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Existence of solutions for a variational unilateral system *

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Abstract

In this work the authors study the existence of weak solutions of the nonlinear unilateral mixed problem associated to the inequalities

$$u_{tt} - M(|\nabla u|^2)\Delta u + \theta \geq f,$$
$$\theta_t - \Delta\theta + u_t \geq g,$$

where f, g, M are given real-valued functions with M positive.

1 Introduction

Let Ω be a bounded and open set of $\mathbb{R}^n$, with smooth boundary $\Gamma = \partial\Omega$, and let T be a positive real number. Let $\mathbb{Q} = \Omega\times]0,T[$ be the cylinder with lateral boundary $\Sigma = \Gamma\times]0,T[$.

We study the variational nonlinear system

$$u_{tt} - M(|\nabla u|^2)\Delta u + \theta \geq f \quad \text{in} \quad Q, \tag{1.1}$$
$$\theta_t - \Delta\theta + u_t \geq g \quad \text{in} \quad Q, \tag{1.2}$$
$$u = \theta = 0 \quad \text{in} \quad \Sigma \tag{1.3}$$
$$u(0) = u_0, \quad u'(0) = u_1, \quad \theta(0) = \theta_0. \tag{1.4}$$

The above system with $M(s) = m_0 + m_1 s$ (m_0 and m_1 positive constants) and $\theta = 0$ is a nonlinear perturbation of the canonical Kirchhof model

$$u_{tt} - \big(m_0 + m_1 \int_\Omega |\nabla u^2 dx\big)\Delta u = f\,. \tag{1.5}$$

This model describes small vibrations of a stretched string when only the transverse component of the tension is considered, see for example, Arosio & Spagnolo [1], Pohozaev [12].

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Several authors have studied (1.5). For Ω bounded, we can cite: D'ancona & Spagnolo [5], Medeiros & Milla Miranda [9], Hosoya & Yamada [7], Lions [8], Medeiros [10], and Matos [9]. For Ω unbounded, we can cite Bisiguin [2], Clark & Lima [4], and Matos [9]. The system (1.1)–(1.4) was studied also in the case when (1.1) and (1.2) are equations, see for example [3].

In the present work we show the existence of a weak solution for the variational nonlinear system (1.1)–(1.4), under appropriate assumptions on M, f and g. We employ Galerkin's approximation method and the penalization method used by Frota & Lar'kin [6].

2 Notation and main result

We represent the Sobolev space of order m on Ω by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\, D^\alpha u \in L^p(\Omega), \forall\, |\alpha| \le m\}$$

and its associated norm by

$$\|u\|_{m,p} = \Big(\sum_{|\alpha|\le m} |D^\alpha u|^p_{L^p(\Omega)} \Big)^{1/p}, \quad u \in W^{m,p}(\Omega), \quad 1 \le p < \infty\,.$$

When $p = 2$, we have the usual Sobolev space $H^m(\Omega)$. Let $D(\Omega)$ be the space of the test functions on Ω, and let $W_0^{m,p}(\Omega)$ be the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$. When $p = 2$, we have $W_0^{2,p}(\Omega) = H_0^m(\Omega)$. The dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$, with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. For the rest of this paper we use the symbol $(\cdot,\cdot)$ to indicate the inner product in $L^2(\Omega)$, and $((\cdot,\cdot))$ to indicate the inner product in $H_0^1(\Omega)$.

Let $\mathbb{K} = \{\psi \in W_0^{2,4}(\Omega); |\Delta\psi| \le 1 \text{ and } \psi \ge 0 \text{ a. e. in } \Omega\ \}$. Then we have the following proposition whose proof can be found in [6]

Proposition 2.1 *The set $\mathbb{K}$ is a closed and connected in $W_0^{2,4}(\Omega)$.*

Definition Let V be a Banach space and V' its dual. An operator β from V to V' is called hemicontinous if the function

$$\lambda \to \langle \beta(u + \lambda v), w\rangle$$

is continuous for all $\lambda \in \mathbb{R}$. The operator β is called monotone if

$$\langle \beta(u) - \beta(v), u - v\rangle \ge 0, \quad \forall u, v \in V.$$

We consider the penalization operator $\beta : W_0^{2,4}(\Omega) \to W^{-2,4/3}(\Omega)$ such that $\beta(z) = \beta_1(z) + \beta_2(z)$, $z \in W_0^{2,4}(\Omega)$, where $\beta_1(z)$ and $\beta_2(z)$ are defined by

$$\langle \beta_1(z), v\rangle = -\int_\Omega z^-(x) v(x) dx,$$

$$\langle \beta_2(z), v\rangle = -\int_\Omega (1 - |\Delta z(x)|^2)^- \Delta z(x) \Delta v(x) dx$$

for all v in $W_0^{2,4}(\Omega)$.

Proposition 2.2 *The operator β defined above satisfies the following coditions:*

i) *β is monotone and hemicontinous*

ii) *β is bounded; this is, $\beta(S)$ is bounded in $W^{2,4/3}(\Omega)$ for all bounded set S in $W_0^{2,4}(\Omega)$.*

iii) *$\beta(u) = 0$ if only if u belongs to $\mathbb{K}$.*

The proof of this proposition can be found in [6].

In this article, we assume the following hypotheses:

A1) $M \in C^1[0,\infty)$, $M(s) \geq 0$ for $s \geq 0$, and $\int_0^\infty M(s)ds = \infty$

A2) f, g belong to $H^1(0,T; L^2(\Omega))$.

The main result of this paper is stated as follows.

Theorem 2.3 *Assume A1) and A2). For $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, u_1, θ_0 in the interior of $\mathbb{K}$, there exist functions $u, \theta : \mathbb{Q} \to \mathbb{R}$ such that*

$$u \in L^\infty(0,T;\ H_0^1(\Omega) \cap H^2(\Omega)) \tag{2.1}$$

$$u' \in L^1(0,T;\ W_0^{2,4}(\Omega)) \text{ and } u'(t) \in \mathbb{K} \text{ a.e. in } [0,T] \tag{2.2}$$

$$u'' \in L^\infty(0,T;\ L^2(\Omega)) \tag{2.3}$$

$$\theta \in L^\infty(0,T;\ H_0^1(\Omega)) \text{ and } \theta(t) \in \mathbb{K} \text{ a.e. in } [0,T]\,. \tag{2.4}$$

Also

$$(u''(t) - M(\|u(t)\|^2)\Delta u(t) + \theta(t) - f(t), v - u'(t) \geq 0,\ \forall v \in \mathbb{K}\ \text{ a.e. in } [0,T] \tag{2.5}$$

$$(\theta'(t) - \Delta\theta(t) + u'(t) - g(t), v - \theta(t)) \geq 0\ \forall v \in \mathbb{K}\ \text{ a.e. in } [0,T] \tag{2.6}$$

$$u(0) = u_0,\ u'(0) = u_1,\ \theta(0) = \theta_0\,. \tag{2.7}$$

To obtain the solution $\{u, \theta\}$ of problem (2.1)–(2.4) in Theorem 2.3, we consider the following associated penalized problem. For $0 < \varepsilon < 1$, consider

$$u_\varepsilon''(t) - M(\|u_\varepsilon(t)\|^2)\Delta u_\varepsilon(t) + \theta_\varepsilon(t) + \frac{1}{\varepsilon}\beta(u_\varepsilon'(t)) = f(t) \text{ in } Q \tag{2.8}$$

$$\theta_\varepsilon'(t) - \Delta\theta_\varepsilon(t) + u_\varepsilon' + \frac{1}{\varepsilon}\beta(\theta_\varepsilon(t)) = g(t) \text{ in } Q \tag{2.9}$$

$$u_\varepsilon(0) = u_{0\varepsilon}, u_\varepsilon'(0) = u_{1\varepsilon}, \theta_\varepsilon(0) = \theta_{0\varepsilon} \text{ in } \Omega \tag{2.10}$$

Here β is a penalization operator, M, f, and g are as above. The solution $\{u_\varepsilon, \theta_\varepsilon\}$ of the penalized problem (2.8)–(2.10) are guaranteed by the following theorem.

Theorem 2.4 *Suppose the hypotheses of the Theorem 2.3 hold, and for* $0 < \varepsilon < 1$, *then there exist functions* $\{u_\varepsilon, \theta_\varepsilon\}$ *such that*

$$u_\varepsilon, \theta_\varepsilon \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \tag{2.11}$$

$$u'_\varepsilon \in L^4(0,T; W_0^{2,4}(\Omega)) \tag{2.12}$$

$$u''_\varepsilon \in L^\infty(0,T; L^2(\Omega)) \tag{2.13}$$

$$\theta_\varepsilon \in L^4(0,T; W_0^{2,4}(\Omega)) \tag{2.14}$$

$$\begin{aligned}(u''_\varepsilon(t), v) + M(\|u_\varepsilon(t)\|^2)((u_\varepsilon(t), v)) + (\theta_\varepsilon(t), v) + \frac{1}{\varepsilon}\langle\beta(u'_\varepsilon(t)), v\rangle \\ = (f(t), v) \text{ a.e. in } [0,T] \text{ for all } v \in W_0^{2,4}(\Omega),\end{aligned} \tag{2.15}$$

$$\begin{aligned}(\theta'_\varepsilon(t), v) + ((\theta_\varepsilon(t), v)) + (u'_\varepsilon(t), v) + \frac{1}{\varepsilon}\langle\beta(\theta_\varepsilon(t)), v\rangle \\ = (g(t), v) \text{a.e. in } [0,T] \text{ for all } v \in W_0^{2,4}(\Omega),\end{aligned} \tag{2.16}$$

$$u_\varepsilon(0) = u_{0\varepsilon}, u'_\varepsilon(0) = u_{1\varepsilon}, \theta_\varepsilon(0) = \theta_{0\varepsilon}. \tag{2.17}$$

Proof We will use Galerkin's method and a compactness argument.
First step (Approximated system) Let $w_1, \dots, w_m, \dots$ be an orthonormal base of $W_0^{2,4}(\Omega)$ consisting of eigenfunctions of the Laplacian operator. Let $V_m = [w_1, \dots, w_m]$ the subspace of $W_0^{2,4}(\Omega)$, generated by the first m vectors w_j. We look for a pair of functions

$$u_{\varepsilon m}(t) = \sum_{j=1}^m g_{jm}(t) w_j, \quad \theta_{\varepsilon m}(t) = \sum_{j=1}^m h_{jm}(t) w_j \quad \text{in} \quad V_m$$

with $g_{jm} \in C^2([0,T])$ and $h_{jm} \in C^1([0,T])$, for all $j = 1, \dots, m$. Which are solutions of the following system of ordinary differential equations

$$\begin{aligned}(u''_{\varepsilon m}(t), w_j) + M(\|u_{\varepsilon m}(t)\|^2)((u_{\varepsilon m}(t), w_j)) + (\theta_{\varepsilon m}(t), w_j) + \\ \frac{1}{\varepsilon}\langle\beta(u'_{\varepsilon m}(t)), w_j\rangle = (f(t), w_j),\end{aligned} \tag{2.18}$$

$$\begin{aligned}(\theta'_{\varepsilon m}(t), w_j) + ((\theta_{\varepsilon m}(t), w_j)) + (u'_{\varepsilon m}(t), w_j) + \\ \frac{1}{\varepsilon}\langle\beta(\theta_{\varepsilon m}(t)), w_j\rangle = (g(t), w_j),\end{aligned} \tag{2.19}$$

for $j = 1, \dots, m$, with the initial conditions: $u_{\varepsilon m}(0) = u_{0\varepsilon m}$, $u'_{\varepsilon m}(0) = u_{1\varepsilon m}$, $\theta_{\varepsilon m}(0) = \theta_{0\varepsilon m}$, where

$$\begin{aligned}u_{0\varepsilon m} &= \sum_{j=1}^m (u_{0\varepsilon}, w_j) w_j \to u_0 \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega), \\ u_{1\varepsilon m} &= \sum_{j=1}^m (u_{1\varepsilon}, w_j) w_j \to u_1 \text{ strongly in } H_0^1(\Omega), \\ \theta_{0\varepsilon m} &= \sum_{j=1}^m (\theta_{0\varepsilon}, w_j) w_j \to \theta_0 \text{ strongly in } W_0^{2,4}(\Omega).\end{aligned} \tag{2.20}$$

The system (2.18)–(2.20) contains $2m$ unknowns functions $g_{jm}(t), h_{jm}(t)$; $j = 1, 2, \dots, m$. By Caratheodory's Theorem it follows that (2.18)–(2.20) has a local solution $\{u_{\varepsilon m}(t), \theta_{\varepsilon m}(t)\}$ on $[0, t_m[$. In order to extend these local solution to the interval $[0, T[$ and to take the limit in m, we must obtain some a priori estimates.

Estimate (i) Note that finite linear combinations of the w_j are dense in $W_0^{2,4}(\Omega)$, then we can take $w \in W_0^{2,4}(\Omega)$ in (2.18) and (2.19) instead of w_j. Taking $w = 2u'_{\varepsilon m}(t)$ in (2.18) and $w = 2\theta_{\varepsilon m}(t)$ in (2.19) we obtain

$$\begin{aligned} \frac{d}{dt}|u'_{\varepsilon m}(t)|^2 + \frac{d}{dt}\widehat{M}(\|u_{\varepsilon m}(t)\|^2) + \frac{2}{\varepsilon}\langle\beta(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t)\rangle \\ = 2(f(t), u'_{\varepsilon m}(t)) - 2(\theta_{\varepsilon m}(t), u'_{\varepsilon m}(t)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} \frac{d}{dt}|\theta_{\varepsilon m}(t)|^2 + \|\theta_{\varepsilon m}(t)\|^2 + \frac{2}{\varepsilon}\langle\beta(\theta_{\varepsilon m}(t)), \theta_{\varepsilon m}(t)\rangle \\ = -2(u'_{\varepsilon m}(t), \theta_{\varepsilon m}(t)) + 2\langle g(t), \theta_{\varepsilon m}(t)\rangle\,, \end{aligned} \tag{2.22}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$. Adding (2.21) and (2.22), and integrating from 0 to $t \le t_m$ we have

$$\begin{aligned} |u'_{\varepsilon m}(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \int_0^{\|u_{\varepsilon m}(t)\|^2} M(s)ds + \int_0^t \|\theta_{\varepsilon m}(s)\|^2 ds + \\ \frac{2}{\varepsilon}\int_0^t \langle\beta(u'_{\varepsilon m}(s)), u'_{\varepsilon m}(s)\rangle ds + \frac{2}{\varepsilon}\int_0^t \langle\beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s)\rangle ds \le \\ \int_0^T |f(t)|^2 ds + 3\int_0^t |u'_{\varepsilon m}(s)|^2 ds + 3\int_0^t |\theta_{\varepsilon m}(s)|^2 ds + \\ \int_0^T |g(t)|^2 dt + |\theta_{0\varepsilon m}|^2 + |u_{1\varepsilon m}|^2. \end{aligned} \tag{2.23}$$

From (2.20) and hypothesis (A2) there exists a positive constant C, independently of $\varepsilon > 0$ and m such that

$$\begin{aligned} |u'_{\varepsilon m}(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \int_0^{\|u_{\varepsilon m}(t)\|^2} M(s)ds + \int_0^t \|\theta_{\varepsilon m}(s)\|^2 ds + \\ \frac{2}{\varepsilon}\Big[\int_0^t \langle\beta(u'_{\varepsilon m}(s)), u'_{\varepsilon m}(s)\rangle ds + \int_0^t \langle\beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s)\rangle ds\Big] \le \\ C + 3\int_0^t |u'_{\varepsilon m}(s)|^2 ds + 3\int_0^t |\theta_{\varepsilon m}(s)|^2 ds. \end{aligned} \tag{2.24}$$

Next we analyze the sign of the term $\int_0^t \langle\beta(u'_{\varepsilon m}(s)), u'_{\varepsilon m}(s)\rangle ds$. Note that $-u'_{\varepsilon m}(t) \le u'_{\varepsilon m}(t)^-$. Then, by the definition of β, we have

$$\begin{aligned} \langle\beta(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t)\rangle &= \langle\beta_1(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t)\rangle + \langle\beta_2(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t)\rangle \\ &= -\int_\Omega (u'_{\varepsilon m}(x,t))^- u'_{\varepsilon m}(x,t)dx + \\ &\quad \int_\Omega (1 - |\Delta u'_{\varepsilon m}(t)|^2)^- (\Delta u'_{\varepsilon m}(t))^2 dx \ge 0. \end{aligned}$$

Similarly, we have,

$$\langle \beta(\theta_{\varepsilon m}(t)), \theta_{\varepsilon m}(t)\rangle \geq 0\,.$$

Because $M(s) \geq 0$ for all s, from (2.24) and Gronwall's inequality it follows that

$$|u'_{\varepsilon m}(t)|^2 + |\theta_{\varepsilon m}(t)|^2 \leq C_1, \quad \forall \varepsilon, m, \forall t \in [0, t_m[.$$

Returning to (2.24), we obtain

$$\begin{gathered} |u'_{\varepsilon m}(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \int_0^{\|u_{\varepsilon m}(t)\|^2} M(s)ds + \int_0^t \|\theta_{\varepsilon m}(s)\|^2 ds + \\ \frac{2}{\varepsilon}[\int_0^t \langle \beta(u'_{\varepsilon m}(s)), u'_{\varepsilon m}(s)\rangle ds + \int_0^t \langle \beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s)\rangle ds] \leq C + 3C_1 T\,. \end{gathered} \tag{2.25}$$

Since $\int_0^\infty M(s)ds = \infty$, by (2.25) we can find C_1 such that

$$\|u_{\varepsilon m}(t)\|^2 \leq C_1, \quad \forall \varepsilon, m, \forall t \in [0, t_m[.$$

Thus there exists, other constant $C = C(T)$ independently of ε, m and $t \in [0, t_m[$ such that

$$\begin{gathered} |u'_{\varepsilon m}(t)|^2 + |\theta_{\varepsilon m}(t)|^2 + \|u_{\varepsilon m}(t)\|^2 + \int_0^t \|\theta_{\varepsilon m}(s)\|^2 ds + \\ \frac{2}{\varepsilon}\int_0^t \langle \beta(u'_{\varepsilon m}(s)), u'_{\varepsilon m}(s)\rangle ds + \frac{2}{\varepsilon}\int_0^t \langle \beta(\theta_{\varepsilon m}(s)), \theta_{\varepsilon m}(s)\rangle ds \leq C \end{gathered} \tag{2.26}$$

Estimate (ii) We will obtain a bound for $|u''_{\varepsilon m}(0)|$. For this, we note that u_1 being in the interior of $\mathbb{K}$ and $u_{1\varepsilon m} \to u_1$ imply that $u_{1\varepsilon m}$ is in the interior of $\mathbb{K}$, for m large. Therefore, $|\Delta u_{1\varepsilon m}| \leq 1$ and $u_{1\varepsilon m} \geq 0$ a. e. in Ω. Also we have $(u_{1\varepsilon m})^- = 0$ and $(1 - |\Delta u_{1\varepsilon m}|^2)^- = 0$ a. e. in Ω. Thus

$$\langle \beta(u_{1\varepsilon m}), u''_{\varepsilon m}(0)\rangle = 0 \tag{2.27}$$

Taking $t = 0$ and $v = u''_{\varepsilon m}(0)$ in (2.14), and observing (2.27), we obtain

$$|u''_{\varepsilon m}(0)|^2 + M(\|u_{0\varepsilon m}\|^2)((u_{0\varepsilon m}, u''_{\varepsilon m}(0))) + (\theta_{\varepsilon m}, u''_{\varepsilon m}(0)) = (f(0), u''_{\varepsilon m}(0))$$

which implies

$$|u''_{\varepsilon m}(0)|^2 \leq |f(0)||u''_{\varepsilon m}(0)| + M(\|u_{0\varepsilon m}\|^2)|\Delta u_{0\varepsilon m}||u''_{\varepsilon m}(0)| + |\theta_{0\varepsilon m}||u''_{\varepsilon m}(0)|.$$

From $u_{0\varepsilon m} \to u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$, $\theta_{0\varepsilon m} \to \theta_0$ in $H_0^1(\Omega)$, $M \in C^1[0, \infty)$, and $f \in H^1(0, T; L^2(\Omega)$, we obtain

$$|u''_{\varepsilon m}(0)| \leq C, \tag{2.28}$$

with C independent of ε, m, and $t \in [0, T[$.

Estimate (iii) We obtain estimates for $|\Delta u'_{\varepsilon m}(t)|$, $|\Delta\theta_{\varepsilon m}(t)|$, $\int_0^t |u'_{\varepsilon m}(s)|^3 ds$, and $\int_0^t |\theta'_{\varepsilon m}(s)|^3 ds$. For this, we need the following lemma whose proof can be found in [6].

Lemma 2.5 *Let $h : \Omega \to \mathbb{R}$ be an arbitrary function. Then*

$$h^4 - 1 \le 2(1-h^2)^- h^2.$$

By this lemma, we have

$$(\Delta u'_{\varepsilon m})^4 - 1 \le 2[1-(\Delta u'_{\varepsilon m})^2]^- (\Delta u'_{\varepsilon m})^2\,.$$

Therefore,

$$\begin{aligned}
\|\Delta u'_{\varepsilon m}\|^4_{L^4(Q)} =& \int_0^T \int_\Omega |\Delta u'_{\varepsilon m}(x,t)|^4 dx\,dt \\
\le& 2\int_0^T \int_\Omega (1-\Delta|u'_{\varepsilon m}(x,t)|^2)^- (\Delta u'_{\varepsilon m}(x,t))^2 dx\,dt + \operatorname{meas}(Q) \\
=& 2\int_0^T \langle \beta_2(\Delta u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t)\rangle dx\,dt + \operatorname{meas}(Q) \\
\le& 2\int_0^T (\beta(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t))dt + \operatorname{meas}(Q)\,.
\end{aligned}$$

Using (2.26), we obtain

$$\|\Delta u'_{\varepsilon m}\|^4_{L^4(Q)} \le C\varepsilon + \operatorname{meas}(Q) < C + \operatorname{meas}(Q) \tag{2.29}$$

with C independent of ε, m and $t \in [0,T[$. Analogously, using the Lemma 2.5 with $h = \Delta\theta_{\varepsilon m}$ and (2.26), we obtain

$$\|\Delta\theta_{\varepsilon m}\|^4_{L^4(Q)} \le C + \operatorname{meas}(Q) \tag{2.30}$$

On the other hand, from (2.18) and (2.19), we obtain

$$\begin{aligned}
\frac{1}{\varepsilon}\langle\beta(u'_{\varepsilon m}(t)), v\rangle + \frac{1}{\varepsilon}\langle\beta(\theta_{\varepsilon m}(t)), v\rangle \le C(|f(t)|) + |g(t)|)\|v\| + \\
M(\|u_{\varepsilon m}(t)\|^2)\|u_{\varepsilon m}(t)\|.\|v\| + C(|\theta_{\varepsilon m}(t)|) + |u'_{\varepsilon m}(t)|) \le \\
|f(t)||v| + |g(t)||v| + M(\|u_{\varepsilon m}(t)\|^2)\|u_{\varepsilon m}(t)\|.\|v\||u''_{\varepsilon m}(t)||v| + |\theta_{\varepsilon m}(t)||v| + \\
|\theta'_{\varepsilon m}(t)||v| + \|\theta_{\varepsilon m}(t)\|.\|v\| + |u'_{\varepsilon m}(t)||v| \le \\
C\{|f(t)| + |g(t)| + |u''_{\varepsilon m}(t)| + |\theta_{\varepsilon m}(t)| + |\theta'_{\varepsilon m}(t)| + |u'_{\varepsilon m}(t)|\}\|v\| + \\
(M(\|u_{\varepsilon m}(t)\|^2)\|u_{\varepsilon m}(t)\| + \|\theta_{\varepsilon m}(t)\|)\|v\|.
\end{aligned}$$

Since $f, g \in C^0([0,T]; L^2(\Omega))$, from the inequality above we obtain

$$\frac{1}{\varepsilon}|\langle\beta(u'_{\varepsilon m}(t)), v\rangle| \le C_1\|v\| \quad \forall v \in W_0^{2,4}(\Omega)\,, \tag{2.31}$$

$$\frac{1}{\varepsilon}|\langle\beta(\theta_{\varepsilon m}(t)), v\rangle| \le C_1\|v\| \quad \forall v \in W_0^{2,4}(\Omega), \tag{2.32}$$

independent of ε, m and $t \in [0,T]$; this is,

$$\|\beta(u'_{\varepsilon m})\|_{L^\infty(0,T;W^{2,4/3}(\Omega))} \le C_1\,, \tag{2.33}$$

$$\|\beta(\theta_{\varepsilon m})\|_{L^\infty(0,T;W^{2,4/3}(\Omega))} \le C_1\,. \tag{2.34}$$

To estimate $|\Delta u_{\varepsilon m}(t)|$, we note that

$$\begin{aligned}|\Delta u_{\varepsilon m}(t)|^2 =&|\Delta u_{0\varepsilon m}|^2 + \int_0^t \frac{d}{ds}|\Delta u_{\varepsilon m}(s)|^2 ds \\ =&|\Delta u_{0\varepsilon m}|^2 + 2C\int_0^t |\Delta u_{\varepsilon m}(s)| \|\Delta u'_{\varepsilon m}(s)\| \\ \leq&|\Delta u_{0\varepsilon m}|^2 + C\int_0^t (|\Delta u_{\varepsilon m}(s)|^2 + \|\Delta u'_{\varepsilon m}(s)\|^2) ds\,,\end{aligned}$$

where C is the constant of the embedding from $H^1_0(\Omega)$ into $L^2(\Omega)$. From (2.20), (2.29) and Gronwall's inequality, we obtain

$$|\Delta u_{\varepsilon m}(t)|^2 < C\,, \tag{2.35}$$

where C is a constant independent of ε, m and $t \in [0, T[$.

Next, we obtain an estimate for $\int_0^t \|\Delta u'_{\varepsilon m}(s)\|^3 ds$. Let C represent various positives constants of the embedding in the sequence

$$W_0^{2,4}(\Omega) \hookrightarrow H^2_0(\Omega) \hookrightarrow H^1_0(\Omega)\,.$$

Observing that $W_{H^2(\Omega)} \leq C|\Delta w|$ we obtain

$$\int_0^t \|u'_{\varepsilon m}(s)\|^3 ds \leq C\int_0^t \|u'_{\varepsilon m}(s)\|^3_{H^2(\Omega)} ds \leq C\int_0^t |\Delta u'_{\varepsilon m}(s)|^3 ds, \tag{2.36}$$

independently of ε and m. It follows from Höder's inequality that

$$\int_0^t |\Delta u'_{\varepsilon m}(s)|^3 ds \leq (\int_0^T 1^1 ds)^{1/4}(\int_0^t \|\Delta u'_{\varepsilon m}(s)\|^4 ds)^{3/4}$$

and substituting in (2.36) and observing (2.29), we obtain

$$\int_0^t \|u'_{\varepsilon m}(s)\|^3 ds \leq C, \tag{2.37}$$

independent of ε, m and $t \in [0, T[$.

Estimate (iv) We will obtain the estimative for $|u''_{\varepsilon m}(t)|$. Let us consider the functions

$$\begin{gathered}\Psi_h(t) = \tfrac{1}{h}[u_{\varepsilon m}(t+h) - u_{\varepsilon m}(t)]\,, \\ M_h(t) = \tfrac{1}{h}[M(\|u_{\varepsilon m}(t+h)\|^2) - M(\|u_{\varepsilon m}(t)\|^2)]\,, \\ f_h(t) = \tfrac{1}{h}[f(t+h) - f(t)].\end{gathered}$$

Setting $w = 2\Psi'_h(t)$ in (1.14), we obtain

$$\begin{gathered}2(u''_{\varepsilon m}(t), \Psi'_h(t)) + 2M(\|u_{\varepsilon m}(t)\|^2)((u_{\varepsilon m}(t), \Psi'_h(t)))+ \\ \frac{2}{\varepsilon}\langle\beta(u'_{\varepsilon m}(t)), \Psi'_h(t))\rangle = 2(f(t), \Psi'_h(t))\end{gathered}. \tag{2.38}$$

Substituting t by $t+h \in [0,T]$ in (2.18) and taking $w = 2\Psi_h'(t)$, we set

$$\begin{aligned} 2(u''_{\varepsilon m}(t+h), \Psi'_h(t)) + 2M(\|u_{\varepsilon m}(t+h)\|^2)((u_{\varepsilon m}(t+h), \Psi'_h(t)))+ \\ \frac{2}{\varepsilon}\langle \beta(u'_{\varepsilon m}(t+h)), \Psi'_h(t)\rangle = 2(f(t+h), \Psi'_h(t)). \end{aligned} \tag{2.39}$$

Now, from (2.38) and (2.39) it follows, for $h \neq 0$, that

$$\begin{aligned} 2(\frac{u''_{\varepsilon m}(t+h) - u''_{\varepsilon m}(t)}{h}, \Psi'_h(t)) + \frac{2}{h}M(\|u_{\varepsilon m}(t+h)\|^2)((u_{\varepsilon m}(t+h), \Psi'_h(t)))- \\ \frac{2}{h}M(\|u_{\varepsilon m}(t)\|^2)((u_{\varepsilon m}(t), \Psi'_h(t))) + \frac{2}{h\varepsilon}\langle \beta(u'_{\varepsilon m}(t+h)) - \beta(u'_{\varepsilon m}(t)), \Psi'_h(t)\rangle = \\ 2(\frac{f(t+h)-f(t)}{h}, \Psi'_h(t)), \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt}|\Psi'_h(t)|^2 + \frac{2}{h}M(\|u_{\varepsilon m}(t+h)\|^2)(u_{\varepsilon m}(t+h), \Psi'_h(t))- \\ \frac{2}{h}M(\|u_{\varepsilon m}(t)\|^2)((u_{\varepsilon m}(t), \Psi'_h(t)))+ \\ \frac{2}{h\varepsilon}\langle \beta(u'_{\varepsilon m}(t+h)) - \beta(u'_{\varepsilon m}(t)), \Psi'_h(t)\rangle = 2(f_h(t), \Psi'_h(t)). \end{aligned} \tag{2.40}$$

Nothing that

$$\begin{aligned} \frac{2}{h}M(\|u_{\varepsilon m}(t+h)\|^2)((u_{\varepsilon m}(t+h), \Psi'_h(t))) - \frac{2}{h}M(\|u_{\varepsilon m}(t)\|^2)((u_{\varepsilon m}(t), \Psi'_h(t))) = \\ 2M(\|u_{\varepsilon m}(t+h)\|^2)((\Psi_h(t), \Psi'_h(t))) + \frac{2M(\|u_{\varepsilon m}(t+h)\|^2)}{h}((u_{\varepsilon m}(t), \Psi'_h(t)))- \\ \frac{2M(\|u_{\varepsilon m}(t)\|^2)}{h}((u_{\varepsilon m}(t), \Psi'_h(t))) = \\ M(\|u_{\varepsilon m}(t+h)\|^2)\frac{d}{dt}(\|\Psi_h(t)\|^2) + 2M_h(t)((u_{\varepsilon m}(t), \Psi'_h(t))). \end{aligned}$$

From (2.40) it follows that

$$\begin{aligned} \frac{d}{dt}|\Psi'_h(t)|^2 + M(\|u_{\varepsilon m}(t+h)\|^2)\frac{d}{dt}(\|\Psi_h(t)\|^2)+ \\ \frac{2}{h^2\varepsilon}\langle \beta(u'_{\varepsilon m}(t+h)) - \beta(u'_{\varepsilon m}(t)), u'_{\varepsilon m}(t+h) - u'_{\varepsilon m}(t)\rangle = \\ -2M_h(t)((u_{\varepsilon m}(t), \Psi'_h(t))) + 2(f_h(t), \Psi'_h(t)). \end{aligned}$$

By the monotonicity of the operator β, we obtain

$$\begin{aligned} \frac{d}{dt}|\Psi'_h(t)|^2 + M(\|u_{\varepsilon m}(t+h)\|^2)\frac{d}{dt}(\|\Psi_h(t)\|^2) \\ \leq 2|M_h(t)(\Delta u_{\varepsilon m}(t), \Psi'_h(t))| + 2|(f_h(t), \Psi'_h(t))|. \end{aligned} \tag{2.41}$$

Integrating (2.41) in t we have

$$\begin{gathered}|\Psi_h'(t)|^2+\int_0^t M(\|u_{\varepsilon m}(s+h)\|^2)\frac{d}{ds}(\|\Psi_h(s)\|^2)ds\leq\\ |\Psi_h'(0)|^2+2\int_0^t|M_h(s)(\Delta u_{\varepsilon m}(s),\Psi_h'(s))|ds+2\int_0^t|(f_h(s),\Psi_h'(s))|ds\,.\end{gathered}$$

Taking the limit as $h\to 0$, it follows

$$\begin{gathered}|u_{\varepsilon m}''(t)|^2+\int_0^t M(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}'(s)\|^2ds\leq\\ |u_{\varepsilon m}''(0)|^2+2\int_0^t[M'(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}(s)\|^2]|\Delta u_{\varepsilon m}(s),u_{\varepsilon m}''(s)|ds+\\ 2\int_0^t|(f'(s),u_{\varepsilon m}''(s))|ds\,.\end{gathered}\tag{2.42}$$

Using Assumption (A2) and (2.28), we obtain, from (2.42),

$$\begin{gathered}|u_{\varepsilon m}''(t)|^2+\int_0^t M(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}'(s)\|^2ds\leq\\ C+4\int_0^t|M'(\|u_{\varepsilon m}(s)\|^2)|\|u_{\varepsilon m}'(s)\|\|u_{\varepsilon m}(s)\|\|\Delta u_{\varepsilon m}(s)\|\|u_{\varepsilon m}''(s)\|ds+\\ \int_0^t|u_{\varepsilon m}''(s)|^2ds.\end{gathered}\tag{2.43}$$

From (2.26), (2.35) and (2.37) it follows that there exists a positive constant C such that

$$\|u_{\varepsilon m}(t)\|^2+|\Delta u_{\varepsilon m}(t)|^2+\int_0^t\|u_{\varepsilon m}'(s)\|^2ds\leq C,\quad\forall\varepsilon,m,t.\tag{2.44}$$

Since $M\in C^1([0,\infty))$, we also obtain from (2.44),

$$|M'(\|u_{\varepsilon m}(s)\|^2)|\leq C,\quad\forall\varepsilon,m,t.\tag{2.45}$$

On the other hand, using integration by parts, we get

$$\begin{gathered}\int_0^t M(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}'(s)\|^2ds=\\ M(\|u_{\varepsilon m}(s)\|^2)\|u_{\varepsilon m}'(s)\|^2-M(\|u_{0\varepsilon m}(s)\|^2)\|u_{1\varepsilon m}(s)\|^2-\\ \int_0^t M'(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}'(s)\|^2\|u_{\varepsilon m}'(s)\|^2ds.\end{gathered}$$

Estimates (2.37), (2.44), and (2.45) together imply

$$-\int M'(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u_{\varepsilon m}(s)\|^2\|u_{\varepsilon m}'(s)\|^2ds\geq -C\int_0^t\|u_{\varepsilon m}'(s)\|^3\geq -C,$$

independently of ε, m, and t. Therefore,

$$\int_0^t M(\|u_{\varepsilon m}(s)\|^2)\frac{d}{ds}\|u'_{\varepsilon m}(s)\|^2 ds \geq M(\|u_{\varepsilon m}(t)\|^2)\|u'_{\varepsilon m}(t)\|^2 - C, \tag{2.46}$$

independently of ε, m and t. Here, C denote various positive constants. Making use of inequalities (2.44)–(2.46) in (2.43) we obtain

$$|u''_{\varepsilon m}(t)|^2 + M(\|u_{\varepsilon m}(t)\|^2)\|u'_{\varepsilon m}(s)\|^2 \leq C + C\int_0^t |u''_{\varepsilon m}(s)|^2 ds, \tag{2.47}$$

independently of ε, m, and t. From (2.47) and using Gronwall's inequality, we have

$$|u''_{\varepsilon m}(t)|^2 \leq C, \tag{2.48}$$

independently of ε, m and t.

Passage to the limit By estimates (2.26) and (2.35) we obtain

$$\begin{aligned} (u_{\varepsilon m}) \quad &\text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\\ (u'_{\varepsilon m}) \quad &\text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),\\ (\theta_{\varepsilon m}) \quad &\text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Therefore, we can get subsequences, if necessary, denoted by $(u_{\varepsilon m})$ and $(\theta_{\varepsilon m})$, such that

$$u_{\varepsilon m} \to u_\varepsilon \quad \text{weak star in} \quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{2.49}$$

$$u'_{\varepsilon m} \to u'_\varepsilon \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.50}$$

$$\theta_{\varepsilon m} \to \theta_\varepsilon \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.51}$$

Similarly by (2.48), we obtain

$$u''_{\varepsilon m} \to u''_\varepsilon \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.52}$$

Also, by (2.33) and (2.34), there exist functions $\mathcal{X}_\varepsilon, \phi_\varepsilon \in L^{4/3}(0,T;W^{2,4/3}(\Omega))$ such that

$$\beta(u'_{\varepsilon m}) \to \mathcal{X}_\varepsilon \quad \text{in} \quad L^{4/3}(0,T;W^{2,4/3}(\Omega)), \tag{2.53}$$

$$\beta(\theta_{\varepsilon m}) \to \phi_\varepsilon \quad \text{in} \quad L^{4/3}(0,T;W^{2,4/3}(\Omega)). \tag{2.54}$$

It follows from the embeding $W_0^{2,4}(\Omega)$ into $L^4(\Omega)$ and of (2.29) that

$$|u'_{\varepsilon m}|^4_{L^4(0,T;W_0^{2,4}(\Omega))} \leq C\|\Delta u'_{\varepsilon m}\|^4_{L^4(\Omega)} \leq K.$$

Therefore, there exists a subsequence of $(u_{\varepsilon m})$ such that

$$u'_{\varepsilon m} \to u'_\varepsilon \quad \text{weak star in} \quad L^4(0,T;W_0^{2,4}(\Omega)). \tag{2.55}$$

Analogously, by (2.30) we obtain

$$\theta_{\varepsilon m} \to \theta_\varepsilon \quad \text{weak star in} \quad L^4(0,T;W_0^{2,4}(\Omega)). \tag{2.56}$$

Being the embedding from $H_0^1(\Omega)\cap H^2(\Omega)$ into $H_0^1(\Omega)$ compact, we can set a subsequence, again denoted by $(u_{\varepsilon m})$, such that:

$$u_{\varepsilon m} \to u_\varepsilon \quad \text{strong in} \quad L^2(0,T;H_0^1(\Omega)). \tag{2.57}$$

By assumption (A1) we obtain

$$M(\|u_{\varepsilon m}(t)\|^2) \to M(\|u_\varepsilon(t)\|^2). \tag{2.58}$$

From the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$u'_{\varepsilon m} \to u'_\varepsilon \quad \text{strong in} \quad L^2(0,T;L^2(\Omega)). \tag{2.59}$$

Then taking limit in the system (2.18)–(2.20), when $m\to\infty$, with $w=v\varphi(t)$, $v\in W_0^{2,4}(\Omega)$, $\varphi(t)\in\mathcal{D}(0,T)$ instead of w_j, and using the fact that β is monotone and hemicontinous operator, we obtain that $\{u_\varepsilon,\theta_\varepsilon\}$ is a weak solution of the system (2.18)–(2.20).

The initial conditions (2.19) can be obtained by observing the convergence above and the definition of weak solution; this is,

$$\begin{gathered}
u'_\varepsilon(0) = \lim_{m\to\infty} u_{0\varepsilon m} = \lim_{m\to\infty}\sum_{j=1}^m (u_{0\varepsilon},w_j)w_j = u_0\,,\\
u'_\varepsilon(0) = \lim u_{1\varepsilon m} = \lim_{m\to\infty}\sum_{j=1}^m (u_{1\varepsilon},w_j)w_j = u_1\,,\\
\phi_\varepsilon(0) = \lim_{m\to\infty} \theta_{0\varepsilon m} = \lim_{m\to\infty}\sum_{j=1}^m (\theta_{0\varepsilon},w_j)w_j = \theta_0.
\end{gathered}$$

This concludes the proof of Theorem 2.4

3 Main Result

In this section, we will prove the Theorem 2.3. By Theorem 2.4, there exists functions $u_\varepsilon,\theta_\varepsilon:\mathbb{Q}\to\mathbb{R}$ such that

$$\begin{gathered}
u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\\
u'_\varepsilon,\theta_\varepsilon \in L^4(0,T;W_0^{2,4}(\Omega)),\\
u''_\varepsilon \in L^\infty(0,T;L^2(\Omega)),\\
\theta_\varepsilon \in L^\infty(0,T;L^2(\Omega)),
\end{gathered}$$

satisfying the system

$$(u''_\varepsilon(t),w)+M[\|u_\varepsilon(t)\|^2]((u_\varepsilon(t),w))+\frac{1}{\varepsilon}\langle\beta(u'_\varepsilon(t),w)\rangle=(g(t),w),$$
$$(\theta_\varepsilon(t),w)+((\theta_\varepsilon(t),w))+(u'_\varepsilon(t),w)+\frac{1}{\varepsilon}\langle\beta(\theta_\varepsilon(t),w)\rangle=(g(t),w),$$

a.e. in $[0,T]$, for all $w\in W_0^{2,4}(\Omega)$. $u_\varepsilon(0)=u_0$; $u'_\varepsilon(0)=u_1$, and $\theta_\varepsilon(0)=\theta_0$.

Being the estimates (2.26), (2.29), (2.30), (2.33), (2.34), (2.32) and (2.44) independently of ε, m and t we obtain by Uniform Boundedness Theorem that there exists a positive constant C such that

$$\begin{aligned}|u'_\varepsilon(t)|^2+|\theta_\varepsilon(t)|^2+\|u_\varepsilon(t)\|^2+\int_0^T\|\theta_\varepsilon(t)\|^2ds+\\ \frac{2}{\varepsilon}\int_0^T\langle\beta(u'_\varepsilon(s),u'_\varepsilon(s))\rangle ds+\frac{2}{\varepsilon}\int_0^T\langle\beta(\theta_\varepsilon(s)),\theta_\varepsilon(s))\rangle ds\le\\ C\|\Delta u'_\varepsilon\|^4_{L^4(Q)}\le C\,,\end{aligned}$$

and

$$\|\Delta\theta_\varepsilon\|^4_{L^4(Q)}\le C\,,\quad \|\beta(u'_\varepsilon)\|_{L^{\frac{4}{3}}(0,T;W^{2,4/3}(\Omega))}\le C\,,$$
$$\|\beta(\theta_\varepsilon)\|_{L^{4/3}(0,T;W^{2,4/3}(\Omega))}\le C\,,\quad |\Delta u_\varepsilon(t)|^2\le C\,,\quad |u''_\varepsilon(t)|^2\le C.$$

Consequently, we can find a subnet, which we still represent by (u_ε), (θ_ε) such that

$$\begin{gathered}u_\varepsilon\to u\quad\text{weak star in}\quad L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\\ u'_\varepsilon\to u'\quad\text{weak star in}\quad L^\infty(0,T;L^2(\Omega)),\\ u''_\varepsilon\to u''\quad\text{weak star in}\quad L^\infty(0,T;L^2(\Omega)),\\ \beta(u'_\varepsilon)\to\beta(u')\quad\text{weak in}\quad L^{4/3}(0,T;W^{-2,\frac{4}{3}}(\Omega)),\\ \beta(\theta_\varepsilon)\to\beta(\theta)\quad\text{weak in}\quad L^{4/3}(0,T;W^{-2,\frac{4}{3}}(\Omega)),\\ u'_\varepsilon\to u'\quad\text{weak in}\quad L^4(0,T;W_0^{2,4}(\Omega)),\\ \theta_\varepsilon\to\theta\quad\text{weak in}\quad L^4(0,T;W_0^{2,4}(\Omega)).\end{gathered}$$

By the compactness theorem of Aubin-Lions [8], we obtain

$$u_\varepsilon\to u\quad\text{strongly}\quad L^2(0,T;H_0^1(\Omega)),\tag{3.1}$$
$$u'_\varepsilon\to u'\quad\text{strongly}\quad L^2(0,T;L^2(\Omega)).\tag{3.2}$$

We observe that

$$\begin{gathered}(u''_\varepsilon(t),v(t))+M(\|u_\varepsilon(t)\|^2)((u_\varepsilon(t),v(t)))+(\theta_\varepsilon(t),v(t))+\\ \frac{1}{\varepsilon}\langle\beta(u'_\varepsilon(t)),v(t)\rangle=(f(t),v(t))\,,\\ (\theta'_\varepsilon(t),v(t))+((\theta_\varepsilon(t),v(t)))+(u'_\varepsilon(t),v(t))+\frac{1}{\varepsilon}\langle\beta(\theta_\varepsilon(t)),v(t)\rangle=(g(t),v(t))\,.\end{gathered}$$

is true for all $v \in L^4(0,T;W_0^{2,4}(\Omega))$.

On the other hand, being $u_\varepsilon', \theta_\varepsilon \in L^4(0,T;W_0^{2,4}(\Omega))$ implies

$$(u_\varepsilon''(t), u_\varepsilon'(t)) + M(\|u_\varepsilon(t)\|^2)((u_\varepsilon(t), u_\varepsilon'(t))) + (\theta_\varepsilon(t), u_\varepsilon'(t)) + \frac{1}{\varepsilon}\langle \beta(u_\varepsilon'(t)), u_\varepsilon'(t)\rangle = (f(t), u_\varepsilon'(t))\,,$$

$$(\theta_\varepsilon'(t), \theta_\varepsilon(t)) + ((\theta_\varepsilon(t), \theta_\varepsilon(t))) + (u_\varepsilon'(t), \theta_\varepsilon(t)) + \frac{1}{\varepsilon}\langle \beta(\theta_\varepsilon(t)), \theta_\varepsilon(t)\rangle = (g(t), \theta_\varepsilon(t)).$$

Subtracting the equations of the system above, we obtain

$$\begin{aligned}(u_\varepsilon''(t), v(t) - u_\varepsilon'(t)) + M(\|u_\varepsilon(t)\|^2)((u_\varepsilon(t), v(t) - u_\varepsilon(t))) + \\ (\theta_\varepsilon(t), v - u_\varepsilon'(t)) + \frac{1}{\varepsilon}\langle \beta(u_\varepsilon'(t)), v(t) - u_\varepsilon'(t)\rangle = (f(t), v(t) - u_\varepsilon'(t))\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}(\theta_\varepsilon'(t), v(t) - \theta_\varepsilon(t)) + ((\theta_\varepsilon(t), v(t) - \theta_\varepsilon(t))) + \\ (u_\varepsilon'(t), v(t) - \theta_\varepsilon(t)) + \frac{1}{\varepsilon}\langle \beta(\theta_\varepsilon(t)), v(t) - \theta_\varepsilon(t)\rangle = (g(t), v(t) - \theta_\varepsilon(t)),\end{aligned} \tag{3.4}$$

for all $v \in W_0^{2,4}(\Omega)$.

Let us consider $v(t) \in K$ a. e. in $[0,T]$. Then we obtain $\beta(v(t)) = 0$ and being β a monotone operator, we have

$$\begin{aligned}\langle \beta(u_\varepsilon'(t)) - \beta(v(t)), v(t) - u_\varepsilon'(t)\rangle \le 0\,,\\ \langle \beta(\theta_\varepsilon(t)) - \beta(v(t)), v(t) - \theta_\varepsilon(t)\rangle \le 0\,.\end{aligned}$$

Therefore,

$$\int_0^T (u_\varepsilon''(t) - M(\|u_\varepsilon(t)\|^2)\Delta u_\varepsilon(t) + \theta_\varepsilon(t) - f(t), v(t) - u_\varepsilon'(t))dt \ge 0, \tag{3.5}$$

$$\int_0^T (\theta_\varepsilon(t) - \Delta\theta_\varepsilon + u_\varepsilon' - g(t), v(t) - \theta_\varepsilon(t))dt \ge 0, \tag{3.6}$$

for all $v \in L^4(0,T;W_0^{2,4}(\Omega))$ with $v(t) \in K$ a.e. in $[0,T]$. Now, taking the limit in (3.6) and (3.7), when $\varepsilon \to 0$ and using (3.1)–(3.3) and observing that $\Delta u_\varepsilon \to \Delta u$ weak in $L^2(0,T;L^2(\Omega))$ it follows that u,θ satisfy (1.5) and (1.6) in Theorem 2.3.

To conclude the proof of the existence of a solution, we show that $u'(t), \theta(t) \in \mathbb{K}$ a.e. in $[0,T]$. In fact, by (2.33) and (2.34) we have

$$\begin{aligned}\|\beta(u_\varepsilon')\|_{L^\infty(0,T;W^{2,\frac{4}{3}}(\Omega))} \le C\varepsilon,\\ \|\beta(\theta_\varepsilon)\|_{L^\infty(0,T;W^{2,\frac{4}{3}}(\Omega))} \le C\varepsilon.\end{aligned}$$

Therefore, as $\varepsilon \to 0$, $\beta(u_\varepsilon') \to 0$ and $\beta(\theta_\varepsilon) \to 0$ strong $L^\infty(0,T;W^{2,\frac{4}{3}}(\Omega))$.

On the other hand we have $\beta(u_\varepsilon') \to \beta(u')$ and $\beta(\theta_\varepsilon) \to \beta(\theta)$ weak in $L^{4/3}(0,T;W^{2,4/3}(\Omega))$. Then, $\beta(u'(t)) = \beta(\theta(t)) = 0$ in $L^\infty(0,T;W^{2,4/3}(\Omega))$. Therefore, $u'(t), \theta(t) \in \mathbb{K}$ a.e. in $[0,T]$.

The initial conditions (1.7) can be verified easily. This concludes the proof of Theorem 2.3.

4 Uniqueness

For proving uniqueness of solutions in Theorem 2.3, we consider the restriction

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad u_0(x) \geq 0 \text{ a. e. in } \Omega, \text{ and } \|u_0\| > 0.$$

Consequently $\|u(t)\| > 0$, for all $t \in [0,T]$. In fact, if there exists $t_0 \in [0,T]$ such that $\|u_0\| = 0$, then

$$\int_\Omega |u(x,t_0)|^2 dx \leq C\|u(t_0)\|^2 = 0,$$

where C is the constant of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, $u(x,t_0) = 0$, a.e. in Ω.

Since $u'(t) \in K$ a.e. in $[0,T]$, we have $u'(t) \geq 0$ a.e. in Ω. This implies that

$$u(x,t) \geq u(x,0) = u_0(x) \quad \text{in } \Omega \text{ a.e. in } [0,T]. \tag{4.1}$$

Being $\|u_0\| > 0$, there exists $\Omega' \subset \Omega$ with $\|\Omega'\| > 0$ such that that $u_0(x) > 0$. By (3.1) it follows that $u(x,t_0) > 0$ in Ω. This is a contradiction.

Theorem 4.1 *Under the hypotheses of Theorem 2.3, if*

i) $M(\lambda) > 0$ *for all* $\lambda > 0$, *and* $M(0) = 0$.

ii) $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_0(x) \geq 0$ *a.e. in* Ω, *and* $\|u_0\| > 0$,

Then the solution $\{u, \theta\}$ *of Theorem 2.3 is unique.*

Proof. From (i) and (ii) it follows that

$$m_0 = \min\{M(\|u(t)\|^2);\ t \in [0,T]\} > 0.$$

Suppose we have two pairs of solutions $\{u, \theta\}$ and $\{w, \varphi\}$ satisfying the conditions of Theorem 2.3. Let $\Psi = u - w$ and $\phi = \theta - \varphi$. Thus, Ψ and ϕ satisfy

$$\begin{gathered}(\Psi''(t) - M(\|u(t)\|^2)\Delta\Psi(t) + \{M(\|w(t)\|^2) - M(\|u(t)\|^2)\}\Delta w + \phi(t), \Psi'(t)) \leq 0,\\ (\phi'(t) - \Delta\phi(t) + \Psi'(t), \phi(t)) \leq 0,\end{gathered}$$

which implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{|\Psi'(t)|^2 + |\phi(t)|^2 + \|\phi(t)\|^2\} + M(\|u(t)\|^2)\frac{1}{2}\frac{d}{dt}\|\Psi(t)\|^2 + 2(\phi(t), \Psi'(t)) \leq \\ \{M(\|u(t)\|^2) - M(\|w(t)\|^2)\}(\Delta w(t), \Psi'(t))\,.\end{aligned}$$

Since

$$M(\|u(t)\|^2)\frac{d}{dt}\|\Psi(t)\|^2 = \frac{d}{dt}\{M(\|u(t)\|^2)\|\Psi(t)\|^2\} - \frac{d}{dt}[M(\|u(t)\|^2)]\|\Psi(t)\|^2$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\{|\Psi'(t)|^2+|\phi(t)|^2+\|\phi(t)\|^2\}+\frac{d}{dt}\{M(\|u(t)\|^2)\|\Psi(t)\|^2\}+2(\phi(t),\Psi'(t))\leq \\ \{M(\|u(t)\|^2)-M(\|w(t)\|^2)\}(\Delta w(t),\Psi'(t))+ \\ M'(\|u(t)\|^2)((u'(t),u(t))\|\Psi(t)\|^2.$$

Now, integrating this inequality form 0 to $t<T$, we obtain

$$\begin{aligned}\frac{1}{2}\{|\Psi'(t)|^2+|\phi(t)|^2+\|\phi(t)\|^2\}+M(\|u(t)\|^2)\|\Psi(t)\|^2+ \\ 2\int_0^t(\phi(t),\Psi'(t))ds\leq \\ \int_0^t\{M(\|u(s)\|^2)-M(\|w(s)\|^2)\}(\Delta w(s),\Psi'(s))ds+ \\ \int_o^t M'(\|u(s)\|^2)((u'(s),u(s))\|\Psi(s)\|^2ds.\end{aligned}\tag{4.2}$$

Note that $\|u(t)\|$ and $\|u'(t)\|\in L^\infty(0,T)$. Then there exists a positive constant C_0 such that

$$\|u(t)\|\leq C_0 \quad\text{and}\quad \|u'(t)\|\leq C_0 \quad\text{a.e. in } [0,T].$$

Since $M\in C^1([0,\infty))$, it follows $|M'(\xi)|\leq C_1$, for all $\xi\in[0,C_0]$.

Now, by the Mean Value Theorem, for each $s\in[0,T]$, there exists ξ_s between $\|u(s)\|^2$ and $\|w(s)\|^2$ such that

$$\begin{aligned}|M(\|u(s)\|^2)-M(\|w(s)\|^2)|\leq C_1|\|u(s)\|^2-\|w(s)\|^2|\leq \\ C_2\|u(s)-w(s)\|=C_2\|\Psi(s)\|.\end{aligned}\tag{4.3}$$

Observing that $|\Delta w(s)|\leq C_3$, from (4.2) and (4.3) we obtain that

$$\begin{aligned}|\Psi'(t)|^2+\|\phi(t)\|^2+M(\|u(t)\|^2\|)\Psi(t)\|^2\leq \\ C_4\int_o^t\left\{|\Psi'(s)|^2+\|\Psi(s)\|^2+\|\phi(s)\|^2\right\}ds,\end{aligned}$$

which implies

$$|\Psi'(t)|^2+\|\phi(t)\|^2+\|\Psi(t)\|^2\leq C_5\int_o^t\left\{|\Psi'(t)|^2+\|\Psi(t)\|^2+\|\phi(s)\|^2\right\}ds.$$

where $C_5=C_4/\min\{1,m_0\}$. From the above inequality and Gronwall inequality if follows that $\|\phi(t)\|=\|\Psi(t)\|=0$, i.e., ϕ and Ψ are zero almost everywhere. This completes the proof of uniqueness.

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