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A note on a Liouville-type result for a system of fourth-order equations in $\mathbb{R}^N$ *

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Abstract

We consider the fourth order system $\Delta^2 u = v^\alpha, \Delta^2 v = u^\beta$ in $\mathbb{R}^N$, for $N \geq 5$, with $\alpha \geq 1$, $\beta \geq 1$, where Δ^2 is the bilaplacian operator. For $1/(\alpha+1) + 1/(\beta+1) > (N-4)/N$ we prove the non-existence of non-negative, radial, smooth solutions. For $\alpha, \beta \leq (N+4)/(N-4)$ we show the non-existence of non-negative smooth solutions.

1 Introduction

In this work we consider the fourth order nonlinear system

$$\begin{aligned} \Delta^2 u &= v^\alpha \\ \Delta^2 v &= u^\beta \end{aligned} \tag{1.1}$$

in the whole space $\mathbb{R}^N$. We are interested in Liouville type results, i.e. we want to determine for which positive real values of the exponents α and β is $(u, v) = (0, 0)$ the only non-negative solution (u, v) of the system. In here, the solution is taken in the classical sense, i.e. $u, v \in C^4(\mathbb{R}^N)$.

This type of problems were studied for the Laplacian operator. Mitidieri [7] proved that if $\alpha, \beta > 1$ and

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} > \frac{N-2}{N}, \tag{1.2}$$

then the system

$$\begin{aligned} \Delta u + v^\alpha &= 0 \\ \Delta v + u^\beta &= 0 \end{aligned} \tag{1.3}$$

has no non-negative, radial, C^2 solutions in $\mathbb{R}^N$. Souto in [10] showed that if

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} > \frac{N-2}{N-1}, \quad \alpha, \beta > 0,$$

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then (1.3) has no positive solutions. A further result was given in a paper of Figueiredo and Felmer [2]. The authors proved that if

$$0<\alpha,\beta\le\frac{N+2}{N-2},\ \text{but not both equal to}\ \frac{N+2}{N-2}$$

then system (1.3) has no positive C^2 solutions.

We point out that it is still an open problem to know whether (1.3) has no non-negative solutions under assumption

$$\frac{N-2}{N-1}\ge\frac{1}{\alpha+1}+\frac{1}{\beta+1}>\frac{N-2}{N}\quad\text{and}\quad(\alpha>\frac{N+2}{N-2}\ \text{or}\ \beta>\frac{N+2}{N-2}).$$

Similar questions remain unsolved in the case where the Laplacian operator in (1.3) is replaced by other self–adjoint operators. For example, concerning the single equation for the bilaplacian operator, Mitidieri [7] also proved that the problem

$$\Delta^2 u=u^\alpha,\quad \Delta u\le 0,\quad \text{in } \mathbb{R}^N$$

has no radial C^4 positive solution, if $1<\alpha<(N+4)/(N-4)$. Using the method of the moving planes, for the same range of α, Lin [6] showed that

$$\Delta^2 u=u^\alpha\quad\text{in } \mathbb{R}^N, \tag{1.4}$$

has no C^4 positive solutions. Later on [11], Xu proved the same result using instead the method of the moving spheres.

We recall that if $N\ge 5$ and $\alpha=(N+4)/(N-4)$ then problem (1.4) has a whole family of positive solutions explicitely given by

$$u(x)=\frac{c_1}{(1+c_2|x|^2)^{\frac{N-4}{2}}},$$

where c_1 and c_2 are some appropriate positive constants.

A natural question that rises is to analyse the behaviour of system (1.1). Here we show that the quoted results in [2, 7] for system (1.3) can be extended to system (1.1). Precisely, we prove the following.

Theorem 1.1 *If $N\ge 5$, $\alpha,\beta\ge 1$, but not both equal to 1 are such that*

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}>\frac{N-4}{N}, \tag{1.5}$$

then system (1.1) has no radial non-negative solutions in $C^4(\mathbb{R}^N)$.

Theorem 1.2 **I)** *If $1\le\alpha,\beta\le\frac{N+4}{N-4}$ but not both equal to 1 neither to $\frac{N+4}{N-4}$, then the only non-negative C^4 solution of system (1.1) in the whole of $\mathbb{R}^N$ is the trivial one: $(u,v)=(0,0)$.*

II) *If $\alpha=\beta=\frac{N+4}{N-4}$, then u and v are radially symmetric with respect to some point of $\mathbb{R}^N$.*

Our proofs are strongly motivated by the works of Figueiredo and Felmer [2], Lin [6], Xu [11] and Mitidieri [7]. Indeed, the proof of Theorem 1 presented in section 2 uses an idea of Mitidieri, which relies on the application of a Rellich type identity. Section 3 is devoted to the proof of the fact that if (u, v) is a non-negative, C^4 solution of (1.1) then u and v are super-harmonic; here we extend the result in [6] for the single equation. This result plays a key role in our work. In section 4 we apply the method of the moving planes in order to prove Theorem 2. In our case, the main difficulty in applying the method stems from the fact that the maximum principle cannot be applied directly to (u, v); to overcome this, we follow an idea of Xu [11] for the single equation and apply the moving planes method for both $(-\Delta u, -\Delta v)$ and (u, v). Contrarily to [2] we use no additional change of variables except for Kelvin transforms.

As far as we know this is the first work concerning Liouville type results for a system involving the bilaplacian operator. However, some related questions remain unsolved. For example, it is not clear for us whether Souto's result for system (1.3) can be extended to system (1.1).

2 General auxiliary facts

In this section we state some general results that will be useful in the sequel.

Lemma 2.1 *Let $u \in C^2(\mathbb{R}^N \setminus \{0\})$ be such that $u < 0$ and $\Delta u \geq 0$. Then, for each $\varepsilon > 0$, one has*

$$u(x) \leq M(\varepsilon) := \max\{u(y) : |y| = \varepsilon\}, \quad 0 < |x| \leq \varepsilon.$$

The proof of this lemma is included in the proof of Lemma 1.1 of [2] and is a consequence of the so called Hadamard Three Spheres Theorem ([8]).

Lemma 2.2 *Suppose $y = y(r) \geq 0$ satisfies*

$$y'' + \frac{N-1}{r} y' + \phi(r) \leq 0, \quad r > 0,$$

with ϕ non-negative and non-increasing and y' bounded for r near 0. Then, for $r > 0$,

$$y'(r) \leq 0, \tag{2.1}$$
$$ry'(r) + (N-2)y(r) \geq 0, \tag{2.2}$$
$$y(r) \geq cr^2\phi(r), \tag{2.3}$$

where $c = c(N)$.

Proof. The proofs of (2.2) and (2.3) can be found in [7, Lemma 3.1] and in [9, Lemma 2.7] (see also [11, Lemma 3.1]) respectively. It remains to prove (2.1). Multiplying the inequality in the Lemma by r^{N-1} we get

$$(r^{N-1}y'(r))' \leq -r^{N-1}\phi(r).$$

Integrating from 0 to r we obtain

$$r^{N-1}y'(r) \le -\int_o^r s^{N-1}\phi(s)\,ds \le 0.$$

□

3 Non existence of radial solutions. The superharmonic property of the solutions

In this section we prove that system (1.1) does not admit non-negative radial solutions. For that matter we use the following result.

Theorem 3.1 *If* (u,v) *is a* $C^4(\mathbb{R}^N)$*, non-negative solution of system* (1.1)*, with* $\alpha,\beta \ge 1$*, but not both equal to* 1*, we have:*

$$\Delta u \le 0 \quad \textit{and} \quad \Delta v \le 0.$$

This theorem is the most powerful tool of the present work. We postpone its proof to section 3.

The proof of Theorem 1.1 stated in the Introduction makes use of the following Rellich type identity in a smooth bounded domain Ω, obtained by Mitidieri in [7]:

$$R_2(u,v) = R_1(\Delta u, v) + R_1(u,\Delta v) - \int_{\partial\Omega} \Delta u \Delta v(x,n)d\sigma + N\int_\Omega \Delta u\Delta v\,dx \quad (3.1)$$

where

$$\begin{aligned} R_2(u,v) &= \int_\Omega (\Delta^2 u(x,\nabla v) + \Delta^2 v(x,\nabla u))\,dx,\\ R_1(u,v) &= \int_{\partial\Omega} (\frac{\partial u}{\partial n}(x,\nabla v) + \frac{\partial v}{\partial n}(x,\nabla u) - (\nabla u,\nabla v)(x,n))d\sigma\\ &\quad + (N-2)\int_\Omega (\nabla u,\nabla v)dx, \end{aligned}$$

n is the outward unit normal to $\partial\Omega$ and $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$.

In what follows, for a rotationally symmetric function,

$$\Delta = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$$

and $'$ will denote differentiation with respect to r.

Lemma 3.2 *Let (u, v) be a non-negative, radial, C^4 solution of system* (1.1). *Then for all $r > 0$ we have*

$$
\begin{aligned}
&(\frac{N-4}{2} - \frac{N}{\alpha+1}) \int_0^r v^{\alpha+1}(s) s^{N-1}\, ds + (\frac{N-4}{2} - \frac{N}{\beta+1}) \int_0^r u^{\beta+1}(s) s^{N-1}\, ds \\
&= -\frac{r^N}{\alpha+1} v^{\alpha+1}(r) - \frac{r^N}{\beta+1} u^{\beta+1}(r) + r^N (\Delta u)'(r) v'(r) + r^N (\Delta v)'(r) u'(r) \\
&\quad - \frac{N-4}{2} r^{N-1} (\Delta u)'(r) v(r) + \frac{N-4}{2} r^{N-1} (\Delta v)'(r) u(r) \\
&\quad - r^N (\Delta u)(r)(\Delta v)(r) + \frac{N}{2} r^{N-1} (\Delta v)(r) u'(r) + \frac{N}{2} r^{N-1} (\Delta u)(r) v'(r).
\end{aligned}
$$

Proof. We obtain the desired identity by taking into account that u and v are radial and after an integration by parts in (3.1), where we take $\Omega = B_r(0)$. $\square$

Lemma 3.3 *Let (u, v) be a non-negative, radial, C^4 solution of system* (1.1). *Then there exists a positive constant C such that for all $r > 0$,*

$$
u(r) \le C r^{-\frac{4(\alpha+1)}{\alpha\beta-1}}, \quad v(r) \le C r^{-\frac{4(\beta+1)}{\alpha\beta-1}}, \tag{3.2}
$$

$$
|\Delta u(r)| \le C r^{-2-\frac{4(\alpha+1)}{\alpha\beta-1}}, \quad |\Delta v(r)| \le C r^{-2-\frac{4(\beta+1)}{\alpha\beta-1}} \tag{3.3}
$$

$$
\begin{gathered}
|r^{N-1}(\Delta v)'(r) u(r)|, \quad |r^{N-1}(\Delta v)(r) u'(r)|, \quad |r^{N-1}(\Delta u)'(r) v(r)|, \\
|r^{N-1}(\Delta u)(r) v'(r)|, \quad |r^{N}(\Delta v)'(r) u'(r)|, \quad |r^{N}(\Delta u)'(r) v'(r)| \\
\le C r^{N-4-\frac{4(\alpha+\beta+2)}{\alpha\beta-1}}.
\end{gathered} \tag{3.4}
$$

Proof. Since u and v are radial we can write

$$
\begin{gathered}
u''(r) + \frac{N-1}{r} u'(r) + w(r) = 0, \quad w''(r) + \frac{N-1}{r} w'(r) + v^\alpha(r) = 0 \\
v''(r) + \frac{N-1}{r} v'(r) + z(r) = 0, \quad z''(r) + \frac{N-1}{r} z'(r) + u^\beta(r) = 0 \\
u'(0) = w'(0) = v'(0) = z'(0) = 0,
\end{gathered}
$$

where $w := -\Delta u$ and $z := -\Delta v$. From now on we denote by c some positive constant possibly different from place to place. By Theorem 3.1 w and z are non–negative functions. Then by (2.3),

$$
u(r) \ge c r^2 w(r), \quad w(r) \ge c r^2 v^\alpha(r), \quad v(r) \ge c r^2 z(r), \quad z(r) \ge c r^2 u^\beta(r).
$$

So

$$
u^\beta(r) \le c r^{-2} z(r) \le c r^{-4} v(r) \le c r^{-4-\frac{2}{\alpha}} w^{\frac{1}{\alpha}}(r) \le c r^{-4-\frac{4}{\alpha}} u^{\frac{1}{\alpha}}(r),
$$

which implies that $u(r) \le c r^{-4\frac{\alpha+1}{\alpha\beta-1}}$. Then $w(r) \le c r^{-2-\frac{4(\alpha+1)}{\alpha\beta-1}}$, $v(r) \le c r^{-4\frac{\beta+1}{\alpha\beta-1}}$ and finally we obtain $z(r) \le c r^{-2-\frac{4(\beta+1)}{\alpha\beta-1}}$.

From (2.1), we know that $z' \le 0$. Also, $rz'(r) + (N-2)z(r) \ge 0$ (cf. (2.2)). Multiplying this by $r^{N-2}u$ and using the estimates that we obtained before, yields

$$|r^{N-1}z'(r)u(r)| = -r^{N-1}z'(r)u(r) \le (N-2)r^{N-2}z(r)u(r) \le cr^{N-4-\frac{4(\alpha+\beta+2)}{\alpha\beta-1}}.$$

Now, multiplying by $r^{N-1}u'$ the stated inequality and using the previous estimate we get

$$r^N z'(r)u'(r) \le cr^{N-4-\frac{4(\alpha+\beta+2)}{\alpha\beta-1}}.$$

Again from (2.1), $u' \le 0$, so that

$$|r^N z'(r)u'(r)| \le cr^{N-4-\frac{4(\alpha+\beta+2)}{\alpha\beta-1}}.$$

Using similar arguments we obtain the remaining estimates in the statement of the lemma. $\square$

Proof of Theorem 1.1. We multiply the first equation of system (1.1) by v and integrate in $B_r(0)$, for $r > 0$. We obtain

$$\begin{aligned}
\int_{B_r(0)} v^{\alpha+1}\,dx &= \int_{B_r(0)} \Delta^2 u\, v\,dx \\
&= -\int_{B_r(0)} (\nabla(\Delta u), \nabla v)\,dx + \int_{\partial B_r(0)} \frac{\partial(\Delta u)}{\partial n} v\,d\sigma \\
&= \int_{B_r(0)} \Delta u \Delta v\,dx - \int_{\partial B_r(0)} \frac{\partial v}{\partial n}\Delta u\,d\sigma + \int_{\partial B_r(0)} \frac{\partial(\Delta u)}{\partial n} v\,d\sigma.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^r v^{\alpha+1}(s)s^{N-1}\,ds = \\
&\qquad \int_0^r (\Delta u)(s)(\Delta v)(s)s^{N-1}\,ds - \Delta u(r)v'(r)r^{N-1} + (\Delta u)'(r)v(r)r^{N-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^r u^{\beta+1}(s)s^{N-1}\,ds \\
&\quad = \int_0^r (\Delta v)(s)(\Delta u)(s)s^{N-1}\,ds - (\Delta v)(r)u'(r)r^{N-1} + (\Delta v)'(r)u(r)r^{N-1}.
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^r v^{\alpha+1}(s)s^{N-1}\,ds \\
&\quad = \int_0^r u^{\beta+1}(s)s^{N-1}\,ds + (\Delta v)(r)u'(r)r^{N-1} - (\Delta v)'(r)u(r)r^{N-1} \\
&\qquad -\Delta u(r)v'(r)r^{N-1} + (\Delta u)'(r)v(r)r^{N-1}.
\end{aligned}$$

Now, we observe that the assumption (1.5) can be written as

$$\frac{(\alpha+1)(\beta+1)}{\alpha\beta-1} > \frac{N}{4}.$$

Thus from the estimates (3.2)-(3.4) we see that the boundary terms in (3.5) and also in the identity of Lemma 3.2 all have zero limits as $r \to +\infty$. Using (3.5) in Lemma 3.2 and those facts we conclude that

$$(N-4-\frac{N}{\alpha+1}-\frac{N}{\beta+1})\int_0^r u^{\beta+1}(s)s^{N-1}\,ds = o(1) \quad (r\to+\infty).$$

Passing to the limit we obtain $u = 0$ and, as a consequence, also $v = 0$. □

4 Proof of Theorem 3.1

Let (u, v) be a non-negative solution of system (1.1). We define w and z as follows:

$$w(x) := -\Delta u(x) \quad \text{and} \quad z(x) := -\Delta v(x), \tag{4.1}$$

for $x \in \mathbb{R}^N$. We can write system (1.1) as a system of four second order equations

$$\begin{gathered} \Delta u + w = 0 \\ \Delta w + v^\alpha = 0 \\ \Delta v + z = 0 \\ \Delta z + u^\beta = 0. \end{gathered} \tag{4.2}$$

Let $\bar{u}$ be the spherical average of u, i.e.

$$\bar{u}(r) := \frac{1}{\omega_N N r^{N-1}}\int_{\partial B_r(0)} u\,d\sigma,$$

where ω_N denotes the measure of the unit sphere in $\mathbb{R}^N$. Similarly we define $\bar{v}, \bar{w}, \bar{z}$ the spherical averages of v, w, z respectively. From (4.2) and the Hölder inequality, it is easy to see that

$$\Delta\bar{u} + \bar{w} = 0 \tag{4.3}$$

$$\Delta\bar{w} + \bar{v}^\alpha \le 0 \tag{4.4}$$

$$\Delta\bar{v} + \bar{z} = 0 \tag{4.5}$$

$$\Delta\bar{z} + \bar{u}^\beta \le 0, \tag{4.6}$$

where $\Delta f = f'' + \frac{N-1}{r}f'$, $f' = \frac{df}{dr}$, for $f = \bar{u}$, $\bar{w}$, $\bar{v}$, $\bar{z}$.

In order to prove Theorem 3.1 we begin with some auxiliary lemmas.

Lemma 4.1 *Let $N \geq 2$. Consider $p, q > 0$ such that $pq \neq 1$ and let (l_k) be the sequence defined by $l_0 = 2$ and*

$$l_{k+1} = p(ql_k + 4) + 4, \quad k \geq 0.$$

Then

i) $l_k = 2(pq)^k + 4(p+1)((pq)^k - 1)/(pq - 1)$,

ii) $(pql_k + 4p + N)(pql_k + 4p + 2)(pql_k + 4p + N + 2)(pql_k + 4p + 4) \leq (N + 2 + 2pq + 4p)^{4(k+1)}$, *for* $k = 0, 1, 2, \dots$

iii) If $pq > 1$ *then* $l_k \to +\infty$.

Lemma 4.2 *Let $p, q > 0$ be such that $pq \neq 1$. Suppose $b_0 = 0$ and define the sequence (b_k) inductively by*

$$b_{k+1} = pqb_k + 4p(k+1) + 4(k+1), \quad \text{for } k \geq 0.$$

Then

$$b_k = 4(p+1)\Big[\frac{(pq)^{k+1} - (k+1)pq + k}{(pq-1)^2}\Big]$$

for all non-negative integers k.

Lemma 4.3 *Let $p, q > 0$ be such that $pq \neq 1$. Suppose $n_0 = 0$ and define the sequence (n_k) inductively by $n_{k+1} = pqn_k + p$. Then*

$$n_k = p\frac{(pq)^k - 1}{pq - 1}$$

for all non-negative integers k.

The three lemmas above can be proved by induction.

Lemma 4.4 *Let $N \geq 2$. Consider $p, q > 0$ such that $pq > 1$, let (b_k), (l_k) and (n_k) be the sequences defined in the previous lemmas, c_0 a positive constant and z_0 be a non-negative constant. Define the sequence as follows: $r_0 = 0$ and*

$$r_k = \Big(\frac{z_0 2^{qn_{k-1}+1}(N + 2 + 2pq + 4p)^{qb_{k-1}}(N + ql_{k-1})(2 + ql_{k-1})}{c_0^{p^{k-1}q^k}}\Big)^{\frac{1}{ql_{k-1}+2}}$$

for $k \geq 1$. Then there exists a positive number a such that $\lim_{k\to\infty} r_k = a$.

Proof. We can write $r_k = z_0^{\frac{1}{ql_{k-1}+2}} \cdot {}^1r_k \cdot {}^2r_k \cdot {}^3r_k \cdot {}^4r_k$ where

$${}^1r_k := 2^{\frac{qn_{k-1}+1}{ql_{k-1}+2}}, \ {}^2r_k := (N + 2 + 2pq + 4p)^{\frac{qb_{k-1}}{ql_{k-1}+2}},$$

$${}^3r_k := [(N + ql_{k-1})(2 + ql_{k-1})]^{\frac{1}{ql_{k-1}+2}}, \ {}^4r_k := c_0^{-\frac{p^{k-1}q^k}{ql_{k-1}+2}}.$$

We have

$$\frac{qn_{k-1}+1}{ql_{k-1}+2} \to \frac{p}{2pq+4p+2},$$

and

$$\frac{qb_{k-1}}{ql_{k-1}+2} \to \frac{2pq(p+1)}{(pq-1)^2+2(p+1)(pq-1)}.$$

From this we deduce that both ${}^1r_k$ and ${}^2r_k$ converge. Concerning the third sequence,

$$\lim\ {}^3r_k = \lim[(N+ql_{k-1})(2+ql_{k-1})]^{\frac{1}{ql_{k-1}+2}} = 1.$$

Finally, $\frac{p^{k-1}q^k}{ql_{k-1}+2} \to \frac{(pq-1)}{2(pq-1)+4(p+1)}$ yields

$$\lim\ {}^4r_k := c_0^{-\frac{(pq-1)}{2(pq-1)+4(p+1)}}.$$

Since $l_k \to +\infty$, $z_0^{\frac{1}{ql_{k-1}+2}} \to 1$ and we are done. $\square$

Proof of Theorem 3.1 According to (4.1), we want to prove that $w \geq 0$ and $z \geq 0$. Suppose by contradiction that there exists x_0 such that $w(x_0) < 0$. Without loss of generality suppose that $x_0 = 0$. Multiplying (4.4) by r^{N-1}, we get

$$r^{N-1}\bar{w}_{rr} + (N-1)r^{N-2}\bar{w}_r \leq -r^{N-1}\bar{v}^{\alpha} \leq 0,$$

hence

$$(r^{N-1}\bar{w}_r)_r \leq 0.$$

By integrating the last inequality, we obtain $\bar{w}_r \leq 0$. Then $\bar{w}$ is non-increasing and we have

$$\bar{w}(r) \leq \bar{w}(0) < 0, \text{ for all } r > 0,$$

since, from the definition, $\bar{w}(0) = w(0)$. So, from (4.3),

$$\Delta\bar{u} = -\bar{w}(r) \geq -\bar{w}(0).$$

If we multiply the last inequality by r^{N-1} and integrate twice, we get

$$\bar{u}(r) \geq cr^2, \text{ for all } r > 0, \tag{4.7}$$

where $c := -\bar{w}(0)/(2N) > 0$. Without loss of generality, we can assume that $c < 1$. Next we prove by induction that

$$\bar{u}(r) \geq \frac{c^{(\alpha\beta)^k}}{2^{n_k}(N+2+2\alpha\beta+4\alpha)^{b_k}} r^{l_k}, \text{ for all } r > r_k, \tag{4.8}$$

where l_k, b_k, n_k and r_k are defined in Lemmas 4.1, 4.2, 4.3 and 4.4 with $p = \alpha$, $q = \beta$, $z_0 := \bar{z}(0)$ if $\bar{z}(0) > 0$ and $z_0 := 0$ otherwise and $c_0 := c$.

We have (4.8) for $k = 0$ by (4.7), since $l_0 = 2$, $b_0 = 0$, $n_0 = 0$ and $r_0 = 0$. Assume that (4.8) is true for some k. From (4.6),

$$\Delta \bar{z} \le -\frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k}(N+2+2\alpha\beta+4\alpha)^{\beta b_k}} r^{\beta l_k}.$$

If we multiply both sides by r^{N-1} and integrate twice, we obtain

$$\bar{z}(r) \le -\frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k}(N+2+2\alpha\beta+4\alpha)^{\beta b_k}(\beta l_k+N)(\beta l_k+2)} r^{\beta l_k+2} + \bar{z}(0).$$

Then, for all

$$r \ge \Big(\frac{z_0 2^{\beta n_k+1}(N+2+2\alpha\beta+4\alpha)^{\beta b_k}(\beta l_k+N)(\beta l_k+2)}{c^{\alpha^k \beta^{k+1}}}\Big)^{\frac{1}{\beta l_k+2}}$$

we have

$$\bar{z}(r) \le -\frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k+1}(N+2+2\alpha\beta+4\alpha)^{\beta b_k}(\beta l_k+N)(\beta l_k+2)} r^{\beta l_k+2}.$$

From (4.5) we get

$$\Delta \bar{v} \ge \frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k+1}(N+2+2\alpha\beta+4\alpha)^{\beta b_k}(\beta l_k+N)(\beta l_k+2)} r^{\beta l_k+2}.$$

After multiplying the last inequality by r^{N-1} and integrating twice we obtain

$$\bar{v}(r) \ge \frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k+1} D_k} r^{\beta l_k+4} + \bar{v}(0) \ge \frac{c^{\alpha^k \beta^{k+1}}}{2^{\beta n_k+1} D_k} r^{\beta l_k+4},$$

where

$$D_k := (N+2+2\alpha\beta+4\alpha)^{\beta b_k}(\beta l_k+N)(\beta l_k+2)(\beta l_k+N+2)(\beta l_k+4).$$

From (4.4) we have

$$\Delta \bar{w} \le -\frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha} D_k^{\alpha}} r^{\alpha\beta l_k+4\alpha}.$$

Once more, multiplying both sides of the above inequality and integrating twice, we obtain, since $\bar{w}(0) < 0$,

$$\bar{w}(r) \le -\frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha} D_k^{\alpha}(\alpha\beta l_k+4\alpha+N)(\alpha\beta l_k+4\alpha+2)} r^{\alpha\beta l_k+4\alpha+2}.$$

At last, from (4.3) we have

$$\Delta \bar{u} \ge \frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha} D_k^{\alpha}(\alpha\beta l_k+4\alpha+N)(\alpha\beta l_k+4\alpha+2)} r^{\alpha\beta l_k+4\alpha+2}.$$

Using the same procedure that we used before, we get

$$\bar{u}(r) \geq \frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha} D_k^\alpha E_k} r^{\alpha\beta l_k+4\alpha+4} + \bar{u}(0) \geq \frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha} D_k^\alpha E_k} r^{\alpha\beta l_k+4\alpha+4}$$

where

$$E_k := (\alpha\beta l_k + 4\alpha + N)(\alpha\beta l_k + 4\alpha + 2)(\alpha\beta l_k + 4\alpha + N + 2)(\alpha\beta l_k + 4\alpha + 4).$$

Taking into account that $\alpha \geq 1$,

$$\begin{aligned} D_k \leq &(N + 2 + 2\alpha\beta + 4\alpha)^{\beta b_k} (\alpha\beta l_k + 4\alpha + N)(\alpha\beta l_k + 4\alpha + 2) \\ &\times (\alpha\beta l_k + 4\alpha + N + 2)(\alpha\beta l_k + 4\alpha + 4). \end{aligned}$$

Thus, from Lemma 4.1 ii),

$$\bar{u}(r) \geq \frac{c^{(\alpha\beta)^{k+1}}}{2^{\alpha\beta n_k+\alpha}(N + 2 + 2\alpha\beta + 4\alpha)^{\alpha\beta b_k+4\alpha(k+1)+4(k+1)}} r^{\alpha\beta l_k+4\alpha+4}.$$

According to the definition of (l_k), (b_k) and (n_k) we have (4.8) for $k+1$.

Now, fix k_0 such that $r_k < 2 \lim r_k$, for all $k \geq k_0$. Take $A > 1$ such that

$$2A(N + 2 + 2\alpha\beta + 4\alpha)^{\frac{4(\alpha+1)}{\alpha\beta-1}} > 2 \lim r_k.$$

Taking $r = 2c^{-1}A(N + 2 + 2\alpha\beta + 4\alpha)^{\frac{4(\alpha+1)}{\alpha\beta-1}}$ in (4.8), for all $k \geq k_0$ we get

$$\bar{u}(r) \geq c^{(\alpha\beta)^k - l_k} 2^{l_k - n_k} A^{l_k} (N + 2 + 2\alpha\beta + 4\alpha)^{\frac{4(\alpha+1)}{\alpha\beta-1} l_k - b_k}.$$

The right hand member of the inequality goes to infinity, when $k \to +\infty$, since

$$l_k - n_k = \frac{2(\alpha\beta)^{k+1} + (3\alpha + 2)(\alpha\beta)^k - (3\alpha + 4)}{\alpha\beta - 1} \to +\infty,$$
$$(\alpha\beta)^k - l_k \to -\infty$$

and also since $\frac{4(\alpha+1)}{\alpha\beta-1} l_k - b_k$ equals

$$\frac{4(\alpha + 1)(\alpha\beta)^{k+1}}{(\alpha\beta - 1)^2}[1 + \frac{4\alpha + 2}{\alpha\beta} + \frac{k + 1}{(\alpha\beta)^k} - \frac{4\alpha + 4 + k}{(\alpha\beta)^{k+1}}] \to +\infty.$$

This is a contradiction, since $\bar{u}(r)$ is a constant. Thus $w \geq 0$, as claimed.

The case when there exists y_0 such that $z(y_0) < 0$ can be treated in a similar way, and this concludes the proof of Theorem 3.1. $\square$

For later purposes we prove the following results.

Lemma 4.5 *Let $1 \leq \alpha, \beta \leq \frac{N+4}{N-4}$ be such that $\alpha\beta \neq 1$.*

1) *There exists a sequence (R_i) such that $R_i^3 \bar{w}'(R_i) \to 0$ as $R_i \to +\infty$.*

2) *There exists a sequence $(\widetilde{R}_i)$ such that $\widetilde{R}_i^3 \bar{z}'(\widetilde{R}_i) \to 0$ as $\widetilde{R}_i \to +\infty$.*

Proof. From Theorem 3.1, w and z are non-negative functions. Similarly to Lemma 3.3, (2.3) allows to deduce the existence of a positive constant c such that

$$\bar{w}(r) \le c r^{-2-4\frac{\alpha+1}{\beta\alpha-1}}, \quad \bar{z}(r) \le c r^{-2-4\frac{\beta+1}{\beta\alpha-1}}. \tag{4.9}$$

We prove 1). The proof of 2) is similar thanks to the second inequality in (4.9).

Suppose by contradiction that there exist $\delta_0 > 0$ and $r_0 > 0$ such that for all $R > r_0$ we have $-R^3\bar{w}'(R) \ge \delta_0 > 0$. Then

$$\begin{aligned}\delta_0(R-r_0) &\le -\int_{r_0}^R s^3\bar{w}'(s)ds \\ &= -R^3\bar{w}(R) + r_0^3\bar{w}(r_0) + 3\int_{r_0}^R s^2\bar{w}(s)ds \\ &< 3\int_{r_0}^R s^2\bar{w}(s)ds + C \le 3c\int_{r_0}^R s^{-4\frac{\alpha+1}{\beta\alpha-1}}ds + C.\end{aligned}$$

In case $4\frac{\alpha+1}{\beta\alpha-1} = 1$ (which is only possible if $N = 5$ or $N = 6$), we get

$$\delta_0(R-r_0) < 3c(\log R - \log r_0) + C.$$

Dividing both sides of this inequality by $R^{1/2}$ and passing to the limit as $R \to +\infty$ we get a contradiction. Assume now $4\frac{\alpha+1}{\beta\alpha-1} \neq 1$. Then

$$\delta_0(R-r_0) < c\frac{\beta\alpha-1}{\alpha(\beta-4)-5}\Big(R^{\frac{\alpha(\beta-4)-5}{\beta\alpha-1}} - r_0^{\frac{\alpha(\beta-4)-5}{\beta\alpha-1}}\Big) + C.$$

Dividing both sides of the previous inequality by $R^{\frac{2}{3}}$, we get

$$\delta_0(R^{\frac{1}{3}} - r_0R^{-\frac{2}{3}}) \le C\frac{\beta\alpha-1}{\alpha(\beta-4)-5}\Big(R^{\frac{\alpha(\beta-12)-13}{3\beta\alpha-3}} - r_0^{\frac{\alpha(\beta-4)-5}{\beta\alpha-1}}R^{-\frac{2}{3}}\Big) + CR^{-\frac{2}{3}}. \tag{4.10}$$

Since $\beta \le 9$ for $N \ge 5$, again we obtain a contradiction, thus proving the lemma. □

Lemma 4.6 *Let (u,v) be a non-negative $C^4(\mathbb{R}^N)$ solution of system* (1.1)*, with $1 \le \alpha, \beta \le \frac{N+4}{N-4}$ and $\alpha\beta \neq 1$. Then*

$$|x|^{4-N}v^\alpha(x) \quad \textit{and} \quad |x|^{4-N}u^\beta(x) \in L^1(\mathbb{R}^N \setminus B_1(0)).$$

For the proof of this lemma we proceed as in [11, Proposition 3.5], thanks to Lemma 4.5.

5 Proof of Theorem 1.2

Let (u,v) be a $C^4(\mathbb{R}^N)$ non-negative, nontrivial solution of system (1.1). By Theorem 3.1 and the maximum principle, we have

$$u(x) > 0 \quad \text{and} \quad v(x) > 0 \quad \text{in } \mathbb{R}^N. \tag{5.1}$$

So (u,v) is a positive solution of system (1.1). We introduce their Kelvin transforms

$$u^*(x)=|x|^{4-N}u(\frac{x}{|x|^2}),\quad v^*(x)=|x|^{4-N}v(\frac{x}{|x|^2}),$$

for $x\in\mathbb{R}^N\setminus\{0\}$. Then

$$\Delta u^*(x)=|x|^{-N}\Delta u(\frac{x}{|x|^2})-\frac{4}{|x|^{N-2}}(\nabla u(\frac{x}{|x|^2}),\frac{x}{|x|^2})-2(N-4)|x|^{2-N}u(\frac{x}{|x|^2})$$

and at infinity,

$$\Delta u^*(x)=-2(N-4)|x|^{2-N}u_0-\sum_{j=1}^N\frac{a_j}{|x|^N}x_j+O(\frac{1}{|x|^N}),$$

where $u_0=u(0)$, $a_i=\frac{\partial u}{\partial x_i}(0)$. Consequently, for large $|x|$, $\Delta u^*(x)<0$. Similarly to Δv^*. Without loss of generality, we assume that

$$\Delta u^*(x)\le 0\quad\text{and}\quad\Delta v^*(x)\le 0,\quad\forall x\in\mathbb{R}^N\setminus B_1(0).\tag{5.2}$$

In order to prove that $\Delta u^*(x)$ and $\Delta v^*(x)$ are non-positive in $B_1(0)\setminus\{0\}$, we begin with an auxiliary lemma.

Lemma 5.1 *Let $f\in C^2(\overline{B}_1(0)\setminus\{0\})$ be such that $f=0$ on $\partial B_1(0)$ and*

$$\int_{B_1(0)}f\Delta\varphi\,dx\ge 0,\tag{5.3}$$

for all $\varphi\in C^2(B_1(0))\cap C^1(\overline{B}_1(0))$ such that $\varphi=0$ on $\partial B_1(0)$ and $\varphi\ge 0$ in $B_1(0)$. Then $f\le 0$ in $B_1(0)\setminus\{0\}$.

Proof. Suppose by contradiction that $f>0$ over some ball $B\subset B_1(0)\setminus\{0\}$. Fix any nonzero, non-negative function $\psi\in\mathcal{D}(B)$ and denote by $\tilde\psi$ its extension by zero. Consider $\varphi\in C^2(B_1(0))\cap C^1(\overline{B}_1(0))$, $\varphi\ge 0$, such that $-\Delta\varphi=\tilde\psi$ in $B_1(0)$, $\varphi=0$ on $\partial B_1(0)$. By the maximum principle, $\varphi\ge 0$. Then

$$0<\int_B\psi f\,dx=\int_{B_1(0)}\tilde\psi f\,dx=-\int_{B_1(0)}f\Delta\varphi\,dx.$$

This contradicts (5.3). $\square$

The argument for the proof of the next lemma was partially taken from [6]. Since we could not find a precise reference for the complete proof, we present here a proof pointed to us by Prof. J.Q. Liu, to whom we acknowledge.

Lemma 5.2 *Let (u,v) be a $C^4(\mathbb{R}^N)$ positive solution of system* (1.1). *Then (u^*,v^*) satisfies*

$$\begin{aligned}\Delta^2u^*&=|x|^{\alpha(N-4)-(N+4)}(v^*)^\alpha\\ \Delta^2v^*&=|x|^{\beta(N-4)-(N+4)}(u^*)^\beta\end{aligned}\tag{5.4}$$

in $\mathbb{R}^N\setminus\{0\}$. Moreover

$$\Delta u^*(x)<0\quad\textit{and}\quad\Delta v^*(x)<0,\tag{5.5}$$

for all $x\in\mathbb{R}^N\setminus\{0\}$.

Proof. An easy computation allows us to establish (5.4). From (5.2) we are led to prove the second conclusion in $B_1(0)\backslash\{0\}$. Let $w \in C^2(B_1(0))\cap C^1(\overline{B}_1(0))$ be such that $\Delta w = 0$ in $B_1(0)$, $w = \Delta u^*$ on $\partial B_1(0)$. By the maximum principle, $w \leq 0$ in $B_1(0)$. Let $\varphi \in C^2(B_1(0)) \cap C^1(\overline{B}_1(0))$ be such that $\varphi = 0$ on $\partial B_1(0)$ and $\varphi \geq 0$ in $B_1(0)$ and, for each $\varepsilon > 0$, let $\eta_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$ be such that for $i = 1, 2, 3, 4$,

$$\eta_\varepsilon(x) = 1 \quad \text{for } |x| \geq 2\varepsilon, \quad \eta_\varepsilon(x) = 0 \quad \text{for } |x| \leq \varepsilon, \quad |D^i\eta_\varepsilon| \leq c\varepsilon^{-i},$$

where c is a positive constant independent of ε. From now on we denote $B_s = B_s(0)$, for $s = R, 2\varepsilon, \varepsilon$. Multiplying the first equation of (5.4) by $\varphi\eta_\varepsilon$, we get

$$\begin{aligned}\int_{B_1} \varphi\eta_\varepsilon |x|^{\alpha(N-4)-(N+4)}(v^*)^\alpha(x)\,dx &= \int_{B_1} \varphi\eta_\varepsilon \Delta^2 u^*\,dx \\ &= \int_{B_1} \varphi\eta_\varepsilon \Delta(\Delta u^* - w)\,dx.\end{aligned}$$

After two integrations by parts in the last integral and observing that both $\varphi\eta_\varepsilon$ and $(\Delta u^* - w)$ vanish on the boundary of $B_1(0)$, we obtain

$$\begin{aligned}&\int_{B_1} \varphi\eta_\varepsilon |x|^{\alpha(N-4)-(N+4)}(v^*)^\alpha(x)\,dx \\ &= \int_{B_1} \{\eta_\varepsilon\Delta\varphi + 2(\nabla\varphi, \nabla\eta_\varepsilon) + \varphi\Delta\eta_\varepsilon\}(\Delta u^* - w)\,dx. \quad (5.6)\end{aligned}$$

Let $\psi = 2(\nabla\varphi, \nabla\eta_\varepsilon) + \varphi\Delta\eta_\varepsilon$. So $\psi = 0$ for $|x| \leq \varepsilon$ and $|x| \geq 2\varepsilon$. Then, for small ε,

$$\begin{aligned}\int_{B_1} \psi\Delta u^*\,dx &= \int_{B_1} u^*\Delta\psi\,dx = \int_{B_{2\varepsilon}\backslash B_\varepsilon} u^*\Delta\psi\,dx \\ &= \int_{B_{2\varepsilon}\backslash B_\varepsilon} |x|^{\frac{\beta(N-4)-(N+4)}{\beta}} u^*(x)|x|^{\frac{(N+4)-\beta(N-4)}{\beta}}\Delta\psi\,dx \\ &\leq \Big(\int_{B_{2\varepsilon}\backslash B_\varepsilon} |x|^{\beta(N-4)-(N+4)}(u^*)^\beta(x)\,dx\Big)^{1/\beta} \\ &\quad\times\Big(\int_{B_{2\varepsilon}\backslash B_\varepsilon} |x|^{\frac{(N+4)-\beta(N-4)}{\beta-1}}|\Delta\psi|^{\frac{\beta}{\beta-1}}\,dx\Big)^{(\beta-1)/\beta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{B_1} |x|^{\beta(N-4)-(N+4)}(u^*)^\beta(x)dx &= \int_{B_1} |x|^{-(N+4)}u^\beta(\frac{x}{|x|^2})dx \\ &= \int_{\mathbb{R}^N\backslash B_1} |y|^{-(N+4)}u^\beta(y)dy.\end{aligned}$$

Thus, from Lemma 4.6 there exists a constant c such that

$$|\int_{B_{2\varepsilon}\backslash B_\varepsilon} \psi\Delta u^*\,dx| \leq c\Big(\int_{B_{2\varepsilon}\backslash B_\varepsilon} |x|^{\frac{(N+4)-\beta(N-4)}{\beta-1}}|\Delta\psi|^{\frac{\beta}{\beta-1}}\,dx\Big)^{(\beta-1)/\beta}.$$

Since $|\Delta\psi| \leq c\varepsilon^{-4}$ for ε small, we get

$$|\int_{B_{2\varepsilon}\setminus B_\varepsilon} \psi\Delta u^* \, dx| \leq c\varepsilon^{4/\beta} \to 0 \quad \text{as } \varepsilon \to 0.$$

Then, passing to the limit in (5.6) yields

$$\int_{B_1} \Delta\varphi(\Delta u^* - w)\, dx = \int_{B_1} \varphi |x|^{\alpha(N-4)-(N+4)} (v^*)^\alpha(x)\, dx > 0.$$

From Lemma 5.1 we conclude that $\Delta u^* \leq w \leq 0$ in $B_1(0)\setminus\{0\}$. Since $\Delta^2 u^* > 0$, by the maximum principle we must have $\Delta u^* < 0$ in $\mathbb{R}^N \setminus \{0\}$. One can proceed similarly with Δv^*. □

We now apply the moving planes method. We start by considering planes parallel to $x_1 = 0$, coming from $-\infty$. From now on, for $x \in \mathbb{R}^N$ we write $x = (x_1, x')$ where $x' := (x_2, \cdots, x_N) \in \mathbb{R}^{N-1}$. For each λ we define

$$\Sigma_\lambda := \{x := (x_1, x') \in \mathbb{R}^N : x_1 < \lambda\}, \quad T_\lambda := \partial\Sigma_\lambda.$$

For $x = (x_1, x') \in \Sigma_\lambda$, let $x^\lambda := (2\lambda - x_1, x')$ be the reflected point with respect to T_λ. We also consider

$$e_\lambda := (2\lambda, 0) \quad \text{and} \quad \widetilde{\Sigma}_\lambda := \Sigma_\lambda \setminus \{e_\lambda\}.$$

Finally, for $x \in \widetilde{\Sigma}_\lambda$ we define $U_\lambda(x) = u^*(x^\lambda) - u^*(x)$ and $V_\lambda(x) = v^*(x^\lambda) - v^*(x)$.

In what follows we take $\lambda \leq 0$. Using the invariance of the Laplacian under a reflection together with the mean value theorem and the fact that $|x^\lambda| \leq |x|$, it follows from (5.4) that

$$\begin{aligned} \Delta^2 U_\lambda &\geq c(x,\lambda) V_\lambda(x) \\ \Delta^2 V_\lambda &\geq \hat{c}(x,\lambda) U_\lambda(x), \end{aligned} \tag{5.7}$$

for $x \in \widetilde{\Sigma}_\lambda$, where $c(x;\lambda) = \alpha |x|^{\alpha(N-4)-(N+4)} (\psi(x,\lambda))^{\alpha-1}$, with $\psi(x;\lambda)$ a real number between $v^*(x^\lambda)$ and $v^*(x)$, and similarly

$$\hat{c}(x;\lambda) = \beta |x|^{\beta(N-4)-(N+4)} (\hat{\psi}(x,\lambda))^{\beta-1},$$

with $\hat{\psi}(x;\lambda)$ a real number between $u^*(x^\lambda)$ and $u^*(x)$. From (5.1) we conclude that both $c(x;\lambda)$ and $\hat{c}(x;\lambda)$ are positive.

Our next goal is to see that we can start the process of the moving planes. For that matter we begin with some auxiliary facts.

Definition. Let $m \in \mathbb{N}$. We say that a C^2 function in a neighborhood of infinity f has a *harmonic asymptotic expansion* at infinity if:

$$\begin{aligned} f(x) &= \frac{1}{|x|^m}\Big(a_0 + \sum_{i=1}^{m+2} \frac{a_i x_i}{|x|^2}\Big) + O\Big(\frac{1}{|x|^{m+2}}\Big), \\ f_{x_i}(x) &= -m a_0 \frac{x_i}{|x|^{m+2}} + O\Big(\frac{1}{|x|^{m+2}}\Big), \\ f_{x_i,x_j}(x) &= O\Big(\frac{1}{|x|^{m+2}}\Big) \end{aligned} \tag{5.8}$$

where $a_i \in \mathbb{R}$, for $i = 0, \dots, m+2$.

We observe that both u^* and v^* have harmonic asymptotic expansions at infinity, with $m = N-4$ and $a_0 > 0$. Also $(-\Delta u^*)$ and $(-\Delta v^*)$ have harmonic asymptotic expansions at infinity with $m = N-2$ and $a_0 > 0$.

Lemma 5.3 *Let f be a function in a neighborhood of infinity satisfying the asymptotic expansion* (5.8)*, with $a_0 > 0$. Then there exist constants $\tilde{\lambda}_1 < 0$, $R_1 > 0$ such that, if $\lambda \le \tilde{\lambda}_1$, then*

$$f(x) < f(x^\lambda) \quad \textit{for} \quad x_1 < \lambda, \quad x \notin B_{R_1}(e_\lambda).$$

Lemma 5.4 *Let f be a C^2 positive solution of $-\Delta f = F(x)$ in $|x| > R$, where f has a harmonic asymptotic expansion (5.8) at infinity, with $a_0 > 0$. Suppose that, for some negative λ_0 and for every (x_1, x') with $x_1 < \lambda_0$,*

$$f(x_1, x') < f(2\lambda_0 - x_1, x') \quad \textit{and} \quad F(x_1, x') \le F(2\lambda_0 - x_1, x').$$

Then there exist $\varepsilon > 0$, $S > R$ such that

(i) $f_{x_1}(x_1, x') > 0$ in $|x_1 - \lambda_0| < \varepsilon$, $|x| > S$,

(ii) $f(x_1, x') < f(2\lambda - x_1, x')$ in $x_1 < \lambda_0 - \frac{1}{2}\varepsilon < \lambda$, $|x| > S$,

for all $x \in \Sigma_\lambda$, $\lambda \ge \lambda_1$ with $|\lambda_1 - \lambda_0| < c_0\varepsilon$, where c_0 is a positive number depending on λ_0 and f.

We refer the reader to [1, Lemmas 2.3 and 2.4] for the proof.

Proposition 5.5 *There exists $\lambda_1 < 0$ such that if $\lambda \le \lambda_1$ then $\Delta U_\lambda(x) < 0$, $\Delta V_\lambda(x) < 0$, $U_\lambda(x) > 0$ and $V_\lambda(x) > 0$ in $\widetilde{\Sigma}_\lambda$.*

Proof. From Lemma 5.3, there exists $\tilde{\lambda}_1 < 0$ and $R_1 > 0$ such that $\Delta U_\lambda(x) < 0$, $\Delta V_\lambda(x) < 0$, $U_\lambda(x) > 0$ and $V_\lambda(x) > 0$ in $\widetilde{\Sigma}_\lambda \setminus B_{R_1}(e_\lambda)$, for all $\lambda \le \tilde{\lambda}_1$.

By Lemma 5.2, $\Delta u^*(x) < 0$ in $\mathbb{R}^N \setminus \{0\}$. Since

$$\Delta(\Delta u^*)(x) = |x|^{\alpha(N-4)-(N+4)}(v^*)^\alpha(x) > 0,$$

Lemma 2.1 allows us to conclude that

$$\Delta u^*(x) \le M(R_1) := \max\{\Delta u^*(y) : |y| = R_1\}, \quad \forall x : 0 < |x| < R_1.$$

If $x \in B_{R_1}(e_\lambda) \setminus (e_\lambda)$ then $|x - e_\lambda| = |x^\lambda| < R_1$. So, for $x \in B_{R_1}(e_\lambda) \setminus (e_\lambda)$ we have $\Delta u^*(x^\lambda) \le M(R_1)$. From the fact that $\Delta u^*(x) \to 0$ as $|x| \to +\infty$, we conclude that there exists $R_2 > 0$ such that $\Delta u^*(x) > M(R_1)/2$, for $|x| > R_2$. Let $\bar{\lambda}_1 := \min\{-R_1, -R_2\}$. Then, for all $\lambda < \bar{\lambda}_1$,

$$\Delta U_\lambda(x) = \Delta u^*(x^\lambda) - \Delta u^*(x) < M(R_1) - \frac{M(R_1)}{2} < 0,$$

for $x \in B_{R_1}(e_\lambda) \setminus (e_\lambda)$. Similarly, let $\bar\lambda_2, \bar\lambda_3, \bar\lambda_4 < 0$ be such that

$$\Delta V_\lambda(x) = \Delta v^*(x^\lambda) - \Delta v^*(x) < M'(R_1) - M'(R_1)/2 < 0, \text{ for } x \in B_{R_1}(e_\lambda)$$

for all $\lambda < \bar\lambda_2$,

$$U_\lambda(x) = u^*(x^\lambda) - u^*(x) > m(R_1) - m(R_1)/2 > 0$$

for $x \in B_{R_1}(e_\lambda)$ and all $\lambda < \bar\lambda_3$,

$$V_\lambda(x) = v^*(x^\lambda) - v^*(x) > m'(R_1) - m'(R_1)/2 > 0$$

for $x \in B_{R_1}(e_\lambda)$ and all $\lambda < \bar\lambda_4$, where

$$M'(R_1) := \max\{\Delta v^*(y) : |y| = R_1\}, \quad m(R_1) := \min\{u^*(y) : |y| = R_1\},$$
$$m'(R_1) := \min\{v^*(y) : |y| = R_1\}.$$

By choosing $\lambda_1 = \min\{\bar\lambda_1, \bar\lambda_2, \bar\lambda_3, \bar\lambda_4, \tilde\lambda_1\}$ we get the conclusion. □

Let $\lambda_0 := \sup\{\lambda < 0 : \Delta U_\lambda(x) < 0,\ \Delta V_\lambda(x) < 0,\ U_\lambda(x) > 0, \text{ and } V_\lambda(x) > 0 \text{ in } \widetilde\Sigma_\lambda\}$.

Remark. By continuity, we have $\Delta U_{\lambda_0}(x) \le 0$, $\Delta V_{\lambda_0}(x) \le 0$, $U_{\lambda_0}(x) \ge 0$ and $V_{\lambda_0}(x) \ge 0$ in $\widetilde\Sigma_{\lambda_0}$.

Lemma 5.6 *$U_{\lambda_0} \equiv 0$ if and only if $V_{\lambda_0} \equiv 0$.*

Proof. If $U_{\lambda_0} \not\equiv 0$ and $V_{\lambda_0} \equiv 0$, by (5.7) we have $\hat c(x,\lambda_0) U_{\lambda_0}(x) \le 0$. Since $\hat c(x,\lambda_0) > 0$, then $U_{\lambda_0} \le 0$. Since also $U_{\lambda_0} > 0$, this is a contradiction. □

Proposition 5.7 *If $\lambda_0 < 0$ then $U_{\lambda_0} \equiv 0$ and $V_{\lambda_0} \equiv 0$.*

Proof. Suppose by contradiction that the conclusion of the proposition is not true. By Lemma 5.6 we conclude that $U_{\lambda_0} \not\equiv 0$ and $V_{\lambda_0} \not\equiv 0$. Since

$$\begin{gathered}\Delta U_{\lambda_0} \le 0 \quad \text{in } \widetilde\Sigma_{\lambda_0}\\ U_{\lambda_0} \ge 0, \quad U_{\lambda_0} \not\equiv 0 \quad \text{in } \widetilde\Sigma_{\lambda_0}\\ U_{\lambda_0} = 0 \quad \text{on } T_{\lambda_0}\end{gathered}$$

and since $U_{\lambda_0}(x) \to 0$ when $|x| \to \infty$, by the maximum principle we have that $U_{\lambda_0}(x) > 0$ in $\widetilde\Sigma_{\lambda_0}$. By the same arguments we can prove that $V_{\lambda_0}(x) > 0$ in $\widetilde\Sigma_{\lambda_0}$. Then

$$\Delta^2 U_{\lambda_0} \ge c(x,\lambda_0) V_{\lambda_0} > 0 \quad \text{in } \widetilde\Sigma_{\lambda_0} \tag{5.9}$$

and, as a consequence,

$$\begin{gathered}\Delta^2 U_{\lambda_0} > 0 \quad \text{in } \widetilde\Sigma_{\lambda_0}\\ \Delta U_{\lambda_0} \le 0 \quad \text{in } \widetilde\Sigma_{\lambda_0}\\ \Delta U_{\lambda_0} = 0 \quad \text{on } T_{\lambda_0}.\end{gathered}$$

Since $\Delta U_{\lambda_0}(x) \to 0$ when $|x| \to 0$, by the maximum principle we must have $\Delta U_{\lambda_0}(x) < 0$ in $\widetilde{\Sigma}_{\lambda_0}$. Using the Hopf maximum principle we obtain that

$$\frac{\partial \Delta U_{\lambda_0}}{\partial \nu}(x) > 0 \quad \text{on } T_{\lambda_0},$$

where ν is the outward unit normal to $\widetilde{\Sigma}_{\lambda_0}$. We will prove that this is impossible.

From the definition of λ_0, there exists a sequence of real numbers $\lambda_n \searrow \lambda_0$ and a sequence of points in $\widetilde{\Sigma}_{\lambda_n}$ where ΔU_{λ_n} or ΔV_{λ_n} is positive or U_{λ_n} or V_{λ_n} is negative.

If $\Delta U_{\lambda_n}(x) > 0$ for some $x \in \widetilde{\Sigma}_{\lambda_n}$, then

$$c_1 := \sup_{\widetilde{\Sigma}_{\lambda_n}} \Delta U_{\lambda_n} > 0.$$

We shall see that this supreme is attained. Let

$$c_2 := \max_{\partial B_{\frac{\lambda_0}{2}}(e_{\lambda_0})} \Delta U_{\lambda_0} < 0.$$

With $0 < r < \lambda_0/2$, we define $g(x) = \Delta U_{\lambda_0}(x) + c_2(|x - e_{\lambda_0}|^{2-N} r^{N-2} - 1)$, for $x \in B_{\lambda_0/2}(e_{\lambda_0}) \setminus B_r(e_{\lambda_0})$. It is easy to see that $g(x) \le 0$ in $\partial(B_{\lambda_0/2}(e_{\lambda_0}) \setminus B_r(e_{\lambda_0}))$. Then we have

$$\begin{gathered}\Delta g = \Delta^2 U_{\lambda_0} > 0 \quad \text{in } B_{\lambda_0/2}(e_{\lambda_0}) \setminus B_r(e_{\lambda_0}) \\ g \le 0 \quad \text{on } \partial(B_{\lambda_0/2}(e_{\lambda_0}) \setminus B_r(e_{\lambda_0})).\end{gathered}$$

By the maximum principle, $g(x) \le 0$ for all $x \in B_{\lambda_0/2}(e_{\lambda_0}) \setminus B_r(e_{\lambda_0})$. Since r is arbitrary, we conclude that

$$\Delta U_{\lambda_0}(x) \le c_2 < 0 \quad \text{in } \dot{B}_{\lambda_0/2}(e_{\lambda_0}),$$

where $\dot{B}_s(e_{\lambda_0})$ denotes the punctured ball $B_s(e_{\lambda_0}) \setminus \{e_{\lambda_0}\}$, for $s > 0$. By continuity, we have

$$\Delta U_{\lambda_n}(x) \le \frac{c_2}{2} < 0 \quad \text{in } \dot{B}_{\lambda_0/4}(e_{\lambda_n}),$$

for large n. As $\Delta U_{\lambda_n}(x) \to 0$ when $|x| \to +\infty$, there exists r_n such that, for all $|x| \ge r_n$, $\Delta U_{\lambda_n}(x) < c_1/2$. Thus

$$\sup_{\widetilde{\Sigma}_{\lambda_n}} \Delta U_{\lambda_n} = \sup \big\{ \Delta U_{\lambda_n}(x) : x \in (\widetilde{\Sigma}_{\lambda_n} \setminus B_{\lambda_0/4}(e_{\lambda_n})) \cap B_{r_n}(0) \big\}.$$

Then there exists a sequence $(x_n) \subset (\widetilde{\Sigma}_{\lambda_n} \setminus B_{\lambda_0/4}(e_{\lambda_n})) \cap B_{r_n}(0)$ such that

$$\sup_{\widetilde{\Sigma}_{\lambda_n}} \Delta U_{\lambda_n} = \Delta U_{\lambda_n}(x_n) > 0.$$

Hence

$$\nabla(\Delta U_{\lambda_n})(x_n) = 0 \quad \text{and} \quad \Delta(\Delta U_{\lambda_n})(x_n) \le 0. \tag{5.10}$$

It follows from Lemma 5.3 that (x_n) is bounded. Thus, up to a subsequence, $x_n \to x_0$ with $x_0 \in \overline{\widetilde{\Sigma}}_{\lambda_0}$. Passing (5.10) to the limit we obtain

$$\nabla(\Delta U_{\lambda_0})(x_0) = 0 \quad \text{and} \quad \Delta(\Delta U_{\lambda_0})(x_0) \leq 0. \tag{5.11}$$

Thus from (5.9) and (5.11) we conclude that $x_0 \in T_{\lambda_0}$. Since

$$0 < \frac{\partial \Delta U_{\lambda_0}}{\partial \nu}(x_0) = \frac{\partial \Delta U_{\lambda_0}}{\partial x_1}(x_0) = 0,$$

we have a contradiction.

The case when ΔV_{λ_n} takes positive values and the cases when U_{λ_n} and V_{λ_n} take negative values are proved similarly. □

Proof of Theorem 1.2 completed. **I)** In applying the moving planes method, we must consider two cases.

(a) If $\lambda_0 < 0$, by Proposition 5.7 we have $U_{\lambda_0} \equiv 0$ and $V_{\lambda_0} \equiv 0$, so $u^*(x)$ and $v^*(x)$ are symmetric with respect to the plane T_{λ_0}. Since the bilaplacian is invariant for dilations, from (5.4) we get a contradiction. So $u = v = 0$ in $\mathbb{R}^N$.

(b) If $\lambda_0 = 0$ then $U_0(x) \geq 0$ and $V_0(x) \geq 0$ in $\widetilde{\Sigma}_0$, i.e.

$$u(-x_1, x') \geq u(x_1, x') \text{ and } v(-x_1, x') \geq v(x_1, x') \quad \text{for } x_1 \leq 0. \tag{5.12}$$

Defining $\bar{u}(x_1, x') := u(-x_1, x)$ and $\bar{v}(x_1, x') := v(-x_1, x)$, we have

$$\begin{aligned} \Delta^2 \bar{u} &= \bar{v}^\alpha \\ \Delta^2 \bar{v} &= \bar{u}^\beta \end{aligned}$$

in $\mathbb{R}^N$. By performing the latter procedure, we deduce the existence of a corresponding value $\bar{\lambda}_0 \leq 0$. If $\bar{\lambda}_0 < 0$ then $\bar{u} = \bar{v} = 0$ and consequently $u = v = 0$. If $\bar{\lambda}_0 = 0$ then

$$\bar{u}(-x_1, x') \geq \bar{u}(x_1, x') \quad \text{and} \quad \bar{v}(-x_1, x') \geq \bar{v}(x_1, x') \quad \text{for } x_1 \leq 0.$$

By (5.12) we conclude that u and v are radially symmetric with respect to the origin.

We can perform the Kelvin transform with respect to any point, thus u and v are radially symmetric with respect to any point. This implies that u and v are constant functions. From system (1.1), we get $u = v = 0$.

II) We proceed like Figueiredo and Felmer in [2]. □

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