - 1, 0\}$ and $f = \min\{\eta_0, 0\}$ for (4.1) respectively.

The proof of (ii) is the same as the proof of Property (iv) above.

Using test functions from $C_0^1(\Omega \setminus \overline{\Omega}_H)$ in (4.1), we get

$$\begin{gathered} -\nabla\cdot a^{-1}\nabla\eta_0 + \eta_0 = 0 \quad \text{in } \Omega\setminus\overline{\Omega}_H, \\ \eta_0 = 1 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.10}$$

Fix σ such that $\Omega_k^\sigma \cap \Omega_H = \Omega_k$. If $\eta_0^k = 1$ for some k, let $c_k = \operatorname{ess\,sup}_{\Omega_k^\sigma\setminus\Omega_k^{\sigma/2}} \eta_0$, then $c_k < 1 = \operatorname{ess\,sup}_\Omega \eta_0$ by the maximum principle for (4.10). Use $f = \chi_k^\sigma(\eta_0 - c_k)_+ \not\equiv 0$ as a test function in (4.1),

$$0 = \int_{\Omega_k^\sigma\setminus\overline{\Omega}_k} a^{-1}|\nabla f|^2\,dx + \int_{\Omega_k^\sigma} f\eta_0 dx > 0,$$

a contradiction, so that $\eta_0^k < 1$ for $1 \le k \le n$. (iii) is showed.

Applying the maximum principle for (4.10), η_0 can not achieve the maximum value in $\Omega\setminus\overline{\Omega}_H$, combine with (iii), then $\eta_0 \ne 1$ a.e..

To show $\eta_0 \ne 0$ a.e., we use the same idea as in the proof of Property (v) to show $\operatorname{meas}\{\Gamma_0^j\} = 0$, where $\Gamma_0^j = \{x \in \overline{\Omega} | \eta_0(x) = 0\}$. Let $\Gamma_\delta^j\{x \in \overline{\Omega}|\eta_0(x) \le \delta\}$, then $\Gamma_0^j \subset \Gamma_\delta^j$, and $\operatorname{meas}\{\Gamma_\delta^j \setminus \Gamma_0^j\} \to 0$ as $\delta \to 0$. By $\eta_0 = 1$ on $\partial\Omega$, $\Gamma_\delta^j \cap \partial\Omega = \emptyset$ for all $0 \le \delta < 1$. Use $f_\delta = -\chi_{\Gamma_\delta^j}(\delta - \eta_0)_+ \in H_0^1(\Omega) \cap V$ as a test function in (4.1), then

$$\int_{\Gamma_\delta^j\setminus\overline{\Omega}_H} a^{-1}|\nabla f_\delta|^2\,dx + \int_{\Gamma_\delta^j} \eta_0 f_\delta dx = 0, \text{ i.e., } \int_{\Gamma_\delta^j\setminus\overline{\Omega}_H} a^{-1}|\nabla f_\delta|^2\,dx = \int_{\Gamma_\delta^j} \eta_0 |f_\delta|\,dx\,.$$

By the Sobolev embedding,

$$\begin{aligned}\int_{\Gamma_\delta^j} |f_\delta|^2\,dx &\le c(\Omega)\int_{\Gamma_\delta^j} |\nabla f_\delta|^2\,dx\\ &\le c(a,\Omega)\int_{\Gamma_\delta^j\setminus\overline{\Omega}_H} a^{-1}\,|\nabla f_\delta|^2\,dx\\ &\le c(a,\Omega)\int_{\Gamma_\delta^j} \eta_0\,|f_\delta|\,dx\,.\end{aligned}$$

Therefore,

$$\delta^2\operatorname{meas}\{\Gamma_0^j\}\le\int_{\Gamma_\delta^j\setminus\Gamma_0^j}|f_\delta|^2\,dx\le c(a,\Omega)\delta^2\operatorname{meas}\{\Gamma_\delta^j\setminus\Gamma_0^j\};$$

i.e., $\operatorname{meas}\{\Gamma_0^j\}\le c(a,\Omega)\operatorname{meas}\{\Gamma_\delta^j\setminus\Gamma_0^j\}\to 0$, as $\delta\to 0$. Therefore, $\operatorname{meas}\{\Gamma_0^j\}=0$ and (iv) is proved.

The proof of (v) is the same as the proof of Property (vi) above. To prove (vi), use $(T-\eta_0)_+$ as the test function in (4.1). The rest is similar to the proof of Property (vii) above, use the coarea formula, elliptic estimates and iteration to obtain the desired result. □

Proof of Theorem 4.1. The representation of the h_D follows from the linearity of (3.3) and its uniqueness of solution. The regularity of h_D follows from the regularity of η_0 and η_j, $j=1,2,\dots,n$. □

5. Consequence of the Limit Problem

In this section, we follow [4] closely to give a few applications of the limit problem. Set

$$\begin{gathered}a_{ij}=\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla\eta_i\cdot\nabla\eta_j dx+\int_\Omega \eta_i\eta_j dx, 1\le i,j\le n,\\ b_j=\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla\eta_0\cdot\nabla\eta_j dx+\int_\Omega (\eta_0-1)\eta_j dx,\quad 1\le j\le n,\\ b_0=\int_{\Omega\setminus\overline{\Omega}_H} a^{-1}\nabla\eta_0\cdot\nabla\eta_0 dx+\int_\Omega (\eta_0-1)^2 dx,\end{gathered}$$

and apply the same argument as [4, Theorem 3.4] (also see [3, Lemma 2.2]). We can represent the local minimum energy of J_0 as the follows.

Lemma 5.1. *Fix h_e. If (ψ_D,A_D) minimizes J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$, then*

$$J_0(\psi_D,A_D)=\int_\Omega|\nabla\sqrt{a}|^2dx+\sum_{i=1}^n\sum_{j=1}^n a_{ij}d_id_j+2\sum_{j=1}^n b_jd_jh_e+b_0h_e^2\,. \tag{5.1}$$

For a minimizing sequence of J_ϵ in $\mathcal{M}$, we also prove the following result.

Theorem 5.2. *Fix h_e and a sequence $\epsilon_k\to 0^+$ as $k\to\infty$. Let $(\psi_{\epsilon_k},A_{\epsilon_k})$ minimize J_{ϵ_k} in $H^1(\Omega;\mathbb{C})\times H^1(\Omega;\mathbb{R}^2)$ under gauge (1.8), then $|\psi_{\epsilon_k}|\to\sqrt{a}$ in $C(\overline{\Omega})$, and there is a subsequence $(\psi_{\epsilon_{k_\ell}},A_{\epsilon_{k_\ell}})\to(\psi_D,A_D)$ in $\mathcal{M}$ as $\ell\to\infty$, where (ψ_D,A_D) is a minimizer of J_0 in $\mathcal{M}_0$, and $(\psi_D,A_D)\in H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ for some $D\in\mathbb{Z}^n$. Consequently, $J_{\epsilon_{k_\ell}}(\psi_{\epsilon_{k_\ell}},A_{\epsilon_{k_\ell}})\to J_0(\psi_D,A_D)$ as $\ell\to\infty$, moreover, for any $0<$*

$\sigma < r_1$ and ℓ sufficiently large, $|\psi_{\epsilon_{k_\ell}}|$ is uniformly positive outside $\bigcup_{j=1}^n \overline{\Omega}_j^\sigma$, and the degree of $\psi_{\epsilon_{k_\ell}}$ around $\overline{\Omega_j^\sigma}$ is $d_j, j = 1, 2, \dots, n$.

Proof. Use the same argument as [4, Theorem 4.2], we can prove the first part of the theorem; the second part follows from the definition of the degree in (2.3) and the fact that $(\psi_{\epsilon_{k_\ell}}, A_{\epsilon_{k_\ell}}) \to (\psi_D, A_D)$ in $\mathcal{M}$ as $k_\ell \to \infty$. □

From Theorem 5.2, for sufficiently small ϵ, the vortex set of the minimizers of J_ϵ in $\mathcal{M}$ is forced to close the zero set of $a(x)$, by zero set of $a(x)$ corresponds to the normal impurities in the inhomogeneous superconductor, the vortices of the minimizers of J_ϵ is pinned near the normal impurities, which verifies the effectiveness of the pinning mechanism by adding normal impurities to a superconductor.

Proof of Theorem 1.3. By Theorem 2.3, $H^1_{a,D}$ is both open and closed in H^1_a, we can always find $r > 0$ sufficiently small, such that $\mathcal{B}_r \cap \mathcal{M}_0 = \mathcal{B}_r \cap [H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)]$. Apply the same argument as in [4, Theorem 4.6], we derive Theorem 1.3. □

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6. Appendix

In this part, we list some lengthy proofs omitted in Section II.

Proof of Lemma 2.2. First we can find a sequence of nested C^∞ domains to approximate Ω_H, say they are

$$\Omega_H^1 \supset\supset \Omega_H^2 \supset\supset \dots \supset\supset \Omega_H \quad \text{with } \operatorname{dist}\{\partial\Omega_H^m, \partial\Omega_H\} \to 0, \text{ as } m \to \infty.$$

Then by the proposition of Schoen and Uhlenbeck [23, page 267] (also see [8, Lemma A.11 page 244]), there exists a sequence $\{v^m\}_{m=1}^\infty$, such that

$$v^m \in C^2(\overline{\Omega} \setminus \Omega_H^m; S^1) \cap H^1_{\rm loc}(\Omega \setminus \overline{\Omega}_H), v^m \to v \text{ in } H^1_{\rm loc}(\Omega \setminus \overline{\Omega}_H), \text{ as } m \to \infty. \quad (6.1)$$

From (2.3), for any G_j satisfying (2.1), f_{G_j} satisfying (2.2), the degree of v^m on ∂G_j will converge to the degree of $v = u/\sqrt{a}$ on ∂G_j, i.e. for all m sufficiently large, we have

$$d_j = \deg(v^m, \partial G_j) = \frac{1}{\pi}\int_{G_j \setminus \overline{\Omega}_j} J(v^m f_{G_j})dx = \frac{1}{2\pi i}\int_{\partial G_j} (v^m)^*(v^m)_\tau \, ds\,. \quad (6.2)$$

Here τ is the counterclockwise tangent vector field of ∂G_j, and the most right-hand side is derived by using integral by part, it is the standard definition of the degree (winding number) of C^1 function on ∂G_j. We define a sequence of real two-dimensional vector fields by

$$F^m(x) = -\sum_{j=1}^n d_j \nabla\theta_j + i(v^m)^*\nabla v^m, \quad m = 1, 2, \dots, \ x \in \Omega \setminus \overline{\Omega}_H^m. \quad (6.3)$$

Note that $\nabla\theta_j$ is a single-valued smooth vector field on $\Omega\setminus\overline{\Omega}_H$, and $\int_{\partial G_j} \nabla\theta_j \cdot \tau \, ds = 2\pi$, $1 \le j \le n$, for any G_j as in (2.1). From (6.2), $\oint_C F^m \cdot \tau ds = 0$ for any closed curve $C \subset\subset \Omega \setminus \overline{\Omega}_H$, and all m sufficiently large with $C \subset \Omega \setminus \overline{\Omega}_H^m$. Hence there exists a $\phi^m \in H^1(\Omega \setminus \overline{\Omega}_H^m)$, such that $\nabla\phi^m = F^m$ in $\Omega \setminus \overline{\Omega}_H^m$ for all m

sufficiently large. Use (6.3), then $v^m\nabla\phi^m = -v^m\sum_{j=1}^n d_j\nabla\theta_j + i\nabla v^m$. As a result, $v^m(x) = e^{i\phi^m(x)}\cdot e^{i\sum_{j=1}^n d_j\theta_j(x)} = e^{i[\phi^m(x)+\sum_{j=1}^n d_j\theta_j(x)]}$.

Since $v^m \to v = u/\sqrt{a}$ in $H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, as $m\to\infty$, $\nabla\theta_j \in C^\infty(\overline{\Omega}\setminus\Omega_H)$, it follows that $e^{i\phi^m} \equiv v^m\cdot e^{-i\sum_{j=1}^n d_j\theta_j(x)} \to v\cdot e^{-i\sum_{j=1}^n d_j\theta_j(x)}$ in $H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, as $m\to\infty$, and $\nabla\phi^m \to -\sum_{j=1}^n d_j\nabla\theta_j + iv^*\nabla v$ in $L^2_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, as $m\to\infty$. It follows that, after possibly subtracting constants $2\pi k_m$, $k_m\in\mathbb{Z}, m = 1,2,\dots$, $\phi^m\to\phi$ in $H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, as $m\to\infty$, where $\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, and $v = e^{i(\phi+\sum_{j=1}^n d_j\theta_j)}$ in $H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$. Setting $\Theta(x) = \phi(x) + \sum_{j=1}^n d_j\theta_j$, then $u = \sqrt{a}v = \sqrt{a}e^{i\Theta(x)}$. By $|\nabla\theta_j| \le C(\Omega_H)$, $\quad 1\le j\le n$,

$$\begin{aligned}\int_{\Omega\setminus\overline{\Omega}_H} a|\nabla\phi|^2 =&\le \int_{\Omega\setminus\overline{\Omega}_H}|\nabla u|^2 + C\int_{\Omega\setminus\overline{\Omega}_H} a|\sum_{j=1}^n d_j\nabla\theta_j|^2\\ &\le \int_{\Omega\setminus\overline{\Omega}_H}|\nabla u|^2 + C(\Omega_H, D, a)\,.\end{aligned}$$

To show $\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$ is unique (up to an additive constant $2\pi k$, $k\in\mathbb{Z}$). Assume $\tilde\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, satisfying $u = \sqrt{a}e^{i(\tilde\phi+\sum_{j=1}^n d_j\theta_j)}$, then $e^{i(\phi-\tilde\phi)} = 1$ in $H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, with $\phi-\tilde\phi\in H^1_{\rm loc}(\Omega\setminus\overline{\Omega}_H)$, so $\phi-\tilde\phi = 2\pi k$, for some $k\in\mathbb{Z}$. $\square$

Proof of Lemma 2.5. Our proof is standard. We first construct a family of functions in V to approximate a given $f\in V$, then use a mollifier to smooth them, and apply the diagonal rule to finish the proof. Take $\sigma < r_1$, so that $\Omega_j^\sigma\cap\Omega_k^\sigma = \emptyset, k\ne j, 1\le k,j\le n$. Let

$$\alpha(r) = \begin{cases}1, & 0\le r\le\frac12\\ 2-2r, & \frac12 < r\le 1\\ 0, & r>1\,.\end{cases}$$

Then $|\frac{d}{dr}\alpha|\le 2$ If $x = (\tilde x, x_d)\in\mathcal{U}_k^j\setminus\overline{\Omega}_j$ represented in the local coordinate system of $\mathcal{U}_k^j$, define the shift of x away from $\overline{\Omega}_j$ in sense of the local coordinate as

$$m_\sigma^{kj}(x) = x + \alpha(\frac{\phi_k^j(\tilde x) - x_d}{\sigma})((\tilde x, \phi_k^j(\tilde x)) - x),$$

it can be verified that $\left|\nabla m_\sigma^{kj}(x)\right| \le c(\Omega_H)$ a.e. in $\mathcal{U}_k^j\setminus\overline{\Omega}_j$.

For any $f\in V,\sigma$ small enough, define

$$f_\sigma(x) = \begin{cases}\sum_{k=1}^{n_j}\beta_k^j(x)f(m_\sigma^{kj}(x)) & \text{for } x\in\Omega_j^\sigma\subset\Omega_j^{r_1}, j = 1,2,\dots,n,\\ f(x) & \text{for other } x,\end{cases}$$

where $\{\beta_k^j\}_{k=1}^{n_j}$ is the partition of unity from (2.7). Clearly $f_\sigma\in H^1(\Omega)$, we verify $f_\sigma\in V$. For $x\in\Omega_j^{\sigma/2g}$, by (2.6), $\left|\phi_k^j(\tilde x) - x_d\right| \le gd(x)\le\sigma/2$, $m_\sigma^{kj}(x) = (\tilde x, \phi_k^j(\tilde x))\in\partial\Omega$, so that $f_\sigma(x) = f_j =$ constant, and $\nabla f_\sigma = 0$; from (1.4), $a(x) > C_0(\sigma/2g)^{\alpha_j}$ for $x\in\Omega\setminus\Omega_j^{\sigma/2g}$, then $f_\sigma\in V$. Moreover, calculate in the local coordinates, and sum together the difference of f and f_σ in $V-$norm, we obtain,

$$\|f_\sigma - f\|_V^2 \le C(\Omega_H)\int_{\Omega^\sigma\setminus\overline{\Omega}_H}(a^{-1}\left|\nabla f\right|^2 + f^2)dx\,.$$

Since meas$\{\Omega^\sigma\setminus\overline{\Omega}_H\}\to 0$, as $\sigma\to 0$, we have $\|f_\sigma - f\|_V\to 0$, as $\sigma\to 0$.

Choose a mollifier

$$\rho \in C_0^\infty(B_1(0)) \cap C_0^\infty(\mathbb{R}^d), \quad \rho \geq 0, \quad \text{and} \int_{B_1(0)} \rho(x)dx = 1, \tag{6.4}$$

and set $\rho_m(x) = m^d \rho(mx)$, where $B_1(0)$ is the unit ball in $\mathbb{R}^d$. Then for $m > 6g/\sigma$, $\rho_m * f_\sigma \in C^\infty(\Omega) \cap V$, and $\rho_m * f_\sigma = f_j$ in $\overline{\Omega}_j^{\sigma/3g}$.

From (1.4), $|a(x)| \geq C_0(\sigma/3g)^{\alpha_j} \geq C(\overline{\Omega}_H, \sigma) > 0$ in $\Omega \setminus \overline{\Omega}^{\sigma/3g}$, we obtain

$$\|f_\sigma - \rho_m * f_\sigma\|_V^2 \leq C(\overline{\Omega}_H, \sigma)\|f_\sigma - \rho_m * f_\sigma)\|_{H^1}^2 .$$

Hence by $\rho_m * f_\sigma \to f_\sigma, m \to \infty$ in H^1, we have $\rho_m * f_\sigma \to f_\sigma, m \to \infty$ in V. Apply the diagonal rule, pick up the $\rho_{m_\sigma} * f_\sigma \in C^\infty(\Omega) \cap V$, such that
$\rho_{m_\sigma} * f_\sigma \to f$ in V, as $\sigma \to 0$. $\square$

References

[1] A. Aftalion, S. Alama and L. Bronsard. *Giant Vortex and the breakdown of strong pinning in a rotating Bose-Einstein condensate.* preprint (2004).

[2] A. Aftalion, E. Sandier and S. Serfaty. *Pinning Phenomena in the Ginzburg-Landau Model of Superconductivity.* J. Math. Pures Appl.(9), 80(2001), No.3, 339-372.

[3] S. Alama and L. Bronsard. *Vortices and pinning effects for the Ginzburg-Landau model in multiply connected domains.* preprint (2004).

[4] N. Andre, P. Bauman and D. Phillips. *Vortex pinning with bounded fields for the Ginzburg-Landau equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 20(2003) No.4, 705-729.

[5] F. Bethuel, H. Brezis and F. Hélein *Ginzburg-Landau vortiecs.* Progress in Nonlinear Differential Equations and their Applications,13. Birkhäusser, Boston MA, 1994.

[6] F. Bethuel and T. Riviére, *Vortices for a variational problem related to superconductivity,* Ann. Inst. H. Poincar Anal. Non Linaire 12 (1995), no. 3, 243–303.

[7] H. Brezis *The interplay between analysis and topology in some nonlinear PDE problems .* Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 2, 179–201.

[8] H. Brezis and L. Nirenberg. *Degree theory and BMO; part I: Compact manifolds without boundaries.* Selecta Math. (N. S.) 1(1995), No. 2, 197-263.

[9] H. Brezis and L. Nirenberg. *Degree theory and BMO; part II: Compact manifolds with boundaries.* Selecta Math. (N. S.) 2(1996), No. 3, 309-368.

[10] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice. *A boundary value problem related to the Ginzburg-Landau model.* Comm. Math. Phys. 142(1991), 1-23.

[11] S. J. Chapman and G. Richardson. *Vortex pinning by inhomogeneities in type-II superconductors.* Phys. D 108 (1997), no. 4, 397–407.

[12] J. Du and M. D. Gunzburger. *A model for superconducting thin films having variable thickness.* Phys. D 69 (1993), no. 3-4, 215–231.

[13] J. Du, M. D. Gunzburger and J.S. Peterson. *Analysis and approximation of the Ginzburg-Landau model of superconductivity.* SIAM Rev. 34 (1992), no. 1, 54–81.

[14] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* CRC Press, Boca Raton, FL, 1992.

[15] T. Giorgi and D. Phillips. *The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model.* SIAM J. Math. Anal. 30 (1999) no. 2, 341-359.

[16] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer 1998.

[17] K. Likharev. *Superconducting weak links.* Rev. Mod. Phys., 51, (1979), 101-159.

[18] F. Lin and Q. Du. *Ginzburg-Landau vortices: dynamics, pinning, and hysteresis.* SIAM J. Math. Anal. 28 (1997), no. 6, 1265–1293.

[19] S. Jimbo and Y. Morita. *Ginzburg-Landau equation with magnetic effect: non-simply-connected domains.* SIAM J. Math. Anal. 27(1996) No. 5, 1360-1385.

[20] J. Rubinstein and P. Sternberg. *Homotopy classification of minimizers of the Ginzburg-Landau energy and the existence of permanent currents.* Comm. Math. Phys. 179 (1996), no. 1, 257-263.

[21] E. Sandier and S. Serfaty. *Global Minimizers for the Ginzburge-Landau Functional Below the First Critical Magnetic Field* Ann. Inst. H. Poincaré Anal. Non Linéaire, 17(2000) No.1, 119-145

[22] E. Sandier and S. Serfaty. *On the energy of type-II superconductors in the mixed phase.* Rev. Math. Phys. 12 (2000), no. 9, 1219–1257.

[23] R. Schoen and K. Uhlenbeck *Boundary regularity and the Dirichlet problem for harmonic maps.* J. Differential Geom. 18(1983), No.2, 253-268.

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