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HETEROCLINIC ORBITS OF A SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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ABSTRACT. This article concerns a second-order nonlinear difference equation. By using critical point theory, the existence of two heteroclinic orbits is obtained. The main method used is variational.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbb{Z}$, we define $\mathbb{Z}(a, b) = \{n \in \mathbb{Z} | a < n < b\}$, $\mathbb{Z}[a, b] = \{n \in \mathbb{Z} | a \le n \le b\}$. For a set $M \subset \mathbb{R}, r > 0$, $B_r(M)$ is denoted by

$$B_r(M) = \{u \in \mathbb{R} : \inf_{v\in M} |u - v| < r\}.$$

In this article we consider the existence of heteroclinic orbits of the second-order nonlinear difference equation

$$\Delta^2 u_{n-1} + p_n f(u_n) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\{p_n\}_{n\in\mathbb{Z}}$ is a positive real sequence, $f \in C(\mathbb{R}, \mathbb{R})$. Moreover, p and f satisfy the conditions:

- (A1) $0 < \underline{p} = \inf_{n\in\mathbb{Z}}\{p_n\} \le \bar{p} = \sup_{n\in\mathbb{Z}}\{p_n\} < +\infty$;
- (A2) there exists a function $F \in C^1(\mathbb{R}, \mathbb{R})$ with $F(0) = 0$, $F(u + T) = F(u)$, $F'(u) = f(u)$ and F has a maximum 0 on $\mathbb{R}$. Denote $\Psi = \{u \in \mathbb{R} : F(u) = 0\}$.
- (A3) Ψ consists only of isolated points and $0 \in \Psi$.

As usual, a solution u of (1.1) is called a heteroclinic orbit (or heteroclinic solution) if there exist two constants $\mu, \nu \in \mathbb{R}, \mu \ne \nu$ such that u joins μ to ν, i.e.,

$$u_{-\infty} = \lim_{n\to-\infty} u_n = \mu,$$
$$u_{+\infty} = \lim_{n\to+\infty} u_n = \nu.$$

Such orbits and homoclinic orbits have been found in various models of continuous and discrete dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So the heteroclinic orbits and homoclinic orbits

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have been extensively studied, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

In 1989, Rabinowitz [17] considered the following second-order Hamiltonian system

$$\ddot{q} + V'(q) = 0 \tag{1.2}$$

where $q = (q_1, \dots, q_n)$, V is periodic in $q_i, 1 \leq i \leq n$, and proved the existence and multiple heteroclinic orbits joining maxima of V.

By using variational method and a delicate analysis technique, Xiao and Yu [22] showed that there indeed exist heteroclinic orbits of discrete pendulum equation

$$\Delta^2 u_{n-1} + A \sin u_n = 0, n \in \mathbb{Z}, \tag{1.3}$$

joining every two adjacent points of $\{2k\pi + \pi : k \in \mathbb{Z}\}$.

When $p_n \equiv 1$, Xiao, Long and Shi [21] in 2010 investigated the existence and multiplicity of heteroclinic orbits of the system

$$\Delta^2 u_{n-1} + V'(u_n) = 0, \quad n \in \mathbb{Z}, \tag{1.4}$$

by using the critical point theory. Zhang and Li [24] using variational method proved some existence results of heteroclinic orbits and heteroclinic chains for a second order discrete Hamiltonian system of (1.4).

However, to the best of our knowledge, the results on heteroclinic orbits of discrete systems are very scarce in the literature [21, 22, 24]. The difficulty is the idea of continuous systems depend heavily on the continuity of the solutions and therefore they can not be applied directly to discrete systems. Motivated by the recent papers [3, 6], the purpose of this paper is to consider problem (1.3) in a more general sense. It is obvious that (1.3) is a special of (1.1) with $p_n \equiv A$ and $f(u_n) = \sin u_n$. Our main result is as follows.

Theorem 1.1. *Suppose that* (A1)–(A3) *are satisfied. Then* (1.1) *possesses two heteroclinic orbits joining* 0 *to some* $\tau \in \Psi \backslash \{0\}$*, one of which originates from* 0 *and one of which terminates at* 0.

For basic knowledge of variational methods, we refer the reader to the monographs [14, 18].

2. Variational structure and some lemmas

To apply the critical point theory, we shall establish the corresponding variational functional associated with (1.1) and give some lemmas which will be used in proving our main results. We firstly introduce some basic notation.

Let S be the set of bi-infinite convergent sequences $u = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S := \big\{\{u_n\} | \lim_{n\to+\infty} u_n \text{ and } \lim_{n\to-\infty} u_n \text{ exist}, u_n \in \mathbb{R}, n \in \mathbb{Z}\big\}.$$

Define

$$E := \big\{u \in S : \sum_{n=-\infty}^{+\infty} |\Delta u_n|^2 < +\infty\big\},$$

with the inner product

$$\langle u, v\rangle = \sum_{n=-\infty}^{+\infty} \Delta u_n \Delta v_n + u_0 v_0, \quad \forall u, v \in E. \tag{2.1}$$

Then E is a Hilbert space with the norm

$$\|u\|^2 = \sum_{n=-\infty}^{+\infty} |\Delta u_n|^2 + |u_0|^2, \quad \forall u \in E. \tag{2.2}$$

For $1 < s < +\infty$, the spaces l^s and l^∞ are defined by

$$l^s := \big\{\{u_n\} : \sum_{n=-\infty}^{+\infty} |u_n|^s < +\infty,\ u_n \in \mathbb{R}, n \in \mathbb{Z}\big\},$$
$$l^\infty := \big\{\{u_n\} : \sup_{n\in\mathbb{Z}} |u_n| < +\infty,\ u_n \in \mathbb{R}, n \in \mathbb{Z}\big\}.$$

For any $u \in E$, define the functional J associated with (1.1) on E as follows:

$$J(u) := \frac{1}{2} \sum_{n=-\infty}^{+\infty} |\Delta u_n|^2 - \sum_{n=-\infty}^{+\infty} p_n F(u_n). \tag{2.3}$$

By (A2) and (A3), we have

$$\delta := \frac{1}{3} \inf_{\rho,\varrho\in\Psi,\rho\neq\varrho} |\rho - \varrho| > 0.$$

For $\rho \in \Psi$ and $0 < \epsilon < \delta$, let the set $\Gamma_\epsilon(\rho)$ satisfy

(i) $u_{-\infty}$=0,
(ii) $u_{+\infty} = \rho$,
(iii) $u_n \notin B_\epsilon(\Psi \setminus \{0, \rho\})$ for all $n \in \mathbb{Z}$.

It is easy to see that $\Gamma_\epsilon(\rho)$ is nonempty for all $\rho \in \Psi \setminus \{0\}$ and $0 < \epsilon < \delta$. Denote

$$c_\epsilon(\rho) := \inf_{u\in\Gamma_\epsilon(\rho)} J(u),$$
$$\varphi_\epsilon := \inf_{u\notin B_\epsilon(\Psi)} [-F(u)].$$

Remark 2.1. From (A2) and (A3) it follows that $\varphi_\epsilon > 0$ for all $0 < \epsilon < \delta$. As a matter of fact, $\varphi_\epsilon \neq 0$. If not, there is $v \in \mathbb{R} \notin B_\epsilon(\Psi)$ such that $F(0) = 0$ implies that $v \in \Psi$. This is a contradiction. From $F(u + T) = F(u)$ and $u \notin B_\epsilon(\Psi)$ it follows that $\varphi_\epsilon > 0$.

Lemma 2.2. *For any $a \le b$, assume that $u \in E$ such that $u_n \notin B_\epsilon(\Psi)$, then*

$$\frac{1}{2} \sum_{n=a}^{b} |\Delta u_n|^2 - \sum_{n=a}^{b} p_n F(u_n) \ge \sqrt{2\underline{p}\varphi_\epsilon}|u_{b+1} - u_a|.$$

Proof. By the definition of φ_ϵ and Hölder inequality, we have

$$|u_{b+1} - u_a| \le \sqrt{b + 1 - a}\Big(\sum_{n=a}^{b} |\Delta u_n|^2\Big)^{1/2}.$$

Then

$$\sum_{n=a}^{b} |\Delta u_n|^2 \ge \frac{|u_{b+1} - u_a|^2}{b + 1 - a}.$$

Thus,

$$\frac{1}{2} \sum_{n=a}^{b} |\Delta u_n|^2 - \sum_{n=a}^{b} p_n F(u_n) \ge \frac{1}{2} \sum_{n=a}^{b} |\Delta u_n|^2 + \underline{p} \sum_{n=a}^{b} [-F(u_n)]$$

$$\begin{aligned}&\geq \frac{|u_b-u_a|^2}{2(b+1-a)}+\underline{p}(b+1-a)\varphi_\epsilon\\&\geq \sqrt{2\underline{p}\varphi_\epsilon}|u_{b+1}-u_a|.\end{aligned}$$

The desired results are obtained. □

Remark 2.3. For all $\rho\in\Psi\setminus\{0\}$ and $0<\epsilon<\delta$, it follows immediately from Lemma 2.2 that $c_\epsilon(\rho)>0$.

Lemma 2.4. *Assume that $u\in E$ and $J(u)<+\infty$, then there are two constants $\mu,\nu\in\Psi$ such that $u_{-\infty}=\mu, u_{+\infty}=\nu$.*

Proof. To prove $\mu\in\Psi$, arguing by contradiction, we suppose that there exists $\theta>0$ such that $u_n\notin B_\theta(\Psi)$ for all n near $-\infty$. Then, we have

$$J(u)\geq\sum_{n=-\infty}^{n}[-p_nF(u_n)]\geq\underline{p}\sum_{n=-\infty}^{n}\varphi_\theta,\forall n\in\mathbb{Z},$$

which contradicts with $J(u)<+\infty$. Thus, $\mu\in\Psi$. The proof of $\nu\in\Psi$ is similar to the proof of $\mu\in\Psi$. □

By using the ideas developed in [21, 24], we can easily obtain the following three lemmas, but for the sake of completeness, we give the proofs.

Lemma 2.5. *For any given $\rho\in\Psi\backslash\{0\}$, assume that $\{u^{(k)}\}_{k=1}^{\infty}$ is a minimizing sequence for (1.1) restricted to $\Gamma_\epsilon(\rho)$ such that $u_n^{(k)}\to u\in E$ and $J(u)<+\infty$, then $u\in\Gamma_\epsilon(\rho)$.*

Proof. First, $u_n\notin B_\epsilon(\Psi\setminus\{0,\rho\})$ for all $n\in\mathbb{Z}$. Otherwise, there is n_0 and $\psi\in\Psi\setminus\{0,\rho\}$ such that $u_{n_0}\in B_\epsilon(\psi)$. Therefore, for sufficiently large k, we have

$$|u_{n_0}^{(k)}-\psi|\leq|u_{n_0}^{(k)}-u_{n_0}|+|\psi-u_{n_0}|<\epsilon,$$

which is a contradiction.

Then $u_{-\infty}=\mu\in\{0,\rho\}, u_{+\infty}=\nu\in\{0,\rho\}$. Otherwise, for sufficiently large k_1 and k_2, we have

$$|u_{-k_1}^{(k)}-\mu|\leq|u_{-k_1}^{(k)}-u_{-k_1}|+|u_{-k_1}-\mu|<\epsilon,$$

and

$$|u_{k_2}^{(k)}-\nu|\leq|u_{k_2}^{(k)}-u_{k_2}|+|u_{k_2}-\nu|<\epsilon,$$

which are contradictions.

Next, $u_{-\infty}=0$. From $u^{(k)}\in\Gamma_\epsilon(\rho)$, $u_n^{(k)}\in B_\epsilon(0)$ and $u_n^{(k)}\in\bar{B}_\epsilon(0)$ for $n<0$. Therefore, $\mu\in\bar{B}_\epsilon(0)\cap\{0,\rho\}=\{0\}$.

Finally, $u_{+\infty}=\rho$. Otherwise, $u_{+\infty}=0$. If $u_1^{(k)}\in B_\epsilon(0)$, then $|\Delta u_0^{(k)}|\geq\delta$. Thus,

$$J\big(u^{(k)}\big)\geq\frac{\delta^2}{2}+\frac{1}{2}\sum_{n=2}^{+\infty}|\Delta u_n^{(k)}|^2-\sum_{n=2}^{+\infty}p_nF\big(u_n^{(k)}\big).\tag{2.4}$$

If $u_1^{(k)} \notin B_\epsilon(0)$, then there is an $n^{(k)} \le 1$ such that $u_n^{(k)} \notin B_{\frac{\epsilon}{2}}\{\Psi\}$, $n = n^{(k)}, n^{(k)}+1, \dots, 1$. It follows from Lemma 2.2 that

$$\begin{aligned} &J\big(u^{(k)}\big) \\ &\ge \frac{1}{2}\sum_{n=n^{(k)}}^{1}|\Delta u_n^{(k)}|^2 - \sum_{n=n^{(k)}}^{1} p_n F\big(u_n^{(k)}\big) + \frac{1}{2}\sum_{n=2}^{+\infty}|\Delta u_n^{(k)}|^2 - \sum_{n=2}^{+\infty} p_n F\big(u_n^{(k)}\big) \\ &\ge \frac{\sqrt{2\underline{p}\varphi_{\frac{\epsilon}{2}}}\,\epsilon}{2} + \frac{1}{2}\sum_{n=2}^{+\infty}|\Delta u_n^{(k)}|^2 - \sum_{n=2}^{+\infty} p_n F\big(u_n^{(k)}\big). \end{aligned} \tag{2.5}$$

Set

$$M = \min\Big\{\frac{\delta^2}{2}, \frac{\sqrt{2\underline{p}\varphi_{\frac{\epsilon}{2}}}\,\epsilon}{2}\Big\}.$$

By (2.4) and (2.5), we have

$$J\big(u^{(k)}\big) \ge M + \frac{1}{2}\sum_{n=2}^{+\infty}|\Delta u_n^{(k)}|^2 - \sum_{n=2}^{+\infty} p_n F\big(u_n^{(k)}\big). \tag{2.6}$$

Since $u_{+\infty} = 0$, there is $\tilde{n} \ge 1$ such that

$$u_n^2 \le \frac{M}{16}, \quad \forall n \ge \tilde{n}.$$

For k large enough, we have

$$\Big(u_{\tilde{n}}^{(k)}\Big)^2 \le \frac{M}{12}, \quad \Big(u_{\tilde{n}+1}^{(k)}\Big)^2 \le \frac{M}{12}.$$

Denote

$$v_n^{(k)} = \begin{cases} 0, & n < \tilde{n}+1, \\ u_n^{(k)}, & n \ge \tilde{n}+1. \end{cases}$$

Thus,

$$\begin{aligned} |\Delta v_{\tilde{n}}^{(k)}|^2 &= |u_{\tilde{n}+1}^{(k)}|^2 = |\Delta u_{\tilde{n}}^{(k)} + u_{\tilde{n}}^{(k)}|^2 \\ &\le |\Delta u_{\tilde{n}}^{(k)}|^2 + 4|u_{\tilde{n}}^{(k)}|^2 + 2|u_{\tilde{n}+1}^{(k)}|^2 \le |\Delta u_{\tilde{n}}^{(k)}|^2 + \frac{M}{2}. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned} J\big(v^{(k)}\big) &= \frac{1}{2}\sum_{n=\tilde{n}+1}^{+\infty}|\Delta v_n^{(k)}|^2 - \sum_{n=\tilde{n}+1}^{+\infty} p_n F\big(v_n^{(k)}\big) \\ &\le \frac{1}{2}\sum_{n=2}^{+\infty}|\Delta u_n^{(k)}|^2 - \sum_{n=2}^{+\infty} p_n F\big(u_n^{(k)}\big) + \frac{M}{2} \\ &\le J\big(u^{(k)}\big) - \frac{M}{2}. \end{aligned} \tag{2.8}$$

From (2.8), we have

$$\inf_{s\in\Gamma_\epsilon(\rho)} J(s) \le \inf_{s\in\Gamma_\epsilon(\rho)} J(s) - \frac{M}{2},$$

which is a contradiction. The proof is complete. □

Lemma 2.6. *For any given $\rho \in \Psi\backslash\{0\}$ and $0 < \epsilon < \delta$, there is $\bar{u} = u_{\epsilon,\rho} \in \Gamma_\epsilon(\rho)$ such that $J(u_{\epsilon,\rho}) = c_{\epsilon,\rho}$.*

Proof. Assume that $\{u^{(k)}\}_{k=1}^{\infty}$ is a minimizing sequence for (1.1) restricted to $\Gamma_\epsilon(\rho)$. There is a constant $K>0$ such that $J\big(u^{(k)}\big)\le K$.

On one hand, $\{u_0^{(k)}\}_{k=1}^{\infty}$ is a bounded sequence. If not, $\lim_{i\to\infty} u_0^{(k_i)}=\infty$ and there is $i_0\in\mathbb{N}$ such that $u_0^{(k_i)}\notin B_\epsilon(\rho), i\ge i_0$. Consider $\{u_j^{(k_i)}\}_{i=i_0}^{\infty}$.

Case 1. If $u_j^{(k_i)}\in \bar{B}_\epsilon(\rho)$, then $J(u^{(k_i)})\ge \frac{|u_0^{(k_i)}-\rho-\epsilon|^2}{2}$ and $J(u^{(k_i)})\to\infty$, $i\to+\infty$, which is a contradiction.

Case 2. If $u_j^{(k_i)}\notin \bar{B}_\epsilon(\rho)$. Set

$$n_i=\{n>0: u_{n+j}^{(k_i)}\in \bar{B}_\epsilon(\rho), u_j^{(k_i)}\notin \bar{B}_\epsilon(\rho), \forall j\in\mathbb{Z}[0,n]\}.$$

Then

$$J(u^{(k_i)})\ge\sqrt{2\underline{p}\varphi_\epsilon}|u_0^{(k_i)}-u_{n_i}^{(k_i)}|+\frac{1}{2}|u_{n_i+j}^{(k_i)}-u_{n_i}^{(k_i)}|^2$$

and $J(u^{(k_i)})\to\infty$ as $i\to+\infty$, which is also a contradiction.

By the definition of the norm on E, $\{u^{(k)}\}_{k=1}^{\infty}$ is a bounded sequence. Thus, passing to a subsequence if necessary, there is $\bar{u}\in E$ such that $u^{(k)}$ weakly converges to $\bar{u}$.

On the other hand, $J(\bar{u})<\infty$. As a matter of fact, for $-\infty<a<b<+\infty$, let

$$J(a,b,u)\ge\frac{1}{2}\sum_{n=a}^{b}|\Delta u_n|^2-\sum_{n=a}^{b}p_nF(u_n), u\in E.$$

Thus,

$$J(a,b,\bar{u})\le c_{\epsilon,\rho}\le K,$$

which implies that $J(\bar{u})\le\inf_{u\in\Gamma_\epsilon(\rho)}J(u)$. It follows from Lemma 2.5 that $\bar{u}\in\Gamma_\epsilon(\rho)$. Therefore, $J(u_{\epsilon,\rho})=c_{\epsilon,\rho}$. □

Set

$$c_\epsilon=\inf_{\rho\in\Psi\backslash\{0\}}c_{\epsilon,\rho}.$$

Lemma 2.7. *For any given $\rho\in\Psi\backslash\{0\}$ and $0<\epsilon<\delta$, c_ϵ can be achieved by some $c_{\epsilon,\tau}=J(u_{\epsilon,\tau})$ with $\tau=\tau_\epsilon$ and $u=u_\epsilon=u_{\epsilon,\tau}$ is an interior point of $\Gamma_\epsilon(\tau)$.*

Proof. Let $0<\epsilon^{(i)}<\delta$ is a sequence converging to 0. By $(A3)$, $\{\tau_{\epsilon^{(i)}}\}$ consists of finite elements. Thus, for larger i, $\tau_{\epsilon^{(i)}}=\tau$ independent of i. Denote $u^{(i)}=u_{\epsilon^{(i)},\tau}$. For each $i\in\mathbb{N}$, there is $N_i>0$ such that

$$u_{-n}^{(i)}\in B_\epsilon(0), u_n^{(i)}\in B_{\epsilon^{(i)}}(\tau),\quad \forall n\ge N_i.$$

Assume that for all $i\in\mathbb{N}$, $u^{(i)}$ is not an interior point of $\Gamma_\epsilon(\tau)$. Thus, there is $n^{(i)}\in[-N_i,N_i]$ such that $u_{n^{(i)}}^{(i)}\in\overline{B_{\epsilon^{(i)}}(\Psi\setminus\{0,\tau\})}$. Then, there is $\omega^{(i)}\in\Psi\setminus\{0,\tau\}$ such that $u_{n^{(i)}}^{(i)}\in\overline{B_{\epsilon^{(i)}}(\omega^{(i)})}$ and $\omega_i=\omega$ independent of i. Set

$$\Omega_n^{(i)}=\begin{cases}u_n^{(i)}, & n\le n^{(i)},\\ \omega, & n>n^{(i)}.\end{cases}$$

Therefore, we have $\Omega^{(i)}\in\Gamma_{\epsilon^{(i)}}(\omega)$ and

$$\begin{aligned}&J(u^{(i)})-J(\Omega^{(i)})\\ &=\frac{1}{2}\sum_{n=n^{(i)}+1}^{+\infty}|\Delta u_n^{(i)}|^2-\sum_{n=n^{(i)}+1}^{+\infty}p_nF(u_n^{(i)})-\frac{1}{2}|\omega-u_{n^{(i)}}^{(i)}|^2.\end{aligned}\tag{2.9}$$

If there is $n > n^{(i)}$ such that $|\Delta u_n^{(i)}| > |\omega - u_{n^{(i)}}^{(i)}|$, then $J(\Omega^{(i)}) < J(u^{(i)}) = c_{\epsilon^{(i)}}$ which is a contradiction to the definition of $c_{\epsilon^{(i)}}$. Thus,

$$|\Delta u_n^{(i)}| \le |\omega - u_{n^{(i)}}^{(i)}| \le \epsilon^{(i)}, quad \forall n > n^{(i)}.$$

From $u_\infty^{(i)} = \tau$, there is $m^{(i)}$ such that $u_{m^{(i)}}^{(i)} \in B_{\epsilon^{(0)}}(\tau), m^{(i)} > n^{(i)}$ and $u_n^{(i)} \notin B_{\epsilon^{(0)}}(\Psi), n^{(i)} < n < m^{(i)}$. Since $u_{m^{(i)}}^{(i)} \in B_{\epsilon^{(0)}}(\tau), u_{m^{(i)}-1}^{(i)} \notin B_{\epsilon^{(0)}}(\tau), |\Delta u_n^{(i)}| \le |\omega - u_{m^{(i)}}^{(i)}|$, for i large enough, we have $u_{m^{(i)}}^{(i)} \in B_{\epsilon^{(0)}}(\tau) \setminus \overline{B_{\frac{\epsilon(0)}{2}}(\tau)}$. It follows from (2.9) and Lemma 2.2 that

$$\begin{aligned}
&J(u^{(i)}) - J(\Omega^{(i)}) \\
&\ge \frac{1}{2}\sum_{n=n^{(i)}+1}^{m^{(i)}-1} |\Delta u_n^{(i)}|^2 - \sum_{n=n^{(i)}+1}^{m^{(i)}-1} p_n F(u_n^{(i)}) - \frac{(\epsilon^{(i)})^2}{2} \\
&\ge \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}} \sum_{n=n^{(i)}+1}^{m^{(i)}-1} |\Delta u_n^{(i)}| - \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}} |\Delta u_{m^{(i)}-1}^{(i)}| - \frac{(\epsilon^{(i)})^2}{2} \qquad (2.10)\\
&\ge \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}} |u_{m^{(i)}}^{(i)} - u_{n^{(i)}}^{(i)}| - \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}}\, \epsilon^{(i)} - \frac{(\epsilon^{(i)})^2}{2} \\
&\ge \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}}\, \epsilon^{(0)} - \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}}\, \epsilon^{(i)} - \frac{(\epsilon^{(i)})^2}{2}.
\end{aligned}$$

Since $\epsilon^{(i)}$ is a sequence converging to 0, for i large enough, we have

$$J(\Omega^{(i)}) \le J(u^{(i)}) - \sqrt{2\underline{p}\varphi_{\frac{\epsilon(0)}{2}}}\, \epsilon^{(0)},$$

which contradicts $J(u^{(i)}) = J(u_{\epsilon^{(i)},\tau}) = \inf_{\rho \in \Psi \setminus \{0\}} c_\epsilon(\rho)$. The proof is complete. □

3. Proof of main result

In this section, we proof Theorem 1.1 using a variational method.

Proof of Theorem 1.1. For any $n \in \mathbb{Z}$, it follows from Lemma 2.7 that

$$\frac{d}{du_n} J|_{\Gamma_\epsilon(\tau)}(u) = 0. \qquad (3.1)$$

By (1.1), we have

$$\frac{d}{du_n} J|_{\Gamma_\epsilon(\tau)}(u) = \frac{d}{du_n} J(u) = -\Delta^2 u_{n-1} - p_n f(u_n). \qquad (3.2)$$

From (3.1) and (3.2), we know that $u = u_\epsilon = u_{\epsilon,\tau}$ is a heteroclinic orbit of (1.1) connecting 0 to τ, which originates from 0. And $\omega_{(\cdot)} = u_{(-\cdot)}$ is also a heteroclinic orbit of (1.1) connecting τ to 0, which terminates at τ. The proof is complete. □

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References

[1] Z. AlSharawi, J. M. Cushing, S. Elaydi; *Theory and Applications of Difference Equations and Discrete Dynamical Systems*, Springer, New York, 2014.

[2] P. Chen; *Existence of homoclinic orbits in discrete Hamiltonian systems without Palais-Smale condition*, J. Difference Equ. Appl., 19 (11) (2013), 1781-1794.

[3] P. Chen, T. Cai; *Infinitely many solutions for Schrödinger-Maxwell equations with indefinite sign subquadratic potentials*, Appl. Math. Comput., 226 (2014), 492-502.

[4] P. Chen, X. F. He, X. H. Tang; *Infinitely many solutions for a class of fractional Hamiltonian systems via critical point theory*, Math. Methods Appl. Sci., 39(5) (2016), 1005-1019.

[5] P. Chen, Z. M. Wang; *Infinitely many homoclinic solutions for a class of nonlinear difference equations*, Electron. J. Qual. Theory Differ. Equ., (47) (2012), 1-18.

[6] C. J. Guo, R. P. Agarwal, C. J. Wang, D. O'Regan; *The existence of homoclinic orbits for a class of first order superquadratic Hamiltonian systems*, Mem. Differential Equations Math. Phys., 61 (2014), 83-102.

[7] C. J. Guo, D. O'Regan, Y. T. Xu, R. P. Agarwal; *Existence of homoclinic orbits for a class of first-order differential difference equations*, Acta Math. Sci. Ser. B Engl. Ed., 35(5) (2015), 1077-1094.

[8] C. J. Guo, D. O'Regan, Y. T. Xu, R. P. Agarwal; *Existence of homoclinic orbits of a class of second-order differential difference equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, 20 (2013), 675-690.

[9] C. J. Guo, D. O'Regan, C. J. Wang, R. P. Agarwal; *Existence of homoclinic orbits of superquadratic second-order Hamiltonian systems*, Z. Anal. Anwend., 34(1) (2015), 27-41.

[10] W. N. Huang, X. H. Tang; *The existence of heteroclinic orbits for a class of the second-order Hamiltonian system*, Mediterr. J. Math., 12(1) (2015), 9-20.

[11] M. H. Huang, Z. Zhou; *Ground state solutions of the periodic discrete coupled nonlinear Schrödinger equations*, Math. Methods Appl. Sci., 38(8) (2015), 1682-1695.

[12] M. H. Huang, Z. Zhou; *Standing wave solutions for the discrete coupled nonlinear Schrödinger equations with unbounded potentials*, Abstr. Appl. Anal., 2013 (2013), 1-6.

[13] M. H. Huang, Z. Zhou; *On the existence of ground state solutions of the periodic discrete coupled nonlinear Schrödinger lattice*, J. Appl. Math., 2013 (2013), 1-8.

[14] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[15] P. H. Rabinowitz; *Heteroclinic orbits for a Hamiltonian system of double pendulum type*, Topol. Methods Nonlinear Anal., 9(1) (1997), 41-76.

[16] P. H. Rabinowitz; *Homoclinic and heteroclinic orbits for a class of Hamiltonian systems*, Calc. Var. Partial Differential Equations, 1(1) (1993), 1-36.

[17] P. H. Rabinowitz; *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, Nonlinear Anal., 6(5) (1989), 331-346.

[18] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Amer. Math. Soc., Providence, RI: New York, 1986.

[19] X. H. Tang; *Infinitely many homoclinic solutions for a second-order Hamiltonian system*, Math. Nachr., 289(1) (2016), 116-127.

[20] F. Xia; *Homoclinic solutions for second-order nonlinear difference equations with Jacobi operators*, Electron. J. Differential Equations, 2017(94) (2017), 1-11.

[21] H. F. Xiao, Y. H. Long, H. P. Shi; *Heteroclinic orbits for discrete Hamiltonian systems*, Comm. Math. Anal., 9(1) (2010), 1-14.

[22] H. F. Xiao, J. S. Yu; *Existence of heteroclinic orbits for discrete pendulum equation*, J. Difference Equ. Appl., 17(9) (2011), 1267-1280.

[23] L. W. Yang; *Existence of homoclinic orbits for fourth-order p-Laplacian difference equations*, Indag. Math. (N.S.), 27(3) (2016), 879-892.

[24] H. Zhang, Z. X. Li; *Heteroclinic orbits and heteroclinic chains for a discrete Hamiltonian system*, Sci. China Math., 53(6) (2010), 1555-1564.

[25] Z. Zhou, D. F. Ma; *Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials*, Sci. China Math., 58(4) (2015), 781-790.

[26] Z. Zhou, J. S. Yu; *Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity*, Acta Math. Sin. (Engl. Ser.), 29(9) (2013), 1809-1822.

[27] Z. Zhou, J. S. Yu; *On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems*, J. Differential Equations, 249(5) (2010), 1199-1212.

[28] Z. Zhou, J. S. Yu, Y. M. Chen; *Homoclinic solutions in periodic difference equations with saturable nonlinearity*, Sci. China Math., 54(1) (2011), 83-93.

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