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A HYPERBOLIC PROBLEM WITH NONLINEAR SECOND-ORDER BOUNDARY DAMPING

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Abstract. The initial boundary value problem for the wave equation with nonlinear second-order dissipative boundary conditions is considered. Existence and uniqueness of global generalized solutions are proved.

1. Introduction

In [1], J.L. Lions considers nonlinear problems on manifolds in which the unknown ω satisfies the Laplace equation in a cylinder Q and a nonlinear evolution equation of the form

$$\frac{\partial \omega}{\partial \nu} + \omega_{tt} + |\omega_t|^\rho \omega_t = 0 \tag{1.1}$$

on the lateral boundary Σ of Q. Here ν is an outward normal vector on Σ. This problem models water waves with free boundaries ([2], [3]).

The boundary condition

$$\frac{\partial \omega}{\partial \nu} + |\omega_t|^\rho \omega_t = 0 \tag{1.2}$$

arises when one studies flows of a gas in channels with porous walls [4, 5]. The presence of the second derivative with respect to t in the boundary condition is due to internal forces acting on particles of the medium at the outward boundary.

Motivated by this, we study in the present paper the wave equation

$$u_{tt} - \Delta u = f \quad \text{in} \quad Q \tag{1.3}$$

with the nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} + K(u) u_{tt} + |u_t|^\rho u_t = 0 \quad \text{on} \quad \Sigma \tag{1.4}$$

and with the initial data

$$u(x,0) = u_t(x,0) = 0. \tag{1.5}$$

The term $K(u)u_{tt}$ models internal forces when the density of the medium depends on the displacement.

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In [1] it is shown that (1.1) can be replaced by the evolution equation

$$u_{tt} + A(u) + |u_t|^\rho u_t = 0 \quad \text{on} \quad \Sigma,$$

where A is a linear positive self-adjoint operator. In that sense, the expression (1.2) looks like a semilinear hyperbolic equation on the manifold Σ. Equation (1.4) also behaves as a hyperbolic equation with nonlinear principal operator.

Generally speaking, quasilinear hyperbolic equations do not have global regular solutions. There are examples of "blow-up" at a finite time. (See, for instance, [6].) Nevertheless, the presence of linear damping allows proof of the existence of global solutions for small initial data ([7]). Moreover, a nonlinear damping makes it possible to prove global existence theorems for some quasilinear wave equations without restrictions on a size of the initial conditions ([8], [9]).

Here we use the ideas from [8] to prove the existence of global generalized solutions to the problem (1.3)-(1.5). We exploit the Faedo-Galerkin method, a priori estimates and compactness arguments. Uniqueness is proved in the one-dimensional case.

We consider the classical wave equation only to simplify calculations. Similar results hold for a second-order evolution equation of the form

$$u_{tt} + A(t)u + F(u, u_t) = f,$$

where $A(t)$ is a linear, strictly elliptic operator, and $F(u, u_t)$ is a suitable function of u and u_t. Moreover, hyperbolic-parabolic or elliptic equations also may be considered.

2. The Main Result

For $T > 0$, let Ω be a bounded open set of R^n with sufficiently smooth boundary Γ and $Q = \Omega \times (0, T)$. We consider the hyperbolic problem

$$u_{tt} - \Delta u = f(x, t)\ , \quad (x, t) \in Q; \tag{2.1}$$

$$\left(\frac{\partial u}{\partial \nu} + K(u)u_{tt} + |u_t|^\rho u_t\right)\Big|_{\Sigma_1} = 0; \quad u|_{\Sigma_0} = 0; \tag{2.2}$$

$$u(x, 0) = u_t(x, 0) = 0. \tag{2.3}$$

Here $K(u)$ is a continuously differentiable positive function; ν is the outward unit normal vector on Γ; $\Gamma = \Gamma_0 \cup \Gamma_1$; $\Gamma_0 \cap \Gamma_1 = \varnothing$; $\Sigma_i = \Gamma_i \times (0, T)$ $(i = 0, 1)$; $\rho \in (1, \infty)$.

We denote by $H_1(\Omega)$ the Sobolev space $H^1(\Omega)$ with the condition $u|_{\Gamma_0} = 0$; $(u, v)(t) = \int_\Omega u(x, t)v(x, t)\, dx$; $||u||$ is the norm in $L^2(\Omega)$: $||u||^2(t) = (u, u)(t)$; $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$.

Definition. A function $u(x, t)$ such that

$$\begin{aligned}
&u \in L^\infty(0, T; H_1(\Omega)),\\
&u_t \in L^\infty(0, T; H_1(\Omega)) \cap L^{\rho+2}(\Sigma_1),\\
&u_{tt} \in L^\infty(0, T; L^2(\Omega) \cap L^2(\Gamma_1)),\\
&u(x, 0) = u_t(x, 0) = 0
\end{aligned}$$

is a generalized solution to (2.1)-(2.3) if for any functions $v \in H_1(\Omega) \cap L^{\rho+2}(\Gamma)$ and $\varphi \in C^1(0,T)$ with $\varphi(T)=0$ the following identity holds:

$$\begin{aligned}\int_0^T \left\{ (u_{tt}, v)(t) + (\nabla u, \nabla v)(t) + \int_{\Gamma_1} \left[|u_t|^\rho u_t - K'(u) u_t^2 \right] v \, d\Gamma \right\} \varphi(t)\, dt \\ - \int_0^T \varphi'(t) \int_{\Gamma_1} K(u) u_t v \, d\Gamma \, dt = \int_0^T (f, v) \varphi(t)\, dt \,. \end{aligned} \tag{2.4}$$

We consider functions $K(u)$ satisfying the assumptions

$$0 < K_0 \le K(u) \le C(1 + |u|^\rho), \tag{2.5}$$

$$|K'(u)|^{\frac{\rho}{\rho-1}} \le C(1 + K(u)). \tag{2.6}$$

These conditions mean that the density of the medium can not increase too rapidly as a function of displacement. The condition (2.6) appears quite naturally because functions with polynomial growth, such as $K(u) = 1 + |u|^s$ with $1 \le s \le \rho$, satisfy it. The inequality $K(u) \ge K_0$ means that the vacuum is forbidden.

The main result of this paper is the following.

Theorem. *Let the function $K(u)$ satisfy assumptions (2.5) and (2.6) and suppose $f(x,t) \in H^1(0,T;\, L^2(\Omega))$. Then for all $T > 0$ there exists at least one generalized solution to the problem (2.1)-(2.3). If $n = 1$, this solution is unique.*

Proof. We prove the existence part of the Theorem by the Faedo-Galerkin method. First, we construct approximations of the generalized solution. Then we obtain a priori estimates necessary to guarantee convergence of approximations. Finally, we prove the uniqueness in the one-dimensional case.

3. Approximate solutions

Let $\{w_j(x)\}$ be a basis in $H_1(\Omega) \cap L^{\rho+2}(\Gamma_1)$. We define the approximations

$$u^N(x,t) = \sum_{i=1}^N g_i(t) w_i(x), \tag{3.1}$$

where $g_i(t)$ are solutions to the Cauchy problem

$$(f, w_j)(t) = (u^N_{tt}, w_j)(t) + (\nabla u^N, \nabla w_j)(t) + \int_{\Gamma_1} \left\{ K(u^N) u^N_{tt} + |u^N_t|^\rho u^N_t \right\} w_j \, d\Gamma\,; \tag{3.2}$$

$$g_j(0) = g_j'(0) = 0; \;\; j = 1, ..., N. \tag{3.3}$$

It can be seen that (3.2) is not a normal system of ODE; therefore, we can not apply the Caratheodory theorem directly. To overcome this difficulty, we have to prove that the matrix A defined by

$$(Ag'')_j = g_j''(t) + \int_{\Gamma_1} \left\{ K(u^N) \sum_{i=1}^N g_i''(t) w_i(x) \right\} w_j(x)\, d\Gamma \;; \;\; j = 1, ..., N \tag{3.4}$$

has an inverse. Multiplying (3.2) by $g_j''(t)$ and summing over j, we obtain the quadratic form

$$q\left(g_1'', ..., g_N''\right) = \sum_{j=1}^{N}\left[(g_j'')^2 + \sum_{i=1}^{N}\int_{\Gamma_1} K(u^N) w_i w_j d\Gamma\; g_i'' g_j''\right].$$

The condition $K(u) \geq K_0 > 0$ implies that for any $g''(t) \neq 0$

$$q = \sum_{j=1}^{N}(g_j'')^2 + \int_{\Gamma_1} K(u^N)\left(\sum_{j=1}^{N} g_j'' w_j\right)^2 d\Gamma \geq \sum_{j=1}^{N}(g_j'')^2 + K_0\|u_{tt}^N\|^2_{L^2(\Gamma_1)} > 0.$$

Hence, the quadratic form q is positive definite and all eigenvalues of the symmetric matrix A in (3.4) are positive. Thus, (3.2) can be reduced to normal form and, by the Caratheodory theorem, the problem (3.2),(3.4) has solutions $g_j(t) \in H^3(0, t_N)$ and all the approximations (3.1) are defined in $(0, t_N)$.

4. A priori estimates

Next, we need a priori estimates to show that $t_N = T$ and to pass to the limit as $N \to \infty$. To simplify the exposition, we omit the index N whenever it is unambiguous to do so.

Multiplying (3.2) by $2g_j'$ and summing from $j = 1$ to $j = N$, we obtain

$$\begin{aligned} 2(f, u_t)(t) =& \frac{d}{dt}\left(\|u_t\|^2 + \|\nabla u\|^2\right)(t) + 2\int_{\Gamma_1} |u_t|^{\rho+2} d\Gamma \\ &+ \int_{\Gamma_1}\left\{\frac{d}{dt}\left(K(u)u_t^2\right) - K'(u)(u_t)^3\right\} d\Gamma\,. \end{aligned}$$

Integrating with respect to τ from 0 to t, we get

$$\begin{aligned} 2\int_0^t (f, u_\tau)\, d\tau &= \left(\|u_t\|^2 + \|\nabla u\|^2\right)(t) \\ &\quad + 2\int_0^t\int_{\Gamma_1}\left\{|u_\tau|^{\rho+2} - \frac{1}{2}K'(u)(u_\tau)^3\right\} d\Gamma\, d\tau + \int_{\Gamma_1} K(u)u_t^2\, d\Gamma\,. \end{aligned}$$

Notice that

$$\begin{aligned} & 2\int_0^t\int_{\Gamma_1} |u_\tau|^2\left\{|u_\tau|^\rho - \frac{1}{2}K'(u)u_\tau\right\} d\Gamma\, d\tau \\ & \geq 2\int_0^t\int_{\Gamma_1} |u_\tau|^2\left\{|u_\tau|^\rho - \varepsilon|u_\tau|^\rho - C(\varepsilon)|K'(u)|^{\frac{\rho}{\rho-1}}\right\} d\Gamma\, d\tau\,, \end{aligned}$$

where ε is an arbitrary positive number. From now on, we denote by "C" all constants independent of N.

Fixing $\varepsilon = 1/2$, taking into account (2.6), and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\left(||u_t||^2 + ||\nabla u||^2\right)(t) + \int_0^t \int_{\Gamma_1} |u_\tau|^{\rho+2}\, d\Gamma\, d\tau + \int_{\Gamma_1} K(u) u_t^2\, d\Gamma \\ &\le \int_0^t \left(||f||^2 + ||u_\tau||^2\right)(\tau)\, d\tau + C \int_0^t \int_{\Gamma_1} |u_\tau|^2 (1 + K(u))\, d\Gamma\, d\tau\,. \end{aligned} \tag{4.1}$$

Note that $K(u) \ge C_0(1 + K(u))$ where $2C_0 = \min\{1, K_0\}$. Therefore, for the function

$$E_1(t) = \left(||u_t||^2 + ||\nabla u||^2\right)(t) + C_0 \int_{\Gamma_1} (1 + K(u))|u_t|^2 d\Gamma$$

we have from (4.1) the inequality

$$E_1(t) \le C \left(1 + \int_0^t E_1(\tau)\, d\tau\right).$$

By Gronwall's lemma, we conclude that, for all $t \in (0, T)$ and for all $N \ge 1$,

$$E_1(t) \le C.$$

This and (4.1) give that for all $t \in (0, T)$,

$$\begin{gathered} \int_0^t \int_{\Gamma_1} |u_\tau|^{\rho+2}\, d\Gamma\, d\tau \le C, \\ \int_{\Gamma_1} K(u^N)(u_t^N)^2\, d\Gamma \le C, \end{gathered}$$

where C does not depend on N.

In order to obtain the second a priori estimate, we observe that

$$||u_{tt}||(0) \le ||f||(0); \tag{4.2}$$

$$\int_{\Gamma_1} u_{tt}^2(x, 0) d\Gamma \le ||f||^2 / K(0). \tag{4.3}$$

Indeed, multiplying (3.2) by $g_j''(0)$, summing over j, and setting $t = 0$, we obtain

$$(u_{tt}, u_{tt})(0) + \int_{\Gamma_1} K(0) u_{tt}^2(x, 0)\, d\Gamma = (f, u_{tt})(0)$$

which implies (4.2). Consequently,

$$\int_{\Gamma_1} K(0)u_{tt}^2(x,0)\,d\Gamma \leq ||f||(0)\cdot||u_{tt}||(0) \leq ||f||^2(0),$$

which gives (4.3).

Differentiating (3.2) with respect to t, multiplying by g_j'', and summing over j, we obtain the identity

$$\begin{aligned}(f_t,u_{tt})(t) =&\frac{1}{2}\frac{d}{dt}\left(||u_{tt}||^2+||\nabla u_t||^2\right)(t)\\ &+\int_{\Gamma_1}\left\{K(u)u_{tt}u_{ttt}+K'(u)u_tu_{tt}^2+(\rho+1)|u_t|^\rho u_{tt}^2\right\}\,d\Gamma\,.\end{aligned}$$

Notice that

$$K(u)u_{tt}u_{ttt}=\frac{1}{2}\frac{d}{dt}\left(K\left(u\right)u_{tt}^2\right)-\frac{1}{2}K'(u)u_tu_{tt}^2$$

and

$$\begin{aligned}\left|\int_{\Gamma_1}K'(u)u_tu_{tt}^2d\Gamma\right| \leq&\varepsilon\int_{\Gamma_1}|u_t|^\rho|u_{tt}|^2d\Gamma+C(\varepsilon)\int_{\Gamma_1}|K'(u)|^{\frac{\rho}{\rho-1}}\cdot|u_{tt}|^2\,d\Gamma\\ \leq&\varepsilon\int_{\Gamma_1}|u_t|^\rho|u_{tt}|^2d\Gamma+C(\varepsilon)\int_{\Gamma_1}(1+K(u))|u_{tt}|^2\,d\Gamma\,.\end{aligned}$$

Setting $\varepsilon=\rho$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(||u_{tt}||^2+||\nabla u_t||^2+\int_{\Gamma_1}K(u)u_{tt}^2d\Gamma\right)(t)+\int_{\Gamma_1}|u_t|^\rho|u_{tt}|^2d\Gamma\\ &\qquad\leq\left(||f_t||^2+||u_{tt}||^2\right)(t)+C\int_{\Gamma_1}(1+K(u))|u_{tt}|^2d\Gamma\,.\end{aligned}\tag{4.4}$$

Defining $E_2(t)$ as

$$E_2(t)=\left(||u_{tt}||^2+||\nabla u_t||^2+C_0\int_{\Gamma_1}(1+K(u))|u_{tt}|^2\,d\Gamma\right)(t)$$

and taking into account (4.2), (4.3), we reduce (4.4) to the form

$$E_2(t)\leq C\left(1+\int_0^t E_2(\tau)d\tau\right).$$

By Gronwall's lemma, for all $t\in(0,T),\ N\geq1$ we obtain

$$E_2(t)\leq C\,.$$

Taking into consideration that $u|_{\Sigma_0} = 0$, we obtain the following statements

$$\begin{aligned}
&u^N \in L^\infty(0,T;H^1(\Omega));\\
&u_t^N \in L^\infty(0,T;H^1(\Omega)) \cap L^{\rho+2}(\Sigma) \cap L^\infty(0,T;L^2(\Gamma));\\
&u_{tt}^N \in L^\infty(0,T;L^2(\Omega)\cap L^2(\Gamma));\\
&\frac{\partial}{\partial t}|u_t^N|^{1+\rho/2} \in L^2(\Sigma);\\
&K^{1/2}(u^N)u_{tt}^N \in L^\infty(0,T;L^2(\Gamma)).
\end{aligned} \tag{4.5}$$

5. Passage to the limit

Multiply (3.2) by $\varphi \in C^1(0,T)$ with $\varphi(T) = 0$ and integrate with respect to t from 0 to T. After integration by parts, we obtain

$$\begin{aligned}
&\int_0^T \left\{ (u_{tt}^N, w_j) + (\nabla u^N, \nabla w_j) + \int_{\Gamma_1} |u_t^N|^\rho u_t^N w_j \, d\Gamma \right\} \varphi(t)\, dt\\
&\quad - \int_0^T \varphi'(t) \int_{\Gamma_1} K(u^N) u_t^N w_j(x)\, d\Gamma\, dt + \varphi(t) K(u^N) u_t^N |_0^T\\
&\quad - \int_0^T \varphi(t) \int_{\Gamma_1} K'(u^N)(u_t^N)^2 w_j \, d\Gamma\, dt = \int_0^T (f, w_j)\varphi(t)\, dt.
\end{aligned} \tag{5.1}$$

Because of (4.5) we can extract a subsequence u^μ from u^N such that:

$$\begin{aligned}
&u^\mu \to u \text{ weakly star in } L^\infty(0,T;H_1(\Omega));\\
&u_t^\mu \to u_t \text{ weakly star in } L^\infty(0,T;H_1(\Omega)) \cap L^{\rho+2}(\Sigma);\\
&u_{tt}^\mu \to u_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Omega)\cap L^2(\Gamma));\\
&u^\mu, u_t^\mu \to u, u_t \quad \text{a.e. on } \Sigma.
\end{aligned}$$

Therefore,

$$\begin{gathered}
|u_t^\mu|^\rho u_t^\mu \in L^q(\Sigma), \quad q = (\rho+2)/(\rho+1) > 1, \text{ and converges a.e. on } \Sigma;\\
K(u^\mu) u_t^\mu \in L^q(\Sigma), \quad \text{and converges a.e. on } \Sigma;\\
K'(u^\mu)(u_t^\mu)^2 \in L^q(\Sigma), \quad \text{and converges a.e. on } \Sigma.
\end{gathered}$$

Thus, we are able to pass to the limit in (5.1) to obtain

$$\begin{aligned}
&\int_0^T \left\{ (u_{tt}, w_j) + (\nabla u, \nabla w_j) + \int_{\Gamma_1} \left(|u_t|^\rho u_t - K'(u) u_t^2 \right) w_j \, d\Gamma \right\} \varphi(t)\, dt\\
&\quad - \int_0^T \varphi'(t) \int_{\Gamma_1} K(u) u_t w_j \, d\Gamma\, dt = \int_0^T (f, w_j)\varphi(t)\, dt\,.
\end{aligned} \tag{5.2}$$

It can be seen that all the integrals in (5.2) are defined for any function $\varphi(t) \in C^1(0,T)$, $\varphi(T)=0$. Taking into account that $\{w_j(x)\}$ is dense in $H^1(\Omega)\cap L^{\rho+2}(\Gamma)$, we conclude that (2.4) holds.

If $n=1,2$, one can get more regular solutions. In this case $u \in L^\infty(0,T;L^q(\Gamma))$ for any $q \in [1,\infty)$. Hence, $K(u)u_{tt} \in L^\infty(0,T;L^p(\Gamma))$, where p is an arbitrary number from the interval $[1,2)$. This allows us to rewrite (5.2) in the form

$$\int_0^T (f,w_j)\,dt = \int_0^T \left\{ (u_{tt},w_j) + (\nabla u, \nabla w_j) + \int_{\Gamma_1} (K(u)u_{tt} + |u_t|^\rho u_t)\, w_j\, d\Gamma \right\} \varphi(t)\,dt.$$

Taking into account that almost every point $t \in (0,T)$ is a Lebesgue point and that $w_j(x)$ are dense in $H^1(\Omega)$ and therefore in $L^q(\Gamma)$, we obtain

$$(u_{tt},v)(t) + (\nabla u, \nabla v)(t) + \int_{\Gamma_1} \{K(u)u_{tt} + |u_t|^\rho u_t\} v\, d\Gamma = (f,v)(t),$$

where v is an arbitrary function from $H^1(\Omega)$.

6. Uniqueness

Let $n=1$. Let u and v be two solutions to (2.1)-(2.3), and set $z(x,t)=u(x,t)-v(x,t)$. Then for fixed t, for every function $\phi \in H_1(\Omega)$, we have

$$\begin{aligned}&(z_{tt},\phi)(t) + (\nabla z, \nabla \phi)(t)\\ &+ \int_{\Gamma_1} \{K(u)z_{tt} + v_{tt}(K(u)-K(v)) + |u_t|^\rho u_t - |v_t|^\rho v_t\}\ \phi\, d\Gamma = 0\,.\end{aligned}$$

Since $z_t(x,t) \in L^\infty(0,T;H_1(\Omega))$, we may take $\phi = z_t$, and this equation can be reduced to the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[E(t) + \int_{\Gamma_1} K(u)(z_t)^2\, d\Gamma\right]\\ &+ \int_{\Gamma_1} \left\{ v_{tt} z_t (K(u)-K(v)) - \frac{1}{2}K'(u)u_t(z_t)^2 \right\} d\Gamma \le 0\,.\end{aligned}$$

Here we set $E(t) = \|z_t\|^2(t) + \|\nabla z\|^2(t)$ and use the monotonicity of $|u_t|^\rho u_t$, the differentiability of K, and the regularity of $K(u)u_{tt}$ (see the end of previous section). Condition (2.6) then implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[E(t) + \int_{\Gamma_1} K(u)(z_t)^2\, d\Gamma\right]\\ &\le C \max_{\Gamma_1}(1+K(u))^{\frac{\rho-1}{\rho}} |u_t| \int_{\Gamma_1} (z_t)^2\, d\Gamma + \frac{1}{2}\int_{\Gamma_1} \left\{ |z_t|^2 + |v_{tt}|^2 |K(u)-K(v)|^2 \right\} d\Gamma\\ &\le C \int_{\Gamma_1} |z_t|^2\, d\Gamma + \max_{\Gamma_1} |K(u)-K(v)|^2 \int_{\Gamma_1} |v_{tt}|^2\, d\Gamma\\ &\le C_1 \|z_t\|^2_{L^2(\Gamma_1)} + C_2 \|v_{tt}\|^2_{L^2(\Gamma_1)} \cdot \|z\|^2_{C(\Gamma_1)}\,.\end{aligned}$$

Integrating from 0 to t, using (2.5) and the Sobolev embedding theorem ([10]), we obtain

$$\|z_t\|^2(t) + \|\nabla z\|^2(t) + \|z_t\|^2_{L^2(\Gamma_1)}(t) \le C \int_0^t \left[\|z_t\|^2_{L^2(\Gamma_1)}(\tau) + \|\nabla z\|^2(\tau) \right] d\tau \,.$$

This implies that $\|z\| = 0$ and $u = v$ a.e. in Q. The proof of the Theorem is completed.

Remark. We use homogeneous initial conditions (2.3) for technical reasons. Nonhomogeneous initial data also can be considered without any restrictions on their size ([10]). In fact, suppose that initial conditions are imposed as follows

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega.$$

Using the transformation $v(x,t) = u(x,t) - u_0(x) - u_1(x) \cdot t$, we obtain the problem

$$v_{tt} - \Delta v = F(x,t) \quad \text{in } Q; \tag{6.1}$$

$$\frac{\partial v}{\partial \nu} + \frac{\partial \phi}{\partial \nu} + K(v+\phi)v_{tt} + |v_t + u_1|^\rho (v_t + u_1) = 0 \quad \text{on } \Sigma_1; \tag{6.2}$$

$$v + \phi = 0 \qquad \text{on } \Sigma_0; \tag{6.3}$$

$$v(x,0) = v_t(x,0) = 0 \quad \text{in } \Omega. \tag{6.4}$$

Here $\phi(x,t) = u_0(x) + u_1(x) \cdot t$ and $F(x,t) = (f + \Delta\phi)(x,t)$ are given functions. It is clear that for regular solutions the compatibility conditions

$$\frac{\partial u_0}{\partial \nu} + K(u_0)(f + \Delta u_0) + |u_1|^\rho u_1 \;|_{\Gamma_1} = 0; \quad u_0 \;|_{\Gamma_0} = 0$$

need to be satisfied. This implies that conditions (6.2)-(6.4) are also compatible.

If $(u_0, u_1)(x) \in H^2(\Omega)$, than $F(x,t) \in H^1(0,T; L^2(\Omega))$. Moreover, if $u_1 \in L^{\rho+2}(\Gamma_1)$, then we are able to obtain necessary a priori estimates and to pass to the limit by the method of Sections 4 and 5. Of course, the use of conditions (6.2), (6.3) in place of (2.2) complicates calculations, but does not affect the final result.

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