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GROUND STATE SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATION WITH SINGULAR COEFFICIENTS

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Abstract. In this article, we study the quasilinear Schrödinger equation with the critical exponent and singular coefficients,

$$-\Delta u + V(x)u - \Delta(|u|^2)u = \lambda\frac{|u|^{q-2}u}{|x|^\mu} + \frac{|u|^{22^*(\nu)-2}u}{|x|^\nu} \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $2 < q < 22^*(\mu)$, $2^*(s) = \frac{2(N-s)}{N-2}$, and λ, μ, ν are parameters with $\lambda > 0$, $\mu, \nu \in [0, 2)$. By applying the Mountain Pass Theorem and the Concentration Compactness Principle, we establish the existence of the ground state solutions to the above problem.

1. Introduction

In this article we study the existence of ground state solutions of the quasilinear Schrödinger equations with singular coefficients,

$$-\Delta u + V(x)u - \Delta(|u|^2)u = \lambda\frac{|u|^{q-2}u}{|x|^\mu} + \frac{|u|^{22^*(\nu)-2}u}{|x|^\nu} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $2 < q < 22^*(\mu)$, $2^*(s) = \frac{2(N-s)}{N-2}$, and λ, μ, ν are parameters with $\lambda > 0$, $\mu, \nu \in [0, 2)$. The corresponding energy functional for (1.1) is

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V|u|^2dx \\ &\quad - \frac{\lambda}{q}\int_{\mathbb{R}^N}\frac{|u|^q}{|x|^\mu}dx - \frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|u|^{22^*(\nu)}}{|x|^\nu}dx. \end{aligned} \tag{1.2}$$

Throughout this article, we assume that potential satisfies:

- (A0) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ with $\inf V(x) = V_0 > 0$, and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

By the ground state solution of (1.1), we mean that $u \neq 0$ and its energy is minimal among the energy of all nontrivial solutions to (1.1).

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The question addressed in this paper is motivated by analogous results for the ground state solutions of the Schrödinger equation

$$i\partial_t\phi = -\Delta\phi + W(x)\phi - f(|\phi|^2)\phi - k\Delta h(|\phi|^2)h'(|\phi|^2)\phi, \tag{1.3}$$

where $\phi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, k is a real constant and $f, h : \mathbb{R}^+ \to \mathbb{R}$ are suitable functions. In particular, we consider the model with $h(s) = s$,

$$f(s) = \sqrt{\lambda\frac{|s|^{q-2}}{|x|^\mu} + \frac{|s|^{22^*(\nu)-2}}{|x|^\nu}},$$

$W(x) = V(x) + \beta$. A stationary equation of the desired form is obtained by considering standing wave solutions, $\phi(t, x) = \exp(-i\beta t)u(x)$. Once we substitute the formula of standing wave solutions into (1.3) with the special choices of $h(s)$, $f(s)$ and $W(x)$ as pointed above, we can immediately obtain (1.1). According to this substitution, u is a solution of (1.1) if and only if ϕ is standing wave solution to (1.3).

Schrödinger equations of this type have appeared in many physical models. It can be used to describe different physical phenomena due to the variety of the nonlinear term h. When $h(s) = s$, (1.3) was used to discuss the time evolution of the condensate wave function of super-fluid film equation in plasma physics [14, 16]. When $h(s) = (1 + s)^{1/2}$, (1.3) models the self-channeling of a high-power ultra short laser in matter, see [4, 8, 10, 22]. Equation (1.3) also appears in the theory of Heisenberg ferromagnets and magnons [13] and in condensed matter theory [20]. For further physical backgrounds and applications, we refer readers to [6, 3, 17, 21] and references therein.

Poppenberg-Schmitt-Wang [21] considered the eigenvalue problem

$$-\Delta u + V(x)u - (\Delta|u|^2)u = \lambda|u|^{q-2}u, \tag{1.4}$$

with bounded potential $V(x)$ and $q > 2$, $\lambda > 0$. They showed the existence of positive ground state solutions for one dimensional case via the constrained variational method. For the same equation mentioned in (1.4), Liu-Wang-Wang [18] also proved the existence of positive solution with unbounded/bounded/periodic potential $V(x)$ when $4 < q < 22^*$ and $\lambda > 0$ by a change of variables and an Orlicz space, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. In a recent work of do Ó-Miyagaki-Soares [11], the critical exponent problem is studied for the following quasilinear equation

$$-\Delta u + V(x)u - (\Delta|u|^2)u = |u|^{22^*-2}u + |u|^{q-2}u, \tag{1.5}$$

where $4 < q < 22^*$, 2^* is again the critical Sobolev exponent, $N \geq 3$. Applying a change of variables and the Moutain Pass Theorem, they showed that there is a positive solution for (1.5) with bounded/periodic potential $V(x)$. Liu-Liu-Wang [15] extended the method in [11] to investigate the more general Schrödinger equations with critical growth

$$\begin{aligned}
&-\sum_{i,j=1}^N D_j(a_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^N D_sa_{ij}(u)D_iuD_ju + V(x)u \\
&= |u|^{22^*-2}u + |u|^{q-2}u,
\end{aligned} \tag{1.6}$$

where $4 < q < 22^*$, $N \geq 3$. It is easy to see that (1.6) can be transferred into (1.5) with $a_{ij}(u) = (1 + 2u^2)\delta_{i,j}$. They also obtained the existence for positive solution with bounded potential well by the Nehari method. Later on, via classical variation techniques, Wu-Zhou [25] improved the results of [11] with unbounded potential $V(x)$, and they relax the restriction $q > 4$ to $q > 2$. Moreover, Bae-Choi-Pahk [2] studied the existence of nodal radial solutions to the elliptic equations:

$$-\Delta u = \lambda \frac{|u|^{q-2}u}{|x|^{\mu}} + \frac{|u|^{2^*(\nu)-2}u}{|x|^{\nu}} \quad \text{in } B_1, \tag{1.7}$$

with $\mu, \nu > -2$, $\lambda > 0$. The singular terms addressed in our model are similar to the one in (1.7). Therefore, motivated by [2] and [25], we notice that the existence of ground state solutions for (1.1) depends not only on the range of q, but also the parameter λ, for the case when $\mu \neq 0$ and $\nu \neq 0$. The main theorem of our paper is as follows:

Theorem 1.1. *Let q and λ be positive parameters, for every fixed μ, $\nu \in [0, 2)$, we have the following statements:*

- (ii) *if $\frac{2(N+2-2\mu)}{N-2} < q < 22^*(\mu)$, there exists a ground state solution of* (1.1) *for any $\lambda > 0$;*
- (ii) *if $2 < q \leq \frac{2(N+2-2\mu)}{N-2}$, there exists a constant $\lambda^* > 0$, such that for $\lambda > \lambda^*$,* (1.1) *has a ground state solution.*

We now briefly mention the main difficulties of this problem. As observed in [19], for each fixed ν, the number $22^*(\nu)$ behaves like a critical exponent for the embedding $X \hookrightarrow L^{22^*(\nu)}(\mathbb{R}^N, |x|^{-\nu})$, where $X = \{u \in H^1(\mathbb{R}^N) :\ u^2 \in H^1(\mathbb{R}^N), \sqrt{|V|}u \in L^2(\mathbb{R}^N)\}$ is the domain of the energy functional corresponding to (1.1). The main tool of this problem is a variant version of the Mountain Pass Theorem [23], which introduced a so-called Cerami sequence. We denote it as $(C)_c$ sequence for convenience. The action of the $(C)_c$ sequence in the modified Mountain Pass Theorem is similar to the Palais-Smale sequence in the classical Mountain Pass Theorem [1]. Due to lack of compactness of the embedding $X \hookrightarrow L^{22^*(\nu)}(\mathbb{R}^N, |x|^{-\nu})$, it complicates the process of verifying the existence and nonvanishing of the weak limit of a $(C)_c$ sequence. On the other hand, X is not even a vector space, which causes that the usual variation techniques cannot be applied directly. Therefore, the choice of a suitable function space is also important for our discussion.

The plan of this paper is as follows: Section 2 states some preliminary results and establishes the Mountain Pass geometry structure, Section 3 covers the compactness of $(C)_c$ sequence. Section 4 is devoted to the proof of Theorem 1.1.

In what follows, C denotes the universal positive constant unless specified, $L^q(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm $\|u\|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{1/q}$, $1 \leq q < \infty$.

2. Preliminaries

Let

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

be endowed with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}^N} \big(\nabla u \nabla v + uv\big) dx,$$

and the norm $\|u\|_H^2 = \langle u, u\rangle_H$. We define

$$E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \big[\nabla u \nabla v + V(x)uv\big] dx,$$

and the associated norm $\|u\|^2 = \langle u, u\rangle$. It is easy to see that both $H^1(\mathbb{R}^N)$ and E are Hilbert spaces. Fortunately, thanks to [7] and [26], we can obtain the following lemma.

Lemma 2.1. *Let $0 \le \sigma < 2$. The embedding $E \hookrightarrow L^s(\mathbb{R}^N, |x|^{-\sigma})$ is continuous for $2 \le s \le 2^*(\sigma)$ and compact for $2 \le s < 2^*(\sigma)$ when $V(x)$ satisfies the condition (A0).*

Proof. By [7] and [26], the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $2 \le p \le 2^*$ and compact for $2 \le p < 2^*$ under the condition (A0).

Let $u_n \to 0$ in E as $n \to \infty$. By the Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u_n|^s}{|x|^\sigma} dx &= \int_{\mathbb{R}^N} \frac{|u_n|^\sigma}{|x|^\sigma} \cdot |u_n|^{s-\sigma} dx \\
&\le \Big(\int_{\mathbb{R}^N} \frac{u_n^2}{|x|^2} dx\Big)^{\sigma/2} \Big(\int_{\mathbb{R}^N} |u_n|^{\frac{2(s-\sigma)}{2-\sigma}} dx\Big)^{\frac{2-\sigma}{2}}.
\end{aligned} \tag{2.1}$$

Since $2 \le s \le 2^*(\sigma)$ and $0 \le \sigma < 2$, it follows that $2 \le \frac{2(s-\sigma)}{2-\sigma} \le 2^*$. Hence, applying the Hardy's and Sobolev inequalities to (2.1), we can obtain $\int_{\mathbb{R}^N} \frac{|u_n|^s}{|x|^\sigma} dx \le C\|u_n\|^s$, which gives that $u_n \to 0$ in $L^s(\mathbb{R}^N, |x|^{-\sigma})$ for $2 \le s \le 2^*(\sigma)$ as $n \to \infty$. Therefore, the first part of this lemma is proved.

Secondly, let $\{u_n\}$ be a bounded sequence in E. It is clear that, up to a subsequence, $u_n \rightharpoonup u$ in E as $n \to \infty$. Employing the Hölder's and Hardy's inequalities, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|u_n - u|^s}{|x|^\sigma} dx &= \int_{\mathbb{R}^N} \frac{|u_n - u|^\sigma}{|x|^\sigma} \cdot |u_n - u|^{s-\sigma} dx \\
&\le \Big(\int_{\mathbb{R}^N} \frac{|u_n - u|^2}{|x|^2} dx\Big)^{\sigma/2} \Big(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(s-\sigma)}{2-\sigma}} dx\Big)^{\frac{2-\sigma}{2}} \\
&\le \|u_n - u\|^\sigma \Big(\int_{\mathbb{R}^N} |u_n - u|^{\frac{2(s-\sigma)}{2-\sigma}} dx\Big)^{\frac{2-\sigma}{2}}.
\end{aligned} \tag{2.2}$$

Since $2 \le s < 2^*(\sigma)$ and $0 \le \sigma < 2$, we have $2 \le \frac{2(s-\sigma)}{2-\sigma} < 2^*$. It then follows that $u_n \to u$ in $L^{\frac{2(s-\sigma)}{2-\sigma}}(\mathbb{R}^N)$. Therefore, we can obtain $u_n \to u$ in $L^s(\mathbb{R}^N, |x|^{-\sigma})$ for $2 \le s < 2^*(\sigma)$ from (2.2). Thus $E \hookrightarrow L^s(\mathbb{R}^N, |x|^{-\sigma})$ is compact for $2 \le s < 2^*(\sigma)$. We complete our proof. □

The purpose of this section is to establish the variational structure of (1.1). And the main difficulty arises from the function space where the energy functional (1.2) is not well defined. To overcome this difficulty, and motivated by [18] and [9], we define a $\mathcal{C}^\infty$ function $f(t)$ as below:

$$f(-t) = -f(t) \text{ on } (-\infty, 0], \quad f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \text{ on } [0, +\infty).$$

We also need some properties on f.

Proposition 2.2. *The function $f(t)$ has the following properties:*

(A1) *f is a uniquely defined, invertible $\mathcal{C}^\infty$-function;*
(A2) *$0 < f'(t) \le 1$ for all $t \in \mathbb{R}$;*
(A3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
(A4) *$|f(t)| \le 2^{\frac{1}{4}}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
(A5) *There exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}$$

(A6) *$\frac{f(t)}{2} \le tf'(t) \le f(t)$ for $t \ge 0$;*
(A7) *$|f(t)f'(t)| \le \frac{1}{\sqrt{2}}$ for all $t \in \mathbb{R}$.*

We observe that a direct calculation implies that $\frac{f(t)}{t}$ is decreasing for $t > 0$ from (A6). The proof of this proposition may be found in [9], [18, 11, 12].

After using the same change of variables $v = f^{-1}(u)$ as in [18] and [9], and the definition of f mentioned above, $I(u)$ can be transferred into a new functional $J(v)$:

$$\begin{aligned} J(v) = \frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla v|^2 + V(x)f^2(v)\Big]dx - \frac{\lambda}{q}\int_{\mathbb{R}^N}\frac{|f(v)|^q}{|x|^\mu}dx \\ - \frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(v)|^{22^*(\nu)}}{|x|^\nu}dx, \end{aligned} \tag{2.3}$$

which is well defined on E. Moreover, a standard argument shows that $J \in \mathcal{C}^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'(v), \varphi\rangle = \int_{\mathbb{R}^N}\Big[\nabla v\nabla\varphi + V(x)f(v)f'(v)\varphi \\ - \lambda\frac{|f(v)|^{q-2}}{|x|^\mu}f(v)f'(v)\varphi - \frac{|f(v)|^{22^*(\nu)-2}}{|x|^\nu}f(v)f'(v)\varphi\Big]dx \end{aligned} \tag{2.4}$$

for all v, $\varphi \in E$. Therefore, the nontrivial critical points of the functional J are also the nontrivial weak solutions of the following equation

$$-\Delta v = f(v)f'(v)\Big[\lambda\frac{|f(v)|^{q-2}}{|x|^\mu} + \frac{|f(v)|^{22^*(\nu)-2}}{|x|^\nu} - V(x)\Big] \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

According to this change of variables (see [9, 25]), we notice that if v is a solution of (2.5), $u = f(v)$ is also a solution of (1.1).

Now show that the functional J exhibits the Mountain Pass geometry structure. Let

$$B(\rho) = \{v \in E : \int_{\mathbb{R}^N}\big[|\nabla v|^2 + V(x)f^2(v)\big]dx < \rho^2\}.$$

Lemma 2.3. *For any fixed $\mu, \nu \in [0, 2)$, the functional J satisfies*

(i) *there exist positive constants α and ρ_0, such that $J(v) \ge \alpha$ for all $v \in \partial B(\rho_0)$,*
(ii) *there exists $w \in E$ such that $J(w) < 0$.*

Proof. (i) From (A7), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla f^2(v)|^2 dx &= \int_{\mathbb{R}^N} |2f(v)f'(v)\nabla v|^2 dx\\ &\le 2\int_{\mathbb{R}^N} |\nabla v|^2 dx\\ &\le 2\int_{\mathbb{R}^N} \Big[|\nabla v|^2 + V(x)f^2(v)\Big]dx.\end{aligned} \tag{2.6}$$

For fixed $\mu \in [0,2)$, for any $\varepsilon > 0$ and $2 < q < 22^*(\mu)$, there exists a constant $C(\varepsilon) > 0$ such that $\frac{|t|^q}{|x|^\mu} \le \varepsilon \frac{|t|^2}{|x|^\mu} + C(\varepsilon)\frac{|t|^{22^*(\mu)}}{|x|^\mu}$. Thus, for any $v \in \partial B(\rho)$, by using Sobolev-Hardy inequality and (2.6), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f(v)|^{22^*(\mu)}}{|x|^\mu} dx &\le C\Big(\int_{\mathbb{R}^N} \big|\nabla(f^2(v))\big|^2 dx\Big)^{2^*(\mu)/2}\\ &\le C\Big[\int_{\mathbb{R}^N} \big(|\nabla v|^2 + V(x)f^2(v)\big)\, dx\Big]^{2^*(\mu)/2}\\ &\le C\rho^{2^*(\mu)}\,.\end{aligned} \tag{2.7}$$

Similarly,

$$\int_{\mathbb{R}^N} \frac{|f(v)|^{22^*(\nu)}}{|x|^\nu} dx \le C\rho^{2^*(\nu)}. \tag{2.8}$$

By (2.7), it follows from (A3) and Sobolev-Hardy inequality again that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f(v)|^q}{|x|^\mu} dx &\le \varepsilon \int_{\mathbb{R}^N} \frac{|f(v)|^2}{|x|^\mu} dx + C(\varepsilon)\int_{\mathbb{R}^N} \frac{|f(v)|^{22^*(\mu)}}{|x|^\mu} dx\\ &\le C\varepsilon \int_{\mathbb{R}^N} |\nabla v|^2 dx + C\cdot C(\varepsilon)\rho^{2^*(\mu)}\\ &\le C\varepsilon\rho^2 + C\cdot C(\varepsilon)\rho^{2^*(\mu)}.\end{aligned} \tag{2.9}$$

Thus, from (2.7) and (2.8), we obtain

$$\begin{aligned}J(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \Big[|\nabla v|^2 + V(x)f^2(v)\Big]dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} \frac{|f(v)|^q}{|x|^\mu} dx\\ &\quad - \frac{1}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|f(v)|^{22^*(\nu)}}{|x|^\nu} dx\\ &\ge [\frac{1}{2} - \frac{\lambda}{q}C\varepsilon]\rho^2 - \frac{\lambda}{q}C\cdot C(\varepsilon)\rho^{2^*(\mu)} - \frac{1}{22^*(\nu)}C\rho^{2^*(\nu)}\\ &\ge \frac{1}{4}\rho^2 - C\cdot C(\varepsilon)\rho^{2^*(\mu)} - C\rho^{2^*(\nu)},\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Choose $\rho_0 > 0$ with $\frac{1}{4}\rho_0^2 - C\cdot C(\varepsilon)\rho_0^{2^*(\mu)} - C\cdot\rho_0^{2^*(\nu)} =: \alpha > 0$. Then we have $J(v) \ge \alpha$ for all $v \in \partial B(\rho_0)$.

(ii) For fixed $\nu \in [0,2)$, given $\psi \in E \cap L^{22^*(\nu)}(\mathbb{R}^N)$ with $0 < \psi \le 1$, we can show that

$$J(t\psi) \to -\infty \quad \text{as } t \to \infty.$$

Indeed, since $0 < t\psi(x) \le t$ for $t > 0$, and (A6) in Proposition 2.2, it follows that

$$\frac{f(t\psi(x))}{t\psi(x)} \ge \frac{f(t)}{t} \Rightarrow f(t\psi(x)) \ge f(t)\psi(x). \tag{2.10}$$

Thus, for $t \geq 1$, by (2.10), (A3) and (A5), we have

$$
\begin{aligned}
J(t\psi) &\leq \frac{1}{2}\int_{\mathbb{R}^N}\Big[|\nabla(t\psi)|^2 + V(x)f^2(t\psi)\Big]dx - \frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(t\psi)|^{22^*(\nu)}}{|x|^\nu}dx \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^N}\Big[|\nabla\psi|^2 + V(x)\psi^2\Big]dx - \frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(t)\psi|^{22^*(\nu)}}{|x|^\nu}dx \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N}\Big[|\nabla\psi|^2 + V(x)\psi^2\Big]dx - Ct^{2^*(\nu)}\int_{\mathbb{R}^N}\frac{|\psi|^{22^*(\nu)}}{|x|^\nu}dx \\
&\to -\infty, \quad \text{as } t \to +\infty,
\end{aligned}
$$

since $2^*(\nu) > 2$ for $\nu \in [0, 2)$. This implies that there exists a t large and positive such that $w = t\psi$, $J(w) < 0$. $\square$

It is well known that the minimization problem

$$
S = \inf\Big\{\int_{\mathbb{R}^N}|\nabla v|^2dx : v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N}\frac{|v|^{2^*(\nu)}}{|x|^\nu}dx = 1\Big\}
$$

has a solution given by

$$
w_\epsilon(x) = \frac{[(N-\nu)(N-2)\epsilon]^{\frac{N-2}{4(2-\nu)}}}{[\epsilon + |x|^{2-\nu}]^{\frac{N-2}{2(2-\nu)}}}.
$$

Let $0 < R < 1$, and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N, [0, 1])$ be a smooth cut-off function, such that $\varphi(x) = 1$ for $|x| \leq R$, $0 < \varphi(x) < 1$ for $R < |x| < 2R$, and $\varphi(x) = 0$ for $|x| \geq 2R$. For any $\epsilon > 0$, it is known that $-\Delta(w_\epsilon^2) = \frac{w_\epsilon^{\frac{2(N+2-2\nu)}{N-2}}}{|x|^\nu}$ and S can be attained by w_ϵ^2. Set $u_\epsilon = \varphi w_\epsilon$. By a similar computation to that in [2, 5], we have:

$$
\begin{gathered}
\int_{\mathbb{R}^N}|\nabla(u_\epsilon^2)|^2dx = S^{\frac{N-\nu}{2-\nu}} + O\big(\epsilon^{\frac{N-2}{2-\nu}}\big), \\
\int_{\mathbb{R}^N}\frac{|u_\epsilon|^{22^*(\nu)}}{|x|^\nu}dx = S^{\frac{N-\nu}{2-\nu}} + O\big(\epsilon^{\frac{N-\nu}{2-\nu}}\big),
\end{gathered}
\tag{2.11}
$$

$$
\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2dx \leq O\big(\epsilon^{\frac{N-2}{2(2-\nu)}}|\ln\epsilon|\big), \quad \int_{\mathbb{R}^N}u_\epsilon^2dx = O\big(\epsilon^{\frac{N-2}{2(2-\nu)}}\big), \tag{2.12}
$$

$$
\int_{\mathbb{R}^N}\frac{|u_\epsilon|^q}{|x|^\mu}dx = O\big(\epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q}\big) \quad \text{for } \frac{2(N-\mu)}{N-2} < q < \frac{4(N-\mu)}{N-2}. \tag{2.13}
$$

As usual, we define the Mountain Pass level c of J to be

$$
c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)), \tag{2.14}
$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0\}$. And it is easy to see that $c > 0$ by Lemma 2.3.

Since

$$
J(f^{-1}(0)) = I(0) = 0,\ J(f^{-1}(tu_\epsilon)) = I(tu_\epsilon) \to -\infty \quad \text{as } t \to \infty,
$$

it follows that there exists $t_0 \neq 0$ such that $J(f^{-1}(t_0u_\epsilon)) < 0$. Let

$$
\gamma_1(t) = f^{-1}(tt_0u_\epsilon),
$$

we have

$$
\gamma_1(0) = 0, \gamma_1(1) = f^{-1}(t_0u_\epsilon) \neq 0, J(\gamma_1(1)) = J(f^{-1}(t_0u_\epsilon)) < 0.
$$

Therefore, from the definition of the Mountain Pass level, it follows that

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)) \le \sup_{t\in[0,1]} J(\gamma_1(t)) \\ &= \sup_{t\in[0,1]} J(f^{-1}(tt_0u_\epsilon)) = \sup_{t\ge 0} J(f^{-1}(tu_\epsilon)) \\ &= \sup_{t\ge 0} I(tu_\epsilon). \end{aligned}$$

Taking u_ϵ as a test function, from the following lemma, we can check that

$$c < \frac{2-\nu}{4(N-\nu)} S^{\frac{N-\nu}{2-\nu}}. \tag{2.15}$$

Lemma 2.4. *Let μ, $\nu \in [0,2)$ be fixed, we have*

(i) *if $\frac{2(N+2-2\mu)}{N-2} < q < 22^*(\mu)$, then (2.15) holds for any $\lambda > 0$;*

(i) *if $2 < q \le \frac{2(N+2-2\mu)}{N-2}$, there exists a positive constant λ^*, such that (2.15) still holds for $\lambda > \lambda^*$.*

Proof. (i) Since $I(0) = 0$, $\lim_{t\to\infty} I(tu_\epsilon) = -\infty$, there exists $t_\epsilon > 0$ such that $I(t_\epsilon u_\epsilon) = \max_{t\ge 0} I(tu_\epsilon)$. We claim that there exist positive constants t_1 and t_2 such that $t_1 \le t_\epsilon \le t_2$ for $\epsilon \in (0,\epsilon_0)$. In fact, by (2.11)-(2.13), there is a small $\epsilon_2 > 0$ such that

$$\begin{aligned} I(tu_\epsilon) &\le \frac{t^2}{2}\int_{\mathbb{R}^N} \big[|\nabla u_\epsilon|^2 + V(x)u_\epsilon^2\big]dx + \frac{t^4}{4}\int_{\mathbb{R}^N} |\nabla u_\epsilon^2|^2 dx \\ &\quad - \frac{t^{22^*(\nu)}}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^{22^*(\nu)}}{|x|^\nu}dx \\ &\le \frac{t^2}{2} + \frac{t^4}{2}S^{\frac{N-\nu}{2-\nu}} - \frac{t^{22^*(\nu)}}{4\cdot 2^*(\nu)}S^{\frac{N-\nu}{2-\nu}} \end{aligned} \tag{2.16}$$

for all $\epsilon \in (0,\epsilon_2)$. Hence

$$\frac{t_\epsilon^{22^*(\nu)}}{2\cdot 2^*(\nu)}S^{\frac{N-\nu}{2-\nu}} \le t_\epsilon^2 + t_\epsilon^4 S^{\frac{N-\nu}{2-\nu}}$$

which implies that there exists a constant $t_2 > 0$ such that $t_\epsilon \le t_2$ for all $\epsilon \in (0,\epsilon_2)$.

Since $2^*(\mu) < \frac{2(N+2-2\mu)}{N-2} < q < 22^*(\mu)$, it follows from (2.11)-(2.13) that there exists $\epsilon_1 \in (0,\epsilon_2)$ such that

$$\begin{aligned} I(tu_\epsilon) &\ge \frac{t^4}{4}\int_{\mathbb{R}^N} |\nabla(u_\epsilon^2)|^2 dx - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^q}{|x|^\mu}dx - \frac{t^{22^*(\nu)}}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^{22^*(\nu)}}{|x|^\nu}dx \\ &\ge \frac{1}{8}S^{\frac{N-\nu}{2-\nu}}t^4 - \lambda C\epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q}t^q - \frac{1}{2^*(\nu)}S^{\frac{N-\nu}{2-\nu}}t^{22^*(\nu)}, \end{aligned}$$

for all $\epsilon \in (0,\epsilon_1)$. Let

$$\chi = \max_{0\le t\le 1}\big[\frac{1}{8}t^4 - \frac{1}{2^*(\nu)}t^{22^*(\nu)}\big]S^{\frac{N-\nu}{2-\nu}}.$$

Then $\chi > 0$. Since $\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q > 0$, we can find a small $\epsilon_0 < \epsilon_1$ with $\lambda C\epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q} \le \frac{\chi}{2}$ for all $\epsilon \in (0,\epsilon_0)$. Thus

$$I(t_\epsilon u_\epsilon) \ge \max_{0\le t\le 1}\Big\{\frac{1}{8}S^{\frac{N-\nu}{2-\nu}}t^4 - \lambda C\epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q}t^q - \frac{1}{2^*(\nu)}S^{\frac{N-\nu}{2-\nu}}t^{22^*(\nu)}\Big\} \ge \frac{\chi}{2}.$$

Combining the above inequality with (2.16), we deduce that

$$\frac{\chi}{2} \le \frac{t_\epsilon^2}{2} + \frac{t_\epsilon^4}{2} S^{\frac{N-\nu}{2-\nu}} - \frac{t_\epsilon^{22^*(\nu)}}{4\cdot 2^*(\nu)} S^{\frac{N-\nu}{2-\nu}},$$

which implies that there exists a $t_1 > 0$ such that $t_\epsilon \ge t_1$ for all $\epsilon \in (0, \epsilon_0)$. Hence we prove our claim.

For $\epsilon \in (0, \epsilon_0)$, from (2.11)-(2.13), we have

$$\begin{aligned}
I(t_\epsilon u_\epsilon) &\le \frac{t_\epsilon^4}{4}\int_{\mathbb{R}^N} |\nabla(u_\epsilon^2)|^2 dx - \frac{t_\epsilon^{22^*(\nu)}}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_\epsilon|^{22^*(\nu)}}{|x|^\nu} dx \\
&\quad - \frac{\lambda}{q} t_1^q \int_{\mathbb{R}^N} \frac{|u_\epsilon|^q}{|x|^\mu} dx + \frac{t_2^2}{2}\int_{\mathbb{R}^N} \big[|\nabla u_\epsilon|^2 + V(x)u_\epsilon^2\big] dx \\
&\le \Big(\frac{t_\epsilon^4}{4} - \frac{t_\epsilon^{22^*(\nu)}}{22^*(\nu)}\Big) S^{\frac{N-\nu}{2-\nu}} + O\left(\epsilon^{\frac{N-2}{2(2-\nu)}} |\ln \epsilon|\right) - C\lambda \epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q} \\
&\le \frac{2-\nu}{4(N-\nu)} S^{\frac{N-\nu}{2-\nu}} + O\left(\epsilon^{\frac{N-2}{2(2-\nu)}} |\ln \epsilon|\right) - C\lambda \epsilon^{\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q} \\
&< \frac{2-\nu}{4(N-\nu)} S^{\frac{N-\nu}{2-\nu}},
\end{aligned}$$

for $\epsilon > 0$ small enough and $\frac{N-\mu}{2-\nu} - \frac{N-2}{4(2-\nu)}q < \frac{N-2}{2(2-\nu)}$. Therefore we can find a small $\bar\epsilon > 0$ such that

$$\sup_{t\ge 0} J(f^{-1}(tu_{\bar\epsilon})) = \sup_{t\ge 0} I(tu_{\bar\epsilon}) = I(t_{\bar\epsilon} u_{\bar\epsilon}) < \frac{2-\nu}{4(N-\nu)} S^{\frac{N-\nu}{2-\nu}}.$$

Moreover, from (2.16), we conclude that $J(f^{-1}(tu_{\bar\epsilon})) = I(tu_{\bar\epsilon}) \to -\infty$ as $t \to +\infty$, which shows that there exists a $\bar t > 0$ such that $J(f^{-1}(\bar t u_{\bar\epsilon})) < 0$. By taking $\bar\gamma(t) = f^{-1}(t\bar t u_{\bar\epsilon})$, we have $\bar\gamma \in \Gamma$ and $c \le \max_{t\in[0,1]} J(\bar\gamma(t)) < \frac{2-\nu}{4(N-\nu)} S^{\frac{N-\nu}{2-\nu}}$ for any $\lambda > 0$.

(ii) In the proof of this part, we first rewrite I to be I_λ. Let $u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $u_0 \ne 0$ and define $t_\lambda > 0$ such that $I_\lambda(t_\lambda u_0) = \sup_{t\ge 0} I_\lambda(tu_0)$. We claim that $t_\lambda \to 0$ as $\lambda \to +\infty$.

We will prove our claim by contradiction. Suppose that the claim is not true. Then there exists a constant $t_0 > 0$ and a sequence $\{\lambda_n\}$ such that $t_{\lambda_n} \ge t_0$ as $\lambda_n \to +\infty$ for all n. Without loss of generality, we may assume that $\lambda_n \ge 1$ for all n. Let $t_n = t_{\lambda_n}$ and $I_1 = I_\lambda|_{\lambda=1}$, then $0 \le I_{\lambda_n}(t_n u_0) \le I_1(t_n u_0)$ for all n. Then it follows that

$$\begin{aligned}
I_1(t_n u_0) &= \frac{t_n^4}{4}\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \frac{t_n^2}{2}\int_{\mathbb{R}^N} \big[|\nabla u_0|^2 + V(x)u_0^2\big]\, dx \\
&\quad - \frac{t_n^q}{q}\int_{\mathbb{R}^N} \frac{|u_0|^q}{|x|^\mu} dx - \frac{t_n^{22^*(\nu)}}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_0|^{22^*(\nu)}}{|x|^\nu} dx \\
&\le \frac{t_n^4}{4}\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \frac{t_n^2}{2}\int_{\mathbb{R}^N} \big[|\nabla u_0|^2 + V(x)u_0^2\big]\, dx \\
&\quad - \frac{t_n^{22^*(\nu)}}{22^*(\nu)}\int_{\mathbb{R}^N} \frac{|u_0|^{22^*(\nu)}}{|x|^\nu} dx \to -\infty, \quad \text{as } t_n \to \infty,
\end{aligned}$$

which gives us a contradiction since $22^*(\nu) > 4 > 2$. Thus t_n is bounded from above. Moreover, we also have

$$
\begin{aligned}
&I_{\lambda_n}(t_n u_0)\\
&\le \frac{t_n^4}{4}\int_{\mathbb{R}^N}|\nabla u_0^2|^2dx + \frac{t_n^2}{2}\int_{\mathbb{R}^N}\big[|\nabla u_0|^2 + V(x)u_0^2\big]dx - \lambda_n\frac{t_n^q}{q}\int_{\mathbb{R}^N}\frac{|u_0|^q}{|x|^\mu}dx\\
&\le C - \lambda_n\frac{t_0^q}{q}\int_{\mathbb{R}^N}\frac{|u_0|^q}{|x|^\mu}dx \to -\infty
\end{aligned}
\tag{2.17}
$$

as $n\to\infty$, which contradicts $I_{\lambda_n}(t_n u_0)\ge 0$. Hence our claim holds. Since $t_\lambda \to 0$ as $\lambda\to+\infty$ and $I_\lambda(t_\lambda u_0) \le \frac{t_\lambda^4}{4}\int_{\mathbb{R}^N}|\nabla u_0^2|^2dx + \frac{t_\lambda^2}{2}\int_{\mathbb{R}^N}\big[|\nabla u_0|^2+V(x)u_0^2\big]dx$, we can obtain that $I_\lambda(t_\lambda u_0)\to 0$ as $\lambda\to+\infty$. Therefore, there exists $\lambda^*>0$ such that $\sup_{t\ge 0} I_\lambda(tu_0) < \frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}}$ for any $\lambda>\lambda^*$. This implies that $c < \frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}}$ for all $\lambda>\lambda^*$. The proof is complete. □

3. Compactness of the $(C)_c$ sequence

Since the compactness of $(C)_c$ sequence plays an important role in our process, we will pay much more attention on the $(C)_c$ sequence of J in this section. Recall that $\{v_n\}$ is a $(C)_c$ sequence of J if $J(v_n)\to c$ and $(1+\|v_n\|)J'(v_n)\to 0$ as $n\to\infty$.

Lemma 3.1. *Any $(C)_c$ sequence $\{v_n\}\subset E$ of J is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{v_n\}\subset E$ be a $(C)_c$ sequence of J at level c, that is,

$$J(v_n)\to c \quad\text{and}\quad (1+\|v_n\|)J'(v_n)\to 0 \quad\text{as } n\to\infty.$$

Choosing $\varphi_n = \frac{f(v_n)}{f'(v_n)}$, it is easy to see that $\varphi_n\in E$ since $v_n\in E$ and the definition of f'. Then, by (A3) and (A4), we obtain $\|\varphi_n\|\le 5\|v_n\|$ and $\langle J'(v_n),\varphi_n\rangle\to 0$ as $n\to\infty$.

By defining two real functions

$$\psi(t,x) = \lambda\frac{|t|^{q-2}t}{|x|^\mu} + \frac{|t|^{22^*(\nu)-2}t}{|x|^\nu}, \quad \Psi(t,x) = \int_0^t \psi(s,x)ds,$$

choosing $\sigma=\max\{\mu,\nu\}$, we find that there exists a constant $\tau\in(4,22^*(\nu))$ such that

$$\lim_{t\to 0}\frac{|x|^\sigma[t\psi(t,x)-\tau\Psi(t,x)]}{t^2} = 0$$

and

$$\lim_{|t|\to+\infty}\frac{|x|^\sigma[t\psi(t,x)-\tau\Psi(t,x)]}{t^\tau} = +\infty \quad\text{uniformly for } x\in\mathbb{R}^N.$$

Therefore, there exists $r>0$ such that

$$t\psi(t,x)-\tau\Psi(t,x)\ge 0, \quad \text{for any } |t|>r,\ x\in\mathbb{R}^N. \tag{3.1}$$

Moreover, for any $\varepsilon>0$, there exists a positive constant $C(\varepsilon)$ such that

$$|t\psi(t,x)-\tau\Psi(t,x)| \le \varepsilon\frac{|t|^2}{|x|^\sigma} + C(\varepsilon)\frac{|t|^{22^*(\nu)}}{|x|^\sigma}, \quad \forall t\in\mathbb{R},\ x\in\mathbb{R}^N. \tag{3.2}$$

With φ_n defined before, we can deduce from (3.1) that

$$
\begin{aligned}
c+o(1) &= J(v_n)-\frac{1}{\tau}\langle J'(v_n),\varphi_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2dx-\frac{1}{\tau}\int_{\mathbb{R}^N}\Big(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\Big)|\nabla v_n|^2dx \\
&\quad +\big(\frac{1}{2}-\frac{1}{\tau}\big)\int_{\mathbb{R}^N}V(x)f^2(v_n)dx \\
&\quad +\int_{\mathbb{R}^N}\big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\big]dx \\
&\geq \big(\frac{1}{2}-\frac{2}{\tau}\big)\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\big(\frac{1}{2}-\frac{1}{\tau}\big)\int_{\mathbb{R}^N}V(x)f^2(v_n)dx \\
&\quad +\int_{B}\big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\big]dx,
\end{aligned}
\tag{3.3}
$$

where $B=\{x\in\mathbb{R}^N:|f(v_n)|\leq r\}$. By (3.2), there is a constant $M>V_0$ such that

$$
\big|\frac{1}{\tau}t\psi(t,x)-\Psi(t,x)\big|\leq\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\frac{t^2}{|x|^\sigma},\quad\text{for any } |t|\leq r,\ x\in\mathbb{R}^N, \tag{3.4}
$$

where V_0 is given in (A0).

Let $A=\{x\in\mathbb{R}^N:V(x)\leq M\}$. By (3.4) and (A0), we obtain

$$
\begin{aligned}
&\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{\mathbb{R}^N}V(x)f^2(v_n)dx+\int_B\big[\frac{1}{\tau}\psi(f(v_n),x)f(v_n)-\Psi(f(v_n),x)\big]dx \\
&\geq\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{B\cap\{x\in\mathbb{R}^N:|x|>1\}}V(x)f^2(v_n)dx-\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\int_B\frac{|f(v_n)|^2}{|x|^\sigma}dx \\
&=\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{B\cap\{x\in\mathbb{R}^N:|x|>1\}}V(x)f^2(v_n)dx \\
&\quad -\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\int_{B\cap\{x\in\mathbb{R}^N:|x|>1\}}\frac{|f(v_n)|^2}{|x|^\sigma}dx \\
&\quad -\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\int_{B\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{|f(v_n)|^2}{|x|^\sigma}dx \\
&\geq\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{B\cap\{x\in\mathbb{R}^N:|x|>1\}}\big(V(x)-M\big)f^2(v_n)dx \\
&\quad -\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\int_{B\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{|f(v_n)|^2}{|x|^\sigma}dx \\
&\geq\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{A\cap B\cap\{x\in\mathbb{R}^N:|x|>1\}}\big(V_0-M\big)r^2dx \\
&\quad -\big(\frac{1}{4}-\frac{1}{2\tau}\big)M\int_{B\cap\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{r^2}{|x|^\sigma}dx \\
&\geq\big(\frac{1}{4}-\frac{1}{2\tau}\big)\big(V_0-M\big)r^2\,\mathrm{meas}\big(A\cap B\cap\{x\in\mathbb{R}^N:|x|>1\}\big) \\
&\quad -\big(\frac{1}{4}-\frac{1}{2\tau}\big)Mr^2\int_{\{x\in\mathbb{R}^N:|x|\leq1\}}\frac{1}{|x|^\sigma}dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \big(\frac{1}{4}-\frac{1}{2\tau}\big)\big(V_0-M\big)r^2\operatorname{meas}(A)-\big(\frac{1}{4}-\frac{1}{2\tau}\big)r^2M\int_0^1\rho^{N-\sigma-1}d\rho \\
&\geq \big(\frac{1}{4}-\frac{1}{2\tau}\big)\big(V_0-M\big)r^2\operatorname{meas}(A)-\big(\frac{1}{4}-\frac{1}{2\tau}\big)r^2M\,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\big(\frac{1}{2}-\frac{2}{\tau}\big)\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\big(\frac{1}{4}-\frac{1}{2\tau}\big)\int_{\mathbb{R}^N}V(x)f^2(v_n)dx \\
&\leq \big(\frac{1}{4}-\frac{1}{2\tau}\big)r^2\left[\big(M-V_0\big)\operatorname{meas}(A)+M\right]+c+o(1).
\end{aligned}
\tag{3.5}
$$

Therefore,

$$
\int_{\mathbb{R}^N}\Big[|\nabla v_n|^2+V(x)f^2(v_n)\Big]dx\leq C\,,
$$

since $\operatorname{meas}(A)$ is finite according to the assumption (A0).

Moreover, by (A5) and using Sobolev inequality again, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^2dx&=\int_{\{|v_n|\leq 1\}}|v_n|^2dx+\int_{\{|v_n|>1\}}|v_n|^2dx \\
&\leq C\int_{\mathbb{R}^N}V(x)f^2(v_n)dx+\int_{\mathbb{R}^N}|v_n|^{2^*}dx \\
&\leq C\int_{\mathbb{R}^N}V(x)f^2(v_n)dx+C\Big[\int_{\mathbb{R}^N}|\nabla v_n|^2dx\Big]^{2^*/2}<+\infty,
\end{aligned}
$$

where $2^*=2N/(N-2)$ is the critical Sobolev exponent. Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, which completes our proof. $\square$

Lemma 3.2. *Let $\{v_n\}\subset E$ be a $(C)_c$ sequence of J. If $c<\frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}}$, there exist positive constants R and ξ, and a sequence $\{y_n\}\subset\mathbb{R}^N$, such that*

$$
\limsup_{n\to\infty}\int_{B_R(y_n)}|v_n|^2dx\geq\xi.
$$

Proof. Suppose that the conclusion is not true. It then follows from [24, Lemma 1.21] that $v_n\to 0$ in $L^s(\mathbb{R}^N)$ for all $2<s<2^*$. By Hölder's inequality, Hardy's inequality and Lemma 3.1, we have

$$
\int_{\mathbb{R}^N}\frac{|v_n|^q}{|x|^\mu}dx\leq C\Big(\int_{\mathbb{R}^N}|v_n|^{\frac{2(q-\mu)}{2-\mu}}dx\Big)^{\frac{2-\mu}{2}}.
$$

Since $2<q<2^*(\mu)$ and $0\leq\mu<2$, then $2<\frac{2(q-\mu)}{2-\mu}<2^*$, we can obtain $v_n\to 0$ in $L^{\frac{2(q-\mu)}{2-\mu}}(\mathbb{R}^N)$, hence

$$
v_n\to 0\quad\text{in } L^q(\mathbb{R}^N,|x|^{-\mu}),\ \text{for } 2<q<2^*(\mu).
$$

Then by (A4), Lemma 3.1 and the interpolation, we deduce that

$$
f(v_n)\to 0\quad\text{in } L^q(\mathbb{R}^N,|x|^{-\mu}),\ \text{for } 2<q<22^*(\mu).
\tag{3.6}
$$

By passing to a subsequence of $\{v_n\}$ and Lemma 3.1 we may assume that

$$
\begin{gathered}
\int_{\mathbb{R}^N}\Big(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\Big)|\nabla v_n|^2dx+\int_{\mathbb{R}^N}V(x)f^2(v_n)dx\to b, \\
\int_{\mathbb{R}^N}\frac{|f(v_n)|^{22^*(\nu)}}{|x|^\nu}dx\to d.
\end{gathered}
$$

On the other hand,

$$
\begin{aligned}
&S\Big(\int_{\mathbb{R}^N}\frac{|f(v_n)|^{22^*(\nu)}}{|x|^\nu}dx\Big)^{\frac{2}{2^*(\nu)}}\\
&\le \int_{\mathbb{R}^N}|\nabla f^2(v_n)|^2dx=\int_{\mathbb{R}^N}\frac{4f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2dx\\
&\le \int_{\mathbb{R}^N}\Big(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\Big)|\nabla v_n|^2dx+\int_{\mathbb{R}^N}V(x)f^2(v_n)dx.
\end{aligned}
$$

By passing to a subsequence of $\{v_n\}$ to the both sides of the above inequality, we obtain $Sd^{\frac{2}{2^*(\nu)}}\le b$. On the other hand, it can be deduced from (3.6) that $0=\lim_{n\to\infty}\langle J'(v_n),w_n\rangle=b-d$, where $w_n=\frac{f(v_n)}{f'(v_n)}$. Therefore, $b=d\ge S^{\frac{N-\nu}{2-\nu}}$. And, by (3.6) again, we obtain

$$
\begin{aligned}
c&=\lim_{n\to\infty}J(v_n)\\
&=\lim_{n\to\infty}\Big[\frac12\int_{\mathbb{R}^N}\Big(|\nabla v_n|^2+V(x)f^2(v_n)\Big)dx-\frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(v_n)|^{22^*(\nu)}}{|x|^\nu}dx\Big]\\
&\ge\lim_{n\to\infty}\Big[\frac14\int_{\mathbb{R}^N}\Big(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\Big)|\nabla v_n|^2dx+\frac14\int_{\mathbb{R}^N}V(x)f^2(v_n)dx\\
&\quad-\frac{1}{22^*(\nu)}\int_{\mathbb{R}^N}\frac{|f(v_n)|^{22^*(\nu)}}{|x|^\nu}dx\Big]\\
&=\Big(\frac14-\frac{1}{22^*(\nu)}\Big)d\\
&\ge\frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}},
\end{aligned}
$$

which gives us a contradiction since $c<\frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}}$. The proof is complete. $\square$

4. Proof of the main result

Proof of Theorem 1.1. Let c be the Mountain Pass level given in (2.14). From Lemma 2.3, Lemma 2.4 and the modified Mountain Pass Theorem [23], J has a $(C)_c$ sequence $\{v_n\}\subset E$. By Lemma 3.1, we may assume that $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $f(v_n)\rightharpoonup f(v)$ in E, under the assumption (A0), for any $0\le\sigma<2$, by Lemma 2.1 the embedding $E\hookrightarrow L^r(\mathbb{R}^N,|x|^{-\sigma})$ is continuous for $2\le r\le 2^*(\sigma)$, it is also compact for $2\le r<2^*(\sigma)$, which implies

$$
\begin{gathered}
f(v_n)\to f(v)\quad\text{in } L^s(\mathbb{R}^N,|x|^{-\mu}),\ \text{for } 2\le s<22^*(\mu),\\
f(v_n)\rightharpoonup f(v)\quad\text{in } L^{22^*(\nu)}(\mathbb{R}^N,|x|^{-\nu}),
\end{gathered}
$$

with $\mu,\nu\in[0,2)$. Hence, we have $\langle J'(v_n),\varphi\rangle\to\langle J'(v),\varphi\rangle=0$ for any $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$, that is, v is a weak solution of (2.5). We conclude from Lemma 2.4 that for any $\mu,\nu\in[0,2)$, $c<\frac{2-\nu}{4(N-\nu)}S^{\frac{N-\nu}{2-\nu}}$ holds when either of the following statement holds: (1) $\frac{2(N+2-2\mu)}{N-2}<q<22^*(\mu)$ and each $\lambda>0$; (2) $2<q\le\frac{2(N+2-2\mu)}{N-2}$ and each $\lambda>\lambda^*$ for some positive constant λ^*. Moreover, by Lemma 3.2, there exists a constant $\xi>0$ such that

$$
\int_{\mathbb{R}^N}|v|^2dx=\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^2dx\ge\xi>0,
$$

which implies that v is a nontrivial solution of problem (2.5). Hence $u = f(v)$ is a nontrivial solution of problem (1.1).

Finally, letting $e = \inf\{J(v) : v \in E, v \neq 0, J'(v) = 0\}$, it is easy to see that e is attained by the lower semi-continuity. The proof is complete. $\square$

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