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SEMIPOSITONE m-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. We study the m-point nonlinear boundary-value problem

$$-[p(t)u'(t)]' = \lambda f(t, u(t)), \quad 0 < t < 1,$$
$$u'(0) = 0, \quad \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1),$$

where $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\alpha_i > 0$ for $1 \le i \le m-2$ and $\sum_{i=1}^{m-2} \alpha_i < 1$, $m \ge 3$. We assume that $p(t)$ is non-increasing continuously differentiable on $(0, 1)$ and $p(t) > 0$ on $[0, 1]$. Using a cone-theoretic approach we provide sufficient conditions on continuous $f(t, u)$ under which the problem admits a positive solution.

1. Introduction

In this note we consider the nonlinear m-point eigenvalue problem

$$-[p(t)u'(t)]' = \lambda f(t, u(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u'(0) = 0, \quad \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1), \tag{1.2}$$

where $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\alpha_i > 0$ for $1 \le i \le m-2$, $\sum_{i=1}^{m-2} \alpha_i < 1$. We also assume that the function $p(t)$ is non-increasing continuously differentiable on $(0, 1)$ and $p(t) > 0$ on $[0, 1]$. The inhomogeneous term in (1.1) is allowed to change its sign. Other assumptions on $f(t, u(t))$ will be made later.

The study of multi-point boundary-value problems was initiated by Il'in and Moiseev in [7, 8]. Many authors since then considered nonlinear multi-point boundary-value problems (see, e.g., [2, 4, 5, 6, 9, 14, 15, 16, 17] and the references therein). In particular, Ma studied in [15] positive solutions to the three-point nonlinear boundary-value problem

$$-u''(t) = a(t)f(u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

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where $0 < \alpha$, $0 < \eta < 1$ and $\alpha\eta < 1$. The results of [15] were complemented in the works of Webb [17], Kaufmann [9], Kaufmann and Kosmatov [10], and Kaufmann and Raffoul [11].

Among the studies dealing with semipositone multi-point boundary-value problems, we mention the papers by Cao and Ma [3] and Liu [13]. Cao and Ma considered the boundary-value problem

$$\begin{gathered} -u''(t) = \lambda a(t) f(u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1). \end{gathered}$$

The authors applied the Leray-Schauder fixed point theorem to obtain an interval of eigenvalues for which at least one positive solution exists. Liu applied a fixed point index method to obtain such an interval for

$$\begin{gathered} -u''(t) = \lambda a(t) f(u(t)), \quad 0 < t < 1, \\ u'(0) = 0, \quad \alpha u(\eta) = u(1). \end{gathered}$$

Our approach is based on Krasnosel'skiĭ's cone-theoretic theorem [12] and enables us to show the existence of a positive solution for the semipositone problem (1.1), (1.2). Other applications of Krasnosel'skiĭ's fixed point theorem to semipositone problems can, for example, be found in [1].

2. Preliminaries

We now proceed with the auxiliaries. Consider the equation

$$-[p(t)u'(t)]' = g(t), \quad 0 < t < 1, \tag{2.1}$$

with the boundary conditions (1.2).

For convenience we set $\alpha = \sum_{i=1}^{m-2} \alpha_i$. Recall that $\alpha < 1$.

Lemma 2.1. *If $g \in C[0,1]$ and $g(t) \geq 0$ on $[0,1]$, then*

$$\begin{aligned} u(t) = &-\int_0^t \Big(\int_s^t \frac{d\tau}{p(\tau)}\Big) g(s)\, ds + \frac{1}{1-\alpha} \int_0^1 \Big(\int_s^1 \frac{d\tau}{p(\tau)}\Big) g(s)\, ds \\ &- \frac{1}{1-\alpha} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} \left(\int_s^{\eta_i} \frac{d\tau}{p(\tau)}\right) g(s)\, ds \end{aligned} \tag{2.2}$$

is the unique nonnegative solution on $[0,1]$ of the problem (2.1), (1.2).

Proof. Integration of (2.1) from 0 to t with the use of the boundary condition (1.2) at 0 yields

$$u'(t) = -\frac{1}{p(t)} \int_0^t g(s)\, ds \leq 0.$$

Integrating again we get

$$u(t) = -\int_0^t \frac{1}{p(s)} \Big(\int_0^s g(\tau)\, d\tau\Big)\, ds + A = -\int_0^t \Big(\int_s^t \frac{d\tau}{p(\tau)}\Big) g(s)\, ds + A.$$

Using the multi-point condition in (1.2) we determine A and obtain (2.2). Since $u'(t) \le 0$,

$$
\begin{aligned}
u(t) &\ge u(1)\\
&= \frac{\alpha}{1-\alpha}\int_0^1 \Big(\int_s^1 \frac{d\tau}{p(\tau)}\Big) g(s)\,ds - \frac{1}{1-\alpha}\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i}\Big(\int_s^{\eta_i}\frac{d\tau}{p(\tau)}\Big) g(s)\,ds\\
&= \frac{1}{1-\alpha}\sum_{i=1}^{m-2}\alpha_i\Big[\int_0^1 \Big(\int_s^1 \frac{d\tau}{p(\tau)}\Big) g(s)\,ds - \int_0^{\eta_i}\Big(\int_s^{\eta_i}\frac{d\tau}{p(\tau)}\Big) g(s)\,ds\Big] \ge 0
\end{aligned}
$$

on $[0,1]$ and the proof is complete. □

For $g(t) = 1$ on $[0,1]$, we denote by $u_0(t)$ the unique solution (2.2). Then we have

$$
\begin{aligned}
C &= \max_{t\in[0,1]} u_0(t) = u_0(0)\\
&= \frac{1}{1-\alpha}\int_0^1 \Big(\int_s^1 \frac{d\tau}{p(\tau)}\Big) g(s)ds - \frac{1}{1-\alpha}\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i}\Big(\int_s^{\eta_i}\frac{d\tau}{p(\tau)}\Big) g(s)ds.
\end{aligned}
$$

The Green's function for $-[p(t)u'(t)]' = 0$ with (1.2) is given by

$$
\begin{aligned}
G(t,s) &= \frac{1}{1-\alpha}\int_s^1 \frac{d\tau}{p(\tau)}\\
&\quad - \begin{cases} \int_s^t \frac{d\tau}{p(\tau)}, & s \le t\\ 0, & s > t\end{cases} - \begin{cases} \frac{1}{1-\alpha}\sum_{i=1}^{m-2}\alpha_i\chi_i(s)\int_s^{\eta_i}\frac{d\tau}{p(\tau)}, & s \le \eta_{m-2}\\ 0, & s > \eta_{m-2},\end{cases}
\end{aligned}
$$

where

$$
\chi_i(s) = \begin{cases} 1, & s \le \eta_i\\ 0, & s > \eta_i.\end{cases}
$$

Note that

$$
\max_{t\in[0,1]}\int_0^1 G(t,s)\,ds = C. \tag{2.3}
$$

The integral operator $T\colon \mathcal{B} \to \mathcal{B}$ associated with (1.1), (1.2) is defined by

$$
Tu(t) = \int_0^1 G(t,s) f(s,u(s))\,ds
$$

A routine argument shows that T is completely continuous.

Definition 2.2. Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be closed and nonempty. Then $\mathcal{C}$ is said to be a cone if

(1) $\alpha u + \beta v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$ and for all $\alpha, \beta \ge 0$, and
(2) $u, -u \in \mathcal{C}$ implies $u \equiv 0$.

Our Banach space, $\mathcal{B}$, is the space $C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. We will show now that the unique solution (2.2) satisfies

$$
\min_{t\in[0,1]} u(t) \ge \gamma\|u\|, \tag{2.4}
$$

where

$$
\gamma = \max_{1\le i\le m-2}\frac{\alpha_i(1-\eta_i)}{1-\alpha_i\eta_i}.
$$

To this end, note that the solution (2.2) is concave, since $g(t) \ge 0$ and $u'(t), p'(t) \le 0$ on $[0,1]$. By concavity and since $u(1) > \alpha_i u(\eta_i)$ for each $1 \le i \le m-2$,

$$\begin{aligned}
\|u\| &= u(0) \\
&\le u(1) + \frac{u(1) - u(\eta_i)}{1-\eta_i}(0-1) \\
&< u(1)\frac{1-\alpha_i\eta_i}{\alpha_i(1-\eta_i)} \\
&= \frac{1-\alpha_i\eta_i}{\alpha_i(1-\eta_i)} \min_{t\in[0,1]} u(t)
\end{aligned}$$

and hence (2.4) holds.

The estimate (2.4) is used for defining our cone $\mathcal{C} \subset \mathcal{B}$ by

$$\mathcal{C} = \{u(t) \in \mathcal{B} : u(t) \ge 0 \text{ on } [0,1],\ \min_{t\in[0,1]} u(t) \ge \gamma\|u\|\}. \tag{2.5}$$

It turns out that our operator T is cone-preserving. Fixed points of T are solutions of (1.1), (1.2). The existence of a fixed point of T follows from a fixed point theorem due to Krasnosel'skiĭ [12], which we now state.

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{C} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T\colon \mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{C}$$

be a completely continuous operator such that either

- (i) $\|Tu\| \le \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$, *or*
- (ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{C} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The following assumptions will stand throughout the remainder of this note:

- (A1) $f(t,z)$ is a continuous function on $[0,1] \times [0,\infty)$
- (A2) There exists $M > 0$ such that $f(t,z) + M \ge 0$ on $[0,1] \times [0,\infty)$
- (A3) There exist continuous nonnegative nondecreasing on $[0,\infty)$ functions $\psi_a(z)$ and $\psi_b(z)$ with $\psi_b(z) \le f(t,z) + M \le \psi_a(z)$ on $[0,1] \times [0,\infty)$.

3. Positive solutions

We now state our main results.

Theorem 3.1. *Let the assumptions (A1)-(A3) be satisfied. Assume, in addition, that*

$$\lim_{z\to 0^+} \frac{\psi_a(z)}{z} = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{\psi_b(z)}{z} = \infty.$$

Then, for a sufficiently small $\lambda > 0$, the problem (1.1), (1.2) has a positive solution.

Proof. Consider the equation

$$-[p(t)u'(t)]' = \lambda f_p(t, u(t) - u_\lambda(t)), \quad 0 < t < 1, \tag{3.1}$$

with the boundary conditions (1.2), where

$$f_p(t,z) = \begin{cases} f(t,z) + M, & z \ge 0 \\ f(t,0) + M, & z \le 0 \end{cases}$$

and $u_\lambda(t) = \lambda M u_0(t)$ ($u_0(t)$ is given by (2.2) for $g \equiv 1$). Our objective is to show that the problem (3.1), (1.2) has a positive solution.

Our completely continuous and cone-preserving operator associated with (3.1), (1.2) is defined by

$$T_\lambda u(t) = \lambda \int_0^1 G(t,s) f_p(s, u(s) - u_\lambda(s))\, ds$$

Since $\lim_{z\to 0^+} \frac{\psi_a(z)}{z} = 0$, there exists $R_1 > 0$ such that

$$\psi_a(z) \le \frac{1}{\lambda C} z$$

for all $z \le R_1$.

Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < R_1\}$, then for $u \in \mathcal{C} \cap \partial\Omega_1$ we have

$$\psi_a(u(s)) \le \psi_a(\|u\|) \le \frac{1}{\lambda C} R_1 \tag{3.2}$$

for all $s \in [0,1]$, since $\psi_a(z)$ is nondecreasing. Now, if $u(s) \ge u_\lambda(s)$ for $s \in [0,1]$, then

$$f_p(s, u(s) - u_\lambda(s)) = f(s, u(s) - u_\lambda(s)) + M \le \psi_a(u(s) - u_\lambda(s)) \le \psi_a(u(s)).$$

If $u(s) \le u_\lambda(s)$, then

$$f_p(s, u(s) - u_\lambda(s)) = f(s, 0) + M \le \psi_a(0) \le \psi_a(u(s))$$

(we know that $u(s) \ge 0$ as an element of $\mathcal{C}$). Combining both cases and using (3.2) and (2.3), we get

$$\begin{aligned}
\|T_\lambda u\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, u(s) - u_\lambda(s))\, ds \\
&\le \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) \psi_a(u(s))\, ds \\
&\le \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds\, \frac{1}{\lambda C} R_1 = R_1,
\end{aligned}$$

that is, $\|T_\lambda u\| \le \|u\|$ on $\mathcal{C} \cap \partial\Omega_1$.

Since $\lim_{z\to\infty} \frac{\psi_b(z)}{z} = \infty$, then also $\lim_{z\to\infty} \frac{\psi_b(\gamma z - \lambda M C)}{z} = \infty$. Thus, there exists $R_2 > 0$ large enough (so that $R_2 > \frac{\lambda M C}{\gamma}$ and $R_2 > R_1$) such that

$$\psi_b(\gamma z - \lambda M C) \ge \frac{1}{\lambda C} z$$

for all $z \ge R_2$. In fact,

$$\psi_b(\gamma R_2 - \lambda M C) \ge \frac{1}{\lambda C} R_2. \tag{3.3}$$

Define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < R_2\}$, then for $u \in \mathcal{C} \cap \partial\Omega_2$ we have

$$u(s) - u_\lambda(s) \ge \gamma\|u\| - \lambda M u_0(s) \ge \gamma R_2 - \lambda M C > 0.$$

Now, for all $s \in [0,1]$,

$$f_p(s, u(s) - u_\lambda(s)) = f(s, u(s) - u_\lambda(s)) + M \ge \psi_b(u(s) - u_\lambda(s)) \ge \psi_b(\gamma R_2 - \lambda M C),$$

since $\psi_b(z)$ is nondecreasing. Therefore, by (3.3) and (2.3),

$$\begin{aligned}\|T_\lambda u\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f_p(s, u(s) - u_\lambda(s))\, ds \\ &\geq \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)\psi_b(\gamma R_2 - \lambda MC)\, ds \\ &\geq \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds\, \frac{1}{\lambda C} R_2 = R_2,\end{aligned}$$

that is, $\|T_\lambda u\| \leq \|u\|$ on $\mathcal{C} \cap \partial\Omega_2$.

Since the assumptions of Theorem 2.3 are satisfied, we conclude that the problem (3.1), (1.2) has a positive solution in $\mathcal{C} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which we denote by u_p.

Let λ be small enough so that $R_1 > \frac{\lambda MC}{\gamma}$. Now we have $u_p(t) \geq \gamma\|u_p\| \geq \gamma R_1 > \lambda MC \geq u_\lambda(t)$ for all $t \in [0,1]$. Set $u(t) = u_p(t) - u_\lambda(t)$, then

$$\begin{aligned}-[p(t)u'(t)]' &= -[p(t)u_p'(t)]' - \lambda M \\ &= \lambda f_p(t, u_p(t) - u_\lambda(t)) - \lambda M \\ &= \lambda(f(t, u_p(t) - u_\lambda(t)) + M) - \lambda M \\ &= \lambda f(t, u(t)),\end{aligned}$$

which shows that $u(t)$ is a positive solution of (1.1), (1.2). The proof is complete. $\square$

Example. To illustrate our main result, we consider the inhomogeneous term in the form of the function

$$f(t,z) = -1 + z^2(2 + \sin(4\pi z(1+t^3))).$$

The function $f(t,z)$ is continuous and, setting $M = 1$, we get $f(t,z) + M \geq 0$ on $[0,1] \times [0,\infty)$. In addition, for $\psi_b(z) = z^2$ and $\psi_a(z) = 3z^2$, we have $\psi_b(z) \leq f(t,z) + M \leq \psi_a(z)$ and

$$\lim_{z\to 0^+} \frac{\psi_a(z)}{z} = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{\psi_b(z)}{z} = \infty.$$

Thus, Theorem 3.1 applies.

With only minor adjustments to the argument above one can prove our next theorem.

Theorem 3.2. *Let the assumptions (A1)-(A3) be satisfied. Assume, in addition, that*

$$\lim_{z\to 0^+} \frac{\psi_a(z)}{z} = \infty \quad \textit{and} \quad \lim_{z\to\infty} \frac{\psi_b(z)}{z} = 0.$$

Then, for a sufficiently small $\lambda > 0$, the problem (1.1), (1.2) has a positive solution.

Remark. If problem (1.1), (1.2) has a positive solution for some $\lambda_1 > 0$, there is also a positive solution for each $\lambda \in (0, \lambda_1]$.

We say that a function $\psi(z)$ is sublinear if

$$\lim_{z\to 0^+} \frac{\psi(z)}{z} = \infty \quad \text{and} \quad \lim_{z\to\infty} \frac{\psi(z)}{z} = 0.$$

On the other hand, if

$$\lim_{z\to 0^+} \frac{\psi(z)}{z} = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{\psi(z)}{z} = \infty,$$

then the function ψ is called superlinear.

If in the assumption (A3) we take $\psi_a(z) = \psi_b(z)$, then the following corollary to Theorems 3.1 and 3.2 becomes immediate.

Corollary 3.3. *Let the assumptions (A1)-(A3) be satisfied. Assume, in addition, that $\psi_a(z) = \psi_b(z)$ is either sublinear or superlinear. Then, for a sufficiently small $\lambda > 0$, the problem (1.1), (1.2) has a positive solution.*

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