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A SINGULAR THIRD-ORDER 3-POINT BOUNDARY-VALUE PROBLEM WITH NONPOSITIVE GREEN'S FUNCTION

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Abstract. We find a Green's function for the singular third-order three-point BVP

$$u'''(t) = -a(t)f(t, u(t)), \quad u(0) = u'(1) = u''(\eta) = 0$$

where $0 \le \eta < 1/2$. Then we apply the classical Krasnosel'skii's fixed point theorem for finding solutions in a cone. Although this problem Green's function is not positive, the obtained solution is still positive and increasing. Our techniques rely on a combination of a fixed point theorem and the properties of the corresponding vector field.

1. Introduction

Ma in [17], proved the existence of a positive solution of the three-point nonlinear boundary-value problem

$$\begin{gathered} -u''(t) = q(t)f(u(t)), \\ u(0) = 0, \quad \alpha u(\eta) = u(1). \end{gathered}$$

Recently Infante and Webb in [10], studied the three-point nonlinear boundary-value problem

$$\begin{gathered} -u''(t) = q(t)f(u(t)), \\ u'(0) = 0, \quad \alpha u'(1) + u(\eta) = 0. \end{gathered}$$

The main result was the loss of positivity of its solutions, as α decreases.

Since Chazy's attempt [3] to completely classify all third-order differential equations of certain form, analysts were fascinated by the study of third-order differential equations in the pure but also in the applied sense, as in Gamba and Jüngel [6]. The singular third-order boundary value problem

$$\begin{gathered} y'''(x) = (1-y)^{\lambda} g(y), \quad 0 < x < +\infty \quad (\lambda > 0) \\ y(0) = 0, \quad \lim_{x\to+\infty} y(x) = 1, \quad \lim_{x\to+\infty} y'(x) = \lim_{x\to+\infty} y''(x) = 0. \end{gathered} \tag{1.1}$$

arises in the study of draining and coating flows. Jiang and Agarwal [11], established, among other things, the uniqueness and existence of solutions of (1.1).

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In a recent paper Sun [20], proved the existence of infinite positive solutions of the BVP

$$\begin{gathered} u'''(t)=\lambda\alpha(t)f(t,u(t)), \quad 0<t<1 \\ u(0)=u'(\eta)=u''(1)=0, \quad \eta\in(1/2,1) \end{gathered} \tag{1.2}$$

mainly under sub or superlinearity on the nonlinearity f

$$\begin{gathered} f(t,x)\le\frac{r}{\lambda A_0}, \quad \forall(t,x)\in[0,1]\times[0,r] \\ f(t,x)\ge\frac{R}{\lambda B_0}, \quad \forall(t,x)\in[0,1]\times[\theta R,R], \end{gathered}$$

for positive constants θ, R,r, A_0 and B_0 where $R\neq r$. Sun, in order to obtain the existence results, applied also the Krasnosel'skii fixed-point theorem on a cone expansion-compression type. Furthermore, in order to prove a result concerning the multiplicity of solutions, he assumed monotonicity of the nonlinearity with respect to the second variable.

Lately, Agarwal [1], Anderson et al. [2], Hopkins and Kosmatov [9], Li [13], Liu et al. [14, 15, 16], Guo et al. [8], Du et al. [5] and Kang et al. [18] also considered third-order problems. Graef and Yang [7] and Wong [21] considered three-point focal problems, while Palamides and Smyrlis [19] considered the three-point boundary conditions

$$u'''(t)=a(t)f(t,u(t)), \quad x(0)=x''(\eta)=x(1)=0.$$

In all these papers, in order to obtain a positive solution, the corresponding Green's function was assumed positive. In the present paper, mainly motivated by Sun [20] and Anderson et al. [2], we study the singular BVP

$$\begin{gathered} u'''(t)=-a(t)f(t,u(t)), \quad 0<t<1, \\ u(0)=u'(1)=u''(\eta)=0, \quad 0\le\eta<1/2. \end{gathered} \tag{1.3}$$

More precisely the corresponding Green's function $G(t,s)$ is constructed , *which is not a definite sign function for* $(t,s)\in[0,1]\times[0,1]$. The solution $u(t)=\int_0^1 G(t,s)a(s)f(s,u(s))ds$ of (1.3), may still be positive; i.e., if its initial values $u'(0)$ and $u''(0)$ are positive. This observation is based on an analysis of the corresponding vector field on the phase-plane (u',u''), proposed by Palamides and Smyrlis in [19] and in some references therein.

However, it is worth noticing that a positive and increasing solution is obtained. Our approach is based on the well-known Krasnosel'skii's fixed point theorem applied on a new cone. The choice of this cone is devised by the solution's properties of the BVP (1.3), whenever the nonlinearity is constant. We also note that, in contrast to the usual case where for similar problems $\eta\in(1/2,1)$ (see [20]), in our case $\eta\in[0,1/2)$.

2. Preliminaries

Consider the third-order nonlinear singular boundary-value problem (1.3), where we assume that $\eta\in[0,1/2)$, the continuous function $\alpha(t)$, $t\in(0,1)$ is nonnegative and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$.

Then, a vector field with crucial properties for our study is defined. More precisely, considering the (u',u'') phase semi-plane $(u'>0)$, we easily check that, if

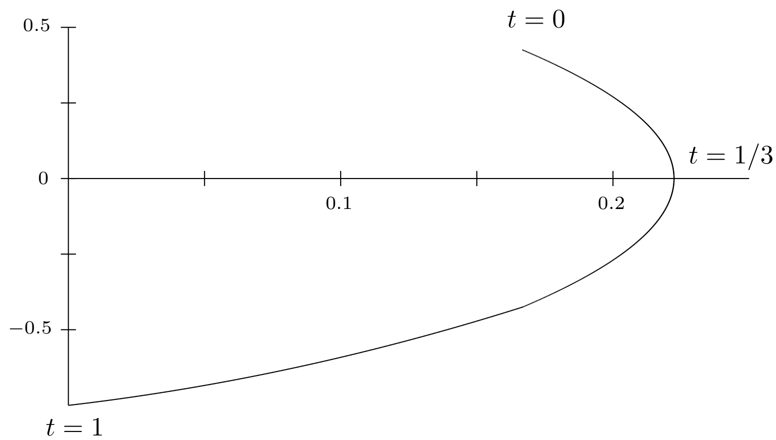

FIGURE 1. The (u', u'') phase space

$u(t) \geq 0$, then $u'''(t) = -\alpha(t)f(t, u(t)) \leq 0$. Thus, any trajectory $(u'(t), u''(t))$, $t \geq 0$, emanating from any point in the fourth quadrant

$$\{(u', u'') : u' \geq 0,\ u'' \geq 0\}$$

"evolutes" in a natural way, when $u''(t) > 0$, towards the positive u'−semi-axis and then, when $u''(t) \leq 0$ "evolutes" towards the negative u''-semi-axis. By assuming a certain growth rate on f (e.g. a sublinearity), we can control the vector field in a way that assures the existence of a trajectory satisfying the boundary conditions of (1.3). These properties, which from now on will be referred as *the nature of the vector field*, combined with the Krasnosel'skii's fixed point principle, are the main tools that we will employ in our study. The above thoughts are illustrated in Fig. 1, where the graph of the solution of the BVP (1.3), where $\eta = 1/3$ and $a(t)f(t, u(t)) = 1$, $0 \leq t \leq 1$, is presented.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set K_0 is called *a cone* of E if it satisfies the following conditions

(1) $x \in K_0$, $\lambda \geq 0$ imply $\lambda x \in K_0$;
(2) $x \in K_0$, $-x \in K_0$ imply $x = 0$.

Consider the Banach space $C[0, 1]$ equipped with the norm

$$\|y\| = \max\{|y(t)| : 0 \leq t \leq 1\}$$

and let

$$K_0 = \{y \in C[0, 1] : y(t) \geq 0,\ y'(t) \geq 0,\ t \in [0, 1];\ y''(t) \leq 0,\ t \in [\eta, 1]\},$$

where $C[0, 1]$ denotes the family of continuous functions. It is obvious that K_0 is a cone in $C[0, 1]$.

Consider now the homogeneous third-order nonlinear singular boundary-value problem,

$$\begin{gathered} u'''(t) = 0, \quad 0 \leq t \leq 1 \\ u(0) = u'(1) = u''(\eta) = 0, \end{gathered} \tag{2.1}$$

Lemmma 2.2. *The boundary value problem* (2.1) *has only the trivial solution*

The proof is trivial and is omitted. Now consider also the BVP

$$\begin{gathered} u'''(t) = -y(t), \quad 0 \le t \le 1, \\ u(0) = u'(1) = u''(\eta) = 0 \end{gathered} \tag{2.2}$$

and let its Green's function be

$$\text{for } s > \eta, \quad G(t,s) = \begin{cases} t(1-s), & t \le s \\ t - \frac{t^2}{2} - \frac{s^2}{2}, & t \ge s \end{cases}$$

$$\text{for } s \le \eta, \quad G(t,s) = \begin{cases} \frac{t^2}{2} - ts, & t \le s \\ -\frac{s^2}{2}, & s \le t. \end{cases}$$

Then for $s \ge \eta$,

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} 1-s, & t \le s \\ 1-t, & t \ge s \end{cases} \qquad \frac{\partial^2}{\partial t^2} G(t,s) = \begin{cases} 0, & t \le s \\ -1, & t \ge s \end{cases}$$

and for $s \le \eta$,

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} t-s, & t \le s \\ 0, & s \le t, \end{cases} \qquad \frac{\partial^2}{\partial t^2} G(t,s) = \begin{cases} 1, & t \le s \\ 0, & s \le t. \end{cases}$$

Thus we obtain

$$\begin{gathered} G(t,s) \le 0 \quad \text{and} \quad \frac{\partial}{\partial t} G(t,s) \le 0 \quad \text{for } 0 \le s \le \eta, \\ G(t,s) \ge 0 \quad \text{and} \quad \frac{\partial}{\partial t} G(t,s) \ge 0, \quad \text{for } \eta \le s \le 1. \end{gathered}$$

Also for $s \ge \eta$, we have

$$\max G(t,s) = G(1,s) = \begin{cases} 1-s \le 1-\eta, & t \le s \\ \frac{1-s^2}{2} \le \frac{1-\eta^2}{2} \le 1-\eta, & t \ge s \end{cases}$$

and for $s \le \eta$, we have

$$\max |G(t,s)| = -\min G(t,s) = -G(0,s) = \begin{cases} 0, & t \le s \\ \frac{s^2}{2} \le \frac{\eta^2}{2}, & s \le t. \end{cases}$$

Consequently,

$$|G(t,s)| \le \max\{1-\eta, \frac{\eta^2}{2}\} = 1-\eta, \quad (t,s) \in [0,1] \times [0,1]. \tag{2.3}$$

Remark 2.3. Consider the special case $y(t) = 1$, $0 \le t \le 1$. Then the BVP

$$\begin{gathered} u'''(t) = -1, \quad 0 \le t \le 1, \\ u(0) = u'(1) = u''(\eta) = 0 \end{gathered} \tag{2.4}$$

admits the unique solution

$$u(t) = \int_0^1 G(t,s) ds.$$

Indeed, we may proceed by cases on the two branches of the above Green's function.

- For $t \leq \eta$,

$$u(t) = -\int_0^t \frac{s^2}{2} ds + \int_t^\eta (\frac{t^2}{2} - st) ds - \int_\eta^1 t(s-1) ds = \frac{1}{2}t - t\eta - \frac{1}{6}t^3 + \frac{1}{2}t^2\eta$$

$$u'(t) = -\int_0^1 \frac{\partial}{\partial t} G(t,s) ds = \frac{d}{dt}(\frac{1}{2}t - t\eta - \frac{1}{6}t^3 + \frac{1}{2}t^2\eta) = t\eta - \eta - \frac{1}{2}t^2 + \frac{1}{2}$$

$$u''(t) = -\int_0^1 \frac{\partial^2}{\partial t^2} G(t,s) ds = \frac{d}{dt}(t\eta - \eta - \frac{1}{2}t^2 + \frac{1}{2}) = \eta - t$$

- For $\eta \leq t$,

$$u(t) = -\int_0^\eta \frac{s^2}{2} ds + \int_\eta^t (\frac{t^2}{2} - t + \frac{s^2}{2}) ds + \int_t^1 t(s-1) ds = \frac{1}{2}t - t\eta - \frac{1}{6}t^3 + \frac{1}{2}t^2\eta.$$

Hence we obtain

$$u(0) = u'(1) = u''(\eta) = 0, \quad u'''(t) = -1.$$

Consider now the unique solution $u = u(t)$, $0 \leq t \leq 1$ of (2.4). Then, recalling that

$$u(t) = \frac{1}{2}t - t\eta - \frac{1}{6}t^3 + \frac{1}{2}t^2\eta, \quad 0 \leq t \leq 1,$$

it is not difficult to show that $u(t) \geq 0$ and $0 \leq t \leq 1$, since $\eta \in (0, 1/2)$. Indeed,

$$u(t) \geq 0 \Leftrightarrow \phi(t) = t^2 - 3\eta t + (6\eta - 3) \leq 0.$$

The fact that $\phi(t)$ is decreasing on $[0, \frac{3}{2}\eta]$ and increasing on $[\frac{3}{2}\eta, 1]$, yields $\phi(0) \leq 0$ for $\eta \in [0, 1/2]$ and $\phi(1) \leq 0$ for $\eta \in [0, 2/3]$; that is $\phi(t) \leq 0$ or $u(t) \geq 0$, $t \in [0, 1]$.

For example, if $\eta = 1/3$, then $u(t) = \frac{1}{6}t - \frac{1}{6}t^3 + \frac{1}{6}t^2 > 0$, $0 < t \leq 1$, and its graph

$$Gr(u) = \{(u'(t), u''(t)) = (\frac{1}{3}t - \frac{1}{2}t^2 + \frac{1}{6}, \frac{1}{3} - t),\ 0 \leq t \leq 1\}$$

on the phase-plane is presented in Fig. 1.

On the other hand, for $\eta = 2/3$,

$$u(t) = -\frac{1}{6}t - \frac{1}{2}t^3 + \frac{1}{3}t^2$$

and its graph at the phase-plane (u', u'') is presented in Fig. 2. In this case, we notice that $u(t) \leq 0$, $u'(t) \leq 0$ for $0 \leq t \leq 1$, and $u''(t) \leq 0$ for $2/3 \leq t \leq 1$.

Finally for $\eta = 1/2$, we have the limited "semi-periodic" solution $u = u(t) = -\frac{1}{6}t^3 + \frac{1}{4}t^2 \geq 0$, $0 \leq t \leq 1$, in the sense that $u'(0) = u'(1) = 0$ (its graph is represented by the thin curve, in Figure 2).

The next result is very useful

Lemmma 2.4. *Let $y \in K_0$. Then, the BVP* (2.2) *admits the unique solution*

$$u(t) = \int_0^1 G(t,s) y(s)\, ds \in K_0$$

in K_0, which is monotonic.

Proof. By the definition of the kernel $G(t,s)$ and the fact that $y \in K_0$ and $\eta \in [0, 1/2)$, we obtain

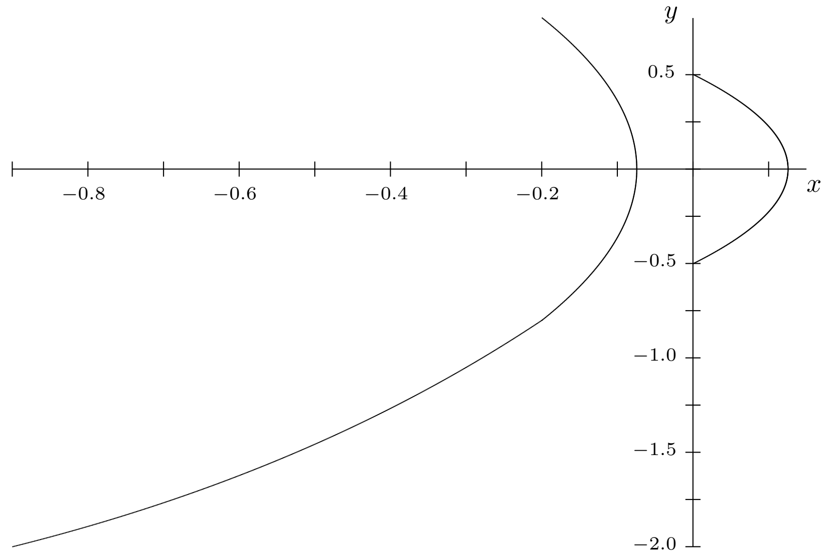

FIGURE 2. Semiperiodic solution (to the right)

- If $0 \le t \le \eta$,

$$\begin{aligned} u'(t) &= \int_0^t 0y(s)ds + \int_t^\eta (t-s)y(s)ds + \int_\eta^1 (1-s)y(s)ds \\ &\ge \max_{t\le s\le\eta} y(s)\int_t^\eta (t-s)ds + \min_{\eta\le s\le 1} y(s)\int_\eta^1 (1-s)ds \\ &= \max_{t\le s\le\eta} y(s)(t\eta - \frac{1}{2}t^2 - \frac{1}{2}\eta^2) + \min_{\eta\le s\le 1} y(s)(\frac{1}{2}\eta^2 - \eta + \frac{1}{2}) \\ &= y(\eta)[t\eta - \eta - \frac{1}{2}t^2 + \frac{1}{2}] \\ &= y(\eta)(t-1)(\eta - \frac{t+1}{2}) \ge 0 \end{aligned}$$

- If $\eta \le t \le 1$,

$$\begin{aligned} u'(t) &= \int_0^\eta 0y(s)ds + \int_\eta^t (1-t)y(s)ds + \int_t^1 t(1-s)y(s)ds \\ &\ge \min_{t\le s\le 1} y(s)[\int_\eta^t (1-t)ds + \int_t^1 t(1-s)ds] \\ &= \min_{t\le s\le 1} y(s)[\frac{3}{2}t - \eta + t\eta - 2t^2 + \frac{1}{2}t^3] \\ &\ge \frac{1}{2}y(\eta)(1-t)(-t^2+3t-2\eta) \ge 0. \end{aligned}$$

Obviously $u(0) = 0$. This results $u(t) \ge 0$, $0 \le t \le 1$. Moreover

$$u''(t) = \int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)y(s)ds = \begin{cases} \int_t^\eta y(s)ds \ge (\eta - t)y(0) \ge 0, & 0 \le t \le \eta, \\ -\int_\eta^t y(s)ds \le (\eta - t)y(\eta) \le 0, & \eta \le t \le 1 \end{cases}$$

and
$$u'''(t) = -y(t), \quad 0 \le t \le 1.$$
Thus, we obtain $u \in K_0$. $\square$

The solution of BVP
$$y'''(x) = -9y(x), \quad 0 \le x \le 1,$$
$$y(0) = y'(1) = y''(\eta) = 0,$$
can be approximated numerically by using the NDSolve command of the software package Mathematica and applying the shooting method. For the initial values
$$y[0] = 0, \quad y'[0] = 1.3, \quad y''[0] = 0.8,$$
we obtain the next plot (Figure 3) of the functions $y(t)$, $y'(t)$ and $y''(t)$, $0 \le t \le 1$.

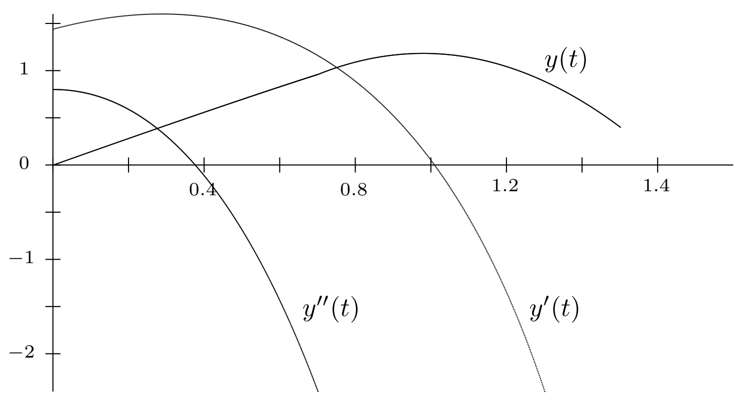

Figure 3. Graph of the solution and its derivatives

Note that these graphs yield "good" approximating relations $y(0) = 0$, $y(t) > 0$, $y'(t) > 0$, $y'(1) \simeq 0$, for $0 < t < 1$, and $y''(0.413) \simeq 0$. This is in agreement with our theoretical approach.

For the interested reader, we present the Mathematica commands:
NDSolve[$\{y'''[x] + 9y[x] == 0, y[0] == 0, y'[0] == 1.3, y''[0] == 0.8\}, y, \{x, 0, 1\}$]
Plot[Evaluate[$\{y[x], y'[x], y''[x]\}$ /. %],{x, 0, 1.4}]

In the same manner, the command
NDSolve[$\{y'''[x] + 9y[x] == 0, y[0] == 0, y'[0] == 1.3, y''[0] == 0.8\}, y, \{x, 0, 1\}$]
ParametricPlot[Evaluate[$\{y'[x], y''[x]\}$] /.%, $\{x, 0, 1\}$, PlotRange − > All],
yields a graph similar to the one in Fig. 1

Lemmma 2.5. *For any $y \in K_0$, the unique solution $u(t)$ of (E_y) belongs also to the cone K_0 and furthermore it satisfies*
$$\min_{t\in[\theta,1-\theta]} u(t) \ge \frac{(\theta-\eta)}{1-\eta}\|u\|,$$
where $\theta \in (\eta, 1/2)$ is arbitrary.

Proof. Taking into account that $u \in K_0$, (Lemma 2.4), we obtain $u(t) \ge 0$, $0 \le t \le 1$ and
$$u''(t) \le 0, \quad \eta \le t \le 1.$$

Hence, the function $u = u(t) \le 0$, $\eta \le t \le 1$ is concave. As a result, for any $t_1, t_2 \in [\eta, 1]$ and $\lambda \in [0, 1]$,

$$u(\lambda t_1 + (1-\lambda)t_2) \ge \lambda u(t_1) + (1-\lambda)u(t_2).$$

Moreover, the fact that the function $u = u(t) \ge 0$, $0 \le t \le 1$ is increasing, implies that $\|u\| = u(1)$. Therefore

$$\frac{u(1)-u(\eta)}{1-\eta} \le \frac{u(t)-u(\eta)}{t-\eta}, \quad t \in [\eta, 1],$$

that is

$$u(t) \ge \frac{t-\eta}{1-\eta}u(1) = \frac{t-\eta}{1-\eta}\|u\|, \quad t \in [\eta, 1].$$

Consequently,

$$\min_{t\in[\theta,1-\theta]} u(t) = u(\theta) \ge \theta^* \|u\|,$$

where $\theta^* = \frac{\theta-\eta}{1-\eta}$. □

3. Main Results

Consider the boundary-value problem

$$\begin{gathered} u'''(t) = -a(t)f(s, u(s)), \quad 0 < t < 1, \\ u(0) = u'(1) = u''(\eta) = 0, \end{gathered} \tag{3.1}$$

where the next two conditions are assumed

- (H1) $a \in C((0,1), (0, +\infty))$ and $0 < \int_\eta^{1-\eta} a(s)ds \le \int_0^1 a(s)ds < +\infty$;
- (H2) $f \in C([0,1] \times [0, +\infty), [0, +\infty))$.

We define the cone

$$K = \{u \in K_0 : y(0) = 0, \min_{t\in[\theta,1-\theta]} u(t) \ge \theta^* \|u\|\}$$

and the operator

$$\mathcal{T}u(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds.$$

By Lemmas 2.4 and 2.5, the BVP (3.1) has a positive solution $u = u(t)$, if and only if u is a fixed point of $\mathcal{T}$ in K.

Lemmma 3.1. *Assume that conditions* (H1)–(H2) *hold. Then, for each $u \in K_0$, the map*

$$t \mapsto \int_0^1 G(t,s)a(s)f(s,u(s))ds$$

has image in the cone K, that is $\mathcal{T}u \in K$.

Proof. Let u be a function in K_0. It is a straightforward consequence of (H1)-(H2) that the function $y(s) = a(s)f(s, u(s))$, $s \in (0, 1)$, is positive and continuous. Hence, $y \in K_0$. Moreover, Lemma 2.4 implies that the map $t \mapsto \int_0^1 G(t,s)a(s)f(s,u(s))\,ds$ is continuous and the integrand $y(s) \in C(0,1)$. Thus, the conclusion of Lemma 3.1 follows immediately from Lemmas 2.4 and 2.5. □

We set

$$A = (1-\eta)\int_0^1 a(s)ds, \quad B = \theta(1-\theta)\int_\theta^{1-\theta} a(s)ds.$$

which are positive because $0 \le \eta < \theta < 1/2$. Moreover, recalling that an operator

$$\mathcal{T} : K \to C([0,1])$$

is called *completely continuous* if it is continuous and maps bounded sets into precompact sets, we state the next well-known result.

Proposition 3.2. *Assume that* (H1)–(H2) *hold. Then $\mathcal{T} : K \to K$ is completely continuous.*

Proof. It is sufficient to show that $\mathcal{T}(K) \subset K$, which follows directly from Lemma 3.1, since $K \subset K_0$. □

We will employ the following fixed point theorem due to Krasnosel'skii [12].

Thmeorem 3.3. *Let E be a Banach space, $P \subseteq E$ a cone, and assume that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathcal{T} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either one of the following two conditions is satisfied*

(i) $\|\mathcal{T}u\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|\mathcal{T}u\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|\mathcal{T}u\| \le \|u\|$, $u \in P \cap \partial\Omega_2$.

Then, $\mathcal{T}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Next, we will prove the existence of at least one positive and increasing solution of the BVP (1.3).

Thmeorem 3.4. *Assume that* (H1)–(H2) *hold and that there exist positive constants $r \ne R$ such that*

A1) $f(t,x) \le \frac{r}{A}$ *for* $(t,x) \in [0,1] \times [0,r]$;

(A2) $f(t,x) \ge \frac{R}{B}$ *for* $(t,x) \in [0,1] \times [\theta^* R, R]$.

Then, the boundary-value problem (3.1) *admits a positive strictly increasing solution $u = u(t)$, $0 \le t \le 1$, where*

$$\min\{r,R\} \le \|u\| \le \max\{r,R\}.$$

Moreover, the obtained solution $u = u(t)$ is convex on the interval $[0,\eta]$ and concave for $\eta \le t \le 1$.

Proof. Assuming first that $r < R$, we consider the open balls

$$\Omega_1 = \{u \in C([0,1]) : \|u\| < r\} \quad \text{and} \quad \Omega_2 = \{u \in C([0,1]) : \|u\| < R\}.$$

Let $u \in K \cap \partial\Omega_1$ be any function. From (2.3) and the sign of nonlinearity, the assumption (A1) yields

$$\begin{aligned}\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)a(s)f(s,u(s))ds\Big| = \int_0^1 \max_{0\le t\le 1} G(t,s)a(s)f(s,u(s))ds \\ &\le (1-\eta)\int_0^1 a(s)f(s,u(s))ds \le (1-\eta)\int_0^1 a(s)\frac{r}{A}ds = r = \|u\|.\end{aligned}$$

Therefore, the first part of assumption (i) in Theorem 3.3 is fulfilled.

Similarly, for every $u \in K \cap \partial\Omega_2$, Lemmas 2.5-3.1 yield $\theta^* R \le u(s) \le R$, $\theta \le s \le 1-\theta$.

Taking into account that both functions $t \mapsto \int_0^1 G(t,s)a(s)f(s,u(s))\,ds$ and $G(.,s)$, $\theta \le s \le 1-\theta$, are positive and the second is also increasing, for $t \in [\eta, 1]$, assumption (A2) implies that

$$
\begin{aligned}
\|\mathcal{T}u\| &= \max_{0\le t\le 1} |\int_0^1 G(t,s)a(s)f(s,u(s))ds| \\
&= \max_{0\le t\le 1} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
&\ge \max_{0\le t\le 1} \int_\theta^{1-\theta} G(t,s)a(s)f(s,u(s))ds \\
&\ge \max_{\theta\le t\le 1-\theta} \int_\theta^{1-\theta} G(t,s)a(s)f(s,u(s))ds \\
&= \int_\theta^{1-\theta} G(1-\theta,s)a(s)f(s,u(s))ds \\
&= \int_\theta^{1-\theta} \Big(1-\theta - \frac{(1-\theta)^2}{2} - \frac{s^2}{2}\Big)a(s)f(s,u(s))ds \\
&= \frac{1}{2}\int_\theta^{1-\theta} (1-\theta^2 - s^2)a(s)f(s,u(s))ds \\
&\ge \frac{1}{2}\int_\theta^{1-\theta} (1-\theta^2 - (1-\theta)^2)a(s)f(s,u(s))ds \\
&= \theta(1-\theta)\int_\theta^{1-\theta} a(s)f(s,u(s))ds \\
&\ge \theta(1-\theta)\int_\theta^{1-\theta} a(s)\frac{R}{B}ds = R = \|u\|.
\end{aligned}
$$

Therefore,

$$
\|\mathcal{T}u\| \ge \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2.
$$

Finally, we may apply Theorem 3.3, to obtain a positive solution $u = u(t)$, $0 \le t \le 1$, of the BVP (3.1). The definition of $K \subset K_0$ and the fact that $u \in K_0$, gives that:

$$
u'(t) \ge 0, \quad 0 \le t \le 1,
$$

that is $u(t)$ is a positive and strictly increasing solution. On the other hand, since $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, it is obvious that

$$
r \le \|u\| \le R.
$$

Assuming now that $r > R$, consider the open balls

$$
\Omega_1 = \{u \in C([0,1]) : \|u\| < R\} \quad \text{and} \quad \Omega_2 = \{u \in C([0,1]) : \|u\| < r\}.
$$

Then, if $u \in K \cap \partial\Omega_1$, by Lemma 2.5, we obtain

$$
\min_{t\in[\theta,1-\theta]} u(t) \ge \frac{(\theta-\eta)}{1-\eta}\|u\| = \theta^*\|u\| = \theta^* R.
$$

On the other hand, assumption (A2) gives

$$
\begin{aligned}
\|\mathcal{T}u\| &= \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)a(s)f(s,u(s))ds\Big| \\
&= \max_{0\le t\le 1} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
&\ge \max_{0\le t\le 1} \int_\theta^{1-\theta} G(t,s)a(s)f(s,u(s))ds \\
&\ge \max_{\theta\le t\le 1-\theta} \int_\theta^{1-\theta} G(t,s)a(s)f(s,u(s))ds \\
&= \int_\theta^{1-\theta} G(1-\theta,s)a(s)f(s,u(s))ds \\
&\ge \theta(1-\theta)\int_\theta^{1-\theta} a(s)\frac{R}{B}ds = R = \|u\|,
\end{aligned}
$$

Working similarly, if $u \in K\cap\partial\Omega_2$, then $0 \le u(s) \le r$, $0 \le s \le 1$. Thus (A1) implies

$$
\begin{aligned}
\|\mathcal{T}u\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)a(s)f(s,u(s))ds \\
&\le (1-\eta)\int_0^1 a(s)\frac{r}{A}ds = r = \|u\|.
\end{aligned}
$$

Therefore, it is clear that the existence result holds. □

Corollary 3.5. *Assume that* (H1)–(H2) *hold and*

(A3) *The nonlinearity is superlinear at both points $x = 0$ and $t = +\infty$; i.e.,*

$$\lim_{x\to 0+}\max_{0\le t\le 1}\frac{f(t,x)}{x} = 0+ \quad \text{and} \quad \lim_{x\to+\infty}\min_{0\le t\le 1}\frac{f(t,x)}{x} = +\infty;$$

or

(A4) *The nonlinearity is sublinear at both points $x = 0$ and $x = +\infty$; i.e.,*

$$\lim_{x\to 0+}\min_{0\le t\le 1}\frac{f(t,x)}{x} = +\infty \quad \text{and} \quad \lim_{x\to+\infty}\max_{0\le t\le 1}\frac{f(t,x)}{x} = 0+$$

Then, the boundary value problem (3.1) *admits a positive, strictly increasing, convex on the interval $[0,\eta]$ and concave for $[\eta,1]$ solution $u = u(t)$, $0 \le t \le 1$.*

Proof. The superlinearity of f ensures the existence of an $r > 0$, such that $\frac{f(t,x)}{x} \le \frac{1}{A}$, for all $(t,x) \in [0,1]\times[0,r]$, This yields assumption (A1) of Theorem 3.4. Similarly taking into account the superlinearity at $+\infty$, we get an $R > r$ such that $\frac{f(t,x)}{x} \ge \frac{1}{\theta^* B}$, for all $(t,x) \in [0,1]\times[\theta^* R, R]$. Hence, Theorem 3.4 can be applied.

On the other hand, when the nonlinearity is sublinear,

- If f is bounded, say by $M > 0$, we may choose any $R \ge AM$. Thus,

$$
\begin{aligned}
\|\mathcal{T}u\| &\le \max_{0\le t\le 1} \Big|\int_0^1 G(t,s)a(s)f(s,u(s))ds\Big| \\
&\le (1-\eta)\int_0^1 a(s)f(s,u(s))ds \\
&= MA \le R = \|u\|, \quad \text{for } u \in K \text{ with } \|u\| = R
\end{aligned}
\tag{3.2}
$$

• If f is unbounded, let R be positive and large enough such that

$$\frac{f(t,R)}{R} \leq \frac{1}{A}, \quad f(t,u) \leq f(t,R), \quad \text{for } (t,u) \in [0,1] \times [0,R].$$

Then

$$f(t,u) \leq f(t,R) \leq \frac{R}{A}, \quad (t,u) \in [0,1] \times [0,R].$$

Consequently,

$$\begin{aligned} \|\mathcal{T}u\| &\leq \max_{0\leq t\leq 1} |\int_0^1 G(t,s)a(s)f(s,u(s))ds| \\ &\leq (1-\eta)\int_0^1 a(s)f(s,u(s))ds \\ &\leq \frac{R}{A}A = \|u\| \quad \text{for } u \in K \text{ with } \|u\| = R. \end{aligned}$$

We know that $u \in K$, where $\|u\| = r$ implies $r \geq u(s) \geq \theta^*\|u\| = \theta^* r$, $\theta \leq s \leq 1-\theta$. Moreover, by the sublinearity of f at $u = 0$, there exists an $r < R$ such that for any $u \in K$ where $\|u\| = r$.

$$f(s,u(s)) \geq \frac{u(s)}{\theta(1-\theta)B\theta^*} \geq \frac{\theta^* r}{\theta(1-\theta)B\theta^*}, \quad (s,u(s)) \in [0,1] \times [\theta^* r, r].$$

Hence, for any $u \in K$ where $\|u\| = r$, we similarly get

$$\begin{aligned} \|\mathcal{T}u\| &= \max_{0\leq t\leq 1} |\int_0^1 G(t,s)a(s)f(s,u(s))ds| \\ &\geq \max_{\theta\leq t\leq 1-\theta} \int_\theta^{1-\theta} G(t,s)a(s)f(s,u(s))ds \\ &= \int_\theta^{1-\theta} G(1-\theta,s)a(s)f(s,u(s))ds \\ &\geq \theta(1-\theta)\int_\theta^{1-\theta} a(s)f(s,u(s))ds \\ &\geq \theta(1-\theta)\int_\theta^{1-\theta} a(s)\frac{r}{B}ds = r = \|u\|, \end{aligned}$$

and this clearly completes the proof. □

Example 3.6. Consider the boundary-value problem

$$\begin{gathered} u'''(t) = -\frac{1}{\sqrt{t}}\sqrt[3]{u(t)+t}, \quad 0 < t \leq 1 \\ u(0) = u'(1) = u''(\eta) = 0. \end{gathered}$$

The function $a(t) = \frac{1}{\sqrt{t}}$ is integrable on $[0,1]$ and the nonlinearity $f(t,u) = -\sqrt[3]{u+t}$ sublinear. Hence, Corollary 3.5 guarantees the existence of a positive increasing solution of the above BVP.

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