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EXISTENCE OF SOLUTIONS TO N-DIMENSIONAL PENDULUM-LIKE EQUATIONS

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Abstract. We study the elliptic boundary-value problem

$$\Delta u + g(x,u) = p(x) \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0,$$

where g is T-periodic in u, and $\Omega \subset \mathbb{R}^n$ is a bounded domain. We prove the existence of a solution under a condition on the average of the forcing term p. Also, we prove the existence of a compact interval $I_p \subset \mathbb{R}$ such that the problem is solvable for $\tilde{p}(x) = p(x) + c$ if and only if $c \in I_p$.

1. Introduction

Existence and multiplicity of periodic solutions to the one-dimensional pendulum like equation

$$u'' + g(t,u) = p(t) \tag{1.1}$$
$$u(0) - u(T) = u'(0) - u'(T) = 0 \tag{1.2}$$

where g is T-periodic in u have been studied by many authors; see e.g. [4] and for the history and a survey of the problem see [6, 7]. In this work, we consider a generalization of this problem to higher dimensions. With this aim, note that the boundary condition (1.2) can be written as

$$u(0) = u(T) = c, \quad \int_0^T u'' = 0$$

where c is a non-fixed constant. Thus, by the divergence Theorem, (1.1)-(1.2) can be generalized to a boundary-value problem for an elliptic PDE in the following way:

$$\begin{gathered}\Delta u + g(x,u) = p(x) \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0,\end{gathered} \tag{1.3}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain. We shall assume that $p \in L^2(\Omega)$, and that $g \in L^\infty\big(\Omega \times \mathbb{R}\big)$ is T-periodic in u. For simplicity we shall assume also that $\frac{\partial g}{\partial u} \in L^\infty(\Omega \times \mathbb{R})$.

This kind of problems have been considered for example in [2], where the authors study a model describing the equilibrium of a plasma confined in a toroidal cavity. Under appropriate conditions this model can be reduced to the nonhomogeneous boundary-value problem

$$\begin{gathered} \Delta u + h(x,u) = 0 \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = \text{constant}, \quad -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = I. \end{gathered} \tag{1.4}$$

The authors prove the existence of at least one solution $u \in H^2$ of the problem for any h satisfying the following assumptions:

- (A1) $h : \overline{\Omega} \times \mathbb{R} \to [0, +\infty)$ is continuous, nondecreasing on u, with $h(x,u) = 0$ for $u \le 0$.
- (A2) $\lim_{u\to+\infty} \int_\Omega h(x,u)dx > I$.
- (A3) $\lim_{u\to+\infty} \frac{h(x,u)}{u^r} = 0$ for some $r \in \mathbb{R}$ (with $r \le \frac{n}{n-2}$ when $n > 2$).

On the other hand, for the particular case $h(x,u) = [u]_+^p$ and $\Omega = B_1(0)$, Ortega has proved in [9] that if $n > 2$ and $p \ge \frac{n}{n-2}$ then there exists a finite constant I_p such that the problem has no solutions for $I > I_p$.

In the second section we obtain a solution of (1.3) by variational methods under a condition on the average of the forcing term p.

In the third section we prove by topological methods that for a given p there exists a nonempty closed and bounded interval I_p such that problem (1.3) is solvable for $\tilde{p} = p + c$ if and only if $c \in I_p$. A similar result for the one-dimensional case has been proved by Castro [3], using variational methods, and by Fournier and Mawhin [4], using topological methods.

2. Solutions by variational methods

For fixed $x \in \Omega$, define $a_g(x)$ as the average of g with respect to u, namely:

$$a_g(x) = \frac{1}{T}\int_0^T g(x,u)du\,.$$

For $\varphi \in L^1(\Omega)$ denote by $\overline{\varphi}$ the average of φ, i.e.

$$\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi(x)dx.$$

Theorem 2.1. *If*

$$\overline{p} = \overline{a_g}, \tag{2.1}$$

then (1.3) *admits at least one solution* $u \in H^2(\Omega)$.

Proof. Let $\mathbb{R} + H_0^1(\Omega) = \{u \in H^1(\Omega) : u\big|_{\partial\Omega} = \text{constant}\}$, and consider the functional $\mathcal{I} : \mathbb{R} + H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \int_\Omega \Big(\frac{|\nabla u(x)|^2}{2} - G(x,u(x)) + p(x)u(x)\Big)dx,$$

where

$$G(x,u) = \int_0^u g(x,s)ds.$$

By standard results, $\mathcal{I}$ is weakly lower semicontinuous in $\mathbb{R}+H_0^1(\Omega)$. We remark that u is a critical point of $\mathcal{I}$ if and only if

$$\int_\Omega (\nabla u.\nabla\varphi - g(x,u)\varphi + p\varphi)dx = 0 \tag{2.2}$$

for any $\varphi \in \mathbb{R}+H_0^1(\Omega)$. In this case, if $c = u\big|_{\partial\Omega}$ then u is a weak solution of the problem

$$\Delta u + g(x,u) = p(x), \quad u\big|_{\partial\Omega} = c. \tag{2.3}$$

It follows that $u \in H^2(\Omega)$. We claim that $\int_{\partial\Omega}\frac{\partial u}{\partial\nu} = 0$. Indeed, taking $\varphi \equiv 1$ in (2.2) we obtain:

$$\int_\Omega g(x,u)dx = \int_\Omega p(x)dx.$$

Integrating (2.3) over Ω, we deduce that

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu} = \int_\Omega \Delta u = 0.$$

Thus, any critical point of $\mathcal{I}$ is a weak solution of (1.3).

To prove the existence of critical points of $\mathcal{I}$, let $\{u_n\} \subset \mathbb{R}+H_0^1(\Omega)$ be a minimizing sequence, and let $c_n = u_n\big|_{\partial\Omega}$. For any $u \in \mathbb{R}+H_0^1(\Omega)$ it holds that

$$\mathcal{I}(u+T) - \mathcal{I}(u) = T\int_\Omega p(x)dx - \int_\Omega [G(x,u+T) - G(x,u)]dx.$$

For fixed x, we have

$$G(x,u(x)+T) - G(x,u(x)) = \int_{u(x)}^{u(x)+T} g(x,s)ds = \int_0^T g(x,s)ds = Ta_g(x),$$

and from (2.1) we deduce that $\mathcal{I}(u+T) = \mathcal{I}(u)$. Hence, we may assume that $c_n \in [0,T]$. By Poincaré's inequality we have that

$$\|u_n - c_n\|_{L^2} \le C\|\nabla u_n\|_{L^2},$$

and then

$$I(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \int_\Omega pu_n dx - \int_\Omega G(x,u_n)dx \ge \frac{1}{2}\|\nabla u_n\|_{L^2}^2 - r\|\nabla u_n\|_{L^2} - s$$

for some constants r, s. Thus, $\{u_n\}$ is bounded, and by classical results $\mathcal{I}$ has a minimum on $\mathbb{R}+H_0^1(\Omega)$. □

3. The maximal interval I_p

Fix $p \in L^2(\Omega)$ such that $\overline{p} = \overline{a_g}$ and consider the problem

$$\begin{gathered} \Delta u + g(x,u) = p(x) + c \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = \text{constant} \quad \int_{\partial\Omega}\frac{\partial u}{\partial\nu} = 0 \end{gathered} \tag{3.1}$$

with $c \in \mathbb{R}$. It is easy to establish a necessary condition on c for the solvability of (3.1): indeed, if u is a solution of (3.1) then

$$\frac{1}{|\Omega|}\int_\Omega g(x,u(x))dx = \overline{p} + c.$$

Thus, if we define $g_u(x) = g(x,u(x))$, we obtain:

$$c = \overline{g_u} - \overline{a_g}.$$

Furthermore, if

$$g_+(x) = \sup_{0\le u\le T} g(x,u), \quad g_-(x) = \inf_{0\le u\le T} g(x,u),$$

it follows that $\overline{g_-} \le \overline{g_u} \le \overline{g_+}$, and hence

$$\overline{g_-} - \overline{a_g} \le c \le \overline{g_+} - \overline{a_g}.$$

In particular,

$$\inf_{[0,T]\times\mathbb{R}} g - \overline{a_g} \le c \le \sup_{[0,T]\times\mathbb{R}} -\overline{a_g}.$$

In the next theorem we obtain also a sufficient condition. More precisely, if we define

$$I_p = \{c\in\mathbb{R} :\ (3.1) \text{ admits a solution in } H^2(\Omega)\},$$

we shall prove that I_p is a nonempty compact interval. From Theorem 2.1, it follows that

$$I_p = [\alpha_p, \beta_p],$$

where

$$\overline{g_-} - \overline{a_g} \le \alpha_p \le 0 \le \beta_p \le \overline{g_+} - \overline{a_g}.$$

Theorem 3.1. *Assume that $\overline{p} = \overline{a_g}$ and define*

$$E = \{u\in\mathbb{R} + H^2\cap H_0^1(\Omega) : \Delta u + g(x,u) = p + \overline{g_u} - \overline{a_g}\}.$$

Then the set

$$E_g := \{\overline{g_u} : u\in E\} \subset \mathbb{R}$$

is a nonempty compact interval. Furthermore, $E_g = \overline{a_g} + I_p$.

For the proof of this theroem, we need Lemmas 3.2, 3.3, 3.4, 3.6, 3.7 and Theorem 3.8 below.

Lemma 3.2 (Poincaré-Wirtinger inequality)**.** *There exists a constant $c\in\mathbb{R}$ such that*

$$\|u-\overline{u}\|_{L^2} \le c\|\nabla u\|_{L^2}$$

for all $u\in H^1(\Omega)$.

The proof of the above lemma can be found in [5].

Lemma 3.3. *Assume that $\overline{p} = \overline{a_g}$. Then for any $r\in\mathbb{R}$ the problem*

$$\Delta u + g(x,u) = p + \overline{g_u} - \overline{a_g}$$
$$u\big|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial\nu} = 0$$

admits at least one solution u such that $\overline{u} = r$.

Proof. For $u\in H^1(\Omega)$ define $Tu = v$ as the unique solution of the problem

$$\begin{gathered}\Delta v = p + \overline{g_u} - \overline{a_g} - g(x,u)\\ v\big|_{\partial\Omega} = \text{constant}, \quad \overline{v} = r.\end{gathered} \tag{3.2}$$

Then $T: H^1(\Omega)\to H^1(\Omega)$ is well defined and compact. Indeed, if u_0 is the unique element of $H^2\cap H_0^1(\Omega)$ such that

$$\Delta u_0 = p + \overline{g_u} - \overline{a_g} - g(x,u),$$

it is clear that $v = u_0 - \overline{u_0} + r$ is the unique solution of (3.2), and compactness follows immediately from the compactness of the mapping $u \to u_0$. Moreover, integrating the equation, it is immediate that

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} = \int_{\Omega} \Delta v = 0.$$

Then

$$\int_{\Omega} \Delta v(v-r) + \int_{\Omega} |\nabla v|^2 = (v\big|_{\partial\Omega} - r) \int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0,$$

and we deduce that

$$\|v - r\|_{H^1} \le c\|\Delta v\|_{L^2} \le C$$

for some constant C. Thus, the proof follows from Schauder Theorem. $\square$

Lemma 3.4. *Let p, E, E_g be as in Theorem 3.1 and*

$$E_T = \{u \in E : u\big|_{\partial\Omega} \in [0,T]\}.$$

Then:

(1) $E_T \subset \mathbb{R} + H_0^1(\Omega)$ *is compact.*
(2) $E_g = \{\overline{g_u} : u \in E_T\}$.

Proof. Let $\{u_n\} \subset E_T$ and $c_n = u_n\big|_{\partial\Omega} \in [0,T]$. ¿From standard elliptic estimates it follows that $\|u_n\|_{H^2} \le C$ for some constant C. Taking a subsequence we may assume that $u_n \to u$ in $\mathbb{R} + H_0^1(\Omega)$. ¿From the equalities

$$\Delta u_n = p + \overline{g_{u_n}} - \overline{a_g} - g(x, u_n)$$

it follows easily that $u \in E_T$, and (1) is proved. Moreover, for any $u \in E$ there exists $k \in \mathbb{Z}$ such that $u_T := u + kT \in E_T$. As $g_{u_T} = g_u$, the proof of (2) follows. $\square$

To complete the proof of Theorem 3.1, it suffices to show that I_p is connected. Indeed, it is clear that u is a solution of (3.1) if and only if $u \in E$ with $c = \overline{g_u} - \overline{a_g}$, and by continuity of the mapping $u \to \overline{g_u}$ it follows that I_p is compact.

Remark 3.5. ¿From Lemma 3.3, E is infinite. In particular, if $I_p = \{0\}$ then (1.3) admits a continuum of solutions.

To apply the method of upper and lower solutions to our problem, we shall first prove an associated maximum principle:

Lemma 3.6. *Let $\lambda > 0$ and assume that $u \in H^2(\Omega)$ satisfies:*

$$\Delta u - \lambda u \ge 0,$$
$$u\big|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \le 0.$$

Then $u \le 0$.

Proof. If $u\big|_{\partial\Omega} = c \le 0$ the result follows by the classical maximum principle. If $c > 0$, let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ and $u^+(x) = \max\{u(x), 0\}$. Then

$$0 \le \int_{\Omega} \lambda u.u^+ \le \int_{\Omega} \Delta u.u^+ = -\int_{\Omega^+} |\nabla u|^2 + c\int_{\partial\Omega} \frac{\partial u}{\partial \nu} < 0\,,$$

a contradiction. $\square$

Lemma 3.7. *Let $\theta \in L^2(\Omega)$ and $\lambda > 0$. Then the problem*

$$\Delta u - \lambda u = \theta \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0$$

admits a unique solution $u_\theta \in H^2(\Omega)$. Furthermore, the mapping $\theta \to u_\theta$ is continuous.

Proof. Let $\mathcal{J} : \mathbb{R} + H_0^1(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{J}(u) = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{\lambda u^2}{2} + \theta u.$$

It is immediate that $\mathcal{J}$ is weakly lower semicontinuous and coercive, then it has a minimum u. Furthermore, $u \in H^2(\Omega)$ and $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0$. Integrating the equation, we also obtain that $-\lambda \overline{u} = \overline{\theta}$.

By standard elliptic estimates and Lemma 3.2, there exists a constant c such that

$$\|w - \overline{w}\|_{H^2} \le c\|\Delta w - \lambda w\|_{L^2}$$

for any $w \in H^2 \cap (\mathbb{R} + H_0^1)$ such that $\int_{\partial\Omega} \frac{\partial w}{\partial \nu} = 0$; thus, uniqueness follows. Finally, if θ_1, $\theta_2 \in L^2(\Omega)$ then

$$\|u_{\theta_1} - u_{\theta_2}\|_{H^2} \le |\Omega|.|\overline{\theta}_1 - \overline{\theta}_2| + c\|\theta_1 - \theta_2\|_{L^2},$$

and the proof is complete. □

Now we have the following result.

Theorem 3.8. *If $\varphi \in L^2(\Omega)$ and there exist $\alpha, \beta \in H^2(\Omega)$ with $\alpha \le \beta$ such that*

$$\Delta\beta + g(\cdot, \beta) \le \varphi(x) \le \Delta\alpha + g(\cdot, \alpha),$$
$$\beta\big|_{\partial\Omega} = \quad \text{constant}, \quad \alpha\big|_{\partial\Omega} = \quad \text{constant},$$
$$\int_{\partial\Omega} \frac{\partial \beta}{\partial \nu} \ge 0 \ge \int_{\partial\Omega} \frac{\partial \alpha}{\partial \nu},$$

then the problem

$$\Delta u + g(x, u) = \varphi(x)$$
$$u\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0$$

admits at least one solution $u \in H^2(\Omega)$ such that $\alpha \le u \le \beta$.

Proof. Let $\lambda \ge R$, where $R = \|\frac{\partial g}{\partial u}\|_{L^\infty}$. For fixed $v \in L^2(\Omega)$ define $Tv = u$ as the unique solution of the problem

$$\Delta u - \lambda u = \varphi - g(x, v) - \lambda v \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0.$$

By the lemmas above, the mapping $T : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. Moreover for $\alpha \le v \le \beta$, we have

$$\Delta u - \lambda u = \varphi - g(x, v) - \lambda v \ge \varphi - g(x, \beta) - \lambda\beta \ge \Delta\beta - \lambda\beta.$$

Hence,

$$\Delta(u - \beta) - \lambda(u - \beta) \ge 0$$

and

$$(u-\beta)\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial(u-\beta)}{\partial\nu} \le 0.$$

From Lemma 3.6, we deduce that $u \le \beta$. In the same way, we obtain that $u \ge \alpha$ and the result follows by Schauder Theorem. □

Proof of Theorem 3.1. Let $P \in H^2(\Omega)$ be any solution of the problem

$$\Delta P = p - \overline{a_g}$$
$$P\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial P}{\partial\nu} = 0.$$

Taking $v = u - P$, problem (3.1) is equivalent to the problem

$$\Delta v + \tilde{g}(x,v) = c + \overline{a_g}$$
$$P\big|_{\partial\Omega} = \quad \text{constant}, \quad \int_{\partial\Omega} \frac{\partial P}{\partial\nu} = 0\,,$$

where $\tilde{g}(x,v) := g(x, v + P(x))$ is continuous and T-periodic in v. Thus, we may assume without loss of generality that p is continuous. Let $c_1, c_2 \in I_p$, $c_1 < c_2$, and take $u_1, u_2 \in E$ such that $\overline{g_{u_i}} = c_i - \overline{a_g}$. As $u_i \in C(\overline{\Omega})$, adding kT for some integer k if necessary, we may suppose that $u_1 \le u_2$. For $c \in [c_1, c_2]$ we have that

$$\Delta u_1 + g(x,u_1) = p + c_1 - \overline{a_g} \le p + c - \overline{a_g} \le p + c_2 - \overline{a_g} = \Delta u_2 + g(x,u_2).$$

From the previous theorem, there exists $u \in E$ such that $\overline{g_u} = c - \overline{a_g}$. The proof is complete. □

Remark 3.9. Using fixed point methods, Lemma 3.7 can be generalized; thus, it is easy to see that Theorem 3.1 is still valid for the more general problem

$$\Delta u + \langle b(x), \nabla u\rangle + g(x,u) = p(x) \quad \text{in } \Omega$$
$$u\big|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} \frac{\partial u}{\partial\nu} = 0\,,$$

where b is a C^1-field such that div $b = 0$. However, for $b \neq 0$ the problem is no longer variational, and then the claim of Theorem 2.1 is not necessarily true. Indeed, in the particular case $n = 1$, it is well known that for the pendulum equation

$$u'' + au' + b\ \sin u = f(t),$$

where a is a positive constant, there exists a family of T-periodic functions f such that $\int_0^T f = 0$ for which the equation has no periodic solutions (see [1, 8, 10]).

Remark 3.10. As in [4], it can be proved that for any c in the interior of I_p there exist at least two solutions of (3.1) which are essentially different (i.e. not differing by a multiple of T).

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