Electronic Journal of Differential Equations, Vol. 2005(2005), No. 05, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLICITY OF SYMMETRIC SOLUTIONS FOR A NONLINEAR EIGENVALUE PROBLEM IN $\mathbb{R}^n$

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Abstract. In this paper, we study the nonlinear eigenvalue field equation
$$-\Delta u + V(|x|)u + \varepsilon(-\Delta_p u + W'(u)) = \mu u$$
where u is a function from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with $n \geq 3$, ε is a positive parameter and $p > n$. We find a multiplicity of solutions, symmetric with respect to an action of the orthogonal group $O(n)$: For any $q \in \mathbb{Z}$ we prove the existence of finitely many pairs (u, μ) solutions for ε sufficiently small, where u is symmetric and has topological charge q. The multiplicity of our solutions can be as large as desired, provided that the singular point of W and ε are chosen accordingly.

1. Introduction

In this paper, we find infinitely many solutions of the nonlinear eigenvalue field equation
$$-\Delta u + V(|x|)u + \varepsilon(-\Delta_p u + W'(u)) = \mu u\,, \tag{1.1}$$
where u is a function from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with $n \geq 3$, ε is a positive parameter and $p \in \mathbb{N}$ with $p > n$.

The choice of the nonlinear operator $-\Delta_p + W'$ is very important. The presence of the p-Laplacian comes from a conjecture by Derrick (see [14]). He was looking for a model for elementary particles, which extended the features of the sine-Gordon equation in higher dimension; he showed that equation
$$-\Delta u + W'(u) = 0$$
has no nontrivial stable localized solutions for any $W \in C^1$ on $\mathbb{R}^n$ with $n \geq 2$. He proposed then to consider a higher power of the derivatives in the Lagrangian function and this has been done for the first time in [6]. So the p-Laplacian is responsible for the existence of nontrivial solutions. As concerns W', it denotes the gradient of a function W, which is singular in a point: this fact constitutes a sort of topological constraint and permits to characterize the solutions of (1.1) by a topological invariant, called topological charge (see [6]).

The free problem
$$-\Delta u - \varepsilon\Delta_6 u + W'(u) = 0$$

2000 *Mathematics Subject Classification.* 35Q55, 45C05.

Key words and phrases. Nonlinear Schrödinger equations; nonlinear eigenvalue problems.

Submitted October 22, 2004. Published January 2, 2005.
Supported by M.U.R.S.T., project "Metodi variazionali e topologici nello studio di fenomeni non lineari".

has been studied in [6], while the concentration of the solutions has been considered in [1]. In [7] and [8] the authors have studied problem (1.1) respectively in a bounded domain and in $\mathbb{R}^n$. In [3] the authors have proved the existence of infinitely many solutions of the free problem, which are symmetric with respect to the action of the orthogonal group $O(n)$.

In this paper, we find a multiplicity of solutions, symmetric with respect to the action considered in [3], of problem (1.1) in $\mathbb{R}^n$: For any $q \in \mathbb{Z}$ we prove the existence of finitely many pairs (u, μ) solutions of problem (1.1) for ε sufficiently small, where u is symmetric and has topological charge q. The multiplicity of the solutions can be as large as one wants, provided that the singular point $\xi_\star = (\xi_0, 0)$ ($\xi_0 \in \mathbb{R}$, $0 \in \mathbb{R}^n$) of W and ε are chosen accordingly.

The basic idea is to consider problem (1.1) as a perturbation of the linear problem when $\epsilon = 0$. In terms of the associated energy functionals, one passes from the non-symmetric functional J_ϵ (defined in (2.10)) to the symmetric functional J_0. Non-symmetric perturbations of a symmetric problem, in order to preserve critical values, have been studied by several authors. We recall only [2], which seems to be the first work on the subject, and the papers [10] and [11].

In fact, the existence result is a result of preservation for the functional J_ϵ of some critical values of the functional J_0, constrained on the unitary sphere of $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$.

Since the topological charge divides the domain Λ of the energy functional J_ϵ into connected components Λ_q with $q \in \mathbb{Z}$, the solutions are found in each connected component and in two different ways: as minima and as min-max critical points of the energy functional constrained on the unitary sphere of $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$. More precisely we can state:

Given $q \in \mathbb{Z}$, for any $\xi_\star = (\xi_0, 0)$ (with $\xi_0 > 0$ and $0 \in \mathbb{R}^n$) and for any $\varepsilon > 0$, there exist $\mu_1(\varepsilon)$ and $u_1(\varepsilon)$ respectively eigenvalue and eigenfunction of the problem (1.1)*, such that the topological charge of $u_1(\varepsilon)$ is q.*

Moreover, given $q \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$, we consider $\xi_\star = (\xi_0, 0)$ with ξ_0 large enough and $0 \in \mathbb{R}^n$. Let λ_j be the eigenvalues of the linear problem (1.1) *with $\epsilon = 0$. Then for ε sufficiently small and for any $j \le k$ with $\lambda_{j-1} < \lambda_j$, there exist $\mu_j(\varepsilon)$ and $u_j(\varepsilon)$ respectively eigenvalue and eigenfunction of the problem* (1.1)*, such that the topological charge of $u_j(\varepsilon)$ is q.*

2. Functional setting

Statement of the problem. We consider from now on the field equation

$$-\Delta u + V(|x|)u + \varepsilon^r(-\Delta_p u + W'(u)) = \mu u\,, \tag{2.1}$$

where u is a function from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with $n \ge 3$, ϵ is a positive parameter and p, $r \in \mathbb{N}$ with $p > n$ and $r > p - n$ (for technical reasons we prefer to re-scale the parameter ϵ). The function V is real and we denote with W' the gradient of a function $W : \mathbb{R}^{n+1} \setminus \{\xi_\star\} \to \mathbb{R}$, where $\xi_\star$ is a point of $\mathbb{R}^{n+1}$, different from the origin, which for simplicity we choose on the first component:

$$\xi_\star = (\xi_0, 0)\,, \tag{2.2}$$

with $\xi_0 \in \mathbb{R}$, $\xi_0 > 0$ and $0 \in \mathbb{R}^n$.

Throughout the paper, we assume the following hypotheses on the function $V : [0, +\infty) \to \mathbb{R}$:

(V1) $\lim_{r\to+\infty} V(r) = +\infty$
(V2) $V(|x|)e^{-|x|} \in L^p(\mathbb{R}^n, \mathbb{R})$
(V3) $\operatorname{ess\,inf}_{r\in[0,+\infty)} V(r) > 0$

The assumptions on the function $W : \mathbb{R}^{n+1} \setminus \{\xi_\star\} \to \mathbb{R}$ are as follows:

(W1) $W \in C^1(\mathbb{R}^{n+1} \setminus \{\xi_\star\}, \mathbb{R})$
(W2) $W(\xi) \geq 0$ for all $\xi \in \mathbb{R}^{n+1} \setminus \{\xi_\star\}$ and $W(0) = 0$
(W3) There exist two constants c_1, $c_2 > 0$ such that
$$\xi \in \mathbb{R}^{n+1},\ 0 < |\xi| < c_1 \Longrightarrow W(\xi_\star + \xi) \geq \frac{c_2}{|\xi|^{\frac{np}{p-n}}}$$
(W4) There exist two constants c_3, $c_4 > 0$ such that
$$\xi \in \mathbb{R}^{n+1},\ 0 \leq |\xi| < c_3 \Longrightarrow |W'(\xi)| \leq c_4|\xi|\,.$$
(W5) For all $\xi \in \mathbb{R}^{n+1} \setminus \{\xi_\star\}$, $\xi = (\xi^1, \tilde{\xi})$ with $\xi^1 \in \mathbb{R}$, $\tilde{\xi} \in \mathbb{R}^n$ and for all $g \in O(n)$, there holds
$$W(\xi^1, g\tilde{\xi}) = W(\xi^1, \tilde{\xi})\,.$$

The space E. We define the following functional spaces:
$\Gamma(\mathbb{R}^n, \mathbb{R})$ is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with respect to the norm
$$\|z\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R})} = \int_{\mathbb{R}^n} [V(|x|)\,|z(x)|^2 + |\nabla z(x)|^2]dx \tag{2.3}$$
$\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm
$$\|u\|^2_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})} = \int_{\mathbb{R}^n} [V(|x|)\,|u(x)|^2 + |\nabla u(x)|^2]dx\,. \tag{2.4}$$
For $s \geq 1$, we set
$$\|\nabla u\|^s_{L^s} = \int_{\mathbb{R}^n} |\nabla u|^s dx = \sum_{i=1}^{n+1} \|\nabla u^i\|^s_{L^s(\mathbb{R}^n,\mathbb{R}^n)} \tag{2.5}$$
with $u = (u^1, u^2, \dots, u^{n+1})$.

The spaces $\Gamma(\mathbb{R}^n, \mathbb{R})$ and $\Gamma(\mathbb{R}^n, \mathbb{R}^{n+1})$ are Hilbert spaces, with scalar products
$$(z_1, z_2)_{\Gamma(\mathbb{R}^n,\mathbb{R})} = \int_{\mathbb{R}^n} [V(|x|)\, z_1 z_2 + \nabla z_1 \cdot \nabla z_2]\, dx\,, \tag{2.6}$$
$$(u_1, u_2)_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})} = \int_{\mathbb{R}^n} [V(|x|)\, u_1 \cdot u_2 + \nabla u_1 \cdot \nabla u_2]\, dx\,. \tag{2.7}$$
We recall a compact embedding theorem (see for example [4]) into L^2.

Theorem 2.1. *The embedding of the space $\Gamma(\mathbb{R}^n, \mathbb{R})$ into the space $L^2(\mathbb{R}^n, \mathbb{R})$ is compact.*

We define the Banach space E as the completion of the space $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1})$ with respect to the norm
$$\|u\|_E = \|u\|_{\Gamma(\mathbb{R}^n,\mathbb{R}^{n+1})} + \|\nabla u\|_{L^p}\,. \tag{2.8}$$
The space E satisfies some useful properties which are listed in the next proposition. They follow from Sobolev embedding theorem and from [3, Proposition 8].

Proposition 2.2. *The Banach space E has the following properties:*

(1) *It is continuously embedded into $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for $2 \leq s \leq +\infty$;*

(2) *It is continuously embedded into* $W^{1,p}(\mathbb{R}^n,\mathbb{R}^{n+1})$*;*
(3) *There exist two constants* C_0*,* $C_1>0$ *such that for every* $u\in E$

$$\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}\le C_0\|u\|_E\,,$$
$$|u(x)-u(y)|\le C_1|x-y|^{1-\frac{n}{p}}\|u\|_{W^{1,p}(\mathbb{R}^n,\mathbb{R}^{n+1})}\,;$$

(4) *If* $u\in E$ *then* $\lim_{|x|\to\infty}u(x)=0$*.*

The energy functional J_ϵ. In the space E, by Proposition 2.2, it is possible to consider the open subset

$$\Lambda=\{u\in E:\xi_\star\notin u(\mathbb{R}^n)\}\,. \tag{2.9}$$

On Λ we consider the functional

$$J_\epsilon(u)=\int_{\mathbb{R}^n}\big[\frac{1}{2}|\nabla u|^2+\frac{1}{2}V(|x|)|u|^2+\frac{\epsilon^r}{p}|\nabla u|^p+\epsilon^rW(u)\big]\,dx\,, \tag{2.10}$$

which is the energy functional associated to the problem (2.1).

It is easy to verify the following lemma (see Lemma 2.3 of [8]).

Lemma 2.3. *The functional* J_ϵ *is of class* C^1 *on the open set* Λ *of* E*.*

The topological charge. On the open set Λ a topological invariant can be defined. Let Σ be the sphere of center $\xi_\star$ and radius ξ_0 in $\mathbb{R}^{n+1}$. Let P be the projection of $\mathbb{R}^{n+1}\setminus\{\xi_\star\}$ onto Σ:

$$P(\xi)=\xi_\star+\frac{\xi-\xi_\star}{|\xi-\xi_\star|}\,. \tag{2.11}$$

Definition 2.4. For any $u\in\Lambda$, $u=(u^1,\dots,u^{n+1})$ the open and bounded set

$$K_u=\{x\in\mathbb{R}^n:u^1(x)>\xi_0\}$$

is called support of u. Then the topological charge of u is the number

$$\operatorname{ch}(u)=\deg(P\circ u,K_u,2\xi_\star)\,.$$

To use some properties of the topological charge, we need to recall the following result, whose proof can be found in [6].

Proposition 2.5. *If a sequence* $\{u_m\}\subset\Lambda$ *converges to* $u\in\Lambda$ *uniformly on* $A\subset\mathbb{R}^n$*, then also* $P\circ u_m$ *converges to* $P\circ u$ *uniformly on* A*.*

This proposition permits to prove the continuity of the charge with respect to the uniform convergence:

Theorem 2.6. *For every* $u\in\Lambda$ *there exists* $r=r(u)>0$ *such that, for every* $v\in\Lambda$

$$\|v-u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}\le r\Longrightarrow\operatorname{ch}(v)=\operatorname{ch}(u)\,.$$

The connected components of Λ. The topological charge divides the open set Λ into the following sets, each of them associated to an integer number $q\in\mathbb{Z}$:

$$\Lambda_q=\{u\in\Lambda:\operatorname{ch}(u)=q\}\,. \tag{2.12}$$

By Theorem 2.6, we can conclude that the sets Λ_q are open in E. Moreover it is easy to see that

$$\Lambda=\bigcup_{q\in\mathbb{Z}}\Lambda_q\,,\quad \Lambda_p\cap\Lambda_q=\emptyset\text{ if }p\neq q$$

and each Λ_q is a connected component of Λ.

3. Symmetry and compactness properties

Action of $O(n)$. We consider the following action of the orthogonal group $O(n)$ on the space of the continuous functions $C(\mathbb{R}^n, \mathbb{R}^{n+1})$:

$$\begin{array}{rccc} T: & O(n) \times C(\mathbb{R}^n, \mathbb{R}^{n+1}) & \longrightarrow & C(\mathbb{R}^n, \mathbb{R}^{n+1}) \\ & (g, u) & \longmapsto & T_g u \end{array} \tag{3.1}$$

where

$$T_g u(x) = (u^1(gx), g^{-1}\tilde{u}(gx)) , \tag{3.2}$$

with

$$u(x) = (u^1(x), \tilde{u}(x)) = (u^1(x), u^2(x), \dots, u^{n+1}(x)) . \tag{3.3}$$

In particular $O(n)$ acts on the space E and so one can prove the following result.

Lemma 3.1. *The open subset $\Lambda \subset E$ and the energy functional J_ϵ are invariant with respect to the action (3.1-3.3).*

Remark 3.2. More precisely every connected component Λ_q of Λ is invariant with respect to the action (3.1-3.3) of the orthogonal group $O(n)$. Moreover for any $u \in E$ and for any $g \in O(n)$

$$\|T_g u\|_E = \|u\|_E .$$

Let F denote the subspace of the fixed points with respect to the action (3.1-3.3) of $O(n)$ on E:

$$F = \{u \in E : \forall g \in O(n)\ T_g u = u\} . \tag{3.4}$$

Remark 3.3. The set F is a closed subspace.

The set

$$\Lambda^F = \Lambda \cap F$$

is a natural constraint for the energy functional J_ϵ. In fact, if $u \in \Lambda^F$ is a critical point for $J_\epsilon\big|_{\Lambda^F}$, it is a global critical point (see [3]):

Lemma 3.4. *For every $u \in \Lambda^F$ and $v \in E$, we have*

$$J'_\epsilon(u)(v) = J'_\epsilon(u)(Pv) ,$$

being P the projection of E onto F.

We denote by Λ_q^F the subset of the invariant functions of topological charge q:

$$\Lambda_q^F = \Lambda_q \cap F .$$

Results of compactness. Next proposition provides a compact embedding for the subspace of the invariant functions of E into $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$:

Proposition 3.5. *The space F equipped with the norm $\|\cdot\|_E$ is compactly embedded into $L^s(\mathbb{R}^n, \mathbb{R}^{n+1})$ for every $s \in [2, \frac{2n}{n-2})$.*

The proof is a consequence of [3, Proposition 4] and of Theorem 2.1.

We set

$$S = \{u \in E : \|u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} = 1\} . \tag{3.5}$$

To get some critical points of the functional J_ϵ on the C^2 manifold $\Lambda \cap S$ we use the following version of Palais-Smale condition. For $J_\epsilon \in C^1(\Lambda, \mathbb{R})$, the norm of the derivative at $u \in S$ of the restriction $\hat{J}_\epsilon = J_\epsilon|_{\Lambda \cap S}$ is defined by

$$\|\hat{J}'_\epsilon(u)\|_\star = \min_{t \in \mathbb{R}} \|J'_\epsilon(u) - t g'(u)\|_{E^*} ,$$

where $g : E \to \mathbb{R}$ is the function defined by $g(u) = \int_{\mathbb{R}^n} |u(x)|^2 dx$.

Definition 3.6. The functional J_ϵ is said to satisfy the Palais-Smale condition in $c \in \mathbb{R}$ on $\Lambda \cap S$ (on $\Lambda_q \cap S$, for $q \in \mathbb{Z}$) if, for any sequence $\{u_i\}_{i\in\mathbb{N}} \subset \Lambda \cap S$ ($\{u_i\}_{i\in\mathbb{N}} \subset \Lambda_q \cap S$) such that $J_\epsilon(u_i) \to c$ and $\|\hat{J}'_\epsilon(u_i)\|_\star \to 0$, there exists a subsequence which converges to $u \in \Lambda \cap S$ ($u \in \Lambda_q \cap S$).

To obtain the Palais-Smale condition, we need a few technical lemmas (see [8] and [6]).

Lemma 3.7. *Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in Λ_q (with $q \in \mathbb{Z}$) such that the sequence $\{J_\epsilon(u_i)\}_{i\in\mathbb{N}}$ is bounded. We consider the open bounded sets*

$$Z_i = \{x \in \mathbb{R}^n : |u_i(x)| > c_3\}\,. \tag{3.6}$$

Then the set $\cup_{i\in\mathbb{N}} Z_i \subset \mathbb{R}^n$ is bounded.

Lemma 3.8. *Let $\{u_i\}_{i\in\mathbb{N}} \subset \Lambda$ be a sequence weakly converging to u and such that $\{J_\epsilon(u_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is bounded, then $u \in \Lambda$.*

Lemma 3.9. *For any $a > 0$, there exists $d > 0$ such that for every $u \in \Lambda$*

$$J_\epsilon(u) \le a \quad \Rightarrow \quad \inf_{x\in\mathbb{R}^n} |u(x) - \xi_\star| \ge d\,.$$

Now it is possible to prove (see [8]) that the functional J_ϵ satisfies the Palais-Smale condition on $\Lambda \cap S$ for any $c \in \mathbb{R}$ and $0 < \epsilon \le 1$. As a consequence the following proposition holds:

Proposition 3.10. *The functional J_ϵ satisfies the Palais-Smale condition on $\Lambda^F \cap S$ (on $\Lambda^F_q \cap S$ for $q \in \mathbb{Z}$) for any $c \in \mathbb{R}$ and $0 < \epsilon \le 1$.*

Proof. Given a Palais-Smale sequence $\{u_m\}_{m\in\mathbb{N}}$ for J_ϵ on $\Lambda^F \cap S \subset \Lambda \cap S$, it strongly converges to a function $u \in \Lambda \cap S$ by Proposition 2.1 of [8]. As the subspace F is closed (see Remark 3.3), $u \in \Lambda^F$. □

4. Eigenvalues of the Schrödinger operator

Existence of the eigenvalues. We define the following subspace of invariant functions with respect to the action of $O(n)$ (see (3.1-3.3)):

$$\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1}) = \{u \in \Gamma(\mathbb{R}^n, \mathbb{R}^{n+1}) : \forall g \in O(n)\ T_g u = u\}\,. \tag{4.1}$$

By Proposition 3.5 the identical embedding of $\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$ into $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$ is continuous and compact. Then there exists a monotone increasing sequence $\{\tilde{\lambda}_m\}_{m\in\mathbb{N}}$ of eigenvalues

$$0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots \le \tilde{\lambda}_m \overset{m\to\infty}{\longrightarrow} +\infty$$

with

$$\tilde{\lambda}_m = \inf_{E_m\in\mathcal{E}_m} \max_{v\in E_m,\, v\neq 0} \frac{\|v\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|v\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\,,$$

where $\mathcal{E}_m$ is the family of all m-dimensional subspaces E_m of $\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$. Also there exists a sequence $\{\varphi_m\}_{m\in\mathbb{N}} \subset \Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$ of eigenfunctions, orthonormal in $L^2(\mathbb{R}^n, \mathbb{R}^{n+1})$, such that

$$(\varphi_m, v)_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} = \tilde{\lambda}_m(\varphi_m, v)_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}\,, \quad \forall v \in \Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})\,, \quad \forall m \in \mathbb{N}\,.$$

Regularity of the eigenfunctions. The eigenfunctions φ_m have been found in the space $\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$. Nevertheless they possess some more regularity properties, as it can be shown using the following theorem:

Theorem 4.1. *If $V(x) \to +\infty$ as $|x| \to \infty$, then for any $z \in H^1(\mathbb{R}^n, \mathbb{R})$ such that*

$$-\Delta z + V(x)z = \lambda z$$

the following estimate holds:

$$|z(x)| \le C_a e^{-a|x|}, \tag{4.2}$$

where $a > 0$ is arbitrary and $C_a > 0$ depends on a.

For the proof of this theorem, see [9, p. 169].

Proposition 4.2. *The eigenfunctions $\varphi_m \in \Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$ of the Schrödinger operator $-\Delta + V(|x|)$ belong to the Banach space E.*

Proof. We prove the result for the real-valued eigenfunctions e_m so that the statement of the proposition follows immediately. By the regularity result of Agmon-Douglis-Nirenberg, if $z \in \Gamma_F(\mathbb{R}^n, \mathbb{R})$ is such that $-\Delta z - \lambda z = -Vz$ and if $Vz \in L^2(\mathbb{R}^n, \mathbb{R}) \cap L^p(\mathbb{R}^n, \mathbb{R})$, then $z \in W^{2,p}(\mathbb{R}^n, \mathbb{R})$.

So we only have to verify that $Vz \in L^2(\mathbb{R}^n, \mathbb{R}) \cap L^p(\mathbb{R}^n, \mathbb{R})$. By Theorem 4.1 and (V_2) we get

$$\int_{\mathbb{R}^n} |V(|x|)z(x)|^p dx \le C \Big\| V(|x|)e^{-|x|} \Big\|^p_{L^p(\mathbb{R}^n,\mathbb{R})} < +\infty .$$

Moreover, if $R > 0$ is such that for $x \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, R)$ $V(|x|) > 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |V(|x|)z(x)|^2 dx \\
&< C\Big(\int_{B_{\mathbb{R}^n}(0,R)} |V(|x|)|^2 e^{-p|x|} dx + \int_{\mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,R)} |V(|x|)|^p e^{-p|x|} dx \Big) < +\infty .
\end{aligned}$$

$\square$

Useful properties. We give here another variational characterization of the eigenvalues (see for example [13] and [16]) and we introduce the subspaces spanned by the eigenfunctions.

Definition 4.3. For $m \in \mathbb{N}$ we consider the following subspaces of $\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})$:

$$F_m = \operatorname{span}[\varphi_1, \dots, \varphi_m], \tag{4.3}$$

$$F_m^\perp = \{u \in \Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1}) : (u, \varphi_i)_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} = 0 \text{ for } 1 \le i \le m\}. \tag{4.4}$$

Lemma 4.4. *The following properties hold:*

$$\tilde{\lambda}_m = \min_{\substack{u \in \Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1}),\ u \ne 0 \\ (u,\varphi_i)_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} = 0 \\ \forall i=1,\dots,m-1}} \frac{\|u\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \tag{4.5}$$

and

$$(\varphi_i, \varphi_j)_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} = \tilde{\lambda}_i \delta_{ij} \quad \forall i, j \in \mathbb{N}. \tag{4.6}$$

Moreover,

$$u \in F_m, u \ne 0 \implies \tilde{\lambda}_1 \le \frac{\|u\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \le \tilde{\lambda}_m, \tag{4.7}$$

$$u \in F_m^{\perp}\,, u \neq 0 \quad \Longrightarrow \quad \frac{\|u\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \geq \tilde{\lambda}_{m+1}\,. \tag{4.8}$$

5. Min-max values

The functions Φ_ϵ^q. We introduce here a particular class of functions in E, which are invariant with respect to the action of the orthogonal group $O(n)$. Let us consider the functions $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined in the following way (see [12]):

$$\varphi(x) = \begin{cases} \begin{pmatrix} \varphi_1(|x|) \\ \varphi_2(|x|)\frac{x}{|x|} \end{pmatrix} & \text{for } x \neq 0 \\ \begin{pmatrix} \varphi_1(0) \\ 0 \end{pmatrix} & \text{for } x = 0 \end{cases} \tag{5.1}$$

where $\varphi_i : [0, +\infty) \to \mathbb{R}$ for $i = 1, 2$. In fact for any $g \in O(n)$ and $x \in \mathbb{R}^n$

$$T_g\varphi(x) = \varphi(x)\,.$$

By Proposition 4.2, the set F_m defined in (4.3) is a subset of E. Then, for any $m \in \mathbb{N}$, let $S(m)$ denote the m-dimensional sphere:

$$S(m) = F_m \cap S\,, \tag{5.2}$$

where S has been defined in (3.5).

Fixed an integer $k \in \mathbb{N}$, we introduce the number

$$M_k = \sup_{u\in S(k)} \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}\,. \tag{5.3}$$

Then we choose the first coordinate ξ_0 of the point $\xi_\star = (\xi_0, 0)$ in such a way that

$$\xi_0 > 2M_k\,. \tag{5.4}$$

We can now introduce for any $q \in \mathbb{Z} \setminus \{0\}$ the functions $\Phi_a^q : \mathbb{R}^n \to \mathbb{R}^{n+1}$ of type (5.1):

$$\Phi_a^q(x) = \begin{cases} \begin{pmatrix} \Phi_{a,1}^q(|x|) \\ \Phi_{a,2}^q(|x|)\frac{x}{|x|} \end{pmatrix} & \text{for } x \neq 0 \\ \begin{pmatrix} \Phi_{a,1}^q(0) \\ 0 \end{pmatrix} & \text{for } x = 0 \end{cases} \tag{5.5}$$

where

$$\begin{gathered} \Phi_{a,1}^q(|x|) = \begin{cases} 2\xi_0[\cos(\pi|x|) + 1] & \text{for } R_1 \leq |x| \leq R_2 \\ 0 & \text{for } 0 \leq |x| \leq R_1 \text{ or } |x| \geq R_2 \end{cases} \\ \Phi_{a,2}^q(|x|) = a|x|e^{-|x|}\sin(\pm\pi|x|) \end{gathered} \tag{5.6}$$

with

(i) $a > 0$
(ii) The sign in the argument of the sine in $\Phi_{a,2}^q$ is equal to the sign of q,
(iii) R_1 is a constant depending on the parity of q:

$$R_1 = R_1(q) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ 1 & \text{if } q \text{ is even,} \end{cases}$$

(iv) R_2 is a positive constant depending on q:
$$R_2 = R_2(q) = \begin{cases} |q| & \text{if } q \text{ is odd,} \\ |q|+1 & \text{if } q \text{ is even.} \end{cases}$$

Next lemma computes the topological charge of the functions just defined (see [3]).

Lemma 5.1. *For any $q \in \mathbb{Z} \setminus \{0\}$, the functions Φ_a^q defined in (5.5), (5.6), with the hypotheses (i)-(iv), belong to E and have topological charge*
$$\operatorname{ch}(\Phi_a^q) = q\,.$$

Proof. The functions Φ_a^q belong to the space E. If we consider the components
$$f_1(x^1, x^2, \dots, x^n) = \Phi_{a,1}^q(|x|)\,,$$
$$f_i(x^1, x^2, \dots, x^n) = \Phi_{a,2}^q(|x|)\frac{x^i}{|x|}\,,$$
where $2 \le i \le n+1$, we have
$$|\nabla_x f_1|^2 = |{\Phi_{a,1}^q}'(|x|)|^2$$
$$|\nabla_x f_i|^2 \le C\left(|{\Phi_{a,2}^q}'(|x|)|^2 + \frac{|\Phi_{a,2}^q(|x|)|^2}{|x|^2}\right)$$
and consequently
$$\begin{aligned}\sum_{i=1}^{n+1}\|\nabla f_i\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^n)} &\le C\int_0^\infty \left(|{\Phi_{a,1}^q}'(r)|^2 + |{\Phi_{a,2}^q}'(r)|^2 + \frac{|\Phi_{a,2}^q(r)|^2}{r^2}\right) r^{n-1}\,dr \\ &< +\infty \\ \sum_{i=1}^{n+1}\|\nabla f_i\|^p_{L^p(\mathbb{R}^n,\mathbb{R}^n)} &\le C\int_0^\infty \left(|{\Phi_{a,1}^q}'(r)|^p + |{\Phi_{a,2}^q}'(r)|^p + \frac{|\Phi_{a,2}^q(r)|^p}{r^p}\right) r^{n-1}\,dr \\ &< +\infty\end{aligned}$$
Moreover the following inequalities hold:
$$\begin{aligned}&\int_{\mathbb{R}^n} V(|x|)\sum_{i=0}^{n}|f_i(x)|^2 dx \\ &\le C\int_{R_1}^{R_2} V(r)(\Phi_{a,1}^q(r))^2 r^{n-1}dr + \int_{\mathbb{R}^n} V(|x|)(\Phi_{a,2}^q(|x|))^2 dx \\ &\le C' + \|V(|x|)e^{-|x|}\|_{L^p(\mathbb{R}^n,\mathbb{R})}\|a^2|x|^2e^{-|x|}\|_{L^q(\mathbb{R}^n,\mathbb{R})} < +\infty\,,\end{aligned}$$
where $q = \frac{p}{p-1}$.

The functions Φ_a^q belong to the space Λ. In fact, if $\Phi_{a,2}^q(|x|) = 0$, then $|x| \in \mathbb{N}\cup\{0\}$ and hence $\Phi_{a,1}^q(|x|) \in \{0, 4\xi_0\}$, so that $\Phi_a^q(\mathbb{R}^n) \not\ni \xi_\star$.

The functions Φ_a^q have topological charge q. Let P be the projection introduced in (2.11) of $\mathbb{R}^{n+1}$ onto the sphere Σ of center $\xi_\star$ and radius ξ_0 in $\mathbb{R}^{n+1}$; then
$$P \circ \Phi_a^q(x) = \begin{pmatrix} \frac{\Phi_{a,1}^q(|x|)-\xi_0}{\sqrt{(\Phi_{a,1}^q(|x|)-\xi_0)^2 + (\Phi_{a,2}^q(|x|))^2}} + \xi_0 \\ \frac{\Phi_{a,2}^q(|x|)}{\sqrt{(\Phi_{a,1}^q(|x|)-\xi_0)^2 + (\Phi_{a,2}^q(|x|))^2}}\frac{x}{|x|} \end{pmatrix}$$

If $K_{\Phi_a^q}$ is the support of Φ_a^q, we can consider on it the local coordinates obtained by the stereographic projection of the sphere Σ from the origin onto the plane $\Pi = \{\xi^1 = 2\xi_0\}$:

$$\begin{array}{rcccl} p & : & \Sigma & \longrightarrow & \Pi \\ & & (\xi^1, \xi^2, \dots, \xi^{n+1}) & \longmapsto & 2\xi_0 \left(\frac{\xi^2}{\xi^1}, \frac{\xi^3}{\xi^1}, \dots, \frac{\xi^{n+1}}{\xi^1} \right) . \end{array}$$

Then the function Φ_a^q in the new coordinates becomes

$$\overline{\Phi}_a^q(x) = p \circ P \circ \Phi_a^q(x) = f_a^q(|x|) \frac{x}{|x|} \,,$$

where

$$f_a^q(|x|) = \frac{\Phi_{a,2}^q(|x|)}{\Phi_{a,1}^q(|x|) - \xi_0 + \xi_0 \sqrt{(\Phi_{a,1}^q(|x|) - \xi_0)^2 + (\Phi_{a,2}^q(|x|))^2}} \,. \tag{5.7}$$

The topological charge is therefore

$$\mathrm{ch}(\Phi_a^q) = \deg\left(\overline{\Phi}_a^q, K_{\Phi_a^q}, 0\right) .$$

Let δ be a positive parameter, $\delta < \frac{3}{4}$ and let i_1, $i_2 \in \mathbb{N} \cup \{0\}$, with

$$i_1 = R_1 \,, \quad i_2 = \max\{i \in \mathbb{N} \cup \{0\} : 2i + 1 \le R_2\} \,.$$

Then the sets

$$K_i = \{x \in \mathbb{R}^n : 2i - \delta < |x| < 2i + \delta\} \,,$$

for $i \in \mathbb{N} \cup \{0\}$, $i_1 \le i \le i_2$, are disjoint and their union satisfies the inclusion:

$$\bigcup_{i=i_1}^{i_2} K_i \subset K_{\Phi_a^q} \,.$$

Moreover this subset of $K_{\Phi_a^q}$ contains all the zeros of the function $\overline{\Phi}_a^q$, that is:

$$\left\{ x \in K_{\Phi_a^q} : \overline{\Phi}_a^q = 0 \right\} \subset \bigcup_{i=i_1}^{i_2} K_i \,.$$

By the excision and the additive properties of the topological degree we can write

$$\deg(\overline{\Phi}_a^q, K_{\Phi_a^q}, 0) = \sum_{i=i_1}^{i_2} \deg(\overline{\Phi}_a^q, K_i, 0) \,.$$

To conclude we want to prove that

$$\deg(\overline{\Phi}_a^q, K_i, 0) = \begin{cases} \mathrm{sign}(q) & \text{for } i = 0 \,, \\ 2\,\mathrm{sign}(q) & \text{for } i \in \mathbb{N} \,. \end{cases}$$

In fact consider the function

$$v_0(x) = \frac{f_a^q(\delta)}{\delta} x \,,$$

where $f_a^q(|x|)$ is defined in (5.7). Since v_0 coincides with $\overline{\Phi}_a^q$ on the boundary of K_0, i.e. for any $x \in \partial K_0$

$$\overline{\Phi}_a^q(x) = f_a^q(|x|) \frac{x}{|x|} = v_0(x) \,,$$

the degrees of the two functions coincide too, so

$$\deg(\overline{\Phi}_a^q, K_0, 0) = \deg(v_0, K_0, 0) = \mathrm{sign}(q) \,.$$

Finally, for $1 \le i \le i_2$, set
$$K_i^+ = \{x \in \mathbb{R}^n : |x| < 2i + \delta\}, \quad K_i^- = \{x \in \mathbb{R}^n : |x| < 2i - \delta\};$$
then the degrees satisfy
$$\deg\left(\overline{\Phi}_a^q, K_i, 0\right) = \deg\left(\overline{\Phi}_a^q, K_i^+, 0\right) - \deg\left(\overline{\Phi}_a^q, K_i^-, 0\right).$$
Analogously to the previous argument, we introduce the functions:
$$v_i^+(x) = \frac{f_a^q(2i+\delta)}{2i+\delta}x, \quad v_i^-(x) = \frac{f_a^q(2i-\delta)}{2i-\delta}x.$$
As $v_i^\pm$ coincides with $\overline{\Phi}_a^q$ on the boundary of $K_i^\pm$, we conclude that
$$\deg\left(\overline{\Phi}_a^q, K_i^+, 0\right) = \deg(v_i^+, K_i^+, 0) = \operatorname{sign}(q),$$
$$\deg\left(\overline{\Phi}_a^q, K_i^-, 0\right) = \deg(v_i^-, K_i^-, 0) = -\operatorname{sign}(q).$$
This completes the proof. □

The following corollary is now immediate.

Corollary 5.2. *For all $q \in \mathbb{Z}$ the connected component Λ_q^F is not empty.*

Lemma 5.3. *Fixed $q \in \mathbb{Z} \setminus \{0\}$, there exists $\hat{a}_q > 0$ such that for every $a \ge \hat{a}_q$ the functions Φ_a^q have the following properties:*

(*i*) *The distance of Φ_a^q from the point $\xi_\star$ is ξ_0, i.e.*
$$d(\Phi_a^q, \xi_\star) = \inf_{x\in\mathbb{R}^n} |\Phi_a^q(x) - \xi_\star| = \xi_0.$$
(*ii*) *If we expand Φ_a^q of a factor $t \ge 1$, $t\Phi_a^q \in \Lambda^F$ and*
$$d(t\Phi_a^q, \xi_\star) = \inf_{x\in\mathbb{R}^n} |t\Phi_a^q(x) - \xi_\star| = \xi_0.$$

Proof. (*i*) We prove that there exists a sufficiently large such that
$$|\Phi_a^q(x) - \xi_\star| \ge \xi_0$$
for all $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ with $0 \le |x| \le R_1$ or $|x| \ge R_2$, it is immediate that
$$|\Phi_a^q(x) - \xi_\star|^2 = a^2|x|^2e^{-2|x|}\sin^2(\pi|x|) + \xi_0^2 \ge \xi_0^2.$$
As for $x \in \mathbb{R}^n$ with $R_1 \le |x| \le R_2$ there holds:
$$\begin{aligned} |\Phi_a^q(x) - \xi_\star|^2 &= \xi_0^2[2\cos(\pi|x|)+1]^2 + a^2|x|^2e^{-2|x|}\sin^2(\pi|x|) \\ &= \left(4\xi_0^2 - a^2|x|^2e^{-2|x|}\right)\cos^2(\pi|x|) + 4\xi_0^2\cos(\pi|x|) + \xi_0^2 + a^2|x|^2e^{-2|x|}. \end{aligned}$$
Let $f_a : [0, +\infty) \to \mathbb{R}$ be the function
$$f_a(r) = \left(4\xi_0^2 - a^2r^2e^{-2r}\right)\cos^2(\pi r) + 4\xi_0^2\cos(\pi r) + a^2r^2e^{-2r}.$$
We consider the polynomial
$$P(y) = P_\alpha(y) = \left(4\xi_0^2 - \alpha^2\right)y^2 + 4\xi_0^2 y + \alpha^2,$$
where $\alpha = \alpha_a(r) = are^{-r}$, on the interval $[-1, +1]$.

Now, if $\alpha^2 = 4\xi_0^2$, the only zero of $P(y)$ is $y = -1$ and therefore on $[-1, 1]$ $P(y)$ is nonnegative. On the contrary, if $\alpha^2 \neq 4\xi_0^2$, the zeros of $P(y)$ are:
$$y_{1,2} = \frac{-2\xi_0^2 \pm \left(\alpha^2 - 2\xi_0^2\right)}{4\xi_0^2 - \alpha^2} = \begin{cases} -1 \\ \frac{\alpha^2}{\alpha^2 - 4\xi_0^2} \end{cases}$$

For $\alpha^2 > 4\xi_0^2$ we have $y_1 = -1$ and $y_2 > 1$, so $P(y) \geq 0$ on $[-1,1]$. For $2\xi_0^2 \leq \alpha^2 < 4\xi_0^2$, we have $y_2 \leq -1$, so $P(y) \geq 0$ and consequently

$$a^2 r^2 e^{-2r} \geq 2\xi_0^2 \Longrightarrow f_a(r) \geq 0\,.$$

If we consider

$$a \geq \frac{\sqrt{2}\xi_0}{R_2 e^{-R_2}} \tag{5.8}$$

and $R_1 = 1$ (i.e. q even), there always holds $\alpha^2 \geq 2\xi_0^2$.

If on the contrary q is odd and so $R_1 = 0$, for a as in (5.8) $(\alpha_a(r))^2 < 2\xi_0^2$ for $0 \leq r < r_1$, where r_1 is such that

$$(\alpha_a(r_1))^2 = 2\xi_0^2\,. \tag{5.9}$$

We choose a sufficiently large to have $r_1 \leq \frac{1}{2}$: then $\cos(\pi r) \in (0,1]$ for any $r \in [0, r_1)$ and so

$$\min_{r\in[0,r_1)} f_a(r) \geq 0\,.$$

(ii) For any $x \in \mathbb{R}^n$ with $0 \leq |x| \leq R_1$ or $|x| \geq R_2$, it is immediate that

$$|t\Phi_a^q(x) - \xi_\star|^2 = t^2 a^2 |x|^2 e^{-2|x|} \sin^2(\pi|x|) + \xi_0^2 \geq \xi_0^2\,.$$

On the contrary for $x \in \mathbb{R}^n$, $R_1 \leq |x| \leq R_2$, there holds:

$$\begin{aligned}|t\Phi_a^q(x) - \xi_\star|^2 &= \xi_0^2[2t\cos(\pi|x|) + 2t - 1]^2 + t^2 a^2 |x|^2 e^{-2|x|} \sin^2(\pi|x|)\\ &= t^2\left(4\xi_0^2 - a^2|x|^2 e^{-2|x|}\right)\cos^2(\pi|x|) + 4t(2t-1)\xi_0^2\cos(\pi|x|)\\ &\quad + \xi_0^2(2t-1)^2 + t^2 a^2 |x|^2 e^{-2|x|}\,.\end{aligned}$$

As before we consider $\widetilde{f}_a : [0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned}&\widetilde{f}_a(r)\\ &= t^2\left(4\xi_0^2 - a^2 r^2 e^{-2r}\right)\cos^2(\pi r) + 4t(2t-1)\xi_0^2\cos(\pi r) + 4t(t-1)\xi_0^2 + t^2 a^2 r^2 e^{-2r}\,.\end{aligned}$$

The polynomial $P(y)$ becomes

$$\widetilde{P}(y) = \widetilde{P}_\alpha(y) = t^2(4\xi_0^2 - \alpha^2)y^2 + 4t(2t-1)\xi_0^2 y + 4t(t-1)\xi_0^2 + t^2\alpha^2\,.$$

If $\alpha^2 = 4\xi_0^2$, the only zero of $\widetilde{P}(y)$ is $y = -1$ and so on $[-1,1]$ $\widetilde{P}(y)$ is nonnegative. If $\alpha^2 \neq 4\xi_0^2$, the zeros of $\widetilde{P}(y)$ are

$$y_{1,2} = \frac{(2-4t)\xi_0^2 \pm \left(2\xi_0^2 - t\alpha^2\right)}{t(4\xi_0^2 - \alpha^2)} = \begin{cases} -1 \\ \frac{4(1-t)\xi_0^2 - t\alpha^2}{t(4\xi_0^2 - \alpha^2)} \end{cases}$$

For $\alpha^2 > 4\xi_0^2$ we have $y_1 = -1$ and $y_2 > 1$, then $\widetilde{P}(y) \geq 0$ in $[-1,1]$. For $2\xi_0^2 \leq \alpha^2 < 4\xi_0^2$, there holds $y_2 \leq -1$, so $\widetilde{P}(y) \geq 0$ and consequently

$$a^2 r^2 e^{-2r} \geq 2\xi_0^2 \Longrightarrow \widetilde{f}_a(r) \geq 0\,.$$

Now, with the choice of a done in (i) and $R_1 = 1$ (q even), $\alpha^2 \geq 2\xi_0^2$.

Finally, if $R_1 = 0$ and a is as in (i), $\alpha^2 < 2\xi_0^2$ for $0 \leq r < r_1 \leq \frac{1}{2}$ (where r_1 is as in (5.9)),

$$\min_{r\in[0,r_1)} \widetilde{f}_a(r) \geq 0\,.$$

This completes the proof. $\square$

Definition 5.4. For any $q \in \mathbb{Z} \setminus \{0\}$ and for $\hat{a}_q$ as in Lemma 5.3, we define the function

$$\Phi^q = \Phi^q_{\hat{a}_q} \,. \tag{5.10}$$

Evidently for $i = 1, 2$ we pose $(\Phi^q)_i = \Phi^q_{\hat{a}_q, i}$.

Moreover we introduce the rescaled functions Φ^q_ϵ, with $q \in \mathbb{Z} \setminus \{0\}$ and $0 < \epsilon \le 1$:

$$\Phi^q_\epsilon(x) = \Phi^q\Big(\frac{x}{\epsilon}\Big) \,. \tag{5.11}$$

Remark 5.5. (1) The functions Φ^q_ϵ belong to Λ^F_q.
(2) By definition of Φ^q_ϵ and by Lemma 5.3 the image of Φ^q_ϵ does not intersect the point $\xi_\star$ and the distance of the image from the point is ξ_0.
(3) Even if we expand the functions Φ^q_ϵ $(0 < \epsilon \le 1)$ of a factor $t \ge 1$, their image is such that they do not meet the point $\xi_\star$ and the distance is still ξ_0. Hence $t\Phi^q_\epsilon \in \Lambda^F_q$ for all $t \ge 1$ and $\epsilon \in (0, 1]$.

Remark 5.6. The norms of the functions Φ^q_ϵ satisfy the following equalities depending on the parameter ϵ:

$$\|\Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} = \epsilon^n \|\Phi^q\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} \,, \tag{5.12}$$

$$\|\nabla\Phi^q_\epsilon\|^2_{L^2} = \epsilon^{n-2}\|\nabla\Phi^q\|^2_{L^2} \,, \tag{5.13}$$

$$\|\nabla\Phi^q_\epsilon\|^p_{L^p} = \frac{1}{\epsilon^{p-n}}\|\nabla\Phi^q\|^p_{L^p} \,. \tag{5.14}$$

The functions Φ^q_ϵ own some fundamental properties, which are presented in the following lemma.

Lemma 5.7. *Given $q \in \mathbb{Z}\setminus\{0\}$ and $k \in \mathbb{N}$, we consider $\xi_\star = (\xi_0, 0)$ with $\xi_0 > 2M_k$, where*

$$M_k = \sup_{u \in S(k)} \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})} \,,$$

and $0 \in \mathbb{R}^n$. There exist $\hat{\rho}_q > 0$ and $\bar{\epsilon}_q$, with $0 < \bar{\epsilon}_q \le 1$, such that for all $0 < \epsilon \le \bar{\epsilon}_q$ we have

(i) $\|\Phi^q_\epsilon + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} \le 1$ *for all $u \in S(k)$,*
(ii) $\inf_{\substack{\epsilon\in(0,\bar{\epsilon}_q]\\ u\in S(k)}} \|\Phi^q_\epsilon + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} > 0$,
(iii) $\displaystyle\inf_{\substack{x\in\mathbb{R}^n\\ \epsilon\in(0,\bar{\epsilon}_q]\\ u\in S(k)}} \Big|\frac{\Phi^q_\epsilon(x) + \hat{\rho}_q u(x)}{\|\Phi^q_\epsilon + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} - \xi_\star\Big| > \frac{\xi_0}{2}$,
(iv) $\displaystyle\frac{\Phi^q_\epsilon + \hat{\rho}_q u}{\|\Phi^q_\epsilon + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \in \Lambda_q \cap S$ *for all $u \in S(k)$.*

Proof. (i) For any $\rho > 0$ and $0 < \epsilon \le 1$ we have

$$\|\Phi^q_\epsilon + \rho u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} \le \epsilon^{\frac{n}{2}}\|\Phi^q\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} + \rho \,.$$

Let $\bar{\epsilon}_q$ be such that

$$\begin{gathered} \bar{\epsilon}_q < \Big(\frac{1}{\|\Phi^q\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\Big)^{\frac{2}{n}} \,, \\ \bar{\epsilon}_q \le 1 \,. \end{gathered} \tag{5.15}$$

Then there exists $\hat{\rho}_q > 0$ such that $\|\Phi^q_{\bar{\epsilon}_q}\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} + \hat{\rho}_q \le 1$.

(ii) As $\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}\geq \hat\rho_q-\|\Phi^q_\epsilon\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}$, reducing if necessary $\overline{\epsilon}_q$, we get $\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}>0$.
(iii) By (ii) of Lemma 5.3 we deduce that for all $u\in S(k)$

$$\inf_{\substack{x\in\mathbb{R}^n\\ \epsilon\in(0,\overline{\epsilon}_q]}}\Big|\frac{\Phi^q_\epsilon(x)}{\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}-\xi_\star\Big|=\xi_0\,.$$

To get (iii) it is sufficient to prove that, reducing if necessary $\overline{\epsilon}_q$, for all $\epsilon\leq\overline{\epsilon}_q$

$$\sup_{u\in S(k)}\frac{\hat\rho_q\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}<\frac{\xi_0}{2}\,.$$

We observe that

$$\begin{aligned}\sup_{u\in S(k)}\frac{\hat\rho_q\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}&\leq\frac{\hat\rho_q M_k}{\inf_{u\in S(k)}\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\\&\leq\frac{M_k}{1-\frac{\epsilon^{\frac{n}{2}}}{\hat\rho_q}\|\Phi^q\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\,.\end{aligned}$$

Since $M_k<\frac{\xi_0}{2}$, for $\overline{\epsilon}_q$ sufficiently small we have (iii).
(iv) follows immediately from (iii). $\square$

The values $c^q_{\epsilon,j}$. Using the properties of the functions Φ^q_ϵ seen in Lemma 5.7, it is possible to introduce the following subsets of $\Lambda^F\cap S$:

Definition 5.8. Fixed $k\in\mathbb{N}$, $q\in\mathbb{Z}\setminus\{0\}$ and $0<\epsilon\leq\overline{\epsilon}_q$, where $\overline{\epsilon}_q$ is defined in Lemma 5.7, we set

$$\mathcal{M}^q_{\epsilon,j}=\Big\{\frac{\Phi^q_\epsilon+\hat\rho_q u}{\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}:u\in S(j)\Big\}\tag{5.16}$$

with $j\leq k$ and $\hat\rho_q$ defined in Lemma 5.7. We pose by convention $\mathcal{M}^q_{\epsilon,0}=\emptyset$.

Remark 5.9. We outline the following properties of the sets $\mathcal{M}^q_{\epsilon,j}$:

(i) $\mathcal{M}^q_{\epsilon,j-1}\subset\mathcal{M}^q_{\epsilon,j}$;
(ii) $\mathcal{M}^q_{\epsilon,j}\subset\Lambda^F_q\cap S$;
(iii) $\mathcal{M}^q_{\epsilon,j}$ is a compact set;
(iv) $\mathcal{M}^q_{\epsilon,j}$ is a sub-manifold of Λ^F_q for $0<\epsilon\leq\overline{\epsilon}_q$ (see Lemma 5.7).

Next definition introduces the min-max values $c^q_{\epsilon,j}$.

Definition 5.10. Fixed $k\in\mathbb{N}$, for all $q\in\mathbb{Z}\setminus\{0\}$, $j\leq k$ and $0<\epsilon\leq\overline{\epsilon}_q$ ($\overline{\epsilon}_q$ is defined in Lemma 5.7), we define the following values:

$$c^q_{\epsilon,j}=\inf_{h\in\mathcal{H}^q_{\epsilon,j}}\sup_{v\in\mathcal{M}^q_{\epsilon,j}}J_\epsilon\big(h(v)\big)\,,\tag{5.17}$$

where $\mathcal{H}^q_{\epsilon,j}$ are the following sets of continuous transformations:

$$\mathcal{H}^q_{\epsilon,j}=\Big\{h:\Lambda^F_q\cap S\to\Lambda^F_q\cap S: h\,\text{continuous},\ h\big|_{\mathcal{M}^q_{\epsilon,j-1}}=\mathrm{id}_{\mathcal{M}^q_{\epsilon,j-1}}\Big\}\,.$$

We observe that $\mathcal{H}^q_{\epsilon,j+1}\subset\mathcal{H}^q_{\epsilon,j}$.

Lemma 5.11. *Fixed $k\in\mathbb{N}$, for all $q\in\mathbb{Z}\setminus\{0\}$, $j<k$ and $0<\epsilon\leq\overline{\epsilon}_q$, we have*

(i) $c^q_{\epsilon,j}\in\mathbb{R}$,
(ii) $c^q_{\epsilon,j}\leq c^q_{\epsilon,j+1}$.

6. Main results

Minima. We recall now the Deformation Lemma:

Lemma 6.1 (Deformation Lemma). *Let J be a C^1-functional defined on a C^2-Finsler manifold E. Let c be a regular value for J. We assume that:*

(i) *J satisfies the Palais-Smale condition in c on M,*
(ii) *there exists $k > 0$ such that the sublevel J^{c+k} is complete.*

Then there exist $\delta > 0$ and a deformation $\eta : [0,1] \times E \longrightarrow E$ such that:

(a) *$\eta(0,u) = u$ for all $u \in E$,*
(b) *$\eta(t,u) = u$ for all $t \in [0,1]$ and u such that $|J(u) - c| \geq 2\delta$,*
(c) *$J(\eta(t,u))$ is non-increasing in t for any $u \in E$,*
(d) *$\eta(1, J^{c+\delta}) \subset J^{c-\delta}$.*

To apply Lemma 6.1 on each connected component Λ_q^F, with $q \in \mathbb{Z} \setminus \{0\}$, intersected with the unitary sphere S we need the completeness of the sub-levels of the functional J_ϵ. It is simple to verify next:

Lemma 6.2. *For any $q \in \mathbb{Z}$, $\epsilon \in (0,1]$ and $c \in \mathbb{R}$, the subset $\Lambda_q^F \cap S \cap J_\epsilon^c$ of the Banach space E is complete.*

Now we get easily the minimum values of the functional J_ϵ on each set $\Lambda_q^F \cap S$:

Theorem 6.3. *Given $q \in \mathbb{Z}$, for any $\xi_\star = (\xi_0, 0)$ with $\xi_0 > 0$ and $0 \in \mathbb{R}^n$ and for any $\epsilon > 0$, there exists a minimum for the functional J_ϵ on the subset $\Lambda_q^F \cap S$ of $\Lambda \cap S$.*

Proof. For any $t \geq 1$ we have that $t\Phi^q \in \Lambda_q^F$ (see (iii) of Remark 5.5) and in particular the function $\frac{\Phi^q}{\|\Phi^q\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}$ is in $\Lambda_q^F \cap S$. This means that $\Lambda_q^F \cap S$ is not empty for all $q \in \mathbb{Z}$, since it is obvious that $\Lambda_0^F \cap S \neq \emptyset$.

The claim follows by the fact that $\Lambda_q^F \cap S$ is not empty, the functional J_ϵ is bounded from below and satisfies the Palais-Smale condition on $\Lambda_q^F \cap S$ (see Proposition 3.10). $\square$

Remark 6.4. We point out that to have this result there is no need to require that the first coordinate ξ_0 of the point $\xi_\star$ is sufficiently large (see (5.4)). In fact this assumption is necessary to have properties *(iii)* and *(iv)* of Lemma 5.7, while here we only have to show that $\Lambda_q^F \cap S$ is not empty for all $q \in \mathbb{Z}$.

Critical values. The next theorem is an existence and multiplicity result of solutions for the problem $\big(P_\epsilon\big)$.

Theorem 6.5. *Given $q \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$, we consider $\xi_\star = (\xi_0, 0)$ with $\xi_0 > 2M_k$, where*

$$M_k = \sup_{u \in S(k)} \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n+1})},$$

and $0 \in \mathbb{R}^n$.

Then there exists $\hat\epsilon_q \in (0,1]$ such that for any $\epsilon \in (0, \hat\epsilon_q]$ and for any $2 \leq j \leq k$ with $\tilde\lambda_{j-1} < \tilde\lambda_j$, we get that $c^q_{\epsilon,j}$ is a critical value for the functional J_ϵ restricted to the manifold $\Lambda_q^F \cap S$. Moreover $c^q_{\epsilon,j-1} < c^q_{\epsilon,j}$.

The proof of this theorem is similar to the proof of Theorem 3.1 in [8], but for the convenience of the reader we summarize it here.

Proof. We begin with some notation: if $u \in F$ we define the projections

$$P_{F_j} u = \sum_{i=1}^{j} (u, \varphi_i)_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})} \varphi_i \,, \quad Q_{F_j} u = u - P_{F_j} u \,. \tag{6.1}$$

It is immediate that

$$(Q_{F_j} u, \varphi_i)_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})} = \tilde{\lambda}_i (Q_{F_j} u, \varphi_i)_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} = 0 \quad \forall i = 1, \dots, j \,. \tag{6.2}$$

We divide the argument into five steps.
Step 1 *For any $h \in \mathcal{H}^q_{\epsilon,j}$ the intersection of the set $h(\mathcal{M}^q_{\epsilon,j})$ with the set $\{u \in F : (u, \varphi_i)_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})} = 0 \, \forall i = 1, \dots, j-1\}$ is not empty: in fact there exists $v \in \mathcal{M}^q_{\epsilon,j}$ such that $P_{F_{j-1}} h(v) = 0$.*

This is obtained by an argument of degree theory (for the proof see [7]).
Step 2 *We prove that*

$$\sup_{v \in \mathcal{M}^q_{\epsilon,j}} J_\epsilon(v) \le \tilde{\lambda}_j + \sigma(\epsilon) \tag{6.3}$$

$$c^q_{\epsilon,j} \le \tilde{\lambda}_j + \sigma(\epsilon) \tag{6.4}$$

where $\lim_{\epsilon \to 0} \sigma(\epsilon) = 0$*.* First of all we verify that

$$\sup_{v \in \mathcal{M}^q_{\epsilon,j}} J_0(v) \le \tilde{\lambda}_j + \sup_{u \in S(j)} \frac{\|Q_{F_j} \Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})}}{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} + \|Q_{F_j} \Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \,. \tag{6.5}$$

In fact by Definition 5.8, (6.1) and (6.2) we have:

$$\begin{aligned}
\sup_{v \in \mathcal{M}^q_{\epsilon,j}} J_0(v) &= \sup_{u \in S(j)} \Big\| \frac{\Phi^q_\epsilon + \hat{\rho}_q u}{\|\Phi^q_\epsilon + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \Big\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})} \\
&= \sup_{u \in S(j)} \frac{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})} + \|Q_{F_j} \Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})}}{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} + \|Q_{F_j} \Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \\
&\le \sup_{u \in S(j)} \Big(\frac{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})}}{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \\
&\quad + \frac{\|Q_{F_j} \Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})}}{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} + \|Q_{F_j} \Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \Big) \\
&\le \tilde{\lambda}_j + \sup_{u \in S(j)} \frac{\|Q_{F_j} \Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n, \mathbb{R}^{n+1})}}{\|P_{F_j} \Phi^q_\epsilon + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})} + \|Q_{F_j} \Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n, \mathbb{R}^{n+1})}} \,.
\end{aligned}$$

Using the definition of J_ϵ and (6.5), we prove the following inequalities:

$$\begin{aligned}
c^q_{\epsilon,j} &= \inf_{h\in\mathcal{H}^q_{\epsilon,j}} \sup_{v\in\mathcal{M}^q_{\epsilon,j}} J_\epsilon(h(v)) \\
&\le \sup_{v\in\mathcal{M}^q_{\epsilon,j}} J_\epsilon(v) \\
&\le \sup_{v\in\mathcal{M}^q_{\epsilon,j}} J_0(v) + \epsilon^r \sup_{v\in\mathcal{M}^q_{\epsilon,j}} \int_{\mathbb{R}^n} \Big(\frac{1}{p}|\nabla v|^p + W(v)\Big)\,dx \\
&\le \tilde\lambda_j + \sup_{u\in S(j)} \frac{\|Q_{F_j}\Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}}{\|P_{F_j}\Phi^q_\epsilon+\hat\rho_q u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}+\|Q_{F_j}\Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \\
&\quad + \frac{\epsilon^r}{p}\sup_{u\in S(j)} \frac{\|\nabla(\Phi^q_\epsilon+\hat\rho_q u)\|^p_{L^p}}{\|\Phi^q_\epsilon+\hat\rho_q u\|^p_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \\
&\quad + \epsilon^r \sup_{u\in S(j)} \int_{\mathbb{R}^n} W\Big(\frac{\Phi^q_\epsilon+\hat\rho_q u}{\|\Phi^q_\epsilon+\hat\rho_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\Big)\,dx\,.
\end{aligned} \tag{6.6}$$

At this point we note that $\lim_{\epsilon\to 0}\|Q_{F_j}\Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})}=0$, in fact

$$\begin{aligned}
\|Q_{F_j}\Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} &\le \|\Phi^q_\epsilon\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} \\
&= \int_{\mathbb{R}^n} |\nabla\Phi^q_\epsilon(x)|^2dx + \int_{\mathbb{R}^n} V(|x|)|\Phi^q_\epsilon(x)|^2dx\,,
\end{aligned}$$

where the right-hand side tends to zero for $\epsilon\to 0$, because (5.13) holds and

$$\begin{aligned}
\int_{\mathbb{R}^n} V(|x|)|\Phi^q_\epsilon(x)|^2dx &= \int_{\mathbb{R}^n} V(|x|)\Big[\Big|(\Phi^q)_1\Big(\frac{|x|}{\epsilon}\Big)\Big|^2+\Big|(\Phi^q)_2\Big(\frac{|x|}{\epsilon}\Big)\Big|^2\Big]\,dx \\
&\le c\int_{\epsilon R_1}^{\epsilon R_2} V(r)\Big|(\Phi^q)_1\Big(\frac{r}{\epsilon}\Big)\Big|^2 r^{n-1}dr \\
&\quad + \|V(|x|)e^{-|x|}\|_{L^p(\mathbb{R}^n,\mathbb{R})}\Big\|e^{|x|}\Big|(\Phi^q)_2\Big(\frac{|x|}{\epsilon}\Big)\Big|^2\Big\|_{L^q(\mathbb{R}^n,\mathbb{R})} \\
&\le \Big(c\max_{r\in[0,R_2]} V(r)R_2^{n-1}(R_2-R_1)\Big)\epsilon^n \\
&\quad + \Big(\big\|V(|x|)e^{-|x|}\big\|_{L^p(\mathbb{R}^n,\mathbb{R})}\||x|^2e^{-|x|}\|_{L^q(\mathbb{R}^n,\mathbb{R})}\Big)\epsilon^{\frac{n}{q}}
\end{aligned}$$

where q denotes the dual exponent of p.

Moreover by (ii) of Lemma 5.7 we obtain

$$\sup_{0<\epsilon\le\bar\epsilon}\sup_{u\in S(j)} \frac{1}{\|P_{F_j}\Phi^q_\epsilon+\hat\rho_q u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}+\|Q_{F_j}\Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} < +\infty\,,$$

in fact

$$\begin{gathered}
\|P_{F_j}\Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} \le \epsilon^n\|\Phi^q\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}\,, \\
\|Q_{F_j}\Phi^q_\epsilon\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})} \le \epsilon^n\|\Phi^q\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}\,.
\end{gathered}$$

Therefore the second term of the last inequality of (6.6) goes to zero when ϵ goes to zero.

Now we observe that the following inequality holds:

$$\epsilon^r\|\nabla\Phi^q_\epsilon+\hat\rho_q u\|^p_{L^p} \le \Big(\epsilon^{\frac{r-(p-n)}{p}}\|\nabla\Phi^q\|_{L^p}+\epsilon^{\frac{r}{p}}\hat\rho_q\|\nabla u\|_{L^p}\Big)^p\,.$$

Then by this inequality and (ii) of Lemma 5.7 (we recall that $r > p - n$), we have that the third term of the last inequality of (6.6) tends to zero when ϵ tends to zero.

Regarding the last term, we verify that

$$\int_{\mathbb{R}^n} W\left(\frac{\Phi_\epsilon^q + \hat{\rho}_q u}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\right) dx$$

is bounded uniformly with respect to $\epsilon \in (0, \overline{\epsilon}]$ and $u \in S(k)$. In fact by definition of Φ_ϵ^q and by the exponential decay of the eigenfunctions (see Theorem 4.1) there exists a ball $B_{\mathbb{R}^n}(0,R)$ such that, if we write $u = \sum_{m=1}^j a_m \varphi_m$ with $\sum_{m=1}^j a_m^2 = 1$, for all $x \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,R)$ the following inequalities hold

$$\begin{aligned}\left|\frac{\Phi_\epsilon^q(x) + \hat{\rho}_q u(x)}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\right| &= \frac{\hat{\rho}_q |u(x)|}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \\ &\le \frac{C\,\hat{\rho}_q \Big(\sum_{m=1}^j |a_m|\Big) e^{-|x|}}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}} \\ &\le M e^{-|x|} < c_3\end{aligned}$$

where the constant M does not depend on $u \in S(j)$ nor on ϵ for ϵ small enough (see the point (ii) of Lemma 5.7). By $\big(W_4\big)$ we get

$$\left|W\left(\frac{\Phi_\epsilon^q(x) + \hat{\rho}_q u(x)}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\right)\right| \le c_4 \frac{|\Phi_\epsilon^q(x) + \hat{\rho}_q u(x)|^2}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}$$

for any $x \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,R)$. Concluding we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} W\left(\frac{\Phi_\epsilon^q + \hat{\rho}_q u}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\right) dx\right| \\ &\le \quad c_4 + \int_{B_{\mathbb{R}^n}(0,R)} \left|W\left(\frac{\Phi_\epsilon^q + \hat{\rho}_q u}{\|\Phi_\epsilon^q + \hat{\rho}_q u\|_{L^2(\mathbb{R}^n,\mathbb{R}^{n+1})}}\right)\right| dx\end{aligned}$$

where the integral on the right hand side is bounded by (iii) of Lemma 5.7. So we have the claim.

Step 3 *We prove that* $c_{\epsilon,j}^q \ge \tilde{\lambda}_j$. By Step 1 and by the positivity of W we get

$$\begin{aligned}c_{\epsilon,j}^q &\ge \inf_{h \in \mathcal{H}_{\epsilon,j}^q} \sup_{v \in \mathcal{M}_{\epsilon,j}^q} \|h(v)\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} \\ &\ge \inf_{h \in \mathcal{H}_{\epsilon,j}^q} \sup_{\substack{v \in \mathcal{M}_{\epsilon,j}^q \\ P_{F_{j-1}} h(v) = 0}} \|h(v)\|^2_{\Gamma_F(\mathbb{R}^n,\mathbb{R}^{n+1})} \\ &\ge \tilde{\lambda}_j\end{aligned}$$

In fact by Step 1 for all $h \in \mathcal{H}_{\epsilon,j}^q$ we have that the set $h(\mathcal{M}_{\epsilon,j}^q)$ intersects the set $\{u \in F : (u, \varphi_i) = 0\, \forall i = 1, \dots, j-1\}$ and so from (4.8) we get the claim.

Step 4 *If* $\tilde{\lambda}_{j-1} < \tilde{\lambda}_j$, *then for* ϵ *small enough we have:*

$$c_{\epsilon,j-1}^q \quad < \quad c_{\epsilon,j}^q\,, \tag{6.7}$$

$$\sup_{v \in \mathcal{M}_{\epsilon,j-1}^q} J_\epsilon(v) \quad < \quad c_{\epsilon,j}^q\,. \tag{6.8}$$

By Step 2 and 3 we obtain for ϵ small enough

$$c_{\epsilon,j-1}^q \le \tilde{\lambda}_{j-1} + \sigma(\epsilon) < \tilde{\lambda}_j \le c_{\epsilon,j}^q\,,$$

$$\sup_{v\in\mathcal{M}^q_{\epsilon,j-1}} J_\epsilon(v) \le \tilde\lambda_{j-1} + \sigma(\epsilon) < \tilde\lambda_j \le c^q_{\epsilon,j}\,.$$

Step 5 *If $\tilde\lambda_{j-1} < \tilde\lambda_j$, then $c^q_{\epsilon,j}$ is a critical value for the functional J_ϵ on the manifold $\Lambda^F_q \cap S$.*

By contradiction we suppose that $c^q_{\epsilon,j}$ is a regular value for J_ϵ on $\Lambda^F_q \cap S$. By Proposition 3.10 and Lemmas 6.1 and 6.2, there exist $\delta > 0$ and a deformation $\eta : [0,1] \times \Lambda^F_q \cap S \to \Lambda^F_q \cap S$ such that

$$\eta(0,u) = u \quad \forall u \in \Lambda^F_q \cap S\,,$$
$$\eta(t,u) = u \quad \forall t \in [0,1], \forall u \in J_\epsilon^{c^q_{\epsilon,j}-2\delta}\,,$$
$$\eta(1, J_\epsilon^{c^q_{\epsilon,j}+\delta}) \subset J_\epsilon^{c^q_{\epsilon,j}-\delta}\,.$$

By (6.8) we can suppose

$$\sup_{v\in\mathcal{M}^q_{\epsilon,j-1}} J_\epsilon(v) < c^q_{\epsilon,j} - 2\delta\,. \tag{6.9}$$

Moreover by definition of $c^q_{\epsilon,j}$ there exists a transformation $\hat h \in \mathcal{H}^q_{\epsilon,j}$ such that $\sup_{v\in\mathcal{M}^q_{\epsilon,j}} J_\epsilon\big(\hat h(v)\big) < c^q_{\epsilon,j} + \delta$. Now by the properties of the deformation η and by (6.9) we get $\eta\big(1,\hat h(.)\big) \in \mathcal{H}^q_{\epsilon,j}$ and $\sup_{v\in\mathcal{M}^q_{\epsilon,j}} J_\epsilon\big(\eta(1,\hat h(v))\big) < c^q_{\epsilon,j} - \delta$ and this is a contradiction. □

Remark 6.6. (1) In the assumptions of Theorem 6.5

$$\min_{u\in\Lambda^F_q\cap S} J_\epsilon(u) = c^q_{\epsilon,1}\,.$$

Nevertheless the critical point corresponding to the minimum, found in Theorem 6.3, is not attained in the framework of Theorem 6.5. In fact, to conclude that a value $c^q_{\epsilon,j}$ is critical, j must be strictly greater than one.

(2) Provided that we choose suitably $\xi_\star$ and ϵ, it is possible to find as many solutions of $\big(P_\epsilon\big)$ as we want. In fact, let us suppose that we want $K \in \mathbb{N}$ solutions, then, since $\tilde\lambda_j \to \infty$, there exists $k \in \mathbb{N}$, $k > K$, such that among the first k eigenvalues $\tilde\lambda$ there are K "jumps" $\tilde\lambda_j < \tilde\lambda_{j+1}$, so that Theorem 6.5 gives K critical values.

(3) For all $q \in \mathbb{Z}\setminus\{0\}$, $\epsilon \in (0,1]$ the critical values $c^q_{\epsilon,j}$ tend to the eigenvalues $\tilde\lambda_j$ when ϵ tends to zero.

Acknowledgements. This paper arises from the Ph.D. thesis (see [18]) of the author, who heartily thanks her supervisors V. Benci and A. M. Micheletti.

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