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SOLUTIONS TO $\bar{\partial}$-EQUATIONS ON STRONGLY PSEUDO-CONVEX DOMAINS WITH L^p-ESTIMATES

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Abstract. We construct a solution to the $\bar{\partial}$-equation on a strongly pseudo-convex domain of a complex manifold. This is done for forms of type $(0, s)$, $s \geq 1$, with values in a holomorphic vector bundle which is Nakano positive and for complex valued forms of type (r, s), $1 \leq r \leq n$, when the complex manifold is a Stein manifold. Using Kerzman's techniques, we find the L^p-estimates, $1 \leq p \leq \infty$, for the solution.

1. Introduction

The existence of solutions to the equation $\bar{\partial} g = f$, on strongly pseudo-convex domains in $\mathbb{C}^n$, with L^p-estimates when f is a form of type $(0, s)$; $\bar{\partial} f = 0$, $s \geq 1$, and satisfies L^p-estimates, $1 \leq p \leq \infty$, has been a central theme in complex analysis for many years. Øvrelid [5] has obtained a solution with L^p-estimates for this equation. Abdelkader [1] has extended Øvrelid's results to forms of type (n, s) on strongly pseudo-convex domains in an $n-$dimensional Stein manifold. In this paper we extend Abdelkader's results to forms of type (r, s); $0 \leq r \leq n$. For this purpose, we first study the equation $\bar{\partial} g = f$, on strongly pseudo-convex domains in an n-dimensional complex manifold M when f is a form of type $(0, s)$; $\bar{\partial} f = 0$, $s \geq 1$, with values in a holomorphic vector bundle. Then, we apply this results to the vector bundle $\bigwedge^r T^\star(M)$ (the r^{th}-exterior product of the holomorphic cotangent vector bundle $T^\star(M)$) and using the fact that any $\mathbb{C}-$valued differential form of type (r, s) on M is a differential form of type $(0, s)$ on M with values in the vector bundle $\bigwedge^r T^\star(M)$. When $r = n$, the vector bundle $K(M) = \bigwedge^n T^\star(M)$ is the canonical line bundle of M. Therefore it is sufficient in this case to study the equation $\bar{\partial} g = f$ for f with values in a holomorphic line bundle which is the case in [1]. In fact, the main aim of this paper is to establish the following existence theorem with L^p-estimates:

Theorem 1.1 (Global theorem)**.** *Let M be a complex manifold of complex dimension n and let $E \to M$ be a holomorphic vector bundle, of rank N, over M. Let $D \Subset M$ be a strongly pseudo-convex domain with smooth C^4-boundary. Then*

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(1) If the holomorphic vector bundle E is Nakano positive, then, there exists an integer $k_0 = k_0(D) > 0$ such that for any $f \in L^1_{0,s}(D, E^k)$; $\bar\partial f = 0$, $s \geq 1$ and $k \geq k_0$ there is a form $g = T^s_{N^k} f \in L^1_{0,s-1}(D, E^k)$ satisfies $\bar\partial g = f$, where $T^s_{N^k}$ is a bounded linear operator and $E^k = E \otimes E \otimes \cdots \otimes E$ (k-times). Moreover, if $f \in L^p_{0,s}(D, E^k)$; $1 \leq p \leq \infty$, there is a constant C^k_s such that $\|g\|_{L^p_{0,s-1}(D,E^k)} \leq C^k_s \|f\|_{L^p_{0,s}(D,E^k)}$. The constant C^k_s is independent of f and p. If f is C^∞, then g is also C^∞.
(2) If M is a Stein manifold, then, for any $f \in L^1_{r,s}(D)$; $\bar\partial f = 0$, $0 \leq r \leq n$, and $s \geq 1$, there is a form $g = T^s f \in L^1_{r,s-1}(D)$ such that $\bar\partial g = f$, where T^s is a bounded linear operator. Moreover, if $f \in L^p_{r,s}(D)$; $1 \leq p \leq \infty$, we have $\|g\|_{L^p_{r,s-1}(D)} \leq C_s \|f\|_{L^p_{r,s}(D)}$. The constant C_s is independent of f and p. If f is C^∞, then g is also C^∞.

The plan of this paper is as follows: In section 1, we state the main theorem. In section 2, we set the notation and recall some useful facts. In section 3, we prove an existence theorem with L^2- estimates. In section 4, we give local solution for the $\bar\partial$-equation with L^p-estimates for $1 \leq p \leq \infty$. In section 5, we prove the existence theorem with L^p-estimates.

2. Notation and Preliminaries

Let M be an n-dimensional complex manifold and let $\pi : E \to M$ be a holomorphic vector bundle, of rank N, over M. Let $\{u_j\}$; $j \in I$, be an open covering of M consisting of coordinates neighborhoods u_j with holomorphic coordinates $z_j = (z^1_j, z^2_j, \dots, z^n_j)$ over which E is trivial, namely $\pi^{-1}(u_j) = u_j \times \mathbb{C}^N$. The N-dimensional complex vector space $E_z = \pi^{-1}(z)$; $z \in M$, is called the fiber of E over z. Let $h = \{h_j\}$; $h_j = (h_{j\mu\bar\eta})$ be a Hermitian metric along the fibers of E and let $(h^{\mu\bar\eta}_j)$ be the inverse matrix of $(h_{j\mu\bar\eta})$. Let $\theta = \{\theta_j\}$; $\theta_j = (\theta^\nu_{j\mu})$; $\theta^\nu_{j\mu} = \partial \log h_j = \sum_{\alpha=1}^n \sum_{\eta=1}^N h^{\nu\bar\eta}_j \frac{\partial h_{j\mu\bar\eta}}{\partial z^\alpha_j} dz^\alpha_j = \sum_{\alpha=1}^n \Omega^\nu_{j\mu\alpha} dz^\alpha_j$ and $\Theta = \{\Theta_j\}$; $\Theta_j = (\Theta^\nu_{j\mu})$; $\Theta^\nu_{j\mu} = \sqrt{-1}\bar\partial\partial \log h_j = \sqrt{-1}\sum_{\alpha,\beta=1}^n \Theta^\nu_{j\mu\alpha\bar\beta} dz^\alpha_j \wedge d\bar z^\beta_j$ be the connection and the curvature forms associated to the metric h respectively, where $\Theta^\nu_{j\mu\alpha\bar\beta} = -\frac{\partial \Omega^\nu_{j\mu\alpha}}{\partial d\bar z^\beta_j}$, $1 \leq \mu \leq N$; $1 \leq \nu \leq N$. The associated curvature matrix is given by

$$(H_{j\bar\eta\bar\beta,\nu\alpha}) = \Big(\sum_{\mu=1}^N h_{j\mu\bar\eta}\Theta^\mu_{j\nu\alpha\bar\beta}\Big).$$

Let $T(M)$ (resp. $T^\star(M)$) be the holomorphic tangent (resp. cotangent) bundle of M.

Definition 2.1. E is said to be Nakano positive, at $z \in u_j$, if the Hermitian form

$$\sum H_{j\bar\eta\bar\beta,\nu\alpha}(z)\zeta^\nu_\alpha \bar\zeta^\eta_\beta$$

is positive definite for any $\zeta = (\zeta^\nu_\alpha) \in E_z \otimes T_z(M)$; $\zeta \neq 0$.

The notation $X \Subset M$ means that X is an open subset of M such that its closure is a compact subset of M.

Definition 2.2. A domain $D \Subset M$ is said to be strongly pseudo-convex with smooth C^4-boundary if there exist an open neighborhood U of the boundary ∂D of D and a C^4 function $\lambda : U \to \mathbb{R}$ having the following properties:

(i) $D \cap U = \{z \in U; \lambda(z) < 0\}$.
(ii) $\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j^\alpha \partial \bar{z}_j^\beta} \mu_\alpha \bar{\mu}_\beta \geq L(z)|\mu|^2$; $z \in U \cap u_j$, $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{C}^n$ and $L(z) > 0$.
(iii) The gradient $\nabla \lambda(z) = (\frac{\partial \lambda(z)}{\partial x_j^1}, \frac{\partial \lambda(z)}{\partial y_j^1}, \frac{\partial \lambda(Z)}{\partial x_j^2}, \frac{\partial \lambda(z)}{\partial y_j^2}, \dots, \frac{\partial \lambda(z)}{\partial x_j^n}, \frac{\partial \lambda(z)}{\partial y_j^n}) \neq 0$ for $z = (z_j^1, z_j^2, \dots, z_j^n) \in u_j \cap U$; $z_j^\alpha = x_j^\alpha + i y_j^\alpha$.

Let $\gamma = (\mu_1, \nu_1, \dots, \mu_n, \nu_n)$ be any multi-index and $|\gamma| = \sum_{i=1}^{n} (\mu_i + \nu_i)$, where μ_i and ν_i are non-negative integers. Let $D^\gamma = \partial^{|\gamma|} / \partial x_1^{\mu_1} \partial y_1^{\nu_1} \dots \partial x_n^{\mu_n} \partial y_n^{\nu n}$.

Remark 2.3. By shrinking U we can assume that $U \Subset \tilde{U}$, where $\tilde{U}$ is an open, λ is C^4 on $\tilde{U}$ and the properties (i), (ii) and (iii) of Definition 2.2 hold on $\tilde{U}$. Thus, we can choose a neighborhood V of ∂D such that $V \Subset U$ and for any $z \in V$ there exist positive constants L, F and $F^{'}$ satisfy $L(z) > L$, $|\nabla \tilde{\lambda}(z)| \geq F$ and $|D^\gamma \tilde{\lambda}(z)| \leq F' < \infty$ for any multi-index γ with $|\gamma| \leq 4$, where $\tilde{\lambda}$ is the a slight perturbation of λ.

Definition 2.4. Let X be an n-dimensional complex manifold and let Φ be an exhaustive function on X, that is, the sets $X_c = \{z \in X; \Phi(z) < c\} \Subset X$; $c \in \mathbb{R}$ and $X = \cup X_c$. We say that X is weakly 1-complete (resp. Stein) manifold if Φ is a C^∞ plurisubharmonic (resp. strictly plurisubharmonic), that is, if $\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \Phi(z)}{\partial z_j^\alpha \partial \bar{z}_j^\beta} \mu^\alpha \bar{\mu}^\beta$ is positive semi-definite (resp. positive definite) on X for $\mu = (\mu^1, \dots, \mu^n) \in \mathbb{C}^n$; $\mu \neq 0$.

We will use the standard notation of Hörmander [5] for differential forms. Thus a $\mathbb{C}$-valued differential form $\varphi = \{\varphi_j\}$ of type (r, s) on M can be expressed, on u_j, as $\varphi_j(z) = \sum_{A_r, B_s} \varphi_{j A_r B_s}(z) dz_j^{A_r} \wedge d\bar{z}_j^{B_s}$, where A_r and B_s are strictly increasing multi-indices with lengths r and s, respectively. An E-valued differential form φ of type (r, s), on M, is given locally by a column vector ${}^t\varphi_j = (\varphi_j^1, \varphi_j^2, \dots, \varphi_j^N)$ where φ_j^a, $1 \leq a \leq N$, are $\mathbb{C}$-valued differential forms of type (r, s) on u_j. $\Lambda^{r,s}(M)$ denotes the space of $\mathbb{C}$-valued differential forms of type (r, s) and of class C^∞ on M. Let $\Lambda^{r,s}(M, E)$ (resp. $\mathcal{D}^{r,s}(M, E)$) be the space of E-valued differential forms (resp. with compact support) of type (r, s) and of class C^∞ on M.

Let $h^0 = \{h_j^0\}$, $h_j^0 = (h_{j\mu\bar{\eta}}^0)$, be the initial Hermitian metric along the fibers of E and let $\Theta^0 = \{\Theta_j^0\}$ be the associated curvature form. The induced Hermitian metric along the fibers of the line bundle $B = \bigwedge^N E$ is given by the system of positive C^∞ functions $\{a_j^0\}$, where $a_j^0 = \det(h_{j\mu\bar{\eta}}^0)^{-1}$. Hence, the system $\{1/a_j^0\}$ also defines a Hermitian metric along the fibers of B whose curvature matrix $(H_{j\bar{\eta}\bar{\beta},\nu\alpha})$ is given by $(1/a_j^0)(\partial^2 \log a_j^0 / \partial z_j^\alpha \partial \bar{z}_j^\beta)$. If E is Nakano positive, with respect to h^0, then B is positive, with respect to $\{1/a_j^0\}$, that is, the Hermitian matrix $(\partial^2 \log a_j^0 / \partial z_j^\alpha \partial \bar{z}_j^\beta)$ is positive definite. Hence,

$$ds_0^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\bar{\beta}}^0 \, dz_j^\alpha d\bar{z}_j^\beta \ ; \ \ g_{j\alpha\bar{\beta}}^0 = \partial^2 \log a_j^0 / \partial z_j^\alpha \partial \bar{z}_j^\beta$$

defines a Kähler metric on M. For $\varphi, \psi \in \Lambda^{r,s}(M, E)$, we define a local inner product, at $z \in u_j$, by

$$\sum_{\nu,\mu=1}^{N} h_{j\nu\bar{\mu}}^0 \varphi_j^\nu(z) \wedge \star\overline{\psi_j^\mu(z)} = a(\varphi(z), \psi(z)) dv_0, \tag{2.1}$$

where the Hodge star operator $\star$ and the volume element dv_0 are defined by ds_0^2 and $a(\varphi,\psi)$ is a function, on M, independent of j.

Let $L^p_{r,s}(M,E)$ (resp. $L^\infty_{r,s}(M,E)$) be the Banach space of E-valued differential forms f on M, of type (r,s), such that $\|f\|_{L^p_{r,s}(M,E)} = (\int_M |f(z)|^p dv_0)^{1/p} < \infty$ for $1 \le p < \infty$ (resp. $\|f\|_{L^\infty_{r,s}(M,E)} = \operatorname{ess\,sup}_{z\in M}|f(z)| < \infty$), where $|f(z)| = \sqrt{a(f(z),f(z))}$.

The Hermitian metric along the fibers of $E^k = E\otimes E\otimes\cdots\otimes E$, associated to h^0, is defined by $h^{0k} = \{h_j^{0k}\}$, where $h_j^{0k} = h_j^0 h_j^0 \dots h_j^0$ (k-factors). The transition functions of $K(M)$ are the Jacobian determinant

$$k_{ij} = \frac{\partial(z_j^1, z_j^2,\dots,z_j^n)}{\partial(z_i^1, z_i^2,\dots,z_i^n)}$$

on $u_i\cap u_j$. We see that $|k_{ij}|^2 = g_i g_j^{-1}$ on $u_i\cap u_j$, where $g_i = \det(\partial^2\log a_i^0/\partial z_i^\alpha\partial\bar z_i^\beta)$. Therefore, the system of positive C^∞ functions $\{g_j^{-1}\}$ (resp. $g=\{g_j\}$) determines a Hermitian metric along the fibers of $K(M)$ (resp. the dual bundle $K^{-1}(M)$).

3. Existence Theorems with L^2-Estimates

Let $Y \Subset M$ be weakly 1-complete domain of M with respect to a plurisubharmonic function Φ and $\lambda(t)$ be a real C^∞ function on $\mathbb{R}$ such that $\lambda(t)>0$, $\lambda'(t)>0$ and $\lambda''(t)>0$ for $t>0$ and $\lambda(t)=0$ for $t\le 0$. Let $h_j = e^{-\lambda(\Phi)}h_j^0$, on $u_j\cap Y$, and $a_j = \det(h_j)^{-1}$. Thus, the Hermitian matrix $(\partial^2\log a_j/\partial z_j^\alpha\partial\bar z_j^\beta)$ is positive definite on $u_j\cap Y$. Hence,

$$ds^2 = \sum_{\alpha,\beta=1}^n g_{j\alpha\bar\beta}\, dz_j^\alpha d\bar z_j^\beta\ ;\ \ g_{j\alpha\bar\beta} = \partial^2\log a_j/\partial z_j^\alpha\partial\bar z_j^\beta$$

defines a Kähler metric on Y. The Hermitian metrics $h^k = \{h_j^k\}$ and g induce a Hermitian metric $b^k = \{h_j^k g_j\}$; $k\ge 1$, along the fibers of $K^{-1}(M)\otimes E^k|_Y$, where $h_j^k = h_j h_j \dots h_j$ (k-factors).

Let $L^2_{r,s}(Y, K^{-1}(M)\otimes E^k, \mathrm{loc}, gh^{0k}, ds_0^2)$ be the space of all $K^{-1}(M)\otimes E^k-$ valued differential forms of type (r,s) which has measurable coefficients and square integrable on compact subsets of Y with respect to ds_0^2 and gh^{0k}. For $\varphi,\psi\in\Lambda^{r,s}(Y,K^{-1}(M)\otimes E^k)$ we define a local inner product $a(\varphi(z),\psi(z))_k dv$ by replacing $g_j h_j^k$ and ds^2 instead of h_j^0 and ds_0^2, respectively, in (2.1). For φ or $\psi\in\mathcal{D}^{r,s}(Y,K^{-1}(M)\otimes E^k)$, we define a global inner product by

$$\langle\varphi,\psi\rangle_k = \int_Y a(\varphi,\psi)_k\, dv. \tag{3.1}$$

Let $\omega = \sqrt{-1}\sum_{\alpha,\beta=1}^n g_{j\alpha\bar\beta}dz_j^\alpha\wedge d\bar z_j^\beta$ be the fundamental form of ds^2 and let $L=e(\omega)$ be the wedge multiplication by ω. Let $\Gamma:\Lambda^{r,s}(Y,K^{-1}(M)\otimes E^k)\to\Lambda^{r-1,s-1}(Y,K^{-1}(M)\otimes E^k)$ be the operator locally defined by $\Gamma = (-1)^{r+s}\star L\star$, where the $\star$ operator is defined by ds^2. Let ϑ_k be the formal adjoint of $\bar\partial:\Lambda^{r,s}(Y,K^{-1}(M)\otimes E^k)\to\Lambda^{r,s+1}(Y,K^{-1}\otimes E^k)$ with respect to the inner product (3.1) and $\square_k = \bar\partial\vartheta_k+\vartheta_k\bar\partial$ be the Laplace-Beltrami operator. The curvature form associated to b^k is given by

$$\Theta^k = \{\Theta_j^k\}; \Theta_j^k = \sqrt{-1}\bar\partial\partial\log b_j^k = k\Theta_j^0 + \sqrt{-1}(k\partial\bar\partial\lambda(\Phi) - \partial\bar\partial\log g_j).$$

Since the Levi form $\sqrt{-1}\partial\bar{\partial}\lambda(\Phi)$ is positive semi-definite, E is Nakano positive with respect to h^0 and $\bar{Y}$ is compact subset of M, there exists an integer $k_0 = k_0(Y) > 0$ such that $K^{-1}(M) \otimes E^k|_Y$ is Nakano positive, with respect to b^k, for $k \geq k_0$. Hence as in Nakano [4] we can prove the following lemma:

Lemma 3.1. *Let $f \in L^2_{n,s}(Y, K^{-1}(M) \otimes E^k, \mathrm{loc}, gh^{0k}, ds_0^2)$; $k \geq k_0$, $s \geq 1$ be given, then we can choose the function $\lambda(t)$ such that ds^2 is complete, $\langle f, f\rangle_k < \infty$, and there is a constant $c > 0$ such that*

$$\langle \bar{\partial}\varphi, \bar{\partial}\varphi\rangle_k + \langle \vartheta_k\varphi, \vartheta_k\varphi\rangle_k \geq c\langle \varphi, \varphi\rangle_k, \tag{3.2}$$

for any $\varphi \in \mathcal{D}^{n,s}(Y, K^{-1}(M) \otimes E^k)$.

Remark 3.2. *We note that when E is a line bundle Lemma 3.1 is valid for forms in $\mathcal{D}^{r,s}(Y, K^{-1}(M) \otimes E^k)$ with $r + s \geq n + 1$.*

From Lemma 3.1 and the Hilbert space technique of Hörmander [5], as in the proof of [1, Theorem 2.1], we can prove the following theorem:

Theorem 3.3. *Let $Y \Subset M$ be weakly 1-complete domain and let $E \to M$ be a holomorphic vector bundle over M. If E is Nakano positive, over M, then for any $f \in L^2_{n,s}(Y, K^{-1}(M) \otimes E^k, b^k, ds^2)$ with $\bar{\partial}f = 0$, $s \geq 1$ and $k \geq k_0$ there exists a form $g = Tf \in L^2_{n,s-1}(Y, K^{-1}(M) \otimes E^k, b^k, ds^2)$ satisfies $\bar{\partial}g = f$ and two constants $C = C(Y)$ and $c_k = c_k(G, Y)$ such that*

$$\|g\|_{L^2_{n,s-1}(Y,K^{-1}(M)\otimes E^k,b^k,ds^2)} \leq C\|f\|_{L^2_{n,s}(Y,K^{-1}(M)\otimes E^k,b^k,ds^2)},$$
$$\|g\|_{L^2_{n,s-1}(G,K^{-1}(M)\otimes E^k)} \leq c_k\|f\|_{L^2_{n,s}(G,K^{-1}(M)\otimes E^k)},$$

where T is a bounded linear operator and $G \Subset Y$.

4. Local solution for the $\bar{\partial}$-equation with L^p-estimates

Let $D \Subset M$ be a strongly pseudo-convex domain with λ and U of Definition 2.2. Let $x \in \partial D$ be an arbitrary fixed point and let W_a be an open neighborhood of x such that $W_a \Subset u_j \subset U$, for a certain $j \in I$, and $z_j(W_a)$ is the ball $B(0, a) \Subset \mathbb{C}^n$, where (u_j, z_j) is a holomorphic chart. Then, W_a can be considered as strongly pseudo-convex domain in $\mathbb{C}^n$ and the volume element dv_0 can be considered as the Lebesgue measure on $B(0, a)$.

Theorem 4.1 ([5])**.** *Let $G \Subset \mathbb{C}^n$ be a strongly pseudo-convex domain and $u \in L^1_{0,s}(G)$; $s \geq 1$. Then, there exist kernels $K_s(\xi, z)$ such that the integral $\int_G u(\xi) \wedge K_{s-1}(\xi, z)d\mu(\xi)$ is absolutely convergent for almost all $z \in \bar{G}$ and the operator $T^s : L^p_{0,s}(G) \to L^p_{0,s-1}(G)$, defined by $T^s u(z) = \int_G u(\xi) \wedge K_{s-1}(\xi, z)d\mu(\xi)$, with norm $\leq c$; $1 \leq p \leq \infty$. Moreover, if $\bar{\partial}u = 0$, then, there is a form $g = T^s u$ satisfies $\bar{\partial}g = u$, where $d\mu(\xi)$ is the Lebesgue measure on $\mathbb{C}^n$.*

Now, we extend the operator T^s to $L^p_{0,s}(D \cap W_a, E)$. For this purpose, we define an operator $T^s_N : f \in L^1_{0,s}(D \cap W_a, E) \to T^s_N f \in L^1_{0,s-1}(D \cap W_a, E)$; $s \geq 1$, by

$$T^s_N f(z) = \sum_{\lambda=1}^{N} T^s f^\lambda(z)\ b_\lambda(z), \tag{4.1}$$

where $f(z) = \sum_{\lambda=1}^{N} f^\lambda(z)b_\lambda(z)$, that is, $f^\lambda(z)$ are the components of $f|_{u_j}$ with respect to an orthonormal basis $b_\lambda(z)$ on E_z; $z \in u_j$.

We consider the following situation: In the notation of Definition 2.2, from Remark 2.3, let $y \in \partial V^-$, where $V^- = \{z \in V; \tilde{\lambda}(z) < 0\}$ and let W_a be a neighborhood of y such that $W_a \Subset u_j \subset V$, for a certain $j \in I$, and $z_j(W_a)$ is the ball $B(0,a) \subset \mathbb{C}^n$, $a \leq \tilde{a}$, where $\tilde{a}$ depends continuously on L, F, F' and the distance $d(y, CV)$ from y to the complement of V. In the above notation, as the local theorem in [3], we can prove the following theorem:

Theorem 4.2 (Local theorem)**.** *Let $T^s_{N^k}$ be the linear operator defined by* (4.1) *and let $f \in L^1_{0,s}(V^-, E^k)$; $\bar{\partial} f = 0$, where N^k is the rank of E^k. Then, there is a form $g = T^s_N f \in L^1_{0,s-1}(V^- \cap W_a, E^k)$ such that $\bar{\partial} g = f$. If f is C^∞, then so is g. If $f \in L^p_{0,s}(V^-, E^k)$, then $g \in L^p_{0,s-1}(V^- \cap W_a, E^k)$ and satisfies*

$$\|g\|_{L^p_{0,s-1}(V^- \cap W_a, E^k)} \leq C \|f\|_{L^p_{0,s}(V^-, E^k)};\ 1 \leq p \leq \infty,$$

where $C = C(s,k,N)$ is a constant which depends continuously on L, F, F' and a.

5. Global solution for the $\bar{\partial}$-equation with L^p-estimates

The local result yields Lemma 5.1 (An extension lemma) which in turn enables one to solve $\bar{\partial}\eta = \hat{f}$ (with bounds) in a strongly pseudoconvex domain $\hat{D}$ which is larger than D, $\bar{D} \subseteq \hat{D}$. Here we make use of the L^2-estimates for solutions of the $\bar{\partial}-$equation as presented in Theorem 3.3.

Lemma 5.1 (An extension lemma)**.** *Let $D \Subset M$ be a strongly pseudo-convex domain with smooth C^4-boundary. Then, there exists another slightly larger strongly pseudo-convex domain $\hat{D} \Subset M$ with the following properties: $\bar{D} \Subset \hat{D}$, for any $f \in L^1_{0,s}(D, E^k)$ with $s \geq 1$ and $\bar{\partial} f = 0$, there exist two bounded linear operators L_1, L_2, a form $\hat{f} = L_1 f \in L^1_{0,s}(\hat{D}, E^k)$ and a form $u = L_2 f \in L^1_{0,s-1}(D, E^k)$ such that:*

(i) *$\bar{\partial}\hat{f} = 0$ in $\hat{D}$.*
(ii) *$\hat{f} = f - \bar{\partial} u$ in D.*
(iii) *If $f \in L^p_{0,s}(D, E^k)$, then $\hat{f} \in L^p_{0,s}(\hat{D}, E^k)$ and $u \in L^p_{0,s-1}(D, E^k)$ with the estimates*

$$\|\hat{f}\|_{L^p_{0,s}(\hat{D}, E^k)} \leq C_1 \|f\|_{L^p_{0,s}(D, E^k)}, \tag{5.1}$$

$$\|u\|_{L^p_{0,s-1}(D, E^k)} \leq C_2 \|f\|_{L^p_{0,s}(D, E^k)} \quad 1 \leq p \leq \infty, \tag{5.2}$$

where the constants C_1 and C_2 are independent of f and p.

If f is C^∞ in D, then $\hat{f}$ is C^∞ in $\hat{D}$ and u is C^∞ in D.

Since ∂D is compact, we can Cover ∂D by finitely many neighborhoods W_{i,a_i} of $x_i \in \partial D$, $i = 1, 2, \dots, m$, such that for each x_i we have $W_{i,a_i} \Subset u_j \Subset V \Subset U$ for a certain $i \in I$. Put $a = \min_{1 \leq i \leq m} a_i$. Then as Lemma 2.3.3 and the Claim on page 321 in Kerzman [3] (see also [1, Proposition 3.2]), we can prove the following proposition:

Proposition 5.2. *Let $\hat{D}$ be as in the extension lemma and let $W_{i,a}$ be an open set of $\hat{D}$ such that $W_{i,a} \Subset u_j \subset \hat{D}$, for a certain $j \in I$ and $z_j(W_{i,a})$ is the ball $B(0,a) \Subset \mathbb{C}^n$. Then, for any $f \in L^1_{0,s}(W_{i,a}, E^k)$; $\bar{\partial} f = 0$ there is $\alpha = Tf \in$*

$L^1_{0,s-1}(W_{i,a/2}, E^k)$ *such that* $\bar{\partial}\alpha = f$, *where* T *is a bounded linear operator. If* $f \in L^p_{0,s}(W_{i,a}, E^k)$; $1 \le p \le 2$, *then, we have* $\alpha \in L^{p+1/4n}_{0,s-1}(W_{i,a/2}, E^k)$ *and*

$$\|\alpha\|_{L^{p+1/4n}_{0,s-1}(W_{i,a/2},E^k)} \le c\|f\|_{L^p_{0,s}(W_{i,a},E^k)},$$

and for any p, $1 \le p \le \infty$, *we have*

$$\|\alpha\|_{L^p_{0,s-1}(W_{i,a/2},E^k)} \le c\|f\|_{L^p_{0,s}(W_{i,a},E^k)},$$

where $c = c(n, a, k, N)$ *is a constant independent of* f *and* p.

The proof of Proposition 5.2 is purely local. Using Proposition 5.2, as [1, Proposition 3.2], we prove the following proposition:

Proposition 5.3. *Let* $\hat{D}$ *be as in the extension lemma. Then, there is a strongly pseudo-convex domain* $D_1 \Subset \hat{D}$ *such that for every* $\hat{f} \in L^1_{0,s}(\hat{D}, E^k)$; $\bar{\partial}\hat{f} = 0$, *there are two bounded linear operators* L_1 *and* L_2 *and two forms* $f_1 = L_1\hat{f} \in L^1_{0,s}(D_1, E^k)$ *and* $\eta_1 = L_2\hat{f} \in L^1_{0,s-1}(D_1, E^k)$ *such that:*

- (i) $\bar{\partial}f_1 = 0$ *on* D_1,
- (ii) $\hat{f} = f_1 + \bar{\partial}\eta_1$ *on* D_1,
- (iii) $\|f_1\|_{L^{p+1/4n}_{0,s}(D_1,E^k)} \le c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E^k)}$ *for* $\hat{f} \in L^p_{0,s}(\hat{D}, E^k)$; $1 \le p \le 2$,
- (iv) *For every open set* $W \Subset D_1$ *and for every* p, $1 \le p \le \infty$, *we have*
$$\|f_1\|_{L^p_{0,s}(W,E^k)} \le c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E^k)},$$
$$\|\eta_1\|_{L^p_{0,s-1}(W,E^k)} \le c\|\hat{f}\|_{L^p_{0,s}(\hat{D},E^k)},$$
where $c = c(\hat{D}, W, n, k, N)$ *is a constant independent of* $\hat{f}$ *and* p.

Since every strongly pseudo-convex domain is weakly 1-complete and noting that $\Lambda^{n,s}(D, K^{-1}(M) \otimes E^k) \equiv \Lambda^{0,s}(D, E^k)$; $k \ge 1$. Then, using Theorem 3.3, Proposition 5.3, and the interior regularity properties of the $\bar{\partial}$-operator, as [1, Theorem 3.1], we prove the following theorem:

Theorem 5.4. *Let* $\hat{D}$ *be the strongly pseudo-convex domain of the extension lemma and* $W \Subset \hat{D}$. *Then, for any form* $\hat{f} \in L^1_{0,s}(\hat{D}, E^k)$ *with* $\bar{\partial}\hat{f} = 0$, *there exists a form* $\eta \in L^1_{0,s-1}(W, E^k)$, $\eta = T\hat{f}$ *such that* $\bar{\partial}\eta = \hat{f}$, *where* T *is a bounded linear operator. If* $\hat{f} \in L^p_{0,s}(\hat{D}, E^k)$ *with* $1 \le p \le \infty$ *and* $k \ge k_0$, *then* $\eta \in L^p_{0,s-1}(W, E^k)$ *and*

$$\|\eta\|_{L^p_{0,s-1}(W,E^k)} \le C\|\hat{f}\|_{L^p_{0,s}(\hat{D},E^k)}$$

where $C = C(\hat{D}, W, k)$ *is a constant independent of* $\hat{f}$ *and* p. *If* $\hat{f}$ *is* C^∞, *then* η *is* C^∞.

Proof. Proposition 5.3 yields D_1. A new application of Proposition 5.3 to D_1 yields D_2. We iterate $4n$ times and obtain

$$\hat{D} \supseteq D_1 \supseteq D_2 \supseteq \cdots \supseteq D_{4n} \Supset \overline{W}$$

Hence, for any $f \in L^1_{0,s}(\hat{D}, E^k); \bar{\partial}f = 0$, there exist $f_j \in L^1_{0,s}(D_j, E^k)$ and $\upsilon_j \in L^1_{0,s-1}(D_j, E^k)$; $j = 1, 2, \dots, 4n$. Clearly, we have:

$$\hat{f} = f_1 + \bar{\partial}\upsilon_1 = f_2 + \bar{\partial}\upsilon_1 + \bar{\partial}\upsilon_2 = f_3 + \bar{\partial}\upsilon_1 + \bar{\partial}\upsilon_2 + \bar{\partial}\upsilon_3 = \cdots = f_{4n} + \bar{\partial}(\sum_{j=1}^{4n} \upsilon_j)$$

in D_{4n}, $f_{4n} \in L^2_{0,s}(D_{4n}, E^k)$ and $\|f_{4n}\|_{L^2_{0,s-1}(D_{4n},E^k)} \leq K\|\hat{f}\|_{L^1_{0,s}(\hat{D},E^k)}$.

Now we apply Theorem 3.3 with $\hat{D} = D_{4n}$ and $\overline{W} \subset Y \Subset D_{4n}$. Let υ be the solution of $\bar{\partial}\upsilon = f_{4n}$ obtained from Theorem 3.3, with

$$\|\upsilon\|_{L^2_{0,s-1}(Y,E^k)} \leq K\|f_{4n}\|_{L^2_{0,s}(D_{4n},E^k)} \leq K\|\hat{f}\|_{L^1_{0,s}(\hat{D},E^k)}.$$

Set $\eta = \upsilon + \sum_{j=1}^{4n} \upsilon_j$, then we obtain $\bar{\partial}\eta = \bar{\partial}\upsilon + \bar{\partial}(\sum_{j=1}^{4n} \upsilon_j) = f_{4n} + \bar{\partial}(\sum_{j=1}^{4n} \upsilon_j) = \hat{f}$ in Y (hence in W). Using (iv) of Proposition 5.3, collecting estimates and the estimates $\|.\|_{L^1_{0,s}(\hat{D},E^k)} \leq K\|.\|_{L^p_{0,s}(\hat{D},E^k)}$ (since $\hat{D}$ is bounded), we obtain:

$$\|\eta\|_{L^1 0,s-1(Y,E^k)} \leq K\|\hat{f}\|_{L^p_{0,s}(\hat{D},E^k)}, \quad 1 \leq p \leq \infty. \tag{5.3}$$

Finally, an application of the interior regularity properties for solutions of the elliptic $\bar{\partial}-$operator yields

$$\|\eta\|_{L^p_{0,s-1}(W,E^k)} \leq K(\|\eta\|_{L^1_{0,s-1}(Y,E^k)} + \|\hat{f}\|_{L^p_{0,s}(Y,E^k)}), \quad 1 \leq p \leq \infty,$$

which together with (5.3) give the estimates in Theorem 5.4. □

Proof of Theorem 1.1. Let $\hat{D} \supseteq \bar{D}$ be the strongly pseudo-convex domain furnished by Lemma 5.1 (An extension lemma). If $f \in L^1_{0,s}(D, E^k)$ with $s \geq 1$ and $\bar{\partial}f = 0$, then Lemma 5.1 yields a form $\hat{f} = L_1 f \in L^1_{0,s}(\hat{D}, E^k)$ and a form $u = L_2 f \in L^1_{0,s-1}(D, E^k)$ such that: $\bar{\partial}\hat{f} = 0$; $\hat{f} = f - \bar{\partial}u$ in D, and (i), (ii), (iii), (5.1), (5.2) in that lemma are valid.

We solve $\bar{\partial}\eta = \hat{f}$ using Theorem 5.4 (with $W = D$). Hence, $\eta \in L^1_{0,s-1}(D, E^k)$ and

$$\bar{\partial}\eta = \hat{f} = f - \bar{\partial}u \quad \text{in } D.$$

the desired solution is $g = \eta + u$. The estimates in the first part of Theorem 1.1 follows from those in Lemma 5.1 and Theorem 5.4. η and u are linear in f and they are C^∞ if f is C^∞. The first part of Theorem 1.1 is proved. □

Now, we prove the second part of Theorem 1.1. In fact, Theorem 4.2, Lemma 5.1, Proposition 5.2, and Proposition 5.3 are valid if we replace the vector bundle E^k by the vector bundle $\bigwedge^r T^\star(M)$. If M is a Stein manifold, then every strongly pseudo-convex domain of M is also a Stein manifold. Hence, as [2, Theorem 5.2.4], we can prove the following auxiliary theorem:

Theorem 5.5. *Let M be a Stein manifold of complex dimension n and let $D \Subset M$ be strongly pseudo-convex domain. Then, for every $f \in L^2_{r,s}(D, E^k, \mathrm{loc})$ with $\bar{\partial}f = 0$, $0 \leq r \leq n$ and $s \geq 1$ there exists a form $g = Tf \in L^2_{r,s-1}(D, E^k, \mathrm{loc})$; $\bar{\partial}g = f$, and a constant $c = c(D)$ such that*

$$\|g\|_{L^2_{r.s-1}(D,E^k,\mathrm{loc})} \leq c\|f\|_{L^2_{r,s}(D,E^k,\mathrm{loc})},$$

where T is a bounded linear operator. Moreover, for any $G \Subset D$ there exists a constant $c_1 = c_1(G, D)$ such that

$$\|g\|_{L^2_{r,s-1}(G,E^k,\mathrm{loc})} \leq c_1\|f\|_{L^2_{r,s}(D,E^k)}.$$

Then, we can apply the result of Theorem 5.5 instead of that of Theorem 3.3, we conclude that Theorem 5.4 is valid if we replace E^k by $\bigwedge^r T^\star(M)$; $0 \leq r \leq n$. Using this result and the identity

$$\Lambda^{r,s}(M) \equiv \Lambda^{0,s}(M, \wedge^r T^\star(M)), \quad 1 \leq r \leq n$$

we obtain the second part of our results.

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