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REGULARITY OF WEAK SOLUTIONS TO THE LANDAU-LIFSHITZ SYSTEM IN BOUNDED REGULAR DOMAINS

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Abstract. In this paper, we study the regularity, on the boundary, of weak solutions to the Landau-Lifshitz system in the framework of the micromagnetic model in the quasi-static approximation. We establish the existence of global weak solutions to the Landau-Lifshitz system whose tangential space gradient on the boundary is square integrable.

1. Introduction

The Landau-Lifshitz equation models the behavior of ferromagnetic materials, and it is used in crystallography. This nonlinear systems of partial differential equations is

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \wedge \mathbf{h} - \alpha \mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h}). \tag{1.1}$$

with $\mathbf{h}$ (magnetic excitation) depending on $\mathbf{m}$ (magnetization), and $\alpha \in \mathbb{R}_*^+$ (dampening parameter).

Existence of global weak solutions to the Landau-Lifshitz equation have been proved in [14, 1, 9]. However, these solutions are only required to belong to $\mathrm{L}^\infty(\mathbb{R}^+; \mathbb{H}^1(O))$ and to $\mathbb{H}^1(O \times (0,T))$ for all time $T > 0$. Besides, they are not unique. There are existence results for of more regular solutions in the case when $\mathbf{h} = \Delta \mathbf{m}$, see [10, 11]. For 3D and a full characterization of the unique "good" solution in 2D has been obtained by Harpes[8]. These results are not easily generalized to more complicated forms of $\mathbf{h}$ as they often rely on harmonic analysis. Some authors have studied the regularity of stationary maps when the excitation $\mathbf{h}$ has a more complicated form: Critical points of the energy are regular away from a set of zero two-dimensional Hausdorff measure; see Carbou [3] and Hardt and Kinderlehrer [7]. Using standard analysis, one can easily prove that any weak solution $\mathbf{m}$ belongs to $\mathcal{C}([0,+\infty); \mathbb{H}^1_w(O))$, and to the Nikol'skii space $\mathrm{L}^2(0,T; (B^2_{1,\infty}(O))^3)$, and satisfy $\Delta \mathbf{m} \in \mathrm{L}^2(0,T; \mathbb{L}^1(O))$, see [13, §6.2].

In this paper, we establish that for any initial condition, at least one weak solution to the Landau-Lifshitz system has a trace that belongs to $\mathrm{L}^2(0,T; \mathbb{H}^1(\partial O))$.

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This result is not limited to the case $\mathbf{h} = A\Delta\mathbf{m}$ and can be generalized to a wide range of forms of $\mathbf{h}$.

In section 1, we introduce a common notation used throughout this paper. Then, in section 2, we recall briefly the micromagnetic model. We recall known results concerning the existence of weak solutions in section 3. We state and prove our main theorem in section 4.

Notation. Given an open set O, we denote by $\mathrm{L}^p(O)$ the set of all measurable functions u over O such that $\int_O |u|^p \,\mathrm{d}\mathbf{x} < +\infty$. This is a Banach space for the norm

$$\|u\|_{\mathrm{L}^p(O)} = \Big(\int_O |u|^p \,\mathrm{d}\mathbf{x}\Big)^{1/p}.$$

For any integer $m \geq 0$, we denote by $\mathrm{H}^m(O)$, the space of all measurable functions u over O such that for any multi-indices α, $|\alpha| < m$, $\mathrm{D}^\alpha u$ belongs to $\mathrm{L}^2(O)$. This is an Hilbert space for the norm

$$\|u\|_{\mathrm{H}^m(O)} = \Big(\sum_{|\alpha|\leq m} \|\,\mathrm{D}^\alpha u\|^2_{\mathrm{L}^2(O)}\Big)^{1/2}.$$

We set $\mathbb{H}^m(O) = (\mathrm{H}^m(O))^3$ and $\mathbb{L}^p(O) = (\mathrm{L}^p(O))^3$. By $|O|$, we denote the Lebesgue measure of set O.

Given a smooth surface ∂O, ν represents the unit outward normal vector to the surface, $\frac{\partial u}{\partial \nu}$ the normal trace of u on ∂O, and $\nabla_{\mathrm{T}} u$ the tangential gradient of u on ∂O.

In all this paper, Ω is a bounded open set of $\mathbb{R}^3$ with a smooth boundary. We also define $\Omega_T = \Omega \times (0, T)$.

2. The micromagnetic model

In this section, we recall briefly the micromagnetic model. We begin by introducing the more common energies and excitations that model completely the static behavior, then we introduce the nonlinear PDE that models the evolution problem. From now on, Ω represents the domain filled with a ferromagnetic material.

2.1. The static problem.

The magnetic state of a ferromagnetic material is represented by two vector fields: the magnetization $\mathbf{m}$ and the magnetic excitation $\mathbf{h}$. In the micromagnetic model the magnetization must verify a non convex constraint: $|\mathbf{m}| = 1$ in Ω and be null outside Ω. The excitation $\mathbf{h}$ depends on $\mathbf{m}$.

To each interaction p is associated an energy $\mathrm{E}_p(\mathbf{m})$ and an operator $\mathcal{H}_p$ related by:

$$\mathrm{DE}_p(\mathbf{m}) \cdot \mathrm{v} = -\int_\Omega \mathcal{H}_p(\mathbf{m}) \cdot \mathrm{v} \,\mathrm{d}\mathbf{x}, \quad \mathrm{E}_p(\mathbf{0}) = 0.$$

The excitation contributed by p is $\mathbf{h}_p = \mathcal{H}_p(\mathbf{m})$ and the total excitation is $\mathbf{h} = \sum_p \mathbf{h}_p$. In this paper we consider three interactions, see Brown [2] for details:

Exchange: The exchange energy and its associated excitation are given by:

$$\mathrm{E}_e(\mathbf{m}) = \frac{A}{2}\int_\Omega |\nabla\mathbf{m}|^2 \,\mathrm{d}\mathbf{x}, \quad \mathcal{H}_e(\mathbf{m}) = A\Delta\mathbf{m},$$

where $A > 0$.

Anisotropy:

$$\mathrm{E}_a(\mathbf{m}) = \frac{1}{2}\int_\Omega \mathbf{m}\cdot\mathbf{K}\mathbf{m} \,\mathrm{d}\mathbf{x}, \quad \mathcal{H}_a(\mathbf{m}) = -\mathbf{K}\mathbf{m},$$

where $\mathbf{K}$ is a symmetric positive quadratic form.
Demagnetization field:

$$\mathrm{E}_d(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathcal{H}_d(\mathbf{m})|^2 \,\mathrm{d}\mathbf{x}, \quad \mathcal{H}_d(\mathbf{m}) = \mathbf{h}_d,$$

where $\mathbf{h}_d$ is the only solution in $\mathrm{L}^2(\mathbb{R}^3)$ to the magnetostatic system

$$\operatorname{div}(\mathbf{h}_d + \mathbf{m}) = 0, \qquad \operatorname{curl}(\mathbf{h}_d) = 0,$$

in the sense of distributions.

The demagnetization field operator has been extensively studied by Friedman [4, 5, 6]. This operator is symmetric negative, is linear continuous from $\mathbb{L}^p(\mathbb{R}^3)$ to $\mathbb{L}^p(\mathbb{R}^3)$, $1 < p < +\infty$, and satisfies

$$-\int_\Omega \mathcal{H}_d(\mathbf{m}) \cdot \mathbf{m} \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} |\mathcal{H}_d(\mathbf{m})|^2 \,\mathrm{d}\mathbf{x}.$$

The $\mathcal{L}(\mathbb{L}^2(\mathbb{R}^3), \mathbb{L}^2(\mathbb{R}^3))$ norm of $\mathcal{H}_d$ is less than 1. We also define

$$\mathcal{H} = \mathcal{H}_e + \mathcal{H}_a + \mathcal{H}_d, \quad \mathcal{H}_{d,a} = \mathcal{H}_d + \mathcal{H}_a,$$
$$\mathcal{H}_{e,a} = \mathcal{H}_e + \mathcal{H}_a, \quad \mathrm{E} = \mathrm{E}_e + \mathrm{E}_a + \mathrm{E}_d.$$

The solutions to the static problem are the local minimizers of the total energy E satisfying the constraint $|\mathbf{m}| = 1$ a.e. in Ω.

2.2. **The evolution system.** The Landau-Lifshitz system models the evolution of ferromagnetic materials. Let $\alpha > 0$ be a dampening parameter. The Landau-Lifshitz system comprises the nonlinear system

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \wedge \mathcal{H}(\mathbf{m}) - \alpha \mathbf{m} \wedge \big(\mathbf{m} \wedge \mathcal{H}(\mathbf{m})\big) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{2.1a}$$

the initial condition

$$\mathbf{m}(\cdot, 0) = \mathbf{m}_0 \quad \text{in } \Omega, \tag{2.1b}$$

the non convex constraint

$$|\mathbf{m}| = 1 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{2.1c}$$

and the Neumann boundary conditions

$$\frac{\partial \mathbf{m}}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{2.1d}$$

3. Known results

We give the definition of weak solutions to the Landau-Lifshitz system (2.1).

Definition 3.1. Given $\mathbf{m}_0$ in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0| = 1$ a.e. in Ω, we call $\mathbf{m}$ a weak solution to the Landau-Lifshitz system (2.1) if

(1) For all $T > 0$, $\mathbf{m}$ belongs to $\mathbb{H}^1(\Omega \times (0, T))$, and $|\mathbf{m}| = 1$ a.e. in $\Omega \times \mathbb{R}^+$.

(2) For all ϕ in $\mathbb{H}^1(\Omega \times (0,T))$,

$$\begin{aligned}\iint_{\Omega T} \frac{\partial \mathbf{m}}{\partial t} \cdot \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \alpha \iint_{\Omega_T} \Big(\mathbf{m} \wedge \frac{\partial \mathbf{m}}{\partial t}\Big) \cdot \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ = (1+\alpha^2) A \iint_{\Omega_T} \sum_{i=1}^{3} \Big(\mathbf{m} \wedge \frac{\partial \mathbf{m}}{\partial x_i}\Big) \cdot \frac{\partial \phi}{\partial x_i} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - (1+\alpha^2) \iint_{\Omega_T} (\mathbf{m} \wedge \mathcal{H}_{d,a}(\mathbf{m})) \cdot \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned} \tag{3.1a}$$

(3) $\mathbf{m}(\cdot, 0) = \mathbf{m}_0$ in the sense of traces.

(4) For all $T > 0$,

$$\mathrm{E}(\mathbf{m}(T)) + \frac{\alpha}{1+\alpha^2} \iint_{\Omega_T} \big|\frac{\partial \mathbf{m}}{\partial t}\big|^2 \,\mathrm{d}t\,\mathrm{d}\mathbf{x} \le \mathrm{E}(\mathbf{m}(0)), \tag{3.1b}$$

where

$$\mathrm{E}(\mathbf{u}) = \frac{A}{2}\|\nabla \mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\mathbf{K}\mathbf{u}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2}\|\mathcal{H}_d(\mathbf{m})\|^2_{\mathrm{L}^2(\Omega)}.$$

It was proved by Alouges and Soyeur [1] that the Landau-Lifshitz system (2.1) has at least one weak solution when $\mathbf{h} = A\Delta\mathbf{m}$. This result was generalized to the full $\mathbf{h}$ in [9]:

Theorem 3.2. *Let $\mathbf{m}_0$ belongs to $\mathbb{H}^1(\Omega)$, such that $|\mathbf{m}_0| = 1$ a.e. in Ω. Then, there exists at least one solution to the Landau-Lifshitz system $\mathbf{m}$ in the sense of definition 3.1.*

Proof. The proof is based on the study of a penalized system whose solution converges to a weak solution to the Landau-Lifshitz system. The penalized system was:

$$\alpha \frac{\partial \mathbf{m}^k}{\partial t} + \mathbf{m}^k \wedge \frac{\partial \mathbf{m}^k}{\partial t} = (1+\alpha^2)\big(\mathcal{H}(\mathbf{m}^k) - k(|\mathbf{m}^k|^2 - 1)\mathbf{m}\big), \tag{3.2a}$$

$$\frac{\partial \mathbf{m}^k}{\partial \nu} = 0, \tag{3.2b}$$

$$\mathbf{m}^k(\cdot, 0) = \mathbf{m}_0. \tag{3.2c}$$

See [1, 9] for details. See also [12] for a generalization with surface energies. □

We will prove a slightly stronger result. However, we need a uniform $\mathbb{L}^\infty(\Omega \times \mathbb{R}^+)$ bound of $\mathbf{m}^k$. But, the non locality of the $\mathcal{H}_d$ operator prevents us from proving this result. For this reason, we introduce another penalized, less simple, system (4.1) to prove the existence of solutions more regular on the boundary.

4. Existence of a solution with H^1 regularity in space on the boundary

We state our main result as follows.

Theorem 4.1. *Let $\mathbf{m}_0$ belong to $\mathbb{H}^1(\Omega)$, such that $|\mathbf{m}_0| = 1$ a.e. in Ω. Then, there exists at least one solution to the Landau-Lifshitz system $\mathbf{m}$ in the sense of definition 3.1 such that $\gamma\mathbf{m} \in \mathrm{L}^2(0,T;\mathbb{H}^1(\partial\Omega))$.*

This theorem is a consequence of propositions 4.3 and 4.6. The rest of this section is dedicated to the proof of this theorem. We first introduce a penalized system:

$$
\begin{aligned}
\alpha\frac{\partial \mathbf{m}^k}{\partial t}+\mathbf{m}^k\wedge\frac{\partial \mathbf{m}^k}{\partial t} &= (1+\alpha^2)\Big(\mathcal{H}_{e,a}(\mathbf{m}^k)+1_{\{\mathbf{m}^k\neq 0\}}\mathcal{H}_d(\mathbf{m}^k) \\
&\quad -\big(\mathcal{H}_d(\mathbf{m}^k)\cdot\frac{\mathbf{m}^k}{|\mathbf{m}^k|}\big)\frac{\mathbf{m}^k}{|\mathbf{m}^k|} \\
&\quad -k(|\mathbf{m}^k|^2-1)\mathbf{m}^k\Big) \quad \text{in } \Omega\times\mathbb{R}^+,
\end{aligned} \tag{4.1a}
$$

$$\frac{\partial \mathbf{m}^k}{\partial \nu}=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+, \tag{4.1b}$$

$$\mathbf{m}^k(\cdot,0)=\mathbf{m}_0 \quad \text{in } \Omega. \tag{4.1c}$$

We first show that this penalized system has a weak solution that converges to a weak solution to the Landau-Lifshitz system as k tends to $+\infty$. We then show that the $\mathrm{L}^2(0,T;\mathbb{H}^1(\partial\Omega))$ norm of $\mathbf{m}^k$ is bounded independently of k. This requires a uniform $\mathbb{L}^\infty(\Omega\times\mathbb{R}^+)$ bound of $\mathbf{m}^k$ first: something we could not establish for system (3.2), hence the modification of the penalized system.

4.1. **Properties of solutions to the penalized system** (4.1)**.** One can easily adapt the proof of existence of solution to the original penalized system (3.2) found in [9, 1] to the modified penalized system (4.1):

Proposition 4.2. *Let $\mathbf{m}_0$ in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0|=1$ a.e. in Ω. Then, there exists a solution $\mathbf{m}^k$ to system* (4.1) *in $\mathrm{L}^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))\cap\mathrm{H}^1(\Omega\times(0,T))$, i.e. satisfying:*

$$
\begin{aligned}
&\iint_{\Omega_T}\Big(\alpha\frac{\partial \mathbf{m}^k}{\partial t}+\mathbf{m}^k\wedge\frac{\partial \mathbf{m}^k}{\partial t}\Big)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= -(1+\alpha^2)A\iint_{\Omega_T}\nabla\mathbf{m}^k\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+(1+\alpha^2)\iint_{\Omega_T}1_{\{\mathbf{m}^k\neq 0\}}\mathcal{H}_{d,a}(\mathbf{m}^k)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad -(1+\alpha^2)\iint_{\Omega_T}\Big(\frac{(\mathcal{H}_d(\mathbf{m}^k)\cdot\mathbf{m}^k)\mathbf{m}^k}{|\mathbf{m}^k|^2}\Big)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad -(1+\alpha^2)k\iint_{\Omega_T}(|\mathbf{m}^k|^2-1)\mathbf{m}^k\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned} \tag{4.2a}
$$

for all ψ in $\mathbb{H}^1(\Omega\times(0,T))$, and for all time $T>0$, all $\eta>0$:

$$
\begin{aligned}
&\mathrm{E}(\mathbf{m}^k(\cdot,T))+(\frac{\alpha}{1+\alpha^2}-\eta)\int_0^T\|\frac{\partial \mathbf{m}^k}{\partial t}\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}t+\frac{k}{4}\||\mathbf{m}^k(\cdot,T)|^2-1\|^2_{\mathrm{L}^2(\Omega)} \\
&\le \big(|\Omega|+\mathrm{E}(\mathbf{m}_0)\big)\exp(\frac{T}{2k\eta})-|\Omega|.
\end{aligned} \tag{4.2b}
$$

Proof. The proof is the same as the one in [1, 9] with a minor complication arising from the supplementary term.

Let $(w_1,\dots,w_n,\dots)$ be the orthonormal hilbertian basis of $\mathrm{L}^2(\Omega)$ comprising the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions. The w_i belongs to $\mathcal{C}^\infty(\overline{\Omega})$. Let V_n be the subspace of $\mathrm{L}^2(\Omega)$ spanned by functions $(w_1,\dots,w_n)$. Let $\mathcal{P}_n$ be the orthogonal projector on V_n in $\mathrm{L}^2(\Omega)$. The basis $(w_1,\dots,w_n,\dots)$ is also an hilbertian orthogonal basis of $\mathrm{H}(\Omega)$ and $\mathcal{P}_n$

is also an orthogonal projector in $\mathrm{H}^1(\Omega)$. We search for $\mathbf{m}_n^k$ with the form $\mathbf{m}_n^k = \sum_{i=1}^n \phi_i(t) w_i(\mathbf{x})$, where the ϕ_i are in $\mathcal{C}^\infty(\mathbb{R}^+;\mathbb{R}^3)$ such that

$$\begin{aligned}\alpha\frac{\partial \mathbf{m}_n^k}{\partial t} + \mathcal{P}_n(\mathbf{m}_n^k \wedge \frac{\partial \mathbf{m}_n^k}{\partial t}) &= (1+\alpha^2)\mathcal{P}_n\left(\mathcal{H}(\mathbf{m}_n^k) - \frac{(\mathcal{H}_d(\mathbf{m}_n^k)\cdot \mathbf{m}_n^k)\mathbf{m}_n^k}{|\mathbf{m}_n^k|^2 + n^{-1}}\right) \\ &\quad - (1+\alpha^2)k\mathcal{P}_n\big((|\mathbf{m}_n^k|^2-1)\mathbf{m}_n^k\big)\end{aligned} \tag{4.3a}$$

$$\mathbf{m}_n^k(\cdot,0) = \mathcal{P}_n(\mathbf{m}_0). \tag{4.3b}$$

Let $\mathbf{\Phi}_n = (\phi_1,\dots,\phi_n)$. Equation (4.3a) is equivalent to

$$\frac{\mathrm{d}\mathbf{\Phi}_n}{\mathrm{d}t} - \mathbf{A}(\mathbf{\Phi}_n(t))\frac{\mathrm{d}\mathbf{\Phi}_n}{\mathrm{d}t} = F(\mathbf{\Phi}_n(t)),$$

where F is of class $\mathcal{C}^\infty$ (thanks to the n^{-1} in the denominator which serves no other purpose) and $\mathbf{\Phi}_n(t) \mapsto \mathbf{A}(\mathbf{\Phi}_n(t))$ is linear continuous, thus smooth. Moreover, $\mathbf{A}(\mathbf{\Phi})$ is an antisymmetric matrix for all $\mathbf{\Phi}$. So the matrix $\mathbf{I}-\mathbf{A}(\mathbf{\Phi})$ is nonsingular and the function $\mathbf{\Phi} \mapsto (\mathbf{I}-\mathbf{A}(\mathbf{\Phi}))^{-1}$ is of class $\mathcal{C}^\infty$. By Cauchy-Lipshitz, $\mathbf{\Phi_i}(t)$ exists locally in time. Equation (4.3a) can be expressed as

$$\begin{aligned}&\iint_{\Omega_T}\left(\alpha\frac{\partial \mathbf{m}_n^k}{\partial t} + \mathbf{m}_n^k \wedge \frac{\partial \mathbf{m}_n^k}{\partial t}\right)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &= -(1+\alpha^2)A\iint_{\Omega_T}\nabla\mathbf{m}_n^k\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + (1+\alpha^2)\iint_{\Omega_T}\mathcal{H}_{d,a}(\mathbf{m}_n^k)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - (1+\alpha^2)\iint_{\Omega_T}\left(\frac{(\mathcal{H}_d(\mathbf{m}_n^k)\cdot\mathbf{m}_n^k)\mathbf{m}_n^k}{|\mathbf{m}_n^k|^2+n^{-1}}\right)\cdot\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - (1+\alpha^2)k\iint_{\Omega_T}(|\mathbf{m}_n^k|^2-1)\mathbf{m}_n^k\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,\end{aligned} \tag{4.4}$$

for all ψ in $V_n\otimes\mathcal{C}^\infty(\mathbb{R}^+;\mathbb{R}^3)$. In (4.4), we take $\psi = \frac{\partial \mathbf{m}_n^k}{\partial t}$, we obtain for all $T>0$:

$$\begin{aligned}&\mathrm{E}(\mathbf{m}_n^k(\cdot,T)) + \frac{\alpha}{1+\alpha^2}\int_0^T \|\frac{\partial \mathbf{m}_n^k}{\partial t}\|_{\mathbb{L}^2(\Omega)}^2\,\mathrm{d}t + \frac{k}{4}\||\mathbf{m}_n^k(\cdot,T)|^2-1\|_{\mathrm{L}^2(\Omega)}^2 \\ &\le \mathrm{E}(\mathbf{m}_0) + \frac{k}{4}\||\mathcal{P}_n(\mathbf{m}_0)|^2-1\|_{\mathrm{L}^2(\Omega)}^2 - \iint_{\Omega_T}\left(\frac{(\mathcal{H}_d(\mathbf{m}_n^k)\cdot\mathbf{m}_n^k)\mathbf{m}_n^k}{|\mathbf{m}_n^k|^2+n^{-1}}\right)\cdot\frac{\partial\mathbf{m}_n^k}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned}$$

Therefore, for all $\eta>0$, all time $T>0$:

$$\begin{aligned}&\mathrm{E}(\mathbf{m}_n^k(\cdot,T)) + (\frac{\alpha}{1+\alpha^2}-\eta)\int_0^T \|\frac{\partial \mathbf{m}_n^k}{\partial t}\|_{\mathbb{L}^2(\Omega)}^2\,\mathrm{d}t + \frac{k}{4}\||\mathbf{m}_n^k(\cdot,T)|^2-1\|_{\mathrm{L}^2(\Omega)}^2 \\ &\le \mathrm{E}(\mathbf{m}_0) + \frac{k}{4}\||\mathcal{P}_n(\mathbf{m}_0)|^2-1\|_{\mathrm{L}^2(\Omega)}^2 + \frac{1}{8\eta}\int_0^T\int_\Omega\big((|\mathbf{m}_n^k|^2-1)^2+1\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned}$$

By Gronwall, for all $\eta>0$, all time $T>0$:

$$\begin{aligned}&\mathrm{E}(\mathbf{m}_n^k(\cdot,T)) + (\frac{\alpha}{1+\alpha^2}-\eta)\int_0^T \|\frac{\partial \mathbf{m}_n^k}{\partial t}\|_{\mathbb{L}^2(\Omega)}^2\,\mathrm{d}t + \frac{k}{4}\||\mathbf{m}_n^k(\cdot,T)|^2-1\|_{\mathrm{L}^2(\Omega)}^2 \\ &\le \big(|\Omega| + \mathrm{E}(\mathbf{m}_0) + \frac{k}{4}\||\mathcal{P}_n(\mathbf{m}_0)|^2-1\|_{\mathrm{L}^2(\Omega)}^2\big)\exp(\frac{T}{2k\eta}) - |\Omega|.\end{aligned} \tag{4.5}$$

Thus, $\mathbf{m}_n^k$ exists in global time. Since, $\mathcal{P}_n(\mathbf{m}_0)$ tends to $\mathbf{m}_0$ in $\mathbb{H}^1(\Omega)$, $\frac{k}{4}\||\mathcal{P}_n(\mathbf{m}_0)|^2-1\|_{\mathrm{L}^2(\Omega)}^2$ tends to 0 as n tends to $+\infty$. Therefore, $\mathbf{m}_n^k$ is bounded in $\mathbb{H}^1(\Omega_T)$ and in $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$, independently of n. There exists $\mathbf{m}^k$ in $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$ and in

$\mathbb{H}^1(\Omega \times (0,T))$ and v^k in $\mathrm{L}^2(\Omega)$, such that, modulo a subsequence, as n tends to $+\infty$:

$$\mathbf{m}_n^k \to \mathbf{m}^k \quad \text{strongly in } \mathbb{L}^2(\Omega \times (0,T)), \tag{4.6a}$$

$$\mathbf{m}_n^k \to \mathbf{m}^k \quad \text{weakly in } \mathbb{H}^1(\Omega \times (0,T)), \tag{4.6b}$$

$$\mathbf{m}_n^k \to \mathbf{m}^k \quad \text{weakly}-* \text{ in } \mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega)), \tag{4.6c}$$

$$1_{\{\mathbf{m}^k=0\}} \frac{(\mathcal{H}_d(\mathbf{m}_n^k)\cdot \mathbf{m}_n^k)\mathbf{m}_n^k}{|\mathbf{m}_n^k|^2 + n^{-1}} \to \mathrm{v}^k \quad \text{weakly in } \mathbb{L}^2(\Omega \times (0,T)). \tag{4.6d}$$

Also, by Aubin's lemma, for all $1 < p < +\infty$, $1 < q < 6$, $T > 0$,

$$\mathbf{m}_n^k \to \mathbf{m}^k \quad \text{strongly in } \mathrm{L}^p(0,T;\mathbb{L}^q(\Omega)). \tag{4.6e}$$

The limit $\mathbf{m}^k$ has the required properties: $\mathcal{P}_n(\mathbf{m}_0)$ converges to $\mathbf{m}_0$ in $\mathbb{H}^1(\Omega)$. Computing the limit of (4.5), yields (4.2b). Since on the set $\{(\mathbf{x},t)|\mathbf{m}^k = 0\}$, $\frac{\partial \mathbf{m}^k}{\partial t} = 0$ and $\Delta \mathbf{m}^k = 0$ a.e., computing the limit of (4.4) yields first $\mathrm{v}^k = 1_{\{\mathbf{m}^k=0\}}\mathcal{H}_d(\mathbf{m}^k)$, then (4.2a) for all ψ in $\bigcup_{n=1}^\infty V_n \otimes \mathcal{C}^\infty([0,T];\mathbb{R}^3)$. By density, (4.2a) holds for all ψ in $\mathbb{H}^1(\Omega \times (0,T))$. $\square$

Now, we can prove that $\mathbf{m}^k$ converges to a weak solution to the Landau-Lifshitz system.

Proposition 4.3. *Let $\mathbf{m}_0$ be in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0| = 1$ a.e. in Ω. Let $\mathbf{m}^k$ be a weak solution to the penalized system* (4.1). *Then, there exists a subsequence of $\mathbf{m}^k$, that converges to a weak solution $\mathbf{m}$ to the Landau-Lifshitz system* (2.1) *weakly in $\mathbb{H}^1(\Omega \times (0,T))$.*

Proof. By (4.2b), $\mathbf{m}^k$ is bounded in $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$ and in $\mathbb{H}^1(\Omega \times (0,T)$. Besides, $k\||\mathbf{m}^k|^2 - 1\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}$ is bounded. There exists $\mathbf{m}$ in $\mathrm{H}^1(\Omega \times (0,T)$, such that, up to a subsequence,

$$\mathbf{m}^k \to \mathbf{m} \quad \text{strongly in } \mathbb{L}^2(\Omega \times (0,T)), \tag{4.7a}$$

$$\mathbf{m}^k \to \mathbf{m} \quad \text{weakly in } \mathbb{H}^1(\Omega \times (0,T)), \tag{4.7b}$$

$$\mathbf{m}^k \to \mathbf{m} \quad \text{weakly}-* \text{ in } \mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega)), \tag{4.7c}$$

$$|\mathbf{m}^k|^2 - 1 \to 0 \quad \text{strongly in } \mathbb{L}^2(\Omega \times (0,T)), \tag{4.7d}$$

Also, by Aubin's lemma, for all $1 < p < +\infty$, $1 < q < 6$, $T > 0$,

$$\mathbf{m}^k \to \mathbf{m} \quad \text{strongly in } \mathrm{L}^p(0,T;\mathbb{L}^q(\Omega)). \tag{4.7e}$$

Obviously, $\mathbf{m}(\cdot,0) = \mathbf{m}_0$. By (4.7d), $|\mathbf{m}| = 1$.

We compute the limit of (4.2b) as k tends to $+\infty$ as in [1, 9]. Then, we have η tend to 0: energy inequality (3.1b) is satisfied.

It only remains to prove that $\mathbf{m}$ satisfy (3.1a): in (4.2a), we take $\psi = \mathbf{m}^k \wedge \phi$, with ϕ in $\mathcal{C}^\infty(\overline{\Omega \times (0,T)};\mathbb{R}^3)$ and take the limit as k tends to $+\infty$. Since the supplementary term containing $(\mathcal{H}_d(\mathbf{m}^k)\cdot \frac{\mathbf{m}^k}{|\mathbf{m}^k|})\frac{\mathbf{m}^k}{|\mathbf{m}^k|}$ disappears, we can then conclude as in [1, 9] and obtain (3.1a) for all ϕ in $\mathcal{C}^\infty(\overline{\Omega \times (0,T)})$. By density, (3.1a) also holds for all ϕ in $\mathbb{H}^1(\Omega \times (0,T))$. $\square$

We establish an important proposition that we were unable to prove for the original penalized system (3.2) due to the non locality of $\mathcal{H}_d$.

Proposition 4.4. *Let $\mathbf{m}_0$ be in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0| = 1$ a.e. in Ω. Let $\mathbf{m}^k$ be a weak solution to system* (4.1)*. Then, $\mathbf{m}^k$ satisfy $|\mathbf{m}^k| \leq 1$ a.e. in $\Omega \times \mathbb{R}^+$.*

Proof. We follow Alouges-Soyeur [1], and introduce the map $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

We set $\psi(\mathbf{x}, t) = g(|\mathbf{m}^k(\mathbf{x}, t)|^2 - 1)\mathbf{m}^k(\mathbf{x}, t)$. The function ψ belongs to $\mathbb{H}^1(\Omega \times (0, T))$ and

$$\frac{\partial \psi}{\partial x_i} = 2g'(|\mathbf{m}^k|^2 - 1)\big(\mathbf{m}^k \cdot \frac{\partial \mathbf{m}^k}{\partial x_i}\big)\mathbf{m}^k + g(|\mathbf{m}^k|^2 - 1)\frac{\partial \mathbf{m}^k}{\partial x_i}.$$

Reporting ψ in (4.2a) yields

$$\begin{aligned}
&\alpha \iint_{\Omega_T} g(|\mathbf{m}^k(\mathbf{x}, t)|^2 - 1)\mathbf{m}^k(\mathbf{x}, t) \cdot \frac{\partial \mathbf{m}^k}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= -(1 + \alpha^2)A \sum_{i=1}^{3} \iint_{\Omega_T} 2g'(|\mathbf{m}^k|^2 - 1)\Big(\mathbf{m}^k \cdot \frac{\partial \mathbf{m}^k}{\partial x_i}\Big)^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1 + \alpha^2) \iint_{\Omega_T} g(|\mathbf{m}^k|^2 - 1)|\nabla \mathbf{m}^k|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1 + \alpha^2) \iint_{\Omega_T} g(|\mathbf{m}^k(\mathbf{x}, t)|^2 - 1)\mathbf{m}^k \cdot \mathbf{K}\mathbf{m}^k \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1 + \alpha^2)k \iint_{\Omega_T} (|\mathbf{m}^k|^2 - 1)|\mathbf{m}^k|^2 g(|\mathbf{m}^k(\mathbf{x}, t)|^2 - 1) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

Should we have used system (3.2) instead, then the term containing the global operator would have been very difficult, if not outright impossible, to estimate.

Therefore, $\iint_{\Omega_T} g(|\mathbf{m}^k(\mathbf{x}, t)|^2 - 1)\mathbf{m}^k(\mathbf{x}, t) \cdot \frac{\partial \mathbf{m}^k}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq 0$. Thus, for all $T > 0$,

$$\int_\Omega G(|\mathbf{m}^k(\cdot, T)|^2 - 1) \,\mathrm{d}\mathbf{x} \leq \int_\Omega G(|\mathbf{m}_0|^2 - 1) \,\mathrm{d}\mathbf{x} = 0,$$

where $G(x) = \int_0^x g(s) \,\mathrm{d}s$. Since $G \geq 0$, $G(|\mathbf{m}^k(\cdot, T)|^2 - 1) = 0$ a.e. in Ω for all $T > 0$. Therefore, $\mathbf{m}^k \leq 1$ a.e. in $\Omega \times \mathbb{R}^+$. $\square$

As a corollary to proposition 4.4, we have

Corollary 4.5. *Any weak solution $\mathbf{m}^k$ to system* (4.1) *belongs to the space $\mathbb{H}^{2,1}(\Omega \times (0, T))$.*

4.2. **Uniform $\mathbb{H}^1(\partial\Omega)$ bound of the penalized solution.** To prove Theorem 4.1, we only need to prove the following proposition.

Proposition 4.6. *Let $\mathbf{m}_0$ be in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0| = 1$. Let $\mathbf{m}^k$ be a weak solution to the penalized system* (4.1)*. Then, for all time $T > 0$ the quantity $\|\nabla_{\mathrm{T}} \gamma \mathbf{m}^k\|^2_{\mathbb{L}^2(\partial\Omega \times (0,T))}$ is bounded uniformly in k.*

Proof. Let ϕ_i be in $\mathcal{C}^\infty(\overline{\Omega \times (0, T)})$ for any integer i, $1 \leq i \leq 3$. By corollary 4.5 and proposition 4.4, we can multiply (4.1a) by $\frac{\partial \mathbf{m}^k}{\partial x_i}\phi_i$ and integrate over any open set $O \subset \Omega$ with a smooth boundary:

$$
\begin{aligned}
&\alpha \int_{O\times(0,T)} \frac{\partial \mathbf{m}^k}{\partial t} \cdot \frac{\partial \mathbf{m}^k}{\partial x_i} \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{O\times(0,T)} (\mathbf{m}^k \wedge \frac{\partial \mathbf{m}^k}{\partial t}) \cdot \frac{\partial \mathbf{m}^k}{\partial x_i} \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= (1+\alpha^2) \int_{O\times(0,T)} 1_{\{\mathbf{m}^k \neq 0\}} \mathcal{H}_{d,a}(\mathbf{m}^k) \cdot \frac{\partial \mathbf{m}^k}{\partial x_i} \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1+\alpha^2) \int_{O\times(0,T)} (\mathcal{H}_d(\mathbf{m}^k) \cdot \frac{\mathbf{m}^k}{|\mathbf{m}^k|}) \frac{\mathbf{m}^k}{|\mathbf{m}^k|} \cdot \frac{\partial \mathbf{m}^k}{\partial x_i} \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - (1+\alpha^2) \underbrace{k \int_{O\times(0,T)} \big(|\mathbf{m}^k|^2 - 1\big) \mathbf{m}^k \cdot \frac{\partial \mathbf{m}^k}{\partial x_i} \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t}_{I} \\
&\quad + (1+\alpha^2) A \underbrace{\sum_{j=1}^{3} \int_{O\times(0,T)} \Big(\frac{\partial}{\partial x_j} \Big(\frac{\partial \mathbf{m}^k}{\partial x_i} \cdot \frac{\partial \mathbf{m}^k}{\partial x_j} \Big) - \frac{1}{2} \frac{\partial}{\partial x_i} \Big| \frac{\partial \mathbf{m}^k}{\partial x_j} \Big|^2 \Big) \phi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t}_{II}
\end{aligned}
$$

However, in the above equality,

$$
I = -\frac{k}{4} \int_{O\times(0,T)} \big(|\mathbf{m}^k|^2 - 1\big)^2 \frac{\partial \phi_i}{\partial x_i} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{k}{4} \int_{\partial O\times(0,T)} \big(|\mathbf{m}^k|^2 - 1\big)^2 \nu_i \phi_i \,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t,
$$

and

$$
\begin{aligned}
II = &- \sum_{j=1}^{3} \int_{O\times(0,T)} \Big(\frac{\partial \mathbf{m}^k}{\partial x_i} \cdot \frac{\partial \mathbf{m}^k}{\partial x_j} \Big) \frac{\partial \phi_i}{\partial x_j} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{1}{2} \int_{O\times(0,T)} |\nabla \mathbf{m}^k|^2 \frac{\partial \phi_i}{\partial x_i} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_{\partial O\times(0,T)} \Big(\frac{\partial \mathbf{m}^k}{\partial x_i} \cdot \frac{\partial \mathbf{m}^k}{\partial \nu} \Big) \phi_i \,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t - \frac{1}{2} \int_{\partial O\times(0,T)} |\nabla \mathbf{m}^k|^2 \nu_i \phi_i \,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\alpha\int_{O\times(0,T)}\frac{\partial\mathbf{m}^k}{\partial t}\cdot\frac{\partial\mathbf{m}^k}{\partial x_i}\phi_i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_{O\times(0,T)}(\mathbf{m}^k\wedge\frac{\partial\mathbf{m}^k}{\partial t})\cdot\frac{\partial\mathbf{m}^k}{\partial x_i}\phi_i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\int_{O\times(0,T)}1_{\{\mathbf{m}^k\neq 0\}}\mathcal{H}_{d,a}(\mathbf{m}^k)\cdot\frac{\partial\mathbf{m}^k}{\partial x_i}\phi_i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+(1+\alpha^2)\int_{O\times(0,T)}(\mathcal{H}_d(\mathbf{m}^k)\cdot\frac{\mathbf{m}^k}{|\mathbf{m}^k|})\frac{\mathbf{m}^k}{|\mathbf{m}^k|}\cdot\frac{\partial\mathbf{m}^k}{\partial x_i}\phi_i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+(1+\alpha^2)A\sum_{j=1}^{3}\int_{O\times(0,T)}\Big(\frac{\partial\mathbf{m}^k}{\partial x_i}\cdot\frac{\partial\mathbf{m}^k}{\partial x_j}\Big)\frac{\partial\phi_i}{\partial x_j}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\frac{A}{2}\int_{O\times(0,T)}|\nabla\mathbf{m}^k|^2\frac{\partial\phi_i}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\frac{k}{4}\int_{O\times(0,T)}\big(|\mathbf{m}^k|^2-1\big)^2\frac{\partial\phi_i}{\partial x_i}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=-(1+\alpha^2)\frac{k}{4}\int_{\partial O\times(0,T)}\big(|\mathbf{m}^k|^2-1\big)^2\nu_i\phi_i\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t\\
&\quad+(1+\alpha^2)A\int_{\partial O\times(0,T)}\Big(\frac{\partial\mathbf{m}^k}{\partial x_i}\cdot\frac{\partial\mathbf{m}^k}{\partial\nu}\Big)\phi_i\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t\\
&\quad-(1+\alpha^2)\frac{A}{2}\int_{\partial O\times(0,T)}|\nabla\mathbf{m}^k|^2\nu_i\phi_i\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t,
\end{aligned}
\tag{4.8}
$$

for all ϕ_i in $\mathcal{C}^\infty(\overline{\Omega\times(0,T)})$. In (4.8), we choose ϕ_i independent of the time t such that $\phi_i=\nu_i$ on ∂O and sum over i. We obtain, denoting by ϕ the vector valued function (ϕ_1,ϕ_2,ϕ_3),

$$
\begin{aligned}
&\alpha\int_{O\times(0,T)}\frac{\partial\mathbf{m}^k}{\partial t}\cdot(\phi\cdot\nabla)\mathbf{m}^k\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_{O\times(0,T)}(\mathbf{m}^k\wedge\frac{\partial\mathbf{m}^k}{\partial t})\cdot(\phi\cdot\nabla)\mathbf{m}^k\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\int_{O\times(0,T)}1_{\{\mathbf{m}^k\neq 0\}}\mathcal{H}_{d,a}(\mathbf{m}^k)\cdot(\phi\cdot\nabla)\mathbf{m}^k\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+(1+\alpha^2)\int_{O\times(0,T)}(\mathcal{H}_d(\mathbf{m}^k)\cdot\frac{\mathbf{m}^k}{|\mathbf{m}^k|})\frac{\mathbf{m}^k}{|\mathbf{m}^k|}\cdot(\phi\cdot\nabla)\mathbf{m}^k\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+(1+\alpha^2)A\sum_{i,j=1}^{3}\int_{O\times(0,T)}\Big(\frac{\partial\mathbf{m}^k}{\partial x_i}\cdot\frac{\partial\mathbf{m}^k}{\partial x_j}\Big)\frac{\partial\phi_i}{\partial x_j}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\frac{A}{2}\int_{O\times(0,T)}|\nabla\mathbf{m}^k|^2\operatorname{div}\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&-(1+\alpha^2)\frac{k}{4}\int_{O\times(0,T)}\big(|\mathbf{m}^k|^2-1\big)^2\operatorname{div}\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&=-(1+\alpha^2)\frac{k}{4}\int_{\partial O\times(0,T)}\big(|\mathbf{m}^k|^2-1\big)^2\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t\\
&\quad+(1+\alpha^2)A\int_{\partial O\times(0,T)}\Big|\frac{\partial\mathbf{m}^k}{\partial\nu}\Big|^2\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t
\end{aligned}
$$

$$- (1+\alpha^2)\frac{A}{2}\int_{\partial O\times(0,T)} |\nabla \mathbf{m}^k|^2 \,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}t.$$

The left hand-side is bounded uniformly in k. Therefore,

$$\begin{aligned}\Big|\frac{A}{2}\|\nabla_{\mathrm{T}}\gamma\mathbf{m}^k\|^2_{\mathbb{L}^2(\partial O\times(0,T))} - \frac{A}{2}\|\frac{\partial \mathbf{m}^k}{\partial\nu}\|^2_{\mathbb{L}^2(\partial O\times(0,T))}\\ + \frac{k}{4}\||\gamma\mathbf{m}^k|^2-1\|^2_{\mathrm{L}^2(\partial O\times(0,T))}\Big| \le C(O).\end{aligned}$$

Since $\frac{\partial \mathbf{m}^k}{\partial \nu}=0$ on $\partial\Omega$, taking $O=\Omega$ yields the wanted result. $\square$

We derive from the previous proof the following corollary.

Corollary 4.7. *Let $\mathbf{m}_0$ be in $\mathbb{H}^1(\Omega)$, $|\mathbf{m}_0|=1$ a.e. in Ω. Let $\mathbf{m}^k$ be a weak solution to the penalized system* (4.1). *Then, the quantity $k\||\gamma\mathbf{m}^k|^2-1\|^2_{\mathbb{L}^2(\partial\Omega\times(0,T))}$ is bounded uniformly in k.*

Conclusion. In this paper, we have proved the existence of weak solutions to the Landau-Lifshitz system with an $\mathbb{H}^1$ regularity in space on the boundary of the domain. This result holds for very general form of $\mathbf{h}$ and is not limited to the case $\mathbf{h}=A\Delta\mathbf{m}$. These kinds of results are important because there is currently no "perfect" concept of what is a good weak solution to the Landau-Lifshitz system. Any criteria allowing to discriminate among those weak solutions is always welcome. It is natural to prefer among weak solutions those that are more regular. This result is also interesting as it opens the possibility to use first order transmission conditions between adjacent domains.

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References

[1] F. Alouges and A. Soyeur. On global weak solutions for Landau-Lifshitz equations : existence and nonuniqueness. *Nonlinear Analysis. Theory, Methods & Applications*, 18(11):1071–1084, 1992.

[2] W. F. Brown. *Micromagnetics.* Interscience Publishers, 1963.

[3] G. Carbou. Regularity for critical points of a non local energy. *Calc. Var. PDE*, 5:409–433, 1997.

[4] M. J. Friedman. Mathematical study of the nonlinear singular integral magnetic field equation I. *SIAM J. Appl. Math.*, 39(1):14–20, Août 1980.

[5] M. J. Friedman. Mathematical study of the nonlinear singular integral magnetic field equation II. *SIAM J. Numer. Math.*, 18(4):644–653, August 1981.

[6] M. J. Friedman. Mathematical study of the nonlinear singular integral magnetic field equation III. *SIAM J. Math. Anal.*, 12(4):644–653, July 1981.

[7] R. Hardt and D. Kinderlehrer. Some regularity results in micromagnetism. *Communications in Partial Differential Equations*, 25(7–8):1235–1258, 2000.

[8] P. Harpes. Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow. *Calc. Var. Partial Differential Equations*, 20(2):213–229, 2004.

[9] S. Labbé. *Simulation numérique du comportement hyperfréquence des matériaux ferromagnétiques.* PhD thesis, Université Paris 13, Décembre 1998.

[10] C. Melcher. Existence of partially regular solutions for Landau-Lifshitz equations. preprint, Institute for mathematics and its applications, University of Minnesota, November 2003.

[11] R. Moser. Partial regularity for the Landau-Lifshitz equation in small dimensions. Technical Report 26, Max-Planck-Institut fur Mathematik in den Naturwissenschaften, Leipzig, 2002.

[12] K. Santugini. Solutions to the Landau-Lifshitz system with nonhomogenous boundary conditions. preprint http://www-math.math.univ-paris13.fr/prepub/pp2004/pp2004-11.ps.gz, submitted, March 2004.

[13] K. Santugini-Repiquet. *Matériaux ferromagnetiques: influence d'un espaceur mince non magnétique, et homogénéisation d'agencements multicouches, en présence de couplage sur la frontière*. Thése de doctorat, Université Paris 13, Villetaneuse, dec 2004.

[14] A. Visintin. On Landau-Lifshitz equations for ferromagnetism. *Japan J. Appl. Math.*, 2(1):69–84, 1985.

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