Electronic Journal of Differential Equations, Vol. 2006(2006), No. 41, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

POSITIVE SOLUTIONS FOR A FUNCTIONAL DELAY SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

CHUANZHI BAI, XINYA XU

Abstract. We establish criteria for the existence of positive solutions to the three-point boundary-value problems expressed by second-order functional delay differential equations of the form

$$\begin{gathered} -x''(t) = f(t, x(t), x(t-\tau), x_t), \quad 0 < t < 1, \\ x_0 = \phi, \quad x(1) = x(\eta), \end{gathered}$$

where $\phi \in C[-\tau, 0]$, $0 < \tau < 1/4$, and $\tau < \eta < 1$.

1. Introduction

In recent years, many authors have paid attention to the research of boundary-value problems for functional differential equations because of its potential applications (see, for example, [1, 3, 5, 6, 7, 8, 9, 10, 11]). In a recent paper [7], by applying a fixed-point index theorem in cones, Jiang studied the existence of multiple positive solutions for the boundary-value problems of second-order delay differential equation

$$\begin{gathered} y''(x) + f(x, y(x-\tau)) = 0, \quad 0 < x < 1, \\ y(x) = 0, \quad -\tau \le x \le 0, \; y(1) = 0, \end{gathered} \tag{1.1}$$

where $0 < \tau < 1/4$ and $f \in C([0,1] \times [0, +\infty), [0, \infty))$.

For $\tau > 0$, let $C(J)$ be the Banach space of all continuous functions $\psi : [-\tau, 0] =: J \to \mathbb{R}$ endowed with the sup-norm

$$\|\psi\|_J := \sup\{|\psi(s)| : s \in J\}.$$

For any continuous function x defined on the interval $[-\tau, 1]$ and any $t \in I =: [0, 1]$, the symbol x_t is used to denote the element of $C(J)$ defined by

$$x_t(s) = x(t + s), \quad s \in J.$$

Set

$$C^+(J) =: \{\psi \in C(J) : \psi(s) \ge 0, s \in J\}.$$

2000 *Mathematics Subject Classification.* 34K10, 34B10.
Key words and phrases. Functional delay differential equation; boundary value problem; positive solutions; fixed point index.

Submitted August 10, 2005. Published March 26, 2006.
Supported by grant 03KJD110056 from the Natural Science Foundation of Jiangsu Education Office and Jiangsu Planned Projects for Postdoctoral Research Funds.

In this paper, motivated and inspired by [1, 4, 7, 8], we apply a fixed point theorem in cones to investigate the existence of positive solutions for three point boundary-value problems of second-order functional delay differential equation

$$\begin{gathered} -x''(t) = f(t, x(t), x(t-\tau), x_t), \quad 0 < t < 1, \\ x_0 = \phi, \quad x(1) = x(\eta), \end{gathered} \tag{1.2}$$

where $0 < \tau < 1/4$, $\tau < \eta < 1$, $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times C^+(J) \to \mathbb{R}^+$ is a continuous function, and ϕ is an element of the space

$$C_0^+(J) =: \{\psi \in C^+(J) : \psi(0) = 0\}.$$

We need the following well-known lemma. See [2] for a proof and further discussion of the fixed-point index $i(A, K_r, K)$.

Lemma 1.1. *Assume that E is a Banach space, and $K \subset E$ is a cone in E. Let $K_r = \{x \in K : \|u\| < r\}$. Furthermore, assume that $A : K \to K$ is a compact map, and $Ax \neq x$ for $x \in \partial K_r = \{x \in K; \|x\| = r\}$. Then, one has the following conclusions.*

(1) *If $\|x\| \le \|Ax\|$ for $x \in \partial K_r$, then $i(A, K_r, K) = 0$.*
(2) *If $\|x\| \ge \|Ax\|$ for $x \in \partial K_r$, then $i(A, K_r, K) = 1$.*

2. Preliminaries and some lemmas

In the sequel we shall denote by $C_0(I)$ the space all continuous functions $x : I \to \mathbb{R}$ with $x(0) = 0$. This is a Banach space when it is furnished with usual sup-norm

$$\|x\|_I := \sup\{|x(s)| : s \in I\}.$$

We set

$$C_0^+(I) := \{x \in C_0(I) : x(t) \ge 0, t \in I\}.$$

For each $\phi \in C_0^+(J)$ and $x \in C_0^+(I)$ we define

$$x_t(s; \phi) := \begin{cases} \phi(t+s), & t+s \le 0,\ t \in I,\ s \in J, \\ x(t+s), & 0 \le t+s \le 1,\ t \in I,\ s \in J, \end{cases}$$

and observe that $x_t(\cdot; \phi) \in C^+(J)$.

It is easy to check that $\varphi_1(t) = \sin \frac{\pi}{\eta+1} t$ is the eigenfunction related to the smallest eigenvalue $\lambda_1 = \frac{\pi^2}{(\eta+1)^2}$ of the eigenproblem

$$-x'' = \lambda x, \quad x(0) = 0, \quad x(1) = x(\eta).$$

By [4], the Green's function for the three-point boundary-value problem

$$-x'' = 0, \quad x(0) = 0, \quad x(1) = x(\eta),$$

is given by

$$G(t, s) = \begin{cases} t, & t \le s \le \eta, \\ s, & s \le t \text{ and } s \le \eta, \\ \frac{1-s}{1-\eta} t, & t \le s \text{ and } s \ge \eta, \\ s + \frac{\eta - s}{1-\eta} t, & \eta \le s \le t. \end{cases}$$

Lemma 2.1. *Suppose that $G(t, s)$ is defined as above. Then we have the following results:*

(1) $0 \le G(t, s) \le G(s, s), \quad 0 \le t, s \le 1,$

(2) $G(t,s) \ge \eta t G(s,s), \quad 0 \le t, s \le 1.$

Proof. It is easy to see that (1) holds. To show that (2) holds, we distinguish four cases:

- If $t \le s \le \eta$, then
$$G(t,s) = t \ge \eta ts = \eta t G(s,s).$$
- If $s \le t$ and $s \le \eta$, then
$$G(t,s) = s \ge \eta ts = \eta t G(s,s).$$
- If $t \le s$ and $s \ge \eta$, then
$$G(t,s) = \frac{1-s}{1-\eta} t \ge \eta st \frac{1-s}{1-\eta} = \eta t G(s,s).$$
- Finally, if $\eta \le s \le t$, then
$$\begin{aligned} G(t,s) &= s - \frac{s-\eta}{1-\eta} t \ge s - \frac{s-\eta}{1-\eta} = \frac{\eta(1-s)}{1-\eta} \\ &\ge ts \frac{\eta(1-s)}{1-\eta} = \eta t \frac{s(1-s)}{1-\eta} = \eta t G(s,s). \end{aligned}$$

□

Remark 2.2. If $s \le \eta$ and $s \ge \eta$, then $G(s,s) = s$ and $G(s,s) = \frac{s(1-s)}{1-\eta}$, respectively.

For convenience, let
$$x(t;\phi) := \begin{cases} \phi(t), & -\tau \le t \le 0, \\ x(t), & 0 \le t \le 1. \end{cases}$$
Suppose that $x(t)$ is a solution of BVP (1.2), then it can be written as
$$x(t) = \int_0^1 G(t,s) f(s, x(s), x(s-\tau;\phi), x_s(\cdot;\phi)) ds, \quad t \in I.$$
Let $K \subset C_0(I)$ be a cone defined by
$$K = \{x \in C_0^+(I) : x(t) \ge \eta t \|x\|_I, \ \forall t \in I\}.$$
For each $x \in K$ and $t \in I$, we have
$$\begin{aligned} \|x_t(\cdot;\phi)\|_J &= \sup_{s \in [-\tau,0]} |x_t(s;\phi)| \\ &= \max \left\{ \begin{array}{ll} \sup_{s\in[-\tau,0]} |x(t+s)|, & \text{if } t+s \in I, \\ \sup_{s\in[-\tau,0]} |\phi(t+s)|, & \text{if } t+s \le 0 \end{array} \right\} \\ &\le \max\{\|x\|_I, \|\phi\|_J\}, \end{aligned} \tag{2.1}$$
and
$$\|x_t(\cdot;\phi)\|_J \ge \sup_{s\in[-\tau,0]} \{x(t+s) : t+s \in I\} \ge x(t) \ge \eta t \|x\|_I. \tag{2.2}$$
Define an operator $A_\phi : K \to C_0(I)$ as follows:
$$(A_\phi x)(t) := \int_0^1 G(t,s) f(s, x(s), x(s-\tau;\phi), x_s(\cdot;\phi)) ds, \quad t \in I.$$
Firstly, we have the following result.

Lemma 2.3. $A_\phi(K) \subset K$.

Proof. For any $x \in K$, we observe that $(A_\phi x)(0) = 0$. By Lemma 2.1 (1), we have $(A_\phi x)(t) \geq 0$, $t \in I$. It follows from Lemma 2.1 (1) and (2) that

$$\begin{aligned}(A_\phi x)(t) &\geq \eta t \int_0^1 G(s,s) f(s, x(s), x(s-\tau;\phi), x_s(\cdot;\phi)) ds \\ &\geq \eta t \|A_\phi x\|_I, \quad t \in I.\end{aligned}$$

Thus, $A_\phi(K) \subset K$. □

Secondly, similar to the proof of Theorem 2.1 in [6], we get that

Lemma 2.4. *$A_\phi : K \to K$ is completely continuous.*

We formulate some conditions for $f(t,u,v,\psi)$ as follows which will play roles in this paper.

(H1) $\lim\inf_{u+v+\|\psi\|_J \to +\infty} \min_{t\in[0,1]} \dfrac{f(t,u,v,\psi)}{u+v+\|\psi\|_J} = \infty$.

(H2) $\lim\sup_{u+v+\|\psi\|_J \to +\infty} \max_{t\in[0,1]} \dfrac{f(t,u,v,\psi)}{u+v+\|\psi\|_J} = 0$.

(H3) $\lim\inf_{u+v+\|\psi\|_J \to +\infty} \min_{t\in[0,1]} \dfrac{f(t,u,v,\psi)}{u+v+\|\psi\|_J} > \dfrac{1}{3}\dfrac{\pi^2}{(\eta+1)^2}(1+M)$, where

$$M = \frac{\pi^2\tau(\eta-\tau) + 3\pi(1-\eta^2) + \pi\tau(\eta+1) + 3(\eta+1)^2}{\pi\eta(\eta+1)\left(\int_0^{\eta-\tau} t\sin\frac{\pi}{\eta+1}(t+\tau)dt + 2\int_0^\eta t\sin\frac{\pi}{\eta+1}tdt\right)}.$$

(H4) There is a $h_1 > 0$ such that $0 \leq u \leq h_1$, $0 \leq v \leq \max\{h_1, \|\phi\|_J\}$, $0 \leq \|\psi\|_J \leq \max\{h_1, \|\phi\|_J\}$, and $0 \leq t \leq 1$ implies

$$f(t,u,v,\psi) < \mu h_1, \quad \text{where } \mu = \left(\int_0^1 G(s,s)ds\right)^{-1} = \frac{6}{1+\eta+\eta^2}.$$

(H5) There is a $h_2 > 0$ such that $\frac{1}{4}\eta h_2 \leq u \leq h_2$, $(\frac{1}{4}-\tau)\eta h_2 \leq v \leq h_2$, $\frac{1}{4}\eta h_2 \leq \|\psi\|_J \leq h_2$, and $0 \leq t \leq 1$ implies

$$f(t,u,v,\psi) > bh_2, \quad \text{where } b = \left(\int_{1/4}^{3/4} G\left(\frac{1}{2}, t\right) dt\right)^{-1}.$$

In the following, we give some lemmas which will be used in this paper.

Lemma 2.5. *If* (H1) *is satisfied, then there exist* $0 < r_0 < \infty$ *such that*

$$i(A_\phi, K_r, K) = 0, \qquad r \geq r_0.$$

Proof. Choose $L > 0$ such that

$$\eta(\frac{3}{4}-\tau)L\int_{1/4}^{3/4} G(\frac{1}{2}, s)ds > 1.$$

For $u, v \geq 0$ and $\psi \in C^+(J)$, (H1) implies that there is $r_1 > 0$ such that

$$f(t,u,v,\psi) \geq L(u+v+\|\psi\|_J), \quad u+v+\|\psi\|_J \geq r_1, \quad 0 \leq t \leq 1. \tag{2.3}$$

Choose $r_0 > \frac{4r_1}{3(1-4\tau)\eta}$. For $x \in \partial K_r$, $r \geq r_0$, we have by the definition of K and (2.2) that

$$x(t-\tau) \geq \eta(t-\tau)\|x\|_I \geq \eta(\frac{1}{4}-\tau)r > \frac{1}{3}r_1, \quad \frac{1}{4} \leq t \leq \frac{3}{4},$$

$$\|x_t(\cdot;\phi)\|_J \geq x(t) \geq \eta t\|x\|_I \geq \frac{1}{4}\eta r > \frac{1}{3}r_1, \quad \frac{1}{4} \leq t \leq \frac{3}{4}.$$

So by (2.3), we have for such x,

$$\begin{aligned}
(A_\phi x)\big(\frac{1}{2}\big) &\geq \int_{1/4}^{3/4} G\big(\frac{1}{2},t\big)f(t,x(t),x(t-\tau;\phi),x_t(\cdot;\phi))dt \\
&= \int_{1/4}^{3/4} G\big(\frac{1}{2},t\big)f(t,x(t),x(t-\tau),x_t(\cdot;\phi))dt \\
&\geq L\int_{1/4}^{3/4} G\big(\frac{1}{2},t\big)[x(t)+x(t-\tau)+\|x_t(\cdot;\phi)\|_J]dt \\
&\geq \eta\big(\frac{3}{4}-\tau\big)Lr\int_{1/4}^{3/4} G\big(\frac{1}{2},t\big)dt > r = \|x\|_I.
\end{aligned}$$

This shows that

$$\|A_\phi x\|_I > \|x\|_I, \quad \forall x \in \partial K_r.$$

It is obvious that $A_\phi x \neq x$ for $x \in \partial K_r$. Therefore, by Lemma 1.1, we conclude that $i(A_\phi, K_r, K) = 0$. □

Lemma 2.6. *If* (H2) *is satisfied, then there exists* $0 < R_0 < \infty$ *such that*

$$i(A_\phi, K_R, K) = 1 \quad \text{for } R \geq R_0.$$

Proof. By (H2), for any $0 < \varepsilon < \frac{1}{3}\big(\int_0^1 G(s,s)ds\big)^{-1}$, $u, v \geq 0$ and $\psi \in C^+(J)$, there exists $R' > 0$ such that

$$f(t,u,v,\psi) \leq \varepsilon(u+v+\|\psi\|_J), \quad u+v+\|\psi\|_J \geq R', \quad 0 \leq t \leq 1.$$

Putting

$$C := \max_{0\leq t\leq 1}\ \max_{0\leq u,v,\ u+v+\|\psi\|_J\leq R'} |f(t,u,v,\psi)-\varepsilon(u+v+\|\psi\|_J)|+1,$$

then

$$f(t,u,v,\psi) \leq \varepsilon(u+v+\|\psi\|_J)+C, \quad for u,v \geq 0,\ \psi \in C^+(J),\ t \in I. \tag{2.4}$$

Choose

$$R_0 > (C+2\varepsilon\|\phi\|_J)\int_0^1 G(s,s)ds\Big/\Big(1-3\varepsilon\int_0^1 G(s,s)ds\Big).$$

Let $R \geq R_0$ and consider a point $x \in \partial K_R$. By the definition of $x(t;\phi)$, we get

$$x(s-\tau;\phi) \leq \max\{\|x\|_I, \|\phi\|_J\}, \quad \forall s \in I. \tag{2.5}$$

By (2.1), (2.4) and (2.5), for $x \in \partial K_R$, $R \geq R_0$, and $t \in I$,

$$\begin{aligned}
(A_\phi x)(t) &= \int_0^1 G(t,s)f(s,x(s),x(s-\tau;\phi),x_s(\cdot;\phi))ds \\
&\leq \int_0^1 G(s,s)f(s,x(s),x(s-\tau;\phi),x_s(\cdot;\phi))ds \\
&\leq \int_0^1 G(s,s)[\varepsilon(x(s)+x(s-\tau;\phi)+\|x_s(\cdot;\phi)\|_J)+C]ds \\
&\leq \int_0^1 G(s,s)[\varepsilon(\|x\|_I+2\max\{\|x\|_I,\|\phi\|_J\})+C]ds
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 G(s,s)[\varepsilon(3\|x\|_I + 2\|\phi\|_J) + C]ds \\
&= 3\varepsilon R \int_0^1 G(s,s)ds + (C + 2\varepsilon\|\phi\|_J) \int_0^1 G(s,s)ds \\
&< R = \|x\|_I.
\end{aligned}$$

Thus, $\|A_\phi x\|_I < \|x\|_I$ for $x \in \partial K_R$. Hence, by Lemma 1.1, $i(A_\phi, K_R, K) = 1$. $\square$

Lemma 2.7. *If* (H4) *is satisfied, then* $i(A_\phi, K_{h_1}, K) = 1$.

Proof. Let $x \in \partial K_{h_1}$, then we have by (2.1) and (2.5) that

$$0 \le x(t-\tau;\phi) \le \max\{h_1, \|\phi\|_J\}, \quad 0 \le t \le 1,$$

and

$$0 \le \|x_t(\cdot;\phi)\|_J \le \max\{h_1, \|\phi\|_J\}, \quad 0 \le t \le 1.$$

Thus, from (H4) we obtain

$$\begin{aligned}
(A_\phi x)(t) &\le \int_0^1 G(s,s) f(s, x(s), x(s-\tau;\phi), x_s(\cdot;\phi))ds \\
&< \mu h_1 \int_0^1 G(s,s)ds = h_1 = \|x\|_I, \quad 0 \le t \le 1.
\end{aligned}$$

This shows that

$$\|A_\phi x\|_I < \|x\|_I, \quad \forall x \in \partial K_{h_1}.$$

Hence, Lemma 1.1 implies $i(A_\phi, K_{h_1}, K) = 1$. $\square$

Lemma 2.8. *If* (H5) *is satisfied, then* $i(A_\phi, K_{h_2}, K) = 0$.

Proof. For $x \in \partial K_{h_2}$, we have

$$h_2 = \|x\|_I \ge x(t-\tau) \ge \eta(t-\tau)\|x\|_I \ge \eta\big(\frac{1}{4} - \tau\big)h_2, \quad \frac{1}{4} \le t \le \frac{3}{4},$$

and

$$h_2 = \|x\|_I \ge \sup_{s\in[-\tau,0]} x(t+s) = \|x_t(\cdot;\phi)\|_J \ge x(t) \ge \eta t\|x\|_I \ge \frac{1}{4}\eta h_2,$$

for $\frac{1}{4} \le t \le \frac{3}{4}$. It follows from (H5) that

$$\begin{aligned}
(A_\phi x)\big(\frac{1}{2}\big) &\ge \int_{1/4}^{3/4} G\big(\frac{1}{2}, t\big) f(t, x(t), x(t-\tau), x_t(\cdot;\phi))dt \\
&> b h_2 \int_{1/4}^{3/4} G\big(\frac{1}{2}, t\big)dt = h_2 = \|x\|_I.
\end{aligned}$$

This shows that

$$\|A_\phi x\|_I > \|x\|_I, \quad \forall x \in \partial K_{h_2}.$$

Therefore, by Lemma 1.1, we conclude that $i(A_\phi, K_{h_2}, K) = 0$. $\square$

3. Main results

Theorem 3.1. *Assume that* (H3) *and* (H4) *are satisfied, then BVP* (1.2) *has at least one positive solution.*

Proof. According to Lemma 2.7, we have that

$$i(A_\phi, K_{h_1}, K) = 1. \tag{3.1}$$

Fix $m > 1$, and let $g(t,u,v,\psi) = (u + v + \|\psi\|_J)^m$ for $u, v \geq 0$ and $\psi \in C^+(J)$. Then $g(t,u,v,\psi)$ satisfy (H1). Define $B_\phi : K \to K$ by

$$(B_\phi x)(t) := \int_0^1 G(t,s)g(s,x(s),x(s-\tau;\phi),x_s(\cdot;\phi))ds, \quad t \in I.$$

Then B_ϕ is a completely continuous operator. One has from Lemma 2.5 that there exists $0 < h_1 < r_0 < \infty$, such that $r \geq r_0$ implies

$$i(B_\phi, K_r, K) = 0. \tag{3.2}$$

Define $H_\phi : [0,1] \times K \to K$ by $H_\phi(s,x) = (1-s)A_\phi x + sB_\phi x$, then H_ϕ is a completely continuous operator. By the condition (H3) and the definition of g, for $u, v \geq 0$, $\psi \in C^+(J)$, and $t \in I$, there are $\varepsilon > 0$ and $r' > r_0$ such that

$$\begin{gathered}
f(t,u,v,\psi) \geq \frac{1}{3}(\lambda_1(1+M)+\varepsilon)(u+v+\|\psi\|_J), \quad u+v+\|\psi\|_J > r', \\
g(t,u,v,\psi) \geq \frac{1}{3}(\lambda_1(1+M)+\varepsilon)(u+v+\|\psi\|_J), \quad u+v+\|\psi\|_J > r',
\end{gathered}$$

where $\lambda_1 = \frac{\pi^2}{(\eta+1)^2}$. We define

$$\begin{aligned}
C := &\max_{0\leq t\leq 1} \max_{0\leq u,v,u+v+\|\psi\|_J\leq r'} |f(t,u,v,\psi) - \frac{1}{3}[\lambda_1(1+M)+\varepsilon](u+v+\|\psi\|_J)| \\
&+ \max_{0\leq t\leq 1} \max_{0\leq u,v,u+v+\|\psi\|_J\leq r'} |g(t,u,v,\psi) - \frac{1}{3}[\lambda_1(1+M)+\varepsilon](u+v+\|\psi\|_J)| + 1.
\end{aligned}$$

It follows that

$$f(t,u,v,\psi) \geq \frac{1}{3}[\lambda_1(1+M)+\varepsilon](u+v+\|\psi\|_J) - C, \ u,v \geq 0, \ \psi \in C^+(J), \ t \in I, \tag{3.3}$$

$$g(t,u,v,\psi) \geq \frac{1}{3}[\lambda_1(1+M)+\varepsilon](u+v+\|\psi\|_J) - C, \ u,v \geq 0, \ \psi \in C^+(J), \ t \in I. \tag{3.4}$$

We claim that there exists $r_1 \geq r'$ such that

$$H_\phi(s,x) \neq x, \quad \forall s \in [0,1], \quad x \in K, \quad \|x\| \geq r_1. \tag{3.5}$$

In fact, if $H_\phi(s_1,z) = z$ for some $z \in K$ and $0 \leq s_1 \leq 1$, then $z(t)$ satisfies the equation

$$\begin{aligned}
-z''(t) = &(1-s_1)f(t,z(t),z(t-\tau;\phi),z_t(\cdot;\phi)) \\
&+ s_1 g(t,z(t),z(t-\tau;\phi),z_t(\cdot;\phi)), \quad 0 < t < 1,
\end{aligned} \tag{3.6}$$

and the boundary condition

$$z(0) = 0, \quad z(1) = z(\eta). \tag{3.7}$$

From the above condition, there exists $\xi \in (\eta, 1)$ such that $z'(\xi) = 0$. Multiplying left side of (3.6) by $\varphi_1(t) = \sin \frac{\pi}{\eta+1} t$ and then integrating from 0 to ξ, after integrating two times by parts, we get from $z'(\xi) = 0$ that

$$\int_0^\xi -z''(t)\varphi_1(t)dt = \varphi_1'(\xi)z(\xi) + \lambda_1 \int_0^\xi z(t)\varphi_1(t)dt. \tag{3.8}$$

By (3.6) and (3.7), we have that $-z''(t) \geq 0$ for each $t \in I$. Thus we obtain from (3.6), (3.8) and (H3) that

$$\begin{aligned}
\lambda_1 \int_0^1 z(t)\varphi_1(t)dt &\geq \lambda_1 \int_0^\xi z(t)\varphi_1(t)dt \\
&= \int_0^\xi -z''(t)\varphi_1(t)dt - \varphi_1'(\xi)z(\xi) \\
&\geq \int_0^\eta -z''(t)\varphi_1(t)dt - \|\varphi_1'\|_I \|z\|_I \\
&= (1 - s_1) \int_0^\eta f(t, z(t), z(t-\tau;\phi), z_t(\cdot;\phi))\varphi_1(t)dt \\
&\quad + s_1 \int_0^\eta g(t, z(t), z(t-\tau;\phi), z_t(\cdot;\phi))\varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I.
\end{aligned} \tag{3.9}$$

Combining (2.2), (3.3), (3.4) and (3.9), we get

$$\begin{aligned}
&\lambda_1 \int_0^1 z(t)\varphi_1(t)dt \\
&\geq \frac{1}{3}(1-s_1)(\lambda_1(1+M)+\varepsilon) \int_0^\eta [z(t) + z(t-\tau;\phi) + \|z_t(\cdot;\phi)\|_J]\varphi_1(t)dt \\
&\quad - (1-s_1)C \int_0^\eta \varphi_1(t)dt \\
&\quad + \frac{1}{3}s_1(\lambda_1(1+M)+\varepsilon) \int_0^\eta [z(t) + z(t-\tau;\phi) + \|z_t(\cdot;\phi)\|_J]\varphi_1(t)dt \\
&\quad - s_1 C \int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I \\
&= \frac{1}{3}(\lambda_1(1+M)+\varepsilon) \int_0^\eta [z(t) + z(t-\tau;\phi) + \|z_t(\cdot;\phi)\|_J]\varphi_1(t)dt \\
&\quad - C \int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I \\
&\geq \frac{1}{3}(\lambda_1(1+M)+\varepsilon) \int_0^\eta [2z(t) + z(t-\tau;\phi)]\varphi_1(t)dt \\
&\quad - C \int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I \\
&\geq \frac{1}{3}(\lambda_1(1+M)+\varepsilon)\Big(2\int_0^\eta z(t)\varphi_1(t)dt + \int_\tau^\eta z(t-\tau;\phi)\varphi_1(t)dt\Big) \\
&\quad - C \int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}(\lambda_1(1+M)+\varepsilon)\Big(2\int_0^\eta z(t)\varphi_1(t)dt + \int_\tau^\eta z(t-\tau)\varphi_1(t)dt\Big)\\
&\quad - C\int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I\\
&= \frac{1}{3}(\lambda_1(1+M)+\varepsilon)\Big(2\int_0^\eta z(t)\varphi_1(t)dt + \int_0^{\eta-\tau} z(t)\varphi_1(t+\tau)dt\Big)\\
&\quad - C\int_0^\eta \varphi_1(t)dt - \frac{\pi}{\eta+1}\|z\|_I,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&(\lambda_1 M+\varepsilon)\Big(\int_0^{\eta-\tau} z(t)\varphi_1(t+\tau)dt + 2\int_0^\eta z(t)\varphi_1(t)dt\Big)\\
&\le \lambda_1\int_0^{\eta-\tau} z(t)[\varphi_1(t)-\varphi_1(t+\tau)]dt + \lambda_1\int_{\eta-\tau}^1 z(t)\varphi_1(t)dt\\
&\quad + 2\lambda_1\int_\eta^1 z(t)\varphi_1(t)dt + 3C\int_0^\eta \varphi_1(t)dt + \frac{3\pi}{\eta+1}\|z\|_I\\
&\le \lambda_1\tau(\eta-\tau)\|\varphi_1'\|_I\|z\|_I + \lambda_1(1-\eta+\tau)\|\varphi_1\|_I\|z\|_I\\
&\quad + 2\lambda_1(1-\eta)\|\varphi_1\|_I\|z\|_I + 3C\eta\|\varphi_1\|_I + \frac{3\pi}{\eta+1}\|z\|_I\\
&= \lambda_1\Big[\frac{\pi}{\eta+1}\tau(\eta-\tau) + 3(1-\eta) + \tau + \frac{3(\eta+1)}{\pi}\Big]\|z\|_I + 3C\eta.
\end{aligned}
\tag{3.10}
$$

We also have

$$
\begin{aligned}
&\int_0^{\eta-\tau} z(t)\varphi_1(t+\tau)dt + 2\int_0^\eta z(t)\varphi_1(t)dt\\
&\ge \eta\|z\|_I\int_0^{\eta-\tau} t\varphi_1(t+\tau)dt + 2\eta\|z\|_I\int_0^\eta t\varphi_1(t)dt,
\end{aligned}
\tag{3.11}
$$

which together with (3.10) leads to

$$
\begin{aligned}
&(\lambda_1 M+\varepsilon)\eta\Big(\int_0^{\eta-\tau} t\varphi_1(t+\tau)dt + 2\int_0^\eta t\varphi_1(t)dt\Big)\|z\|_I\\
&\le \lambda_1\Big[\frac{\pi}{\eta+1}\tau(\eta-\tau) + 3(1-\eta) + \tau + \frac{3(\eta+1)}{\pi}\Big]\|z\|_I + 3C\eta\\
&= \lambda_1\frac{1}{\pi(\eta+1)}\big[\pi^2\tau(\eta-\tau) + 3\pi(1-\eta^2) + \pi\tau(\eta+1) + 3(\eta+1)^2\big]\|z\|_I + 3C\eta;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\|z\|_I &\le \frac{3C}{\varepsilon\big(\int_0^{\eta-\tau} t\varphi_1(t+\tau)dt + 2\int_0^\eta t\varphi_1(t)dt\big)}\\
&= \frac{3C}{\varepsilon\big(\int_0^{\eta-\tau} t\sin\frac{\pi}{\eta+1}(t+\tau)dt + 2\int_0^\eta t\sin\frac{\pi}{\eta+1}t dt\big)} := \overline{r}.
\end{aligned}
$$

Let $r_1 = 1+\max\{r', \overline{r}\}$. We obtain (3.5) and consequently, by (3.2) and homotopy invariance of the fixed-point index, we have

$$
\begin{aligned}
i(A_\phi, K_{r_1}, K) &= i(H_\phi(0,\cdot), K_{r_1}, K)\\
&= i(H_\phi(1,\cdot), K_{r_1}, K) = i(B_\phi, K_{r_1}, K) = 0.
\end{aligned}
\tag{3.12}
$$

Use (3.1) and (3.12) to conclude that

$$i(A_\phi, K_{r_1} \setminus \overline{K}_{h_1}, K) = -1.$$

Hence, A_ϕ has fixed points x_* in $K_{r_1} \setminus \overline{K}_{h_1}$, which means that $x_*(t)$ is a positive solution of BVP (1.2) and $\|x_*\|_I > h_1$. Thus, the proof is complete. □

By Lemmas 2.6 and 2.8, we have the following result.

Theorem 3.2. *Assume that* (H2) *and* (H5) *are satisfied, then BVP* (1.2) *has at least one positive solution.*

Finally, we obtain from Lemmas 2.7 and 2.8 the following result.

Theorem 3.3. *If* (H4) *and* (H5) *are satisfied, then BVP (1.2) has at least one positive solution.*

References

[1] C. Z. Bai, J.Fang; *Existence of multiple positive solutions for functional differential equations*, Comput. Math. Appl. 45 (2003) 1797-1806.
[2] K. Deimling; *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
[3] L.H. Erbe, Q.K. Kong; *Boundary value problems for singular second order functional differential equations*, J. Comput. Appl. Math. 53 (1994) 377-388.
[4] J. Henderson; *Double solutions of three-point boundary-value problems for second-order differential equations*, Electron. J. of Differential Equations 2004 (2004), No. 115, 1-7.
[5] J. Henderson; *Boundary Value Problems for Functional Differential Equations*, World Scientific, 1995.
[6] C.H. Hong, C.C. Yeh, C.F. Lee, F.H. Wong; *Existence of positive solutions for functional differential equations*, Comput. Math. Appl. 40 (2000) 783-792.
[7] D.Q. Jiang; *Multiple positive solutions for boundary-value problems of second-order delay differential equations*, Appl.Math. Lett. 15 (2002) 575-583.
[8] G. L. Karakostas, K. G. Mavridis, P. Ch. Tsamatos; *Multiple positive solutions for a functional second-order boundary-value problem*, J. Math. Anal. Appl. 282 (2003) 567-577.
[9] J.W. Lee, D. O'Regan; *Existence results for differential delay equations-I*, J. Differential Equations 102 (1993) 342-359.
[10] J.W. Lee, D. O'Regan; *Existence results for differential delay equations-II*, Nonlinear Analysis 17 (1991) 683-7029.
[11] S.K. Ntouyas, Y, Sficas, P.Ch. Tsamatos; *An existence principle for boundary-value problems for second order functional differential equations*, Nonlinear Analysis 20 (1993) 215-222.

Chuanzhi Bai
Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsi 223001, China. and Department of Mathematics, Nanjing University, Nanjing 210093, China
E-mail address: czbai8@sohu.com

Xinya Xu
Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsi 223001, China
E-mail address: xinyaxu@hytc.edu.cn