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STEADY-STATE BIFURCATIONS OF THE THREE-DIMENSIONAL KOLMOGOROV PROBLEM

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Abstract. This paper studies the spatially periodic incompressible fluid motion in $\mathbb{R}^3$ excited by the external force $k^2(\sin kz, 0, 0)$ with $k \geq 2$ an integer. This driving force gives rise to the existence of the unidirectional basic steady flow $u_0 = (\sin kz, 0, 0)$ for any Reynolds number. It is shown in Theorem 1.1 that there exist a number of critical Reynolds numbers such that u_0 bifurcates into either 4 or 8 or 16 different steady states, when the Reynolds number increases across each of such numbers.

Thanks to the Rabinowitz global bifurcation theorem, all of the bifurcation solutions are extended to global branches for $\lambda \in (0, \infty)$. Moreover we prove that when λ passes each critical value, a) all the corresponding global branches do not intersect with the trivial branch (u_0, λ), and b) some of them never intersect each other; see Theorem 1.2.

1. Introduction

The three-dimensional Kolmogorov problem, which was first formulated by Kolmogorov (see [1]), refers to the Navier-Stokes equations defining the spatially periodic fluid motion in the following form:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda(u\cdot\nabla)u + \lambda\nabla p = k^2(\sin kz, 0, 0), \tag{1.1}$$

$$\text{div } u = 0, \tag{1.2}$$

$$u(t, x, y, z) = u(t, x + 2\pi, y, z) = u(t, x, y + 2\pi, z) = u(t, x, y, z + 2\pi), \tag{1.3}$$

$$\int_{\mathbb{T}^3} u \, dx \, dy \, dz = 0 \,. \tag{1.4}$$

Here $u = (u_1, u_2, u_3)$ is the velocity field, p the pressure, $\lambda > 0$ the Reynolds number in this dimensionless formulation, k a positive integer, and $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ $(n = 2, 3)$ the n-dimensional flat torus. In particular $u_0 = (\sin kz, 0, 0)$ solves this problem for all λ.

There have been extensive mathematical and physical studies for the Kolmogorov problem as well as for general fluid equations; see for instance [6, 8, 9, 10, 11, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27], and the references therein. From the bifurcation point of view, most of the literatures are devoted to the existence of secondary

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steady-states of the Navier-Stokes equations, and in particular the detailed rigorous mathematical analysis on the bifurcation of the solutions of the Kolmogorov problem as well as the general Navier-Stokes equations is still in its early stage, due partially to the difficulty in estimating the eigenvalues of the associated linearized operators.

The main objective of this paper is to establish the existence of multi-branches of steady-state solutions bifurcated respectively from u_0 at some critical Reynolds numbers. The multi-branches are obtained in some flow invariant subspaces, which are defined using special Fourier modes in terms of some n–tuple integer vectors.

Let $\mathbb{N}$ be the set of all positive integers, and $\mathbb{Z}$ be the integer set. The following condition for an integer vector (l, j, i, k) will be used often to define some invariant function spaces:

$$\begin{gathered}(l^2 + j^2 + i^2 - k^2)(l^2 + j^2 + (k - i)^2 - k^2) < 0,\\ 0 \le l < k,\ j \in \mathbb{Z},\ 0 \le i < k\,.\end{gathered} \tag{1.5}$$

For such an integer vector (l, j, i, k), we set

$$n_0 = \max\{n \in \mathbb{N} \mid n^2 l^2 + n^2 j^2 + \min\{\{ni\}^2, (k - \{ni\})^2\} - k^2 < 0\}, \tag{1.6}$$

and

$$i_0 = \begin{cases} 4, & \text{when } j = \{n_0 i\} = 0,\\ 8, & \text{when } |j| + \{n_0 i\} > 0 \text{ and } j\{n_0 i\} = 0,\\ 16, & \text{when } j\{n_0 i\} \neq 0.\end{cases} \tag{1.7}$$

Here $\{n\}$ is defined by

$$\{n\} \equiv n \pmod k). \tag{1.8}$$

Let $H^m(\mathbb{T}^3)$ be the usual Sobolev space of scalar and periodic functions endowed with the usual H^m norm $||\cdot||_{H^m}$, and let $\mathbb{H}^m(\mathbb{T}^3) = H^m(\mathbb{T}^3)^3$ be the vectorial Sobolev space. For $m \ge 1$, we use the following function spaces of divergence-free vector fields:

$$\mathbb{H}^m_\sigma = \left\{u \in \mathbb{H}^m(\mathbb{T}^3) \mid \text{ div } u = 0\right\}.$$

Let $\dot{H}^m(\mathbb{T}^3) = H^m(\mathbb{T}^3)/\mathbb{R}$, $\dot{\mathbb{H}}^m(\mathbb{T}^3) = \mathbb{H}^m(\mathbb{T}^3)/\mathbb{R}$, and $\dot{\mathbb{H}}^2_\sigma = \mathbb{H}^2_\sigma/\mathbb{R}$. When $m = 2$, we can use $\|u\|_{H^2_\sigma} = \|\Delta u\|_{L^2}$ as the norm of $\dot{\mathbb{H}}^2_\sigma$.

As we mentioned before, the main objective of this article is to find and possibly classify steady-state bifurcations of the Kolmogorov problem in some subspaces of $\dot{\mathbb{H}}^2_\sigma$ invariant to the Navier-Stokes equations. Now we state two main theorems of this article without specify invariant subspaces, which will be explicitly given in Sections 4–6.

Thanks to the Rabinowitz global bifurcation theorem, we obtain in these two theorems i_0 different global branches of steady-state solutions bifurcating from u_0 at a single critical value. Each of those branches undergoes local supercritical bifurcation around the bifurcation point, whereas half $(i_0/2)$ of those branches never touch each other away from the bifurcation point.

Theorem 1.1. *Let (l, j, i, k) be an integer vector satisfying (1.5). Then there exists a critical Reynolds number $\lambda_{l,j,i,k} > 0$ such that (1.1-1.4) with $0 < \lambda < \infty$ admit i_0 steady-state solutions $u_{l,j,i,k,n,\lambda}$ $(n = 1, ..., i_0)$ branching off $(\lambda_{l,j,i,k}, u_0)$ continuously such that:* [1]

[1] See (6.1–6.5) for specific spaces where these bifurcation solutions are located.

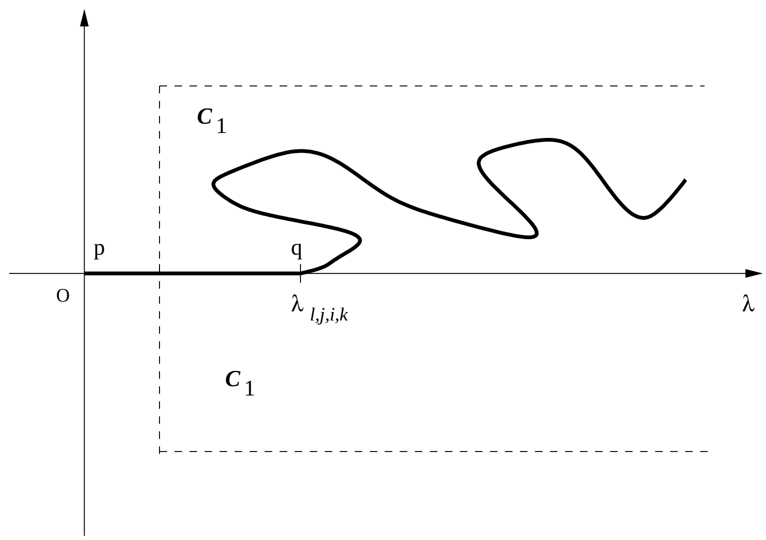

FIGURE 1.1. The global branch $u_{l,j,i,k,n,\lambda}$ in $\mathcal{C}_1$. Here the dotted horizontal lines represent $\|u\|_{H^1} = c_2k^2$, and dotted vertical line is $\lambda = \dfrac{c_1}{k^2}$.

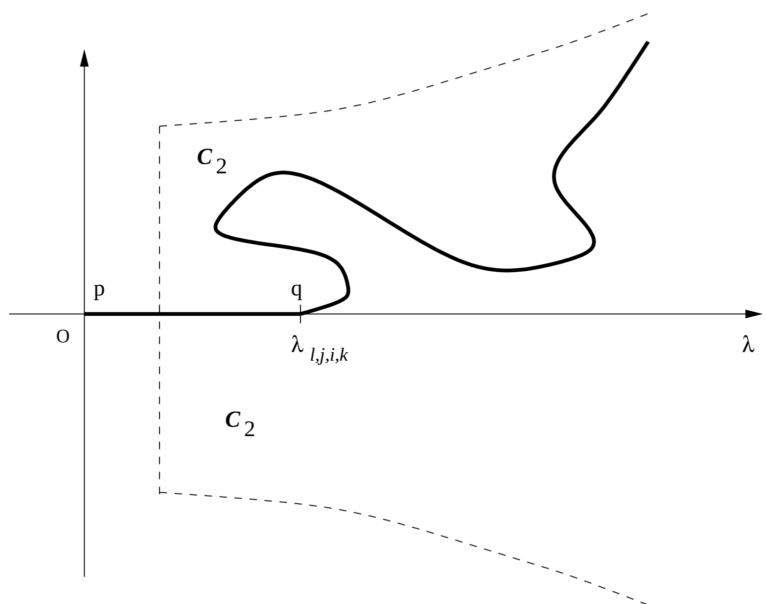

FIGURE 1.2. The global branch $u_{l,j,i,k,n,\lambda}$ in $\mathcal{C}_2$. Here the dotted vertical line is $\lambda = \dfrac{c_1}{k^2}$, while the dotted curves represent $\|u\|_{H^2} = c_3k^2(\lambda^2k^4 + 1)$.

a). *if* $\lambda_{l,j,i,k} < \lambda$, *then*

$$u_{l,j,i,k,n,\lambda} = u_0,$$

b). *if* $\lambda_{l,j,i,k} < \lambda$ *and* $\|\Delta(u_{l,j,i,k,n,\lambda} - u_0)\|_{L^2} + |\lambda - \lambda_{l,j,i,k}| < \epsilon$ *for some small* $\epsilon > 0$, *then*

$$u_0 \neq u_{l,j,i,k,n,\lambda} \neq u_{l,j,i,k,n',\lambda} \neq u_0. \tag{1.9}$$

We remark that for one value of $\lambda \in (0,\infty)$, there might be more than one steady-state solutions on the global bifurcation branch as shown in Figures 1.1 and 1.2. It is easy to see that all steady-state solutions of the Kolmogorov problem lie in both the following subsets of $\dot{\mathbb{H}}^1_\sigma \times (0,\infty)$ and $\dot{\mathbb{H}}^2_\sigma \times (0,\infty)$:

$$\begin{aligned} \mathcal{C}_1 = &\Big\{(u,\lambda) \in \dot{\mathbb{H}}^1_\sigma \times (0,\infty) \quad | \quad \|u\|_{H^1} \le c_2 k^2, \quad \lambda \ge \frac{c_1}{k^2}\Big\} &(1.10)\\ &\cup \Big\{(0,\lambda) \quad | \quad 0 < \lambda \le \frac{c_1}{k^2}\Big\},\\ \mathcal{C}_2 = &\Big\{(u,\lambda) \in \dot{\mathbb{H}}^2_\sigma \times (0,\infty) \quad | \quad \|u\|_{H^2} \le c_3 k^2(\lambda^2 k^4 + 1), \quad \lambda \ge \frac{c_1}{k^2}\Big\} &(1.11)\\ &\cup \Big\{(0,\lambda) \quad | \quad 0 < \lambda \le \frac{c_1}{k^2}\Big\}. \end{aligned}$$

Here the absolute constants c_1, c_2 and c_3 are given in Lemma 3.2 in Section 3. The set $\mathcal{C}_1$ is as shown in Figure 1.1. Since stronger H^2 norm is used in $\mathcal{C}_2$, the schematic picture of $\mathcal{C}_2$ shown in Figure 1.2 is similar to $\mathcal{C}_1$ depicted in Figure 1.1 but with the dotted horizontal lines replaced by quadratic line of λ given by $\|u\|_{H^2} = c_3 k^2(\lambda^2 k^4 + 1)$.

Theorem 1.2. *Let* (l,j,i,k) *be an integer vector satisfying (1.5). Then all bifurcation solutions* $u_{l,j,i,k,n,\lambda}$ *given by Theorem 1.1 satisfying the following properties:*

1. *each branch* $u_{l,j,i,k,n,\lambda}$ *extends to* $\lambda = \infty$ *in* $\mathcal{C}_1 \cap \mathcal{C}_2$ *as shown in Figures 1.1 and 1.2;*
2. *each branch* $u_{l,j,i,k,n,\lambda}$ *intersects with the* λ*–axis only at the line segment pq. In particular,* $u_{l,j,i,k,n,\lambda}$ *never touches the* λ*–axis for* $\lambda > \lambda_{l,j,i,k}$*;*
3. *for* $j \neq 0$ *and for* $i_1 = 0$ *or* $i_0/2$,
$$u_{l,j,i,k,n,\lambda} \neq u_{l,j,i,k,n',\lambda}, \tag{1.12}$$
if
$$\lambda_{l,j,i,k} < \lambda < \infty,\ i_1 + 1 \le n \le i_1 + \frac{i_0}{4} < n' \le i_1 + \frac{i_0}{2}.$$

The above two main theorems are the first ones showing the existence of either 4 or 8 or 16 global branches of steady-state solutions undergoing supercritical pitchfork bifurcations from a single bifurcation point for a Navier-Stokes problem. The method we employ in the proof of these theorems is the Rabinowitz global bifurcation theorem combined with continuous fraction method first introduced by Meshalkin and Sinai [17] and with Fourier analysis.

The central gravity of the proof relies on estimating the eigenvalues of the linearized Navier-Stokes operator associated with (1.1-1.4). One of the main difficulties is that the eigenvalues of the linearized Navier-Stokes operator always have the multiplicity $2m$ for some integer m. Thus careful examination is necessary for the bifurcation analysis. To overcome this difficulty, we reduce the problem in some flow invariant subspaces. More specifically, we find that for such an eigenvalue there exist exactly $2m$ flow invariant subspaces, in each of which the eigenvalue is simple in the algebraic sense. Then the global bifurcation theorem introduced by P. Rabinowitz [21] can be used to complete the bifurcation analysis.

Furthermore, restricting to each flow invariant subspace, the Navier-Stokes equations undergo local supercritical bifurcation around the bifurcation point. For the first time, we obtain in Theorem 5.2 the precise local asymptotic expansions of the bifurcation branches in terms of the Reynolds number λ near the bifurcation point; see (5.6). Technically speaking, the local asymptotic expansion of each bifurcation branch is obtained by studying the projection of Navier-Stokes equations to the unstable direction, and by carefully examining the nonlinear interaction of the corresponding unstable mode given by u^* as in (5.4)).

The two-dimensional (2D) Kolmogorov problem is the 2D Navier-Stokes equations with the special zonal forcing $k^2(\sin ky, 0)$. In this case, the first study of this problem is due to Meshalkin and Sinai [17] using the continuous fraction method. They proved the absence of instability phenomena of the 2D Kolmogorov problem with $k = 1$ regardless the magnitude of the Reynolds number. This result was also confirmed in Marchioro [16] with an alternative approach. The bifurcation analysis to the 2D Kolmogorov problem was first examined by Iudovich [10] in a two-dimensional elongated domain $[0, 2\pi/a] \times [0, 2\pi]$ with $0 < a < 1$, and the supercritical pitchfork bifurcation phenomenon with respect to some critical values $\lambda_{l,k}$ with $l = 0, 1, ..., k-1$ was examined in [6] based on the analysis and numerical computations.

Back to the three-dimensional case, the governing linearized Navier-Stokes system may be reduced to a two-dimensional one via the Squire transformation. Thus the three-dimensional linear instability phenomenon may be observed from a two-dimensional problem; see [12] for details. We note, however, that the Squire transformation is not invertible, and thus is not suitable for bifurcation analysis.

It is not difficult to see that the steady-state solutions not branching off the first bifurcation point are unstable. One may think that unstable steady-state solutions are not of physical interest, but they are necessary in the understanding of the transition to turbulence. Moreover it is normally difficult to give a numerical scheme ensuring the time-dependent discretized solution converging to an unstable steady-state solution as $t \to \infty$; see, for example, [4]. In this article a bifurcated steady-state solution is obtained in a flow invariant space, in which the solution is locally stable at least for λ close to the associated bifurcation value. Therefore one may derive a convergent numerical scheme with respect to every bifurcated steady-state solution above. In addition, noticing (see [3, 25]) that steady-state solutions to a three-dimensional Navier-Stokes system are regular. Thus the local stability of a steady-state solution in a flow invariant subspace shows the global existence of time-dependent regular solutions starting from large initial data near the steady-state solution, although the global existence of a regular time-dependent solution to a three-dimensional Navier-Stokes system remains to be a fundamental open question, and only partial regularity for a solution are available; see e.g. [5, 14, 15, 26].

2. Spectrum of the Linearized Problem

From now on, we consider only the steady-state problem of the Navier-Stokes equations (1.1-1.4), which can be rewritten as

$$-\Delta u + \lambda B(u, u) = k^2(\sin kz, 0, 0), \ u \in \dot{\mathbb{H}}^2_\sigma, \tag{2.1}$$

where $B(u,u) = P[u \cdot \nabla u]$ is the bilinear term, and $P : \dot{\mathbb{H}}^2(\mathbb{T}^3) \to \dot{\mathbb{H}}^2_\sigma$ is the Leray projection operator.

Linearizing (2.1) around the steady-state $u_0 = (\sin kz, 0, 0)$, we derive $-\Delta u + \lambda A u = 0$. Here the linearized operator A is defined by

$$Au = P[(u_0 \cdot \nabla)u + (u \cdot \nabla u_0)] = P\left[\sin kz \frac{\partial u}{\partial x} + u_3 k \cos kz (1,0,0)\right], \tag{2.2}$$

for any $u = (u_1, u_2, u_3) \in \dot{\mathbb{H}}^2_\sigma$.

To obtain the steady-state bifurcation result, we are interested in the real eigenvalues of the linear operator in $\dot{\mathbb{H}}^2_\sigma$. Namely we study the nontrivial solutions of

$$\Delta u - \lambda A u = \rho u. \tag{2.3}$$

For reasons which will be clear later we first introduce some invariant subspaces of $\Delta - \lambda A$. For an integer vector (l, j, i, k) with $l, k \geq 1$, $0 \leq i < k$ and $j \in Z$, we define the following subspaces of $\dot{\mathbb{H}}^2_\sigma$:

$$\mathbb{E}_{l,j,i,k} = \left\{ u \in \dot{\mathbb{H}}^2_\sigma \mid u = \sum_{n\in\mathbb{Z}} (\xi_n, \eta_n, \zeta_n) \sin(lx + jy + iz + nkz) \right\},$$

$$\widetilde{\mathbb{E}}_{l,j,i,k} = \left\{ u \in \dot{\mathbb{H}}^2_\sigma \mid u = \sum_{n\in\mathbb{Z}} (\xi_n, \eta_n, \zeta_n) \cos(lx + jy + iz + nkz) \right\}.$$

The spectrum of this operator in these subspaces is examined in the following theorem.

Theorem 2.1. *Let (l, j, i, k) be an integer vector subject to the condition (1.5). Then there exists an eigenvalue $\rho_{l,j,i,k} : (0, \infty) \to \mathbb{R}$ of (2.3) such that for any $\lambda > 0$*

$$\frac{d\rho_{l,j,i,k}(\lambda)}{d\lambda} > 0,$$
$$\lim_{\lambda\to\infty} \rho_{l,j,i,k}(\lambda) = \infty,$$
$$\lim_{\lambda\to 0^+} \rho_{l,j,i,k}(\lambda) = -l^2 - j^2 - \min\{i^2, (k-i)^2\}.$$

Furthermore for any $\lambda > 0$ and $\rho > -(l^2 + j^2 + \min\{i^2, (k-i)^2\})$,

$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \mathbb{E}_{l,j,i,k} | \ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1,$$
$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \widetilde{\mathbb{E}}_{l,j,i,k} | \ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1,$$
$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \mathbb{E}_{l,j,k-i,k} | \ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1 \quad \text{if } i \neq 0,$$
$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \widetilde{\mathbb{E}}_{l,j,k-i,k} | \ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1 \quad \textit{if } i \neq 0,$$

where the equalities hold if and only if $\rho = \rho_{l,j,i,k}(\lambda)$.

The proof of Theorem 2.1 will be accomplished by converting the eigenvalue problem $\Delta u - \lambda A u = \rho u$ to coupled continuous fraction equations. This approach was first used by Meshalkin and Sinai [17] in a stability problem of fluid flows.

To proceed, we first recall a theorem on continuous fraction [17]. Consider three term recurrence equations:

$$d_n e_n + e_{n-1} - e_{n+1} = 0, \quad n \in \mathbb{Z}, \tag{2.4}$$

where d_n and e_n are complex numbers. Then we have

Theorem 2.2. *Assume*

$$\begin{gathered} Re\, d_n > 0, for\, n \neq 0, \\ \lim_{|n|\to\infty} Re\, d_n = \infty. \end{gathered} \tag{2.5}$$

Then the following two assertions are equivalent:

(i) *There exists a nontrivial solution* $\{e_n\}_{n\in\mathbb{Z}}$ *of (2.4) such that*

$$\sum_{n\in\mathbb{Z}} |e_n|^2 < \infty. \tag{2.6}$$

(ii) *The following equation holds true:*

$$d_0 + \cfrac{1}{d_{-1} + \cfrac{1}{d_{-2} + \cfrac{1}{\ddots}}} = \cfrac{-1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{\ddots}}}. \tag{2.7}$$

Furthermore, the solution obtained in (i) or (ii) is unique up to a constant factor and satisfies that

$$either\ e_n = 0\ for\ all\ n \in \mathbb{Z},\ or\ e_n \neq 0\ for\ all\ n \in \mathbb{Z}. \tag{2.8}$$

Proof. 1. Let $\{e_n\}_{n\in\mathbb{Z}}$ be a nontrivial solution of (2.4). If $e_{n_0} = 0$ for some $n_0 \geq 0$, multiplying the n-th equation of (2.4) by $\bar{e}_n$ and summing up the resultant equations for $n \geq n_0 + 1$ yields

$$\sum_{n\geq n_0+1} \mathrm{Re} d_n |e_n|^2 = 0.$$

This together with (2.5) gives $e_n = 0$ for $n \geq n_0 + 1$, and thus (2.4) shows $e_n = 0$ for all $n \in \mathbb{Z}$. In the same way, the assumption $e_{n_0} = 0$ for some $n_0 \leq -1$ implies $e_n = 0$ for $n \leq n_0 - 1$ and then $e_n = 0$ for all $n \in \mathbb{Z}$. Namely (2.8) holds true.

2. Dividing the nth and $-n$th equation of (2.4) by e_n and e_{-n} respectively yields that for any $n \geq 0$,

$$\begin{gathered} \frac{e_n}{e_{n-1}} = \cfrac{-1}{d_n - \cfrac{e_{n+1}}{e_n}}, \\ \frac{e_{-n}}{e_{-n+1}} = \cfrac{1}{d_{-n} + \cfrac{e_{-n-1}}{e_{-n}}}. \end{gathered}$$

Namely, for any $n \geq 0$,

$$\frac{e_{\pm n}}{e_{\pm(n-1)}} = \cfrac{\mp 1}{d_{\pm n} + \cfrac{1}{d_{\pm(n+1)} + \cfrac{1}{\ddots + \cfrac{1}{d_{\pm(n+m)} \mp \cfrac{e_{\pm(n+m+1)}}{e_{\pm(n+m)}}}}}}.$$

This together with (2.5-2.6) implies that for any $n \geq 0$,

$$\frac{e_{\pm n}}{e_{\pm(n-1)}} = \gamma_{\pm n} \overset{\text{def}}{=} \cfrac{\mp 1}{d_{\pm n} + \cfrac{1}{d_{\pm(n+1)} + \cfrac{1}{d_{\pm(n+2)} + \cfrac{1}{\ddots}}}}. \tag{2.9}$$

In particular, (2.7) follows from $(e_0/e_{-1})^{-1} = e_{-1}/e_0$.

3. Furthermore we infer from (2.9) that for any $n \geq 1$,

$$\begin{aligned} e_n &= e_0 \gamma_1 \cdots \gamma_n, \\ e_{-n} &= e_0 \gamma_{-1} \cdots \gamma_{-n}, \end{aligned}$$

which are uniquely determined up to constants, i.e. an arbitrary choice of e_0.

4. Moreover, if (2.7) is valid, we can also define $\{\gamma_n\}_{n\in\mathbb{Z}}$ and a nontrivial solution $\{e_n\}_{n\in\mathbb{Z}}$ as above. □

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1. We observe that the condition (1.5) is equivalent to either

$$\begin{gathered} l^2 + j^2 + i^2 < k^2, \\ l^2 + j^2 + (k-i)^2 > k^2, \\ 0 \leq i < k/2, \\ j \in \mathbb{Z}, \\ 0 \leq l < k, \end{gathered} \tag{2.10}$$

or

$$\begin{gathered} l^2 + j^2 + i^2 > k^2, \\ l^2 + j^2 + (k-i)^2 < k^2, \\ k/2 < i < k, \\ j \in \mathbb{Z}, \\ 0 \leq l < k. \end{gathered} \tag{2.11}$$

We first consider the case where (2.10) holds true.

Step 2. An equivalent form of the non homogeneous problem $(-\Delta+\rho)u + \lambda A u = (-\Delta + \rho) f$.

For either $u,\ f \in \mathbb{E}_{l,j,i,k}$ or $u,\ f \in \widetilde{\mathbb{E}}_{l,j,i,k}$, we write

$$u = \sum_{n\in\mathbb{Z}} (\xi_n, \eta_n, \zeta_n)\phi_n, \quad f = \sum_{n\in\mathbb{Z}} (a_n, b_n, c_n)\phi_n,$$

where

$$\begin{aligned} \phi_n &= \sin(lx + jy + iz + nkz) \ \text{ if } \ u,\ f \in \mathbb{E}_{l,j,i,k}, \\ \phi_n &= \cos(lx + jy + iz + nkz) \ \text{ if } \ u,\ f \in \widetilde{\mathbb{E}}_{l,j,i,k}. \end{aligned}$$

Here

$$\begin{aligned} (\xi_n, \eta_n, \zeta_n) &= (\xi_n(l,j,i,k,\lambda), \eta_n(l,j,i,k,\lambda), \zeta_n(l,j,i,k,\lambda)), \\ (a_n, b_n, c_n) &= (a_n(l,j,i,k,\lambda), b_n(l,j,i,k,\lambda), c_n(l,j,i,k,\lambda)). \end{aligned}$$

Let

$$\beta_n = \beta_n(l,j,i,k) = l^2 + j^2 + (nk+i)^2. \tag{2.12}$$

Then direct computation yields that $(-\Delta+\rho)u+\lambda Au=(-\Delta+\rho)f$ can be specified as follows:

$$\sum_{n\in\mathbb{Z}}\Big((\beta_n+\rho)(\xi_n-a_n)+\frac{\lambda l}{2}(\xi_{n-1}-\xi_{n+1})+\frac{\lambda k}{2}(\zeta_{n-1}+\zeta_{n+1})\Big)\,\phi_n=-\partial_x p, \tag{2.13}$$

$$\sum_{n\in\mathbb{Z}}\Big((\beta_n+\rho)(\eta_n-b_n)+\frac{\lambda l}{2}(\eta_{n-1}-\eta_{n+1})\Big)\,\phi_n=-\partial_y p, \tag{2.14}$$

$$\sum_{n\in\mathbb{Z}}\Big((\beta_n+\rho)(\zeta_n-c_n)+\frac{\lambda l}{2}(\zeta_{n-1}-\zeta_{n+1})\Big)\,\phi_n=-\partial_z p, \tag{2.15}$$

$$\sum_{n\in\mathbb{Z}}(l\xi_n+j\eta_n+(i+nk)\zeta_n)\phi_n=0, \tag{2.16}$$

$$\sum_{n\in\mathbb{Z}}(la_n+jb_n+(i+nk)c_n)\phi_n=0. \tag{2.17}$$

It follows from (2.13-2.14) that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\Big((\beta_n+\rho)(l\xi_n+j\eta_n-la_n-jb_n)+\frac{\lambda l}{2}(l\xi_{n-1}+j\eta_{n-1}-l\xi_{n+1}-j\eta_{n+1})\Big)\,\phi_n\\&+\sum_{n\in\mathbb{Z}}\frac{\lambda lk}{2}(\zeta_{n-1}+\zeta_{n+1})\phi_n\\&=\quad -l\partial_x p-j\partial_y p.\end{aligned}$$

Applying the operator ∂_z to this equation and the operator $-l\partial_x-j\partial_y$ to (2.15) respectively, and summing the resultant equations, we have

$$\begin{aligned}&\Big((\beta_n+\rho)(l\xi_n+j\eta_n-la_n-jb_n)+\frac{\lambda l}{2}(l\xi_{n-1}+j\eta_{n-1}-l\xi_{n+1}-j\eta_{n+1})\Big)\,(nk+i)\\&+\frac{\lambda lk}{2}(\zeta_{n-1}+\zeta_{n+1})(nk+i)\\&=\quad\Big((\beta_n+\rho)(\zeta_n-c_n)+\frac{\lambda l}{2}(\zeta_{n-1}-\zeta_{n+1})\Big)\,(l^2+j^2),\ n\in\mathbb{Z}.\end{aligned}$$

Moreover, eliminating the pressure function p from (2.13-2.14) yields

$$\begin{aligned}&\Big((\beta_n+\rho)(\xi_n-a_n)+\frac{\lambda l}{2}(\xi_{n-1}-\xi_{n+1})+\frac{\lambda k}{2}(\zeta_{n-1}+\zeta_{n+1})\Big)\,j\\&=\Big((\beta_n+\rho)(\eta_n-b_n)+\frac{\lambda l}{2}(\eta_{n-1}-\eta_{n+1})\Big)\,l,\ n\in\mathbb{Z}.\end{aligned}$$

Thus by (2.16-2.17), we see that the non homogeneous problem

$$(-\Delta+\rho)u+\lambda Au=(-\Delta+\rho)f,\quad u,\ f\in\mathbb{E}_{l,j,i,k}\text{ or }u,\ f\in\widetilde{\mathbb{E}}_{l,j,i,k}$$

is equivalent to the following set of coupled algebraic equations

$$\frac{2\beta_n(\beta_n+\rho)}{\lambda l}\zeta_n + (\beta_{n-1}-k^2)\zeta_{n-1} - (\beta_{n+1}-k^2)\zeta_{n+1} = \frac{2\beta_n(\beta_n+\rho)}{\lambda l}c_n, \tag{2.18}$$

$$\frac{2(\beta_n+\rho)}{\lambda}(j\xi_n - l\eta_n) + l(j\xi_{n-1} - l\eta_{n-1}) - l(j\xi_{n+1} - l\eta_{n+1}) = -kj(\zeta_{n-1}+\zeta_{n+1}) + \frac{2(\beta_n+\rho)}{\lambda}(ja_n - lb_n), \tag{2.19}$$

$$l\xi_n + j\eta_n = -(i+nk)\zeta_n, \tag{2.20}$$

$$la_n + jb_n = -(i+nk)c_n, \tag{2.21}$$

for any $n \in \mathbb{Z}$.

Step 3. Claim: $\{\xi_n\}_{n\in\mathbb{Z}}$ *and* $\{\eta_n\}_{n\in\mathbb{Z}}$ *are uniquely determined by* $\{ja_n - lb_n\}_{n\in\mathbb{Z}}$ *and* $\{\zeta_n\}_{n\in\mathbb{Z}}$ *when* $\rho > -(l^2+j^2+i^2)$.

Let $\mathcal{S}$ be

$$\mathcal{S} = \left\{ \{\tau_n\}_{n\in\mathbb{Z}} \,|\, \sum_{n\in\mathbb{Z}} |\tau_n|^2 < \infty \right\}, \tag{2.22}$$

and $L : \mathcal{S} \to \mathcal{S}$ be a linear operator defined by

$$L\{\tau_n\}_{n\in\mathbb{Z}} = \left\{ \frac{\tau_{n-1}-\tau_{n+1}}{\beta_n+\rho} \right\}_{n\in\mathbb{Z}}.$$

Then (2.19) can be rewritten as

$$\left(\frac{2}{\lambda l} + L\right)\{j\xi_n - l\eta_n\}_{n\in\mathbb{Z}} = -\left\{ \frac{kj(\zeta_{n-1}+\zeta_{n+1})}{(\beta_n+\rho)l} \right\}_{n\in\mathbb{Z}} + \left\{ \frac{2}{\lambda l}(ja_n - lb_n) \right\}_{n\in\mathbb{Z}}.$$

It is easy to see that $L : \mathcal{S} \to \mathcal{S}$ is a compact operator for $\rho > -\beta_0$.

We now show that $-2/\lambda l$ is not an eigenvalue of L or the coupled algebraic equations

$$\frac{2(\beta_n+\rho)}{\lambda l}\tau_n + \tau_{n-1} - \tau_{n+1} = 0,\ n \in \mathbb{Z} \tag{2.23}$$

has no nontrivial solution $\{\tau_n\}_{n\in\mathbb{Z}} \in \mathcal{S}$. Otherwise, let $\{\tau_n\}_{n\in\mathbb{Z}}$ be a nontrivial solution, then by Theorem 2.2,

$$-\frac{2(\beta_0+\rho)}{\lambda l} = \cfrac{1}{\cfrac{2(\beta_1+\rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_2+\rho)}{\lambda l} + \cfrac{1}{\ddots}}} + \cfrac{1}{\cfrac{2(\beta_{-1}+\rho)}{\lambda l} + \cfrac{1}{\cfrac{2(\beta_{-2}+\rho)}{\lambda l} + \cfrac{1}{\ddots}}}.$$

Since $-(\beta_0+\rho)/(\lambda l) < 0$ while the right-hand side of this equation is positive, this leads to a contradiction. Hence (2.23) has no nontrivial solution.

By the Riesz–Schauder theory, it is easy to see that $(2/\lambda l + L)^{-1} : \mathcal{S} \to \mathcal{S}$ is a bounded operator, and

$$\{j\xi_n - l\eta_n\}_{n\in\mathbb{Z}} = -\big(\frac{2}{\lambda l} + L\big)^{-1}\big(\{\frac{kj(\zeta_{n-1}+\zeta_{n+1})}{(\beta_n+\rho)l}\}_{n\in\mathbb{Z}} + \{\frac{2}{\lambda l}(ja_n - lb_n)\}_{n\in bZ}\big).$$

This together with (2.20-2.21) implies the desired assertion.

Step 4. Existence and uniqueness of $\rho = \rho_{l,j,i,k}(\lambda)$ for the eigenvalue problem

$$\Delta u - \lambda A u = \rho u, \;\; u \in \mathbb{E}_{l,j,i,k} \cup \widetilde{\mathbb{E}}_{l,j,i,k}. \tag{2.24}$$

It follows from (2.10) and (2.12) that $\beta_0 - k^2 < 0$ and $\beta_n - k^2 > 0$ for $n \neq 0$. By Steps 2 and 3, (2.24) is uniquely determined by (2.18) with $c_n = 0$ or set of equations

$$\frac{2\beta_n(\beta_n + \rho)}{\lambda l}\zeta_n + (\beta_{n-1} - k^2)\zeta_{n-1} - (\beta_{n+1} - k^2)\zeta_{n+1} = 0, \tag{2.25}$$

which is in the form of (2.4) with $\tau_n = (\beta_n - k^2)\zeta_n$. By Theorem 2.2, it becomes to show the existence and uniqueness of $\rho = \rho_{l,j,i,k}$ in the following algebraic equation

$$\begin{aligned} -\frac{2\beta_0(\beta_0 + \rho)}{\lambda l(\beta_0 - k^2)} \;\; &= \;\; \cfrac{1}{\cfrac{2\beta_1(\beta_1 + \rho)}{\lambda l(\beta_1 - k^2)} + \cfrac{1}{\cfrac{2\beta_2(\beta_2 + \rho)}{\lambda l(\beta_2 - k^2)} + \cfrac{1}{\ddots}}} \\ &\quad + \cfrac{1}{\cfrac{2\beta_{-1}(\beta_{-1} + \rho)}{\lambda l(\beta_{-1} - k^2)} + \cfrac{1}{\cfrac{2\beta_{-2}(\beta_{-2} + \rho)}{\lambda l(\beta_{-2} - k^2)} + \cfrac{1}{\ddots}}}. \end{aligned} \tag{2.26}$$

Multiplying this equation by $-l(\beta_0 - k^2)(\beta_0(\beta_0 + \rho))^{-1}$ gives

$$\frac{2}{\lambda} = \cfrac{1}{\cfrac{g_1(\rho)}{\lambda} + \cfrac{1}{\cfrac{g_2(\rho)}{\lambda} + \cfrac{1}{\ddots}}} + \cfrac{1}{\cfrac{g_{-1}(\rho)}{\lambda} + \cfrac{1}{\cfrac{g_{-2}(\rho)}{\lambda} + \cfrac{1}{\ddots}}}, \tag{2.27}$$

where

$$g_{\pm n}(\rho) = \begin{cases} \dfrac{2\beta_{\pm n}\beta_0(\beta_{\pm n} + \rho)(\beta_0 + \rho)}{l^2(k^2 - \beta_0)(\beta_{\pm n} - k^2)}, & \text{if } n \text{ is odd,} \\[2ex] \dfrac{2\beta_{\pm n}(\beta_{\pm n} + \rho)(k^2 - \beta_0)}{\beta_0(\beta_{\pm n} - k^2)(\beta_0 + \rho)}, & \text{if } n \text{ is even.} \end{cases}$$

Let $G(\lambda, \rho)$ be the right-hand side of (2.27), and for $n \geq 1$ we set

$$G_{\pm n}(\lambda, \rho) = \cfrac{1}{\cfrac{g_{\pm 1}(\rho)}{\lambda} + \cfrac{1}{\cfrac{g_{\pm 2}(\rho)}{\lambda} + \cfrac{1}{\ddots + \cfrac{1}{\cfrac{g_{\pm n}(\rho)}{\lambda}}}}}.$$

We have

$$G(\lambda, \rho) = \lim_{n\to\infty} G_n(\lambda, \rho) + \lim_{n\to\infty} G_{-n}(\lambda, \rho).$$

For any $\lambda > 0$, $\epsilon > 0$ and $M > 0$ arbitrarily large, we see that $\{G_n(\lambda, \cdot)\}_{n\geq 1}$ and $\{G_{-n}(\lambda, \cdot)\}_{n\geq 1}$ are Cauchy sequences of the Banach space $C^1([-\beta_0 + \epsilon, M])$, and thus $G(\lambda,\cdot)\in C^1([-\beta+\epsilon,M])$,

$$\begin{aligned}\frac{\partial G(\lambda,\rho)}{\partial\rho} &= \lim_{n\to\infty}\frac{\partial G_n(\lambda,\rho)}{\partial\rho}+\lim_{n\to\infty}\frac{\partial G_{-n}(\lambda,\rho)}{\partial\rho}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda}\frac{\mathrm{d}g_n(\rho)}{\mathrm{d}\rho}\hat G_1^2(\lambda,\rho)\cdots\hat G_n^2(\lambda,\rho)\\ &\quad+\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda}\frac{\mathrm{d}g_{-n}(\rho)}{\mathrm{d}\rho}\hat G_{-1}^2(\lambda,\rho)\cdots\hat G_{-n}^2(\lambda,\rho),\end{aligned}$$

with

$$\hat G_{\pm n}(\lambda,\rho)=\cfrac{1}{\cfrac{g_{\pm n}(\rho)}{\lambda}+\cfrac{1}{\cfrac{g_{\pm(n+1)}(\rho)}{\lambda}+\cfrac{1}{\ddots}}}.$$

By direct calculation, we infer from (2.10) that for any $n\ge 1$, $\mathrm{d}g_{\pm n}(\rho)/\mathrm{d}\rho>0$ when n is odd, and $\mathrm{d}g_{\pm n}(\rho)/\mathrm{d}\rho<0$ when n is even. Therefore when $\rho>-\beta_0$, we obtain

$$\frac{\partial G(\lambda,\rho)}{\partial\rho}<0. \tag{2.28}$$

Hence the observation

$$\lim_{\rho\searrow-\beta_0}G(\lambda,\rho)=\infty \text{ and } \lim_{\rho\to\infty}G(\lambda,\rho)=0$$

implies the existence and uniqueness of $\rho=\rho_{l,j,i,k}(\lambda)>-\beta_0$ satisfying

$$\frac{2}{\lambda}=G(\lambda,\rho_{l,j,i,k}(\lambda)). \tag{2.29}$$

Step 5. Notice that

$$\lambda G(\lambda,\rho)=\cfrac{1}{\cfrac{g_1(\rho)}{\lambda^2}+\cfrac{1}{g_2(\rho)+\cfrac{1}{\cfrac{g_3(\rho)}{\lambda^2}+\cfrac{1}{\ddots}}}}+\cfrac{1}{\cfrac{g_{-1}(\rho)}{\lambda^2}+\cfrac{1}{g_{-2}(\rho)+\cfrac{1}{\cfrac{g_{-3}(\rho)}{\lambda^2}+\cfrac{1}{\ddots}}}}.$$

Arguing as in the derivation of (2.28), we have

$$\frac{\partial(\lambda G(\lambda,\rho))}{\partial\lambda}>0 \quad \text{for } \lambda>0 \text{ and } \rho>-\beta_0.$$

(2.29) gives for $\rho=\rho_{l,j,i,k}(\lambda)$,

$$\begin{aligned}0 &= \frac{\mathrm{d}(\lambda G(\lambda,\rho))}{\mathrm{d}\lambda}\\ &= \frac{\partial(\lambda G(\lambda,\rho))}{\partial\lambda}+\lambda\frac{\partial G(\lambda,\rho)}{\partial\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}\\ &> \lambda\frac{\partial G(\lambda,\rho)}{\partial\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}.\end{aligned}$$

This shows, by (2.28), that $\mathrm{d}\rho_{l,j,i,k}/\mathrm{d}\lambda>0$.

Furthermore, multiplying (2.26) by λ gives

$$
\begin{aligned}
&-\frac{2\beta_0(\beta_0+\rho_{l,j,i,k}(\lambda))}{l(\beta_0-k^2)} \qquad (2.30)\\
&= \cfrac{1}{\cfrac{2\beta_1(\beta_1+\rho_{l,j,i,k}(\lambda))}{\lambda^2 l(\beta_1-k^2)}+\cfrac{1}{\cfrac{2\beta_2(\beta_2+\rho_{l,j,i,k}(\lambda))}{l(\beta_2-k^2)}+\cfrac{1}{\ddots}}}\\
&\quad+\cfrac{1}{\cfrac{2\beta_{-1}(\beta_{-1}+\rho_{l,j,i,k}(\lambda))}{\lambda^2 l(\beta_{-1}-k^2)}+\cfrac{1}{\cfrac{2\beta_{-2}(\beta_{-2}+\rho_{l,j,i,k}(\lambda))}{l(\beta_{-2}-k^2)}+\cfrac{1}{\ddots}}}.
\end{aligned}
$$

Passing to the limits $\lambda \to 0^+$ and $\lambda \to \infty$ respectively, we obtain immediately

$$
\begin{aligned}
&\lim_{\lambda\to 0^+} \rho_{l,j,i,k}(\lambda) = -l^2 - j^2 - i^2,\\
&\lim_{\lambda\to\infty} \rho_{l,j,i,k}(\lambda) = \infty.
\end{aligned}
$$

In addition, it follows from (2.18) with $\rho = \rho_{l,j,i,k}(\lambda)$ and Theorem 2.2 that

$$
\gamma_{\pm n} = \frac{(\beta_{\pm n} - k^2)\zeta_{\pm n}}{(\beta_{\pm(n-1)} - k^2)\zeta_{\pm(n-1)}} \quad \text{for } n \in N
$$

with ζ_m subject to the following condition

$$
\begin{aligned}
\zeta_0 &= c,\\
\zeta_{\pm n} &= c\frac{(\beta_0-k^2)\gamma_{\pm 1}\cdots\gamma_{\pm n}}{\beta_{\pm n}-k^2}, \quad n \in N.
\end{aligned}
$$

Here c is an arbitrary constant. Notice that $\gamma_{\pm n}$ are uniquely determined by $\rho_{l,i,j,k}(\lambda)$ and $\beta_0, \beta_{\pm 1}, \cdots$. Thus all the eigenfunctions of the spectral problem $\Delta u - \lambda A u = \rho_{l,j,i,k} u$ form a one-dimensional subspace in $\mathbb{E}_{l,j,i,k}$ and $\widetilde{\mathbb{E}}_{l,j,i,k}$ respectively.

Step 6. Claim: $\ker(\Delta - \lambda A - \rho)^m = \ker(\Delta - \lambda A - \rho)$ $(m > 1,\ \rho > -\beta_0)$.

It suffices to consider the operator with $\rho = \rho_{l,j,i,k}$ in the space $\mathbb{E}_{l,j,i,k}$. We start with the case $m = 2$. Let

$$
u = \sum_{n\in\mathbb{Z}} (\xi_n, \eta_n, \zeta_n) \sin(lx + jy + iz + nkz) \in \mathbb{E}_{l,j,i,k}
$$

such that

$$
(\Delta - \lambda A - \rho)^2 u = 0,
$$

or equivalently,

$$
(1 - \lambda(\Delta - \rho)^{-1} A)^2 u = 0.
$$

This implies the existence of

$$
u' = \sum_{n\in\mathbb{Z}} (\xi_n', \eta_n', \zeta_n') \sin(lx + jy + iz + nkz) \in \mathbb{E}_{l,j,i,k}
$$

such that

$$
\begin{aligned}
(-\Delta + \rho)u + \lambda A u &= (-\Delta + \rho)u',\\
(-\Delta + \rho)u' + \lambda A u' &= 0.
\end{aligned} \tag{2.31}
$$

Thanks to (2.18-2.21), the above system (2.31) is equivalent to the following set of coupled algebraic equations:

$$\frac{2\beta_n(\beta_n+\rho)}{\lambda l}\zeta_n+(\beta_{n-1}-k^2)\zeta_{n-1}-(\beta_{n+1}-k^2)\zeta_{n+1} \tag{2.32}$$
$$=\frac{2\beta_n(\beta_n+\rho)}{\lambda l}\zeta_n',$$
$$\frac{2(\beta_n+\rho)}{\lambda}(j\xi_n-l\eta_n)+l(j\xi_{n-1}-l\eta_{n-1})-l(j\xi_{n+1}-l\eta_{n+1}) \tag{2.33}$$
$$=-kj(\zeta_{n-1}+\zeta_{n+1})+\frac{2(\beta_n+\rho)}{\lambda}(j\xi_n'-l\eta_n'),$$
$$l\xi_n+j\eta_n=-(i+nk)\zeta_n, \tag{2.34}$$
$$\frac{2\beta_n(\beta_n+\rho)}{\lambda l}\zeta_n'+(\beta_{n-1}-k^2)\zeta_{n-1}'-(\beta_{n+1}-k^2)\zeta_{n+1}'=0, \tag{2.35}$$
$$\frac{2(\beta_n+\rho)}{\lambda}(j\xi_n'-l\eta_n')+l(j\xi_{n-1}'-l\eta_{n-1}')-l(j\xi_{n+1}'-l\eta_{n+1}') \tag{2.36}$$
$$=-kj(\zeta_{n-1}'+\zeta_{n+1}'),$$
$$l\xi_n'+j\eta_n'=-(i+nk)\zeta_n', \tag{2.37}$$

for any $n\in\mathbb{Z}$.

Thus it amounts to showing that $\zeta_n'=\eta_n'=\xi_n'=0$ for any $n\in\mathbb{Z}$. Indeed, (2.32) gives the following equation in $\mathcal{S}$ defined by (2.22)

$$(1+T)\{(\beta_n-k^2)\zeta_n\}_{n\in\mathbb{Z}}=\{(\beta_n-k^2)\zeta_n'\}_{n\in\mathbb{Z}}, \tag{2.38}$$

with the operator T in the form

$$T\{\tau_n\}_{n\in\mathbb{Z}}=\left\{\frac{\lambda l(\beta_n-k^2)}{2\beta_n(\beta_n+\rho)}(\tau_{n-1}-\tau_{n+1})\right\}_{n\in\mathbb{Z}}.$$

Since $T:\mathcal{S}\mapsto\mathcal{S}$ is compact, by the Riesz-Schauder theory, (2.38) is solvable if and only if

$$\sum_{n\in\mathbb{Z}}(\beta_n-k^2)\zeta_n'\tau_n=0 \text{ for all } \{\tau_n\}_{n\in\mathbb{Z}}\in\mathcal{S} \text{ with } (1+T^*)\{\tau_n\}_{n\in\mathbb{Z}}=0,$$

where T^* denotes the dual operator of T.

To specify the kernel of $1+T^*$, we note that

$$T^*\{\tau_n\}_{n\in\mathbb{Z}}=\left\{-\frac{\lambda l(\beta_{n-1}-k^2)}{2\beta_{n-1}(\beta_{n-1}+\rho)}\tau_{n-1}+\frac{\lambda l(\beta_{n+1}-k^2)}{2\beta_{n+1}(\beta_{n+1}+\rho)}\tau_{n+1}\right\}_{n\in\mathbb{Z}}.$$

Thus $(1+T^*)\{\tau_n\}_{n\in\mathbb{Z}}=0$ becomes, for $n\in\mathbb{Z}$,

$$\frac{2\beta_n(\beta_n+\rho)}{\lambda l(\beta_n-k^2)}\sigma_n+\sigma_{n-1}-\sigma_{n+1}=0,\quad \sigma_m=(-1)^m\frac{\lambda l(\beta_m-k^2)}{2\beta_m(\beta_m+\rho)}\tau_m.$$

or $\{\sigma_n\}_{n\in\mathbb{Z}}\in\ker(1+T)$. If $\{\zeta_n'\}_{n\in\mathbb{Z}}\neq 0$, applying Theorem 2.2 to (2.35) gives $\zeta_n'\neq 0$ for all $n\in\mathbb{Z}$ and

$$\{\sigma_n\}_{n\in\mathbb{Z}}=\left\{(-1)^n\frac{\lambda l(\beta_n-k^2)}{2\beta_n(\beta_n+\rho)}\tau_n\right\}_{n\in\mathbb{Z}}=c\{(\beta_n-k^2)\zeta_n'\}_{n\in\mathbb{Z}}$$

for some constant $c \neq 0$, whenever $\{\tau_n\}_{n\in\mathbb{Z}} \neq 0$. This implies

$$\sum_{n\in\mathbb{Z}}(\beta_n - k^2)\zeta'_n\tau_n = c\sum_{n\in\mathbb{Z}}(-1)^n\frac{2\beta_n(\beta_n+\rho)(\beta_n-k^2)}{\lambda l}{\zeta'}^2_n. \tag{2.39}$$

On the other hand, multiplying by $(\beta_n - k^2)\zeta'_n$ the nth equation of (2.35) and summing up the resultant equations yield

$$\sum_{n\in\mathbb{Z}}\frac{2\beta_n(\beta_n+\rho)(\beta_n-k^2)}{\lambda l}{\zeta'}^2_n = 0.$$

This together with (2.39) implies

$$\sum_{n\in\mathbb{Z}}(\beta_n - k^2)\zeta'_n\tau_n = -c\sum_{n\in\mathbb{Z}}\frac{2\beta_{2n+1}(\beta_{2n+1}+\rho)(\beta_{2n+1}-k^2)}{\lambda l}{\zeta'}^2_{2n+1} \neq 0.$$

This leads to a contradiction, and thus $\{\zeta'_n\}_{n\in\mathbb{Z}} = 0$. From (2.36-2.37) and Step 3, we have $\{\eta'_n\}_{n\in\mathbb{Z}} = 0$ and $\{\xi'_n\}_{n\in\mathbb{Z}} = 0$. Hence we have $u' = 0$ or

$$\ker(\Delta - \lambda A - \rho)^2 = \ker(\Delta - \lambda A - \rho).$$

To reach the case where $m > 2$, suppose $(1 + \lambda(-\Delta+\rho)^{-1}A)^m u = 0$ for some $u \in \mathbb{E}_{l,j,i,k}$. Then there exits a $u' \in \mathbb{E}_{l,j,i,k}$ so that

$$\begin{aligned}(-\Delta+\rho)v + \lambda Av &= (-\Delta+\rho)u', \\ (\Delta - \lambda A - \rho)u' &= 0, \\ (1+\lambda(-\Delta+\rho)^{-1}A)^{m-2}u &= v.\end{aligned}$$

It follows from the argument for the case $m = 2$ that $u' = 0$. Hence, we obtain the case for $m > 2$ by induction.

Step 7. We now consider the case where (2.11) holds true.

Obviously, it follows from the Steps 2 and 3 that spectral problem $\Delta u - \lambda Au = \rho u$ with $u \in \mathbb{E}_{l,j,i,k} \cup \widetilde{\mathbb{E}}_{l,j,i,k}$ is uniquely determined by the system

$$\frac{2\beta_{-n-1}(\beta_{-n-1}+\rho)}{\lambda l}\zeta_{-n-1} + (\beta_{-n-2}-k^2)\zeta_{-n-2} - (\beta_{-n}-k^2)\zeta_{-n} = 0, \quad \text{for } n \in \mathbb{Z}$$

with $\beta_n = \beta_n(l,j,i,k)$. Setting $\beta'_n = \beta_n(l,j,k-i,k)$ and observing that $\beta_{-n-1}(l,j,i,k) = \beta_n(l,j,k-i,k)$, we see that

$$\frac{2\beta'_n(\beta'_n+\rho)}{\lambda l}\zeta_{-n-1} + (\beta'_{n+1}-k^2)\zeta_{-n-2} - (\beta'_{n-1}-k^2)\zeta_{-n} = 0, \quad \text{for } n \in \mathbb{Z}.$$

By setting $\zeta'_n = (-1)^n\zeta_{-n-1}$, we have

$$\frac{2\beta'_n(\beta'_n+\rho)}{\lambda l}\zeta'_n + (\beta'_{n-1}-k^2)\zeta'_{n-1} - (\beta'_{n+1}-k^2)\zeta'_{n+1} = 0, \quad \text{for } n \in \mathbb{Z}.$$

Thus the above argument implies the unique existence of the eigenvalue $\rho_{l,j,i,k}(\lambda)$ so that $\Delta u - \lambda Au = \rho_{l,j,i,k}u$ with $k/2 < i < k$ also form a one-dimensional subspace in $\mathbb{E}_{l,j,i,k}$ and $\widetilde{\mathbb{E}}_{l,j,i,k}$ respectively. In particular,

$$\lim_{\lambda\to 0^+}\rho_{l,j,i,k}(\lambda) = -\beta'_0 = -(l^2+j^2+(k-i)^2).$$

Likewise, we obtain the assertion on the spaces $\mathbb{E}_{l,j,k-i,k}$ and $\widetilde{\mathbb{E}}_{l,j,k-i,k}$. Obviously, we have $\rho_{l,j,i,k}(\lambda) = \rho_{l,j,k-i,k}(\lambda)$.

The proof of Theorem 2.1 is complete. □

For reader's convenience, we state an elementary lemma.

Lemma 2.1. *Let (l, j, i, k) be an integer vector such that $l, k \geq 1$, $0 \leq i < k$ and $j \in \mathbb{Z}$. Then*

$$l^2 + j^2 + i^2 \geq k^2 \quad \text{implies} \quad n^2(l^2 + j^2) + \{ni\}^2 > k^2, \quad \forall n \geq 2,$$

and

$$l^2 + j^2 + (k-i)^2 \geq k^2 \quad \text{implies} \quad n^2(l^2 + j^2) + (k - \{ni\})^2 > k^2, \quad \forall n \geq 2.$$

Proof. Notice that the desired conclusion holds true if $n^2(l^2 + j^2) > k^2$. It suffices to consider the integers n subject to the condition $n^2(l^2 + j^2) \leq k^2$.

In the case where $l^2 + j^2 + i^2 \geq k^2$, we have

$$i^2 \geq k^2 - l^2 - j^2 \geq \frac{(n^2-1)k^2}{n^2} > \frac{(n-1)^2k^2}{n^2},$$

which yields

$$nk > ni = (n-1)k + \{ni\} > (n-1)k.$$

Namely $\{ni\} = ni - (n-1)k > 0$. Hence

$$\begin{aligned} n^2(l^2 + j^2) + \{ni\}^2 &= n^2(l^2 + j^2) + (ni - (n-1)k)^2 \\ &= n^2(l^2 + j^2 + i^2) + n^2k^2 - 2nk^2 - 2n(n-1)ki + k^2 \\ &\geq 2n(n-1)k(k-i) + k^2 > k^2. \end{aligned}$$

For the second case where $l^2 + j^2 + (k-i)^2 \geq k^2$, we have

$$(k-i)^2 \geq k^2 - l^2 - j^2 \geq \frac{(n^2-1)k^2}{n^2} > \frac{(n-1)^2k^2}{n^2},$$

which shows $\{ni\} = ni < k$, and the second part of the lemma can be proved in the same fashion. ☐

We are now in position to state the main result of this section.

Theorem 2.3. *Let an integer vector (l, j, i, k) satisfy (1.5), n_0 be given by (1.6), and $\{ni\}$ be defined as (1.8). Then the following assertions hold true:*

(i). For $\lambda > 0$, $n \in \mathbb{N}$ with $n > n_0$ and $\rho > -(l^2 + j^2 + \min\{i^2, (k-i)^2\})$,

$$\dim\{u \in \mathbb{E}_{nl,nj,\{ni\},k} \cup \widetilde{\mathbb{E}}_{nl,nj,\{ni\},k} |\ \Delta u - \lambda Au - \rho u = 0\} = 0,$$
$$\dim\{u \in \mathbb{E}_{nl,nj,k-\{ni\},k} \cup \widetilde{\mathbb{E}}_{nl,nj,k-\{ni\},k} |\ \Delta u - \lambda Au - \rho u = 0\} = 0, \ \text{if } \{ni\} \neq 0.$$

(ii). There exist n_0 eigenvalues $\rho_{nl,nj,\{ni\},k} : (0, \infty) \to \mathbb{R}$ for $n = 1, ..., n_0$ such that

$$\begin{aligned} &\frac{d\rho_{nl,nj,\{ni\},k}(\lambda)}{d\lambda} > 0, \\ &\lim_{\lambda \to \infty} \rho_{nl,nj,\{ni\},k}(\lambda) = \infty, \\ &\lim_{\lambda \to 0^+} \rho_{nl,nj,\{ni\},k}(\lambda) = -n^2(l^2 + j^2) - \min\{\{ni\}^2, (k - \{ni\})^2\}, \end{aligned}$$

and

$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \mathbb{E}_{nl,nj,\{ni\},k} |\ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1,$$

$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \widetilde{\mathbb{E}}_{nl,nj,\{ni\},k} |\ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1,$$

$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \mathbb{E}_{nl,nj,k-\{ni\},k} |\ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1 \text{ if } \{ni\} \neq 0,$$

$$\dim \bigcup_{m\in\mathbb{N}} \{u \in \widetilde{\mathbb{E}}_{nl,nj,k-\{ni\},k} |\ (\Delta - \lambda A - \rho)^m u = 0\} \leq 1 \text{ if } \{ni\} \neq 0,$$

where the equalities hold if and only if $\rho = \rho_{nl,nj,\{ni\},k}(\lambda)$ *for* $\lambda > 0$.

Proof. (i). Let $n \geq n_0 + 1$. Without loss of generality, we may assume that $\{ni\} < k - \{ni\}$. By the definition (1.6) of n_0, we observe that

$$n^2(l^2 + j^2) + (k - \{ni\})^2 - k^2 > n^2(l^2 + j^2) + \{ni\}^2 - k^2 \geq 0.$$

As in the proof of Theorem 2.1, for $u \in \mathbb{E}_{nl,nj,\{ni\},k} \cup \widetilde{\mathbb{E}}_{nl,nj,\{ni\},k}$, the spectral problem $\Delta u - \lambda Au - \rho u = 0$, is uniquely determined by the following analog of (2.18):

$$\frac{2\beta_m(\beta_m + \rho)}{\lambda nl}\zeta_m + (\beta_{m-1} - k^2)\zeta_{m-1} - (\beta_{m+1} - k^2)\zeta_{m+1} = 0,\ m \in \mathbb{Z}, \tag{2.40}$$

where β_m now equals $\beta_m(nl, nj, \{ni\}, k) = n^2(l^2 + j^2) + (mk + \{ni\})^2$.

Multiplying the m–th equation of (2.40) by $(\beta_m - k^2)\zeta_m$ and summing the resultant equations yield

$$\sum_{m\in\mathbb{Z}} \frac{2\beta_m(\beta_m + \rho)(\beta_m - k^2)}{\lambda nl}|\zeta_m|^2 = 0.$$

Notice that $\beta_m - k^2 > 0$ for $m \neq 0$. Thus $\zeta_m \equiv 0$ for $m \neq 0$, and $\zeta_0 = 0$ as well in view of (2.40).

In the case where $\{ni\} \neq 0$ and $u \in \mathbb{E}_{nl,nj,k-\{ni\},k} \cup \widetilde{\mathbb{E}}_{nl,nj,k-\{ni\},k}$, the spectral problem $\Delta u - \lambda Au - \rho u = 0$ is uniquely determined by

$$\frac{2\beta_m(\beta_m + \rho)}{\lambda nl}\zeta_m + (\beta_{m-1} - k^2)\zeta_{m-1} - (\beta_{m+1} - k^2)\zeta_{m+1} = 0,\ m \in \mathbb{Z},$$

with $\beta_m(nl, nj, k - \{ni\}, k) = n^2(l^2 + j^2) + (mk + k - \{ni\})^2$. Then as before, it is easy to see that $\zeta_m \equiv 0$.

(ii). For $1 \leq n \leq n_0$, it follows from Lemma 2.1 and the definition of the integer n_0 that $(nl, nj, \{ni\}, k)$ satisfies (1.5). Thus Assertion (ii) is a direct consequence of Theorem 2.1. □

3. Invariant Subspaces and Properties of the Steady-State Solutions

In the previous section, we found that the eigenspaces of the spectral problem $\Delta u - \lambda Au = \rho u$ in $\dot{\mathbb{H}}^2_\sigma$ are even-dimensional. Thus it will be difficult to obtain steady-state bifurcations in the whole space $\dot{\mathbb{H}}^2_\sigma$. Fortunately, (1.1-1.4) admit many flow invariant subspaces, in which it will be convenient to examine the bifurcation phenomena.

First, by the divergence free condition dvi$u = 0$, we observe that the bilinear term $B(u,u) = P[u \cdot \nabla u]$ can be alternatively given by

$$\begin{aligned} B(u,u) &= u \cdot \nabla u - \nabla \Delta^{-1}(\partial_x(u \cdot \nabla u_1) + \partial_y(u \cdot \nabla u_2) + \partial_z(u \cdot \nabla u_1)) \\ &= \nabla \cdot (u \otimes u) - \nabla \Delta^{-1} \nabla^2 \cdot (u \otimes u) \end{aligned}$$

with

$$\nabla \cdot (u \otimes u) = \left(\sum_{n=1}^{3} \partial_n (u_n u_1), \sum_{n=1}^{3} \partial_n (u_n u_2), \sum_{n=1}^{3} \partial_n (u_n u_3) \right)$$

and

$$\nabla^2 \cdot (u \otimes u) = \sum_{n=1}^{3} \sum_{m=1}^{3} \partial_n \partial_m (u_n u_m)$$

for $(\partial_1, \partial_2, \partial_3) = (\partial_x, \partial_y, \partial_z)$.

Let the integers $l \geq 0$, $j \in \mathbb{Z}$ and $0 \leq i < k$. We introduce the following subspace of the scalar function space $H^2(\mathbb{T}^3)$:

$$\begin{aligned} H^2_{l,j,i,k}(\mathbb{T}^3) &= \Big\{ v \in H^2(\mathbb{T}^3) \,|\, v = \sum_{n \in \mathbb{N}} \xi_n \sin nkz \\ &\quad + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \eta_{m,n} \sin(mlx + mjy + \{mi\}z + knz) \Big\}, \\ \widetilde{H}^2_{l,j,i,k}(\mathbb{T}^3) &= \Big\{ v \in H^2(\mathbb{T}^3) \,|\, v = \sum_{n \in \mathbb{N}} \xi_n \sin nkz \\ &\quad + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \eta_{m,n} \cos((2m-1)(lx + jy) + \{2mi - i\}z + nkz) \\ &\quad + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \zeta_{m,n} \sin(2m(lx + jy) + \{2mi\}z + nkz) \Big\}. \end{aligned}$$

Here $H^2_{l,j,i,k}(\mathbb{T}^3)$ is a Hilbert space with the norm

$$\begin{aligned} \|v\|_{H^2} &= \|\Delta v\|_{L^2} \\ &= \left(\sum_{n \in \mathbb{N}} n^4 k^4 |\xi_n|^2 + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} (m^2(l^2 + j^2) + (nk + \{mi\})^2)^2 |\eta_{m,n}|^2 \right)^{1/2}, \end{aligned}$$

while $\widetilde{H}^2_{l,j,i,k}(\mathbb{T}^3)$ is a Hilbert space with the norm

$$\begin{aligned} &\|v\|_{H^2} = \|\Delta v\|_{L^2} \\ &= \Big(\sum_{n \in \mathbb{N}} n^4 k^4 |\xi_n|^2 + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} ((2m-1)^2(l^2 + j^2) + (nk + \{2mi - i\})^2)^2 |\eta_{m,n}|^2 \\ &\quad + \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} ((2m)^2(l^2 + j^2) + (nk + \{2mi\})^2)^2 |\zeta_{m,n}|^2 \Big)^{1/2}. \end{aligned}$$

Then we are ready to define the invariant subspaces for the Navier-Stokes equations by

$$\dot{\mathbb{H}}^2_{l,j,i,k} = \Big\{ u = (u_1, u_2, u_3) \in \dot{\mathbb{H}}^2_\sigma |\ u_1,\ u_2,\ u_3 \in H^2_{l,j,i,k}(\mathbb{T}^3) \Big\}, \tag{3.1}$$

$$\tilde{\dot{\mathbb{H}}}^2_{l,j,i,k} = \Big\{ u = (u_1, u_2, u_3) \in \dot{\mathbb{H}}^2_\sigma |\ u_1,\ u_2,\ u_3 \in \widetilde{H}^2_{l,j,i,k}(\mathbb{T}^3) \Big\}. \tag{3.2}$$

Lemma 3.1. *Let $\mathbb{X}$ be either $\dot{\mathbb{H}}^2_{l,j,i,k}$ or $\tilde{\dot{\mathbb{H}}}^2_{l,j,i,k}$ with integers $0 \le l$, $j \in \mathbb{Z}$ and $0 \le i < k$. Then for any $u \in \mathbb{X}$, $\Delta^{-1}B(u,u) \in \mathbb{X}$. Consequently, $\mathbb{X}$ is invariant under the steady-state Navier-Stokes equations (2.1).*

Proof. For $u \in \mathbb{X}$, it follows from Hölder's inequality and Sobolev's inequality that

$$\|B(u,u)\|_{L^2} \le c\|u\|_{L^3}\|\nabla u\|_{L^6} \le c\|\Delta u\|^2_{L^2}.$$

This together with the divergence free condition $\nabla \cdot \Delta^{-1}B(u,u) = 0$ gives $\Delta^{-1}B(u,u) \in \dot{\mathbb{H}}^2_\sigma$.

If $u = (u_1, u_2, u_3) \in \dot{\mathbb{H}}^2_{l,j,i,k}$, we see that u is in the form

$$\begin{aligned}(u_1, u_2, u_3) &= \sum_{n\in\mathbb{N}} (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}) \sin nkz \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} (\eta_{m,n,1}, \eta_{m,n,2}, \eta_{m,n,3}) \sin(mlx + mjy + \{mi\}z + knz).\end{aligned}$$

This gives, for i', $j' = 1$, 2, 3,

$$\begin{aligned}u_{i'}u_{j'} &= \Big(\sum_{n\in\mathbb{N}} \xi_{n,i'} \sin nkz + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \eta_{m,n,i'} \sin(mlx + mjy + \{mi\}z + knz)\Big) \\ &\quad \times \Big(\sum_{n\in\mathbb{N}} \xi_{n,j'} \sin nkz + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \eta_{m,n,j'} \sin(mlx + mjy + \{mi\}z + knz)\Big) \\ &= c_{i',j'} + \sum_{n\in\mathbb{N}} \xi_{n,i',j'} \cos nkz \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \eta_{m,n,i',j'} \cos(mlx + mjy + \{mi\}z + knz)\end{aligned}$$

for some constants $c_{i',j'}$, $\xi_{n,i',j'}$ and $\eta_{m,n,i',j'}$. This yields

$$\begin{aligned}B(u,u) &= \sum_{n\in\mathbb{N}} (\xi'_{n,1}, \xi'_{n,2}, \xi'_{n,3}) \sin nkz \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} (\eta'_{m,n,1}, \eta'_{m,n,2}, \eta'_{m,n,3}) \sin(mlx + mjy + miz + knz).\end{aligned}$$

for some constants $\xi_{n,i'}$ and $\eta'_{m,n,i'}$. We thus have $\Delta^{-1}B(u,u) \in \dot{\mathbb{H}}^2_{l,j,i,k}$.

Likewise, if $u = (u_1, u_2, u_3) \in \tilde{\dot{\mathbb{H}}}^2_{l,j,i,k}$ with

$$\begin{aligned}&(u_1, u_2, u_3) \\ &= \sum_{n\in\mathbb{N}} (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}) \sin nkz \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} (\eta_{m,n,1}, \eta_{m,n,2}, \eta_{m,n,3}) \cos((2m-1)(lx + jy) + \{2mi - i\}z + knz) \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} (\zeta_{m,n,1}, \zeta_{m,n,2}, \zeta_{m,n,3}) \sin(2m(lx + jy) + \{2mi\}z + knz),\end{aligned}$$

we have, for i', $j' = 1, 2, 3$,

$$\begin{aligned} u_{i'}u_{j'} &= \sum_{n\in\mathbb{N}} \xi_{n,i',j'} \cos nkz \\ &\quad + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \eta_{m,n,i',j'} \sin((2m-1)(lx+jy) + \{2mi-i\}z + knz) \\ &\quad + c_{i',j'} + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} \zeta_{m,n,i',j'} \cos(2m(lx+jy) + \{2mi\}z + knz) \end{aligned}$$

for some constants $c_{i',j'}$, $\xi_{n,i',j'}$, $\eta_{m,n,i',j'}$ and $\zeta_{m,n,i',j'}$. It thus readily seen that $\Delta^{-1}B(u,u) \in \tilde{\dot{\mathbb{H}}}^2_{l,j,i,k}$. □

Remark 3.1. Both $\dot{\mathbb{H}}^2_{l,j,i,k}$ and $\tilde{\dot{\mathbb{H}}}^2_{l,j,i,k}$ are also invariant with respect to the time-dependent Navier-Stokes equations (1.1-1.4). Hence, (1.1-1.4) can be restrict to either of the invariant subspaces.

Lemma 3.2. 1. *For any $\lambda \in (0,\infty)$, there is at least one steady-state solution for (2.1).*

2. *There exists an absolute constant c_1 independent of λ and k such that $u_0 = (\sin kz, 0, 0)$ is the unique solution of (2.1) for any $0 < \lambda \le c_1/k^2$.*
3. *For any solution $u \in \dot{\mathbb{H}}^2_\sigma$ of (2.1), there exist two absolute constants independent of λ, k and u such that*

$$\begin{aligned} \|u\|_{H^1} &\le c_2k^2, \\ \|u\|_{H^2} &\le c_3k^2(\lambda^2k^4+1). \end{aligned} \tag{3.3}$$

Proof. The proof of the first two parts of the lemma is standard; we omit the details. The first inequality (3.3) is a direct consequence of the standard energy estimates, while the second inequality follows from the following computation:

$$\begin{aligned} \|\Delta u\|_{L^2} &\le \lambda\|B(u,u)\|_{L^2} + k^2\|u_0\|_{L^2} \\ &\le c\lambda\|u\cdot\nabla u\|_{L^2} + k^2\|u_0\|_{L^2} \\ &\le c\lambda\|u\|_{L^6}\|\nabla u\|_{L^3} + k^2\|u_0\|_{L^2} \\ &\le c\lambda\|\nabla u\|^{3/2}_{L^2}\|\Delta u\|^{1/2}_{L^2} + k^2\|u_0\|_{L^2} \\ &\le c\lambda^2\|\nabla u\|^3_{L^2} + \frac{1}{2}\|\Delta u\|_{L^2} + k^2\|u_0\|_{L^2} \\ &\le c\lambda^2k^6\|u_0\|^3_{L^2} + \frac{1}{2}\|\Delta u\|_{L^2} + k^2\|u_0\|_{L^2}. \end{aligned}$$

□

Lemma 3.3. *Let $\mathbb{X}$ be either $\dot{\mathbb{H}}^2_{l,j,i,k}$ or $\tilde{\dot{\mathbb{H}}}^2_{l,j,i,k}$ with integers $0 \le l$, $j \in \mathbb{Z}$ and $0 \le i < k$. Then $\Delta^{-1}A$ and $\Delta^{-1}B$ are compact and continuous operators from $\mathbb{X}$ into itself.*

Proof. It suffices to notice that for $u \in \dot{\mathbb{H}}^2_\sigma$,

$$\begin{aligned} \|Au\|_{L^2} &\le c\|\nabla u\|_{L^2}, \\ \|B(u)\|_{L^2} &\le c\|u\|_{L^{12}}\|\nabla u\|_{L^{12/5}} \le c\|\nabla u\|^2_{L^{12/5}}. \end{aligned}$$

□

4. Global Branches of Steady-State Solutions

In this section, we consider (2.1) in the flow invariant subspace $\dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k}$ to show the existence two global branches of steady-state solutions bifurcating from the basic flow u_0 at a critical value $\lambda_{l,j,i,k}$.

Theorem 4.1. *Let (l,j,i,k) be an integer vector satisfying (1.5) and n_0 be the integer given in (1.6). Then there exists a critical Reynolds number $\lambda_{l,j,i,k} > 0$ such that problem (2.1) admits two different global branches of steady-state solutions $u^1_{l,j,i,k,\lambda}$, $u^2_{l,j,i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k} \subset \dot{\mathbb{H}}^2_{l,j,i,k} \subset \dot{\mathbb{H}}^2_\sigma$ branching off the bifurcation point $(\lambda_{l,j,i,k}, u_0)$ continuously such that for $m = 1$, 2,*

1. *each branch $u^m_{l,j,i,k,\lambda}$ extends to $\lambda = \infty$ in $\mathcal{C}_1 \cap \mathcal{C}_2$ as shown in Figures 1.1 and 1.2;*
2. *each branch $u^m_{l,j,i,k,\lambda}$ starting from the bifurcation point $(\lambda_{l,j,i,k}, u_0)$ never touches the λ–axis for $\lambda \neq \lambda_{l,j,i,k}$.*

Proof. For the integer n_0 given in (1.6), we see that $\dot{\mathbb{H}}^2_{l,j,i,k} \supset \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k}$, which is in the following form

$$\begin{aligned}\Big\{ & u \in \dot{\mathbb{H}}^2_\sigma \,|\, u = \sum_{n\in\mathbb{N}} (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}) \sin nkz \\ & + \sum_{m\geq 2} \sum_{n\in\mathbb{Z}} (\eta_{m,n,1}, \eta_{m,n,2}, \eta_{m,n,3}) \sin(mn_0lx + mn_0jy + \{mn_0i\}z + knz) \\ & + \sum_{n\in\mathbb{Z}} (\zeta_{n,1}, \zeta_{n,2}, \zeta_{n,3}) \sin(n_0lx + n_0jy + \{n_0i\}z + knz) \Big\}.\end{aligned}$$

It is obvious that the spectral problem

$$\Delta u - \lambda A u = \rho u \quad \text{ for } \rho > -\beta_0 \text{ and } u \in \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k} \tag{4.1}$$

has no eigenfunction in the form

$$\sum_{n\in\mathbb{N}} (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}) \sin nkz.$$

Thus it follows from Theorem 2.3 that (4.1) has a unique eigenvalue $\rho_{n_0l,n_0j,\{n_0i\},k}$ transversal across the imaginary axis at the origin, and the choice of the integer n_0 ensures the simplicity of this eigenvalue.

Let us denote by $\lambda_{l,j,i,k}$ the critical Reynolds number such that

$$\rho_{n_0l,n_0j,\{n_0i\},k}(\lambda_{l,j,i,k}) = 0. \tag{4.2}$$

Taking $u' + u_0$ in place of u in the steady-state equations with respect to (2.1) reduced in $\dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k}$ and omitting the prime, we have

$$u = \lambda\Delta^{-1} A u + \lambda \Delta^{-1} B(u), \quad u \in \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k}, \tag{4.3}$$

of which the linear part

$$u - \lambda \Delta^{-1} A u = 0, \quad u \in \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k} \tag{4.4}$$

has a simple eigenvalue $1/\lambda_{l,j,i,k}$ or a simple characteristic value $\lambda_{l,j,i,k}$ in the notation of Rabinowitz [21]. It follows from [21, Theorem 1.40] that (4.3) admit two continuous branches of solutions $(\lambda, u^1_{l,j,i,k,\lambda} - u_0)$ and $(\lambda, u^2_{l,j,i,k,\lambda} - u_0)$ other than the branch $(\lambda, 0)$ in the space $\mathbb{R} \times \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k}$ under the norm

$\|(\lambda, u)\| = (\lambda^2 + \|\Delta u\|_{L^2})^{1/2}$. Each of those two branches meets $(\lambda_{l,j,i,k}, 0)$ and either

(i) meets ∞ in $\mathbb{R} \times \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}$, or

(ii) meets $(\widetilde{\lambda}, 0)$, where $\widetilde{\lambda} \neq \lambda_{l,j,i,k}$ is a characteristic value of (4.4).

Observe that $(\lambda, 0)$ is the unique solution of (4.3) when $0 < \lambda < c/k^2$. Thus these two branches of solutions are contained in the space $(0, \infty) \times \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}$. Meanwhile, the *a priori* bounds of the the steady-state solutions of (2.1) for every $\lambda > 0$ show that both branches of the bifurcation solutions extends to $\lambda = \infty$ in $\mathcal{C}_1 \cap \mathcal{C}_2$; see Figures 1.1 and 1.2.

Furthermore, the above second alternative is excluded due to the fact that $\lambda_{l,j,i,k}$ is the only characteristic value of (4.4) for $\lambda \in (0, \infty)$. Hence, each branch $u^m_{l,j,i,k,\lambda}$ intersects with the λ–axis only at the line segment pq shown in Figures 1.1 and 1.2. In particular, each branch $u^m_{l,j,i,k,\lambda}$ starting from the bifurcation point $(\lambda_{l,j,i,k}, u_0)$ never touches the λ–axis for $\lambda \neq \lambda_{l,j,i,k}$.

$\square$

5. Supercritical Pitchfork Bifurcation

In order to reach our main results, it is necessary to give the local behavior of the global branches of the steady-state solutions $u^1_{l,j,i,k,\lambda}$ and $u^2_{l,j,i,k,\lambda}$ close to the bifurcation point $(\lambda_{l,j,i,k}, u_0)$. More precisely, we have the following supercritical pitchfork bifurcation theorem.

Theorem 5.1. *Let the steady-state solutions $u^1_{l,j,i,k,\lambda}$, $u^2_{l,j,i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}$ be given in Theorem 4.1. Then for $m = 1, 2$,*

$$\begin{aligned} & u^m_{l,j,i,k,\lambda} = u_0 && \text{for } \lambda < \lambda_{l,j,i,k}, \\ & u_0 \neq u^1_{l,j,i,k,\lambda} \neq u^2_{l,j,i,k,\lambda} \neq u_0 && \text{for } \lambda_{l,j,i,k} < \lambda, \end{aligned} \tag{5.1}$$

provided that

$$\|\Delta(u^m_{l,j,i,k\lambda} - u_0)\|_{L^2} + |\lambda - \lambda_{l,j,i,k}| < \epsilon \tag{5.2}$$

for some small constant $\epsilon > 0$.

For simplicity, we set

$$\lambda_0 = \lambda_{l,j,i,k}, \quad u^+_\lambda = u^1_{l,j,i,k,\lambda}, \quad u^-_\lambda = u^2_{l,j,i,k,\lambda}. \tag{5.3}$$

The proof of Theorem 5.1 will be achieved by analyzing the local asymptotic expansion of $u^\pm_\lambda$ in terms of λ.

First, notice that he bifurcation phenomenon of the Navier-Stokes system (2.1) reduced to the subspace $\dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}$ is excited by the external force $k^2 u_0$ and the unstable subspace of the bifurcation point (λ_0, u_0) is contained in $\mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k}$.

The unstable subspace is given by the nontrivial solution $u^* \in \mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k}$ of the spectral problem $\Delta u^* - \lambda_0 A u^* = 0$:

$$\begin{gathered} u^* = \sum_{n \in \mathbb{Z}} (\xi_n, \eta_n, \zeta_n) \sin(n_0 l x + n_0 j y + \{n_0 i\} z + nkz) \in \mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k}, \\ \zeta_0 = 1, \\ \Delta u^* - \lambda_0 A u^* = 0, \end{gathered} \tag{5.4}$$

By Theorem 2.3, for any n, $\zeta_n \neq 0$. Hence there exists a unique nontrivial solution u^* of (5.4) with $\zeta_0 = 1$.

Then we examine the nonlinear interaction of the unstable direction given by u^*. To this end, we notice that

$$B(u^*, u^*) \in \mathbb{E}_{0,0,0,k} \oplus \mathbb{E}_{2n_0 l, 2n_0 j, \{2n_0 i\}, k}.$$

Hence, we can decompose $B(u^*, u^*)$ as

$$\begin{aligned}
&B(u^*, u^*) = B_0(u^*, u^*) + B_2(u, u) + B_3(u, u),\\
&B_0(u^*, u^*) = \frac{u_0}{4\pi^3} \int_{\mathbb{T}^3} B(u^*, u^*) \cdot u_0 \, dx\, dy\, dz,\\
&B_2(u^*, u^*) \in \mathbb{E}_{0,0,0,k} \quad \text{with} \quad \int_{\mathbb{T}^3} B_2(u^*, u^*) \cdot u_0 \, dx\, dy\, dz = 0,\\
&B_3(u^*, u^*) \in \mathbb{E}_{2n_0 l, 2n_0 j, \{2n_0 i\}, k}.
\end{aligned} \tag{5.5}$$

To prove Theorem 5.1, it suffices for us to prove the following stronger result, providing a detailed local asymptotic expansion of the bifurcation solutions in terms of λ.

Theorem 5.2. *If (5.2) holds true for some small number $\epsilon > 0$, then the two branches of steady-state solutions $(\lambda, u^{\pm}_{\lambda})$ close to the bifurcation point (λ_0, u_0) undergo the supercritical pitchfork bifurcation in the following sense:*

$$u^{\pm}_{\lambda} = \begin{cases} u_0 & \lambda < \lambda_0,\\ u_0 \pm \alpha \frac{\sqrt{\lambda - \lambda_0}}{\lambda} u^* + \frac{\lambda - \lambda_0}{\lambda}[-u_0 + \alpha^2 \Delta^{-1} B_2(u^*, u^*) & \\ +\alpha^2 (-\Delta + \lambda_0 A)^{-1} B_3(u^*, u^*)] + o(\lambda - \lambda_0) & \lambda > \lambda_0\,. \end{cases} \tag{5.6}$$

Here the eigenfunction u^* is given by (5.4) and the constant α is defined

$$\alpha = \frac{2k\pi^{3/2}}{\left(\int_{\mathbb{T}^3} B(u^*, u^*) \cdot u_0 \, dx\, dy\, dz \right)^{1/2}}. \tag{5.7}$$

The definition of the constant α in (5.7) is justified in the following lemma.

Lemma 5.1.

$$\int_{\mathbb{T}^3} B(u^*, u^*) \cdot u_0 \, dx\, dy\, dz > 0.$$

Proof of Lemma 5.1. By the divergence free condition and integration by parts,

$$\begin{aligned}
&\int_{\mathbb{T}^3} B(u^*, u^*) \cdot u_0 \, dx\, dy\, dz \\
&= -\int_{\mathbb{T}^3} B(u^*, u_0) \cdot u^* \, dx\, dy\, dz \\
&= -k \sum_{n,\, m \in \mathbb{Z}} \xi_n \zeta_m \int_{\mathbb{T}^3} \sin(n_0 l x + n_0 j y + \{n_0 i\} z + nkz) \\
&\qquad \times \sin(n_0 l x + n_0 j y + \{n_0 i\} z + mkz) \cos kz \, dx\, dy\, dz \\
&= -2k\pi^3 \sum_{n \in \mathbb{Z}} \xi_n (\zeta_{n-1} + \zeta_{n+1}).
\end{aligned} \tag{5.8}$$

On the other hand, recall from (2.18)-(2.20) that u^* satisfies the following set of coupled algebraic equations

$$\frac{2\beta_n^2}{\lambda_{l,j,i,k}n_0 l}\zeta_n + (\beta_{n-1} - k^2)\zeta_{n-1} - (\beta_{n+1} - k^2)\zeta_{n+1} = 0, \tag{5.9}$$

$$\begin{aligned} &\frac{2\beta_n}{\lambda_{l,j,i,k}}(j\xi_n - l\eta_n) + n_0 l(j\xi_{n-1} - l\eta_{n-1}) - n_0 l(j\xi_{n+1} - l\eta_{n+1}) \\ &\qquad = -kj(\zeta_{n-1} + \zeta_{n+1}), \end{aligned} \tag{5.10}$$

$$-n_0 j\eta_n = n_0 l\xi_n + (\{n_0 i\} + nk)\zeta_n, \tag{5.11}$$

for any $n \in \mathbb{Z}$, where $\beta_n = n_0^2 l^2 + n_0^2 j^2 + (\{n_0 i\} + nk)^2$.

Multiplying (5.9) by ζ_n and summing up the resultant equations yield

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \frac{2\beta_n^2}{\lambda_{l,j,i,k}n_0 l}\zeta_n^2 &= -\sum_{n\in\mathbb{Z}}(\beta_n - \beta_{n+1})\zeta_n\zeta_{n+1} \\ &= \sum_{n\in\mathbb{Z}} k(2\{n_0 i\} + 2nk + k)\zeta_n\zeta_{n+1}. \end{aligned} \tag{5.12}$$

Moreover, multiplying (5.10) by $j\xi_n - l\eta_n$ and summing up the resultant equations yield

$$\sum_{n\in\mathbb{Z}} \frac{2\beta_n}{\lambda_{l,j,i,k}}(j\xi_n - l\eta_n)^2 = -k\sum_{n\in\mathbb{Z}}(j^2\xi_n - lj\eta_n)(\zeta_{n-1} + \zeta_{n+1}). \tag{5.13}$$

We infer then from (5.8), (5.11), (5.12) and (5.13) that

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \frac{2\beta_n}{\lambda_{l,j,i,k}}(j\xi_n - l\eta_n)^2 \\ &= -k\sum_{n\in\mathbb{Z}}(j^2\xi_n + l^2\xi_n + \frac{l}{n_0}(\{n_0 i\} + nk)\zeta_n)(\zeta_{n-1} + \zeta_{n+1}) \\ &= -k(l^2 + j^2)\sum_{n\in\mathbb{Z}}\xi_n(\zeta_{n-1} + \zeta_{n+1}) - \frac{kl}{n_0}\sum_{n\in\mathbb{Z}}(\{n_0 i\} + nk)\zeta_n(\zeta_{n-1} + \zeta_{n+1}) \\ &= \frac{l^2 + j^2}{2\pi^3}\int_{\mathbb{T}^3} B(u^*, u^*)\cdot u_0\,dx\,dy\,dz - \frac{kl}{n_0}\sum_{n\in\mathbb{Z}}(2\{n_0 i\} + 2nk + k)\zeta_n\zeta_{n+1} \\ &= \frac{l^2 + j^2}{2\pi^3}\int_{\mathbb{T}^3} B(u^*, u^*)\cdot u_0\,dx\,dy\,dz - \sum_{n\in\mathbb{Z}} \frac{2\beta_n^2}{\lambda_{l,j,i,k}n_0^2}\zeta_n^2. \end{aligned}$$

We thus have

$$\int_{\mathbb{T}^3} B(u^*, u^*)\cdot u_0\,dx\,dy\,dz = \frac{4\pi^3}{(l^2 + j^2)\lambda_{l,j,i,k}n_0^2}\sum_{n\in\mathbb{Z}}(\beta_n n_0^2(j\xi_n - l\eta_n)^2 + \beta_n^2\zeta_n^2) > 0.$$

□

Now in order to facilitate the understanding of the problem, we consider the infinite mode truncation model by projecting (2.1) onto the unstable space $\mathrm{Span}\{u_0\} \oplus \mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k}$:

$$\begin{aligned} &\text{Find } u \in W = \mathrm{Span}\{u_0\} \oplus \mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k} \text{ such that} \\ &\int_{\mathbb{T}^3} (-\Delta(u - u_0) + \lambda B(u, u))\cdot\phi\,dx\,dy\,dz = 0, \qquad \forall\,\phi \in W. \end{aligned} \tag{5.14}$$

Lemma 5.2. *The parameter $\lambda = \lambda_0$ is a supercritical bifurcation point of (5.14) with the two bifurcation branches given by*

$$u = \begin{cases} u_0 & \text{when } \lambda \le \lambda_0, \\ u_0 \pm \alpha \frac{\sqrt{\lambda-\lambda_0}}{\lambda} u^* - \frac{\lambda-\lambda_0}{\lambda} u_0 & \text{when } \lambda > \lambda_0, \end{cases} \tag{5.15}$$

for $\|\Delta(u-u_0)\|_{L^2} + |\lambda-\lambda_0| < \epsilon$ for some constant $\epsilon > 0$.

Proof of Lemma 5.2. Decompose u as

$$u = \mu u_0 + v \quad \text{with } \mu \in \mathbb{R},\ v \in \mathbb{E}_{n_0 l, n_0 j, \{n_0 i\}, k}.$$

Then (5.14) becomes

$$-\Delta v + \lambda \mu A v = 0, \tag{5.16}$$

$$k^2(\mu - 1) + \frac{\lambda}{4\pi^3} \int_{\mathbb{T}^3} B(v,v) \cdot u_0 \, dx\, dy\, dz = 0, \tag{5.17}$$

It follows from Theorem 2.3 that $u = \mu u_0 + v$ solves (5.16)-(5.17) if and only if

$$\mu\lambda = \lambda_0 \quad \text{and} \quad v = cu^* \quad \text{for some} \quad c \in \mathbb{R},$$

where the eigenfunction u^* is defined by (5.4). Thus (5.17) becomes

$$\frac{4k^2\pi^3(\lambda-\lambda_0)}{\lambda^2} = c^2 \int_{\mathbb{T}^3} B(u^*, u^*) \cdot u_0 \, dx\, dy\, dz.$$

This together with Lemma 5.1 implies

$$c = \begin{cases} 0 & \text{if } \lambda < \lambda_0, \\ \pm\alpha \frac{\sqrt{\lambda-\lambda_0}}{\lambda} & \text{if } \lambda > \lambda_0\,. \end{cases}$$

Thus (5.15) follows, and the lemma is proved. □

Proof of Theorem 5.2. Lemma 5.2 implies that the bifurcation solutions $u_\lambda^\pm$ close to the bifurcation point (λ_0, u_0) are in the following form:

$$u_\lambda^\pm = u_0 + \frac{\sqrt{|\lambda-\lambda_0|}}{\lambda} w_1 + \frac{\lambda-\lambda_0}{\lambda} w + o(|\lambda-\lambda_0|), \tag{5.18}$$

with w_1, $w \in \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}$ independent of λ. Here w can be decomposed as

$$w = -u_0 + w_2 + w_3$$

$$w_2 \in \mathbb{E}_{0,0,0,k}, \quad \int_{\mathbb{T}^3} w_2 \cdot u_0 \, dx\, dy\, dz = 0,$$

$$\int_{\mathbb{T}^3} w_3 \cdot \phi \, dx\, dy\, dz = 0 \qquad \forall \phi \in \mathbb{E}_{0,0,0,k}\,.$$

Thus

$$\begin{aligned}
&\lambda B(u_\lambda^\pm, u_\lambda^\pm)\\
&= \sqrt{|\lambda-\lambda_0|}\Big(B(u_0+\frac{\lambda-\lambda_0}{\lambda}(w_2-u_0),w_1)+B(w_1,u_0+\frac{\lambda-\lambda_0}{\lambda}(w_2-u_0))\Big)\\
&\quad +(\lambda-\lambda_0)\Big(B(u_0+\frac{\lambda-\lambda_0}{\lambda}(w_2-u_0),w_3)+B(w_3,u_0+\frac{\lambda-\lambda_0}{\lambda}(w_2-u_0))\Big)\\
&\quad +\frac{|\lambda-\lambda_0|}{\lambda}B(w_1,w_1)+o(|\lambda-\lambda_0|)\\
&= \frac{\sqrt{|\lambda-\lambda_0|}}{\lambda}\lambda_0(B(u_0,w_1)+B(w_1,u_0))\\
&\quad +\frac{\sqrt{|\lambda-\lambda_0|}}{\lambda}\lambda_0(B(u_0,w_3)+B(w_3,u_0))+\frac{|\lambda-\lambda_0|}{\lambda}B(w_1,w_1)+o(|\lambda-\lambda_0|).
\end{aligned}$$

Hence the stationary Navier-Stokes system

$$-\Delta(u_\lambda^\pm-u_0)+\lambda B(u_\lambda^\pm,u_\lambda^\pm)=0 \quad \text{in } \dot{\mathbb{H}}^2_{n_0l,n_0j,\{n_0i\},k} \tag{5.19}$$

becomes

$$\begin{aligned}
0 &= -\frac{\sqrt{|\lambda-\lambda_0|}}{\lambda}\Delta w_1-\frac{\lambda-\lambda_0}{\lambda}\Delta(w_2-u_0)-\frac{\lambda-\lambda_0}{\lambda}\Delta w_3\\
&\quad +\frac{\sqrt{|\lambda-\lambda_0|}}{\lambda}\lambda_0 Aw_1+\frac{(\lambda-\lambda_0)}{\lambda}\lambda_0 Aw_3+\frac{|\lambda-\lambda_0|}{\lambda}B(w_1,w_1)+o(|\lambda-\lambda_0|).
\end{aligned}$$

This yields

$$\begin{aligned}
-\Delta w_1+\lambda_0 Aw_1 &= 0,\\
(\lambda-\lambda_0)(\Delta(w_2-u_0+w_3)-\lambda_0 Aw_3) &= |\lambda-\lambda_0|B(w_1,w_1).
\end{aligned}$$

By Theorem 2.3, we see $w_1 = cu^* \in \mathbb{E}_{n_0l,n_0j,\{n_0i\},k}$ for some constant c with u^* defined by (5.4). Hence

$$B(w_1,w_1)=c^2B(u^*,u^*)\in \mathbb{E}_{0,0,0,k}\oplus\mathbb{E}_{2n_0l,2n_0j,\{2n_0i\},k},$$

with $B(u^*,u^*)$ decomposed by (5.5).

Then (5.19) yields

$$\begin{aligned}
0 &= -\Delta u^*+\lambda_0 Au^*,\\
(\lambda-\lambda_0)k^2u_0 &= c^2|\lambda-\lambda_0|B_0(u^*,u^*)\\
(\lambda-\lambda_0)\Delta w_2 &= c^2|\lambda-\lambda_0|B_2(u^*,u^*),\\
(\lambda-\lambda_0)(-\Delta w_3+\lambda Aw_3) &= -c^2|\lambda-\lambda_0|B_3(u^*,u^*).
\end{aligned} \tag{5.20}$$

By Lemma 5.1 and (5.20), we obtain that for $\lambda<\lambda_0$, $c=0$. For $\lambda>\lambda_0$, we have

$$\begin{aligned}
0 &= -\Delta u^*+\lambda_0 Au^*,\\
c &= \pm\alpha,\\
w_2 &= c^2\Delta^{-1}B_2(u^*,u^*)\ \in \mathbb{E}_{0,0,0,k},\\
w_3 &= c^2(\Delta-\lambda_0 A)^{-1}B_3(u^*,u^*)\ \in \mathbb{E}_{2n_0l,2n_0j,\{2n_0i\},k}.
\end{aligned} \tag{5.21}$$

Here we have used Theorem 2.3, the definition of n_0 and the Riesz-Schauder theory to obtain the boundedness of the operator $(-\Delta+\lambda_0 A)^{-1}$ in $\mathbb{E}_{2n_0l,2n_0j,\{2n_0i\},k}$. Thus the two branches of steady-state solutions $(\lambda, u_\lambda^\pm)$ close to the bifurcation point (λ_0,u_0) undergo the supercritical pitchfork bifurcation in the sense given by (5.6). The theorem is proved. □

To copy the results of Theorems 4.1 and 5.1 to another space $\widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k}$. we notice that $\dot{\mathbb{H}}^2_\sigma \supset \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k}$, which is in the form

$$\begin{aligned}\Big\{u \in \dot{\mathbb{H}}^2_\sigma \mid u = &\sum_{n\in\mathbb{N}}(\xi_{n,1},\xi_{n,2},\xi_{n,3})\sin nkz\\ &+\sum_{m\geq 2}\sum_{n\in\mathbb{Z}}(\eta_{m,n,1},\eta_{m,n,2},\eta_{m,n,3})\cos((2m-1)(n_0lx+n_0jy)\\ &\qquad + \{2mn_0i - n_0i\}z + knz)\\ &+\sum_{n\in\mathbb{Z}}(\zeta_{n,1},\zeta_{n,2},\zeta_{n,3})\sin(2m(n_0lx+n_0jy)+\{2mn_0i\}z+knz)\\ &+\sum_{n\in\mathbb{Z}}(\eta_{1,n,1},\eta_{1,n,2},\eta_{1,n,3})\cos(n_0lx+n_0jy+\{n_0i\}z+knz)\Big\},\end{aligned}$$

and

$$\widetilde{\dot{\mathbb{H}}}^2_{l,j,i,k} \supset \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k} \text{ for } n_0 \text{ odd }.$$

We see that for each (l,j,i,k) satisfying (1.5), $\widetilde{\mathbb{E}}_{n_0l,n_0j,\{n_0i\},k} \subset \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k}$. It follows from Theorem 2.3 that $\rho_{n_0l,n_0j,\{n_0i\},k}$ is simple and is unique eigenvalue of the spectral problem

$$\Delta u - \lambda Au = \rho u \quad \text{ for } \rho > -\beta_0 \text{ and } u \in \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k}$$

which is transversal across the imaginary axis at the origin. Then the argument as that given in the above proof of Theorems 4.1 and 5.1 shows that there exist two steady-state solutions

$$\widetilde{u}^1_{l,j,i,k,\lambda},\ \widetilde{u}^2_{l,j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k}$$

globally branching off the bifurcation point $(\lambda_{l,j,i,k}, u_0)$, where $\lambda_{l,j,i,k}$ is defined by (4.2). More precisely, we have the following Theorem.

Theorem 5.3. *For the critical Reynolds number $\lambda_{l,j,i,k} > 0$ obtained by (4.2), the problem (2.1) with $0 < \lambda < \infty$ admits also two steady-state solutions*

$$\widetilde{u}^1_{l,j,i,k,\lambda},\ \widetilde{u}^2_{l,j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0l,n_0j,\{n_0i\},k} \subset \dot{\mathbb{H}}^2_\sigma$$

branching off the point $(\lambda_{l,j,i,k}, u_0)$ continuously such that for $m = 1, 2$,

$$\begin{aligned}&\widetilde{u}^m_{l,j,i,k,\lambda} = u_0 && \text{for } \lambda \leq \lambda_{l,j,i,k},\\ &u_0 \neq \widetilde{u}^1_{l,j,i,k,\lambda} \neq \widetilde{u}^2_{l,j,i,k,\lambda} \neq u_0 && \text{for } \lambda > \lambda_{l,j,i,k},\end{aligned}$$

provided that, for some small constant $\epsilon > 0$,

$$\|\Delta(\widetilde{u}^m_{l,j,i,k,\lambda} - u_0)\|_{L^2} + |\lambda - \lambda_0| < \epsilon.$$

Furthermore, both of the global branches of bifurcation solutions $\widetilde{u}^m_{l,j,i,k,\lambda}$ satisfying the following properties:

1. *each branch $\widetilde{u}^m_{l,j,i,k,\lambda}$ extends to $\lambda = \infty$ in $\mathcal{C}_1 \cap \mathcal{C}_2$ as shown in Figures 1.1 and 1.2;*
2. *each branch $\widetilde{u}^m_{l,j,i,k,\lambda}$ intersects with the λ–axis only at the line segment pq. In particular, $\widetilde{u}^m_{l,j,i,k,\lambda}$ never touches the λ–axis for $\lambda > \lambda_{l,j,i,k}$.*

6. Proof of Theorems 1.1 and 1.2

For an given integer vector (l,j,i,k) satisfying (1.5), let $u^m_{l,\pm j,i,k,\lambda}$, $\widetilde{u}^m_{l,\pm j,i,k,\lambda}$, $u^m_{l,\pm j,k-i,k,\lambda}$ and $\widetilde{u}^m_{l,\pm j,k-i,k,\lambda}$ be the bifurcation solutions obtained in Theorems 4.1, 5.1 and 5.3, and let

$$u_{l,j,i,k,n,\lambda} = u^n_{l,j,i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k}, \text{ for } n = 1,2. \tag{6.1}$$

For $\{n(k-i)\} = k - \{n_0 i\}$, we define other bifurcation solutions $u_{l,j,i,k,n,\lambda}$ as follows:

a). *Case* $\{n_0 i\} = j = 0$:

$$u_{l,0,i,k,n,\lambda} = \widetilde{u}^{n-2}_{l,0,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,0,0,k} \text{ for } n = 3,4; \tag{6.2}$$

b). *Case* $\{n_0 i\} = 0, j \neq 0$:

$$u_{l,j,i,k,n,\lambda} = \begin{cases} u^{n-2}_{l,-j,i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l,-n_0 j,0,k} & \text{for } n = 3,4, \\ \widetilde{u}^{n-4}_{l,j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,n_0 j,0,k} & \text{for } n = 5,6, \\ \widetilde{u}^{n-6}_{l,-j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,-n_0 j,0,k} & \text{for } n = 7,8. \end{cases} \tag{6.3}$$

c). *Case* $\{n_0 i\} \neq 0, j = 0$:

$$u_{l,0,i,k,n,\lambda} = \begin{cases} u^{n-2}_{l,0,k-i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l,0,k-\{n_0 i\},k}, & \text{for } n = 3,4, \\ \widetilde{u}^{n-4}_{l,0,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,0,\{n_0 i\},k} & \text{for } n = 5,6, \\ \widetilde{u}^{n-6}_{l,0,k-i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,0,k-\{n_0 i\},k} & \text{for } n = 7,8. \end{cases} \tag{6.4}$$

d). *Case* $\{n_0 i\} \neq 0$, $j \neq 0$:

$$u_{l,j,i,k,n,\lambda} = \begin{cases} u^{n-2}_{l,j,k-i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l,n_0 j,k-\{n_0 i\},k}, & \text{for } n = 3,4, \\ u^{n-4}_{l,-j,i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l,-n_0 j,\{n_0 i\},k}, & \text{for } n = 5,6, \\ u^{n-6}_{l,-j,k-i,k,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l,-n_0 j,k-\{n_0 i\},k}, & \text{for } n = 7,8, \\ \widetilde{u}^{n-8}_{l,j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,n_0 j,\{n_0 i\},k} & \\ \widetilde{u}^{n-10}_{l,j,k-i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,n_0 j,k-\{n_0 i\},k} & \text{for } n = 11,12, \\ \widetilde{u}^{n-12}_{l,-j,i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,-n_0 j,\{n_0 i\},k} & \text{for } n = 13,14, \\ \widetilde{u}^{n-14}_{l,-j,k-i,k,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l,-n_0 j,k-\{n_0 i\},k} & \text{for } n = 15,16. \end{cases} \tag{6.5}$$

Lemma 6.1. *The above defined solutions $u_{l,j,i,k,n,\lambda}$ bifurcate from the same critical Reynolds number $\lambda_{l,j,i,k}$.*

Proof. By Theorem 2.3, the fact $\{n_0(k-i)\} = k - \{n_0 i\}$ in case of $\{n_0 i\} \neq 0$ and the proof of Theorem 2.1, it is easy to see that $\rho_{n_0 l,n_0 j,0,k} = \rho_{n_0 l,-n_0 j,0,k}$ when $\{n_0 i\} = 0$, and $\rho_{n_0 l,n_0 j,\{n_0 i\},k} = \rho_{n_0 l,n_0 j,k-\{n_0 i\},k} = \rho_{n_0 l,-n_0 j,\{n_0 i\},k} = \rho_{n_0 l,-n_0 j,k-\{n_0 i\},k}$ when $\{n_0 i\} \neq 0$. Therefore we infer from (4.2) that $\lambda_{l,j,i,k} = \lambda_{l,-j,i,k}$ when $\{n_0 i\} = 0$, and $\lambda_{l,j,i,k} = \lambda_{l,j,k-i,k} = \lambda_{l,-j,i,k} = \lambda_{l,-j,k-i,k}$ when $\{n_0 i\} \neq 0$. □

Proof of Theorem 1.1. It suffices to prove (1.9). To this end, set

$$(l', j', i') = (n_0 l, n_0 j, \{n_0 i\}).$$

Then we consider the following function spaces:

Case $i' = j = 0$:

$$\mathbb{E}_{l',j',i',k} \subset \dot{\mathbb{H}}^2_{l',j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',i',k};$$

Case $j \neq 0$, $i' = 0$:

$$\begin{aligned} &\mathbb{E}_{l',j',i',k} \subset \dot{\mathbb{H}}^2_{l',j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',i',k},\\ &\mathbb{E}_{l',-j',i',k} \subset \dot{\mathbb{H}}^2_{l',-j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',-j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',-j',i',k};\end{aligned}$$

Case $j = 0$, $i' > 0$:

$$\begin{aligned} &\mathbb{E}_{l',j',i',k} \subset \dot{\mathbb{H}}^2_{l',j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',i',k},\\ &\mathbb{E}_{l',j',k-i',k} \subset \dot{\mathbb{H}}^2_{l',j',k-i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',k-i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',k-i',k};\end{aligned}$$

Case $ji' \neq 0$:

$$\begin{aligned} &\mathbb{E}_{l',j',i',k} \subset \dot{\mathbb{H}}^2_{l',j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',i',k},\\ &\mathbb{E}_{l',-j',i',k} \subset \dot{\mathbb{H}}^2_{l',-j',i',k}, \qquad \widetilde{\mathbb{E}}_{l',-j',i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',-j',i',k}\\ &\mathbb{E}_{l',j',k-i',k} \subset \dot{\mathbb{H}}^2_{l',j',k-i',k}, \qquad \widetilde{\mathbb{E}}_{l',j',k-i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',j',k-i',k},\\ &\mathbb{E}_{l',-j',k-i',k} \subset \dot{\mathbb{H}}^2_{l',-j',k-i',k}, \qquad \widetilde{\mathbb{E}}_{l',-j',k-i',k} \subset \widetilde{\dot{\mathbb{H}}}^2_{l',-j',k-i',k}.\end{aligned}$$

It is easy to see that for any integer vector (l, j, i, k) satisfying (1.5) and for each case,

1. each of these subspaces contains an unstable subspace of (2.1), generating the bifurcation solutions; and
2. these subspaces are orthogonal to each other in $\dot{\mathbb{H}}^2_\sigma$.

Hence (1.9), consequently Theorem 1.1, follows. □

Proof of Theorem 1.2. 1. By Theorems 4.1, 5.1 and 5.2 and (6.1-6.5), it remains to show (1.12). For the integer vector (l, j, i, k) given by (1.5), we set

$$\mathbb{Y} = \Big\{ u \in \dot{\mathbb{H}}^2_{l,j,i,k} \,|\, u = \sum_{n=1}^{\infty} (\xi_n, \eta_n, \zeta_n) \sin knz \Big\}.$$

It is easy to see that $u_0 = (\sin kz, 0, 0)$ is the unique steady-state solution of (2.1) in $\mathbb{Y}$. With this observation in mind, the cases where $\{n_0 i\} = 0$ and $\{n_0 i\} \neq 0$ for (1.12) are shown respectively as follows:

2. In the case where $\{n_0 i\} = 0$ and $j \neq 0$, by (1.7), $i_0 = 4$. Then we need to check, for $i_1 = 0$, 4,

$$u_{l,j,i,k,n,\lambda} \neq u_{l,j,i,k,n',\lambda}, \text{ if } \lambda_{l,j,i,k} < \lambda < \infty,\ i_1 + 1 \leq n \leq i_1 + 2 < n' \leq i_1 + 4. \tag{6.6}$$

Indeed, when $i_1 = 0$ it is obvious that

$$\mathbb{Y} = \dot{\mathbb{H}}^2_{n_0 l, n_0 j, 0, k} \cap \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, 0, k},$$

and from (6.1) and (6.3),

$$u_{l,j,i,k,n,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, n_0 j, 0, k}, \quad u_{l,j,i,k,n',\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, 0, k}.$$

By Assertion 2 of Theorem 1.2,

$$u_{l,j,i,k,n,\lambda} \neq u_0 \neq u_{l,j,i,k,n',\lambda} \text{ for } \lambda > \lambda_{l,j,i,k}.$$

This gives (6.6) with $i_1 = 0$.

For $i_1 = 4$ we see that

$$\mathbb{Y} = \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, 0, k} \cap \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, 0, k},$$

and from (6.3),

$$u_{l,j,i,k,n,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, 0, k},$$

$$u_{l,j,i,k,n',\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, 0, k}.$$

By Assertion 2 of Theorem 1.2,

$$u_{l,j,i,k,n,\lambda} \neq u_0 \neq u_{l,j,i,k,n',\lambda} \text{ for } \lambda > \lambda_{l,j,i,k}.$$

This gives (6.6) with $i_1 = 4$.

3. In the case where $j\{n_0 i\} \neq 0$, we see that $i_0 = 16$. When $i_1 = 0$, we notice that

$$\begin{aligned} \mathbb{Y} &= \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k} \cap \left(\dot{\mathbb{H}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k}\right) \\ &= \dot{\mathbb{H}}^2_{n_0 l, n_0 j, k-\{n_0 i\}, k} \cap \left(\dot{\mathbb{H}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k}\right). \end{aligned}$$

From (6.4) it follows that

$$u_{l,j,i,k,n,\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, n_0 j, \{n_0 i\}, k} \cup \dot{\mathbb{H}}^2_{n_0 l, n_0 j, k-\{n_0 i\}, k} \text{ for } 1 \leq n \leq 4,$$

and

$$u_{l,j,i,k,n',\lambda} \in \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \dot{\mathbb{H}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k} \text{ for } 5 \leq n \leq 8.$$

By Assertion 2 of Theorem 1.2, we thus have

$$u_{l,j,i,k,n,\lambda} \neq u_{l,j,i,k,n',\lambda} \text{ for } 1 \leq n \leq 4 < n' \leq 8.$$

When $i_1 = 8$, we see that

$$\begin{aligned} \mathbb{Y} &= \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, \{n_0 i\}, k} \cap \left(\widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k}\right) \\ &= \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, k-\{n_0 i\}, k} \cap \left(\widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k}\right), \end{aligned}$$

and, by (6.4),

$$u_{l,j,i,k,n,\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, \{n_0 i\}, k} \cup \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, n_0 j, k-\{n_0 i\}, k},$$

$$u_{l,j,i,k,n',\lambda} \in \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, \{n_0 i\}, k} \cup \widetilde{\dot{\mathbb{H}}}^2_{n_0 l, -n_0 j, k-\{n_0 i\}, k}$$

for $9 \leq n \leq 12 < n' \leq 16$. From Assertion 2 of Theorem 1.2 it thus follows that

$$u_{l,j,i,k,n,\lambda} \neq u_{l,j,i,k,n',\lambda} \text{ if } \lambda > \lambda_{l,j,i,k} \text{ and } 9 \leq n \leq 12 < n' \leq 16.$$

Hence, we obtain (1.12) and thus Theorem 1.2 is proved. $\square$

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