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On a nonlinear coupled system with internal damping *

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Abstract

The purpose of this paper is to establish existence, uniqueness, and asymptotic behavior of solutions of a non-linear coupled system with variable coefficients, with coupling of a non-linear wave equation, and a linear heat equation in a smooth-bounded-open domain of $\mathbb{R}^n$.

1 Introduction

The nonlinear wave equation with thermo-elastic coupling is given by the system of equations

$$u''(x,t) - \mu(t)\Delta u(x,t) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}(x,t) + \gamma |u(x,t)|^\rho u(x,t) = 0 \tag{1.1}$$

$$\theta'(x,t) - \Delta\theta(x,t) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i}(x,t) = 0 \quad \text{in} \quad Q, \tag{1.2}$$

with initial and boundary conditions

$$u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad \theta(x,0) = \theta^0(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$u(x,t) = 0, \quad \theta(x,t) = 0 \quad \text{on} \quad \Gamma\times]0,\infty[, \tag{1.4}$$

where u is displacement, θ is absolute temperature, Δ denotes the Laplace operator, μ is a positive real function of t, γ and ρ are positive real numbers, the temporal partial derivative is represented by $'$, Ω is a smooth-bounded-open set in $\mathbb{R}^n$ with C^2 boundary Γ, and $Q = \Omega\times]0,\infty[$.

The non-linearity $|v|^\rho v$ usually appears in relativistic quantum mechanic (see Segal [10] or Schiff [9]), and has been considered by various authors for hyperbolic, parabolic and elliptic equations. Lions [4] studied the wave equation with the same non-linearity, i.e., $|v|^\rho v$, in a smooth-bounded-open domain Ω

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of $\mathbb{R}^n$ with $n \in \mathbb{N}$, and proved existence and uniqueness of solution using both Faedo-Galerkin's and Compactness' methods.

Clark at al [1] investigated the system (1.1)-(1.2) with γ equal to zero and feedback-homogeneous conditions over a part of the boundary Γ. They established global existence of strong and weak solutions by Faedo-Galerkin's method using a particular basis of the space $H_0^1(\Omega) \cap H^2(\Omega)$ intruduced by Medeiros & Milla Miranda [6] and the exponential stability of total energy associated to the weak solution using Komornik-Zuazua's method [2].

Based in the theory developed in the papers of Clark at al [1] and Lions [4] (Theorems 1.1, 1.2 and 1.3), we will prove that the system (1.1)-(1.4) has a unique global strong solution, a unique global weak solution, and the total energy associated to these solutions is asymptotically stable.

The outline of this article is as follows. In Section 2, the basic notations are laid out and global existence of strong and weak solutions are issued, whilst exponential decay is aired in Section 3.

2 Notation and strong-weak solutions

In the sequel $L^p(\Omega)$, $1 \leq p < \infty$, will denote the collection of L-functions which are pth-integrable over Ω. For $m \in \mathbb{N}$, the space $H^m(\Omega)$ is the Sobolev class of the functions of the spatial variable x which along with their first m derivatives belong to $L^2(\Omega)$ (see, for exemple, Sobolev [11] or Medeiros & Milla Miranda [7]) and the closure in $H^m(\Omega)$ of the space $\mathcal{D}(\Omega)$ of test functions on Ω is denoted by $H_0^m(\Omega)$. The inner product and norm of $H_0^m(\Omega)$ and $L^2(\Omega)$ are represented by $((\cdot,\cdot))$, $\|\cdot\|$ and $(\cdot,\cdot)$, $|\cdot|$, respectively. By $W^{m,p}(0,\infty)$ we denote the Sobolev space of the functions of the temporal variable t.

Let X be a Banach space, T a positive real number or $T = +\infty$ and $1 \leq p \leq \infty$, denotes by $L^p(0,T;X)$ the Banach space of all measurable functions $u :]0,T[\longrightarrow X$, such that $t \mapsto \|u(t)\|_X$ is in $L^p(0,T)$, with norm

$$\|v\|_p = \Big(\int_0^T \|v(t)\|_X^p dt \Big)^{1/p}, \quad \text{if} \quad 1 \leq p < \infty,$$

and if $p = \infty$, then

$$\|v\|_\infty = \operatorname*{ess\,sup}_{t \in [0,T]} \|v(t)\|_X .$$

To obtain the existence and uniqueness of global solution of the mixed problem (1.1)-(1.4) we suppose the additional hypotheses about μ and ρ:

$$\mu \in W^{1,1}(0,\infty), \quad \mu(t) \geq \mu_0 > 0 \quad \text{and} \quad \mu'(t) \leq 0. \tag{2.1}$$

ρ is any finite-positive real number if $n = 1, 2$ and

$$\rho \leq \tfrac{2}{n-2} \quad \text{if} \quad n \geq 3. \tag{2.2}$$

Note, for later use, that if ρ satisfies (2.2) then $H_0^1(\Omega)$ has continuous injection into $L^{\rho+2}(\Omega)$.

The constant γ is considered only to get the asymptotic behavior of the total energy of solutions. Thus, it imposes the hypothesis (3.1) fixed in the section 3.

Now we can present the existence results of the initial- and boundary-value problem (1.1)-(1.4). Thus, the global strong solution is guaranteed by

Theorem 2.1 (Strong solution) *Let μ be the fuction defined by (2.1). Let ρ be the given real-positive constant satisfying (2.2) and suppose also γ a real-positive number. If*

$$u_0,\ \theta_0 \ \in\ H_0^1(\Omega)\cap H^2(\Omega) \quad \textit{and} \quad u_1 \ \in\ H_0^1(\Omega),$$

then the system (1.1)-(1.4) has a unique solution $\{u,\theta\}$ such that

$$u \ \ \textit{and} \ \ \theta \in L^\infty\left(0,\infty; H_0^1(\Omega)\cap H^2(\Omega)\right), \tag{2.3}$$

$$u' \in L^\infty\left(0,\infty; H_0^1(\Omega)\right) \quad \textit{and} \quad \theta' \in L^2\left(0,\infty; H_0^1(\Omega)\right), \tag{2.4}$$

$$u'' \in L^\infty\left(0,\infty; L^2(\Omega)\right), \tag{2.5}$$

and the equations (1.1) and (1.2) are given in the sense of $L^\infty\left(0,\infty; L^2(\Omega)\right)$.

Proof. *Existence.* To show the global existence of solution we will use both the Faedo-Galerkin's and Compactness' methods. We consider $(w_j)_{j\in\mathbb{N}}$ an orthonormal basis of $H_0^1(\Omega)\cap H^2(\Omega)$, and denote by $V_m = [w_1, w_2, w_3, \cdots, w_m]$ the subspace of $H_0^1(\Omega)\cap H^2(\Omega)$ spanned by the m first vectors of $(w_j)_{j\in\mathbb{N}}$. In these conditions, the approximated system associated to system (1.1) (1.2) is given by

$$(u''_m(t),v) + \mu(t)((u_m(t),v)) + \sum_{i=1}^{n}\left(\frac{\partial\theta_m}{\partial x_i}(t), v\right) + \gamma\left(|u_m(t)|^\rho u_m(t), v\right) = 0, \tag{2.6}$$

$$(\theta'_m(t),w) + ((\theta_m(t),w)) + \sum_{i=1}^{n}\left(\frac{\partial u'_m}{\partial x_i}(t), w\right) = 0, \tag{2.7}$$

where v and w belong to V_m.

Let $u_m(0) = u_{0m}$, $u'_m(0) = u_{1m}$, and $\theta_m(0) = \theta_{0m}$ be. Hence, u_{0m}, u_{1m}, and θ_{0m} belong to V_m, and satisfy

$$u_{0m} \longrightarrow u_0 \quad \text{strongly} \quad \text{in} \quad H_0^1(\Omega)\cap H^2(\Omega), \tag{2.8}$$

$$\theta_{0m} \longrightarrow \theta_0 \quad \text{strongly} \quad \text{in} \quad H_0^1(\Omega)\cap H^2(\Omega), \tag{2.9}$$

$$u_{1m} \longrightarrow u_1 \quad \text{strongly} \quad \text{in} \quad H_0^1(\Omega). \tag{2.10}$$

Under these conditions, the system (2.6) (2.7) has a local solution $\{u_m(t),\theta_m(t)\}$ over the interval $[0,t_m[$. This interval will be extended to any interval $[0,\infty[$ thanks to the first estimate below.

Estimate I. Substituting v by $2u'_m(t)$ and w by $2\theta_m(t)$ in (2.6) and (2.7) respectively, using Green's formula in the term $\sum_{i=1}^n \left(\frac{\partial u'_m}{\partial x_i}(t), \theta_m(t)\right)$ and integrating over [0,t[, $0 \leq t \leq t_m$, we get

$$\begin{aligned} E_1(t) + 2\int_0^t \|\theta_m(s)\|^2 ds &= \int_0^t \mu'(s)\|u_m(s)\|^2 ds + |u_{1m}|^2 + |\theta_{0m}|^2 \\ &\quad + \mu(0)\|u_{0m}\|^2 + \frac{2\gamma}{\rho+2}\int_\Omega |u_{0m}|^{\rho+2} dx, \end{aligned}$$

where

$$E_1(t) = |u'_m(t)|^2 + |\theta_m(t)|^2 + \mu(t)\|u_m(t)\|^2 + \frac{2\gamma}{\rho+2}\int_\Omega |u_m(t)|^{\rho+2} dx.$$

From (2.8)-(2.10) and (2.1) there is a positive constant k, independent of m such that

$$E_1(t) + 2\int_0^t \|\theta_m(s)\|^2 ds \leq k \quad \text{for all} \quad t \geq 0. \tag{2.11}$$

Hence, we can extent the approximate solutions $\{u_m(t),\, \theta_m(t)\}$ on the whole interval $[0, \infty[$ independent of m and t.

Estimate II. Differentiating the equations (2.6) and (2.7) with respect to t, replacing v by $2u''_m(t)$ and w by $2\theta'_m(t)$, we have according to (2.1) that

$$\begin{aligned} &\frac{d}{dt}\left\{|u''_m(t)|^2 + \mu(t)\|u'_m(t)\|^2 + |\theta'_m(t)|^2\right\} + 2\|\theta'_m(t)\|^2 \\ &\leq \; -2\gamma(\rho+1)\int_\Omega |u_m(t)|^\rho u'_m(t) u''_m(t) dx - 2\mu'(t)((u_m(t), u''_m(t))). \end{aligned} \tag{2.12}$$

Now, let us make an analysis about the term $-2\mu'(t)((u_m(t), u''_m(t)))$. In the equation (2.6) replacing v by $2u''_m(t)$ we can write

$$\begin{aligned} -2\mu(t)((u_m(t), u''_m(t))) &= 2|u''_m(t)|^2 + 2\sum_{i=1}^n \int_\Omega \frac{\partial \theta_m}{\partial x_i}(t) u''_m(t) dx \\ &\quad + 2\gamma\int_\Omega |u_m(t)|^\rho u_m(t) u''_m(t) dx\,. \end{aligned}$$

Multiplying both sides by $\frac{\mu'(t)}{\mu(t)}$ and using (2.1) we obtain

$$\begin{aligned} -2\mu'(t)((u_m(t), u''_m(t))) &\leq 2\frac{\mu'(t)}{\mu(t)}\sum_{i=1}^n \int_\Omega \frac{\partial \theta_m}{\partial x_i}(t) u''_m(t) dx \\ &\quad + 2\gamma\frac{\mu'(t)}{\mu(t)}\int_\Omega |u_m(t)|^\rho u_m(t) u''_m(t) dx. \end{aligned} \tag{2.13}$$

Substituting (2.13) in (2.12) it follows

$$\begin{aligned}\frac{d}{dt}E_2(t)+2\|\theta_m^{'}(t)\|^2 &\leq -2\gamma(\rho+1)\int_\Omega |u_m(t)|^\rho u_m'(t)u_m''(t)dx\\ &\quad+\frac{2\mu'(t)}{\mu(t)}\sum_{i=1}^n\int_\Omega \frac{\partial\theta_m}{\partial x_i}(t)u_m''(t)dx\\ &\quad+\frac{2\gamma\mu'(t)}{\mu(t)}\int_\Omega |u_m(t)|^\rho u_m(t)u_m''(t)dx,\end{aligned}$$

where

$$E_2(t)=|u_m''(t)|^2+\mu(t)\|u_m'(t)\|^2+|\theta_m^{'}(t)|^2.$$

Hence, we can write

$$\begin{aligned}\frac{d}{dt}E_2(t)+2\|\theta_m^{'}(t)\|^2 &\leq 2\gamma(\rho+1)\int_\Omega |u_m(t)|^\rho |u_m'(t)||u_m''(t)|dx\\ &\quad+\frac{2|\mu'(t)|}{\mu(t)}\sum_{i=1}^n\int_\Omega \left|\frac{\partial\theta_m}{\partial x_i}(t)\right||u_m''(t)|dx \qquad (2.14)\\ &\quad+\frac{2\gamma|\mu'(t)|}{\mu(t)}\int_\Omega |u_m(t)|^{\rho+1}|u_m''(t)|dx.\end{aligned}$$

Now, we are going to make an analysis on the second member of (2.14) in order to apply the Gronwall's inequality. By using the Hölder's generalized inequality for the case $\frac{1}{q}+\frac{1}{n}+\frac{1}{2}=1$ we get from hypothesis (2.2) the following continuous injections:

- If $n=1,2$ then $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ for all $q>0$.
- If $n\geq 3$ then $\rho\leq\frac{2}{n-2}$. Thus, $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ where $q=\frac{2n}{n-2}$. Consequently, as $q\geq\rho+2$ then $H_0^1(\Omega)\hookrightarrow L^{\rho+2}(\Omega)$, and since $n\rho\leq q$ then it also follows that $H_0^1(\Omega)\hookrightarrow L^{n\rho}(\Omega)$.

Using the two items above and the estimate I in the first term of right-hand side of (2.14) yields

$$\begin{aligned}&\int_\Omega |u_m(t)|^\rho|u_m'(t)||u_m''(t)|dx\\ &\leq \left[\int_\Omega |u_m(t)|^{\rho n}dx\right]^{1/n}\left[\int_\Omega |u_m'(t)|^q dx\right]^{1/q}\left[\int_\Omega |u_m''(t)|^2dx\right]^{1/2}\\ &= \|u_m(t)\|^\rho_{L^{\rho n}(\Omega)}\|u_m'(t)\|_{L^q(\Omega)}|u_m''(t)|_{L^2(\Omega)} \qquad (2.15)\\ &\leq c_0\|u_m(t)\|^\rho_{H_0^1(\Omega)}\|u_m'(t)\|_{H_0^1(\Omega)}|u_m''(t)|\\ &\leq c_1\|u_m'(t)\||u_m''(t)|.\end{aligned}$$

Substituting (2.15) and applying Cauchy-Schwarz's inequality in the last two integrals in the right-hand side of (2.14) we have

$$\frac{d}{dt}E_2(t)+2\|\theta_m^{'}(t)\|^2 \leq 2c_1\gamma(\rho+1)\|u_m'(t)\||u_m''(t)|$$

$$
\begin{aligned}
&+\frac{2\sqrt{n}}{\mu(t)}|\mu'(t)|\|\theta_m(t)\||u''_m(t)| \\
&+\frac{2\gamma}{\mu(t)}|\mu'(t)|\|u_m(t)\|^{\rho+1}|u''_m(t)|\,.
\end{aligned}
$$

Hence, (2.1), and usual inequalities yields

$$
\begin{aligned}
&\frac{d}{dt}E_2(t)+2\|\theta'_m(t)\|^2 \\
&\le \frac{c_1\gamma(\rho+1)}{\mu_0}\mu(t)\|u'_m(t)\|^2+\frac{c_1\gamma(\rho+1)}{\mu_0}\mu(t)|u''_m(t)|^2+\frac{\sqrt{n}}{\mu_0}|\mu'(t)|\|\theta_m(t)\|^2 \\
&\quad +\frac{\sqrt{n}}{\mu_0}|\mu'(t)||u''_m(t)|^2+\frac{\gamma}{\mu_0}|\mu'(t)|\|u_m(t)\|^{2(\rho+1)}+\frac{\gamma}{\mu_0}|\mu'(t)||u''_m(t)|^2.
\end{aligned}
$$

Let $\nu(t)$ be the function given by

$$
\nu(t)=\frac{1}{\mu_0}\left\{c_1\gamma(\rho+1)\mu(t)+(\sqrt{n}+\gamma)|\mu'(t)|\right\},
$$

then it implies

$$
\begin{aligned}
\frac{d}{dt}E_2(t)+2\|\theta'_m(t)\|^2 &\le \nu(t)\left\{\|u'_m(t)\|^2+|u''_m(t)|^2\right\} \\
&\quad +\frac{\sqrt{n}}{\mu_0}|\mu'(t)|\|\theta_m(t)\|^2+\frac{\gamma}{\mu_0}|\mu'(t)|\|u_m(t)\|^{2(\rho+1)}.
\end{aligned}
$$

From (2.11) there exists a constant c_2 independent of m and t such that

$$
\begin{aligned}
\frac{d}{dt}E_2(t)+2\|\theta'_m(t)\|^2 &\le \nu(t)\left\{\|u'_m(t)\|^2+|u''_m(t)|^2\right\} \\
&\quad +\frac{\sqrt{n}}{\mu_0}|\mu'(t)|\|\theta_m(t)\|^2+\frac{c_2\gamma}{\mu_0}|\mu'(t)|.
\end{aligned}
$$

Now, integrating over the interval $[0,t[$, using again (2.1), (2.11), and observing that $u''_m(0)$ and $\theta'_m(0)$ are bounded in $L^2(\Omega)$ we conclude that there exists a constant c independent of m and t such that

$$
E_2(t)+2\int_0^t\|\theta'_m(s)\|^2ds\le c+\int_0^t\nu(s)\left\{\|u'_m(s)\|^2+|u''_m(s)|^2\right\}ds.
$$

From (2.1) it follows that ν belongs to $L^1(0,\infty)$. Thus, applying the Gronwall's inequality we obtain

$$
E_2(t)+2\int_0^t\|\theta'_m(s)\|^2ds\le k \quad \text{for all} \quad t\ge 0. \tag{2.16}
$$

Limit of the approximated solutions. From (2.11) and (2.16) it is possible to take the limit in the nonlinear system (2.6) (2.7). In fact, from (2.11) we obtain

$$\begin{aligned}
&(u_m) \quad \text{is bounded in} \quad L^\infty\left(0,\infty; H_0^1(\Omega)\right),\\
&(u'_m) \quad \text{is bounded in} \quad L^\infty\left(0,\infty; L^2(\Omega)\right).
\end{aligned}$$

Hence, we have in particular, that the sequences $(u_m)_{m\in\mathbb{N}}$ and $(u'_m)_{m\in\mathbb{N}}$ are bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$ and $L^2\left(0,T;L^2(\Omega)\right)$ respectively. Thus, by compact injection of $H^1(\Omega\times]0,T[)$ into $L^2(\Omega\times]0,T[)$ it follows by Lions-Aubin's theorem that there exists a subsequence of (u_m), which we denote as the original sequence such that

$$u_m \longrightarrow u \quad \text{strong in} \quad L^2\left(0,T;L^2(\Omega)\right),$$

whence, $u_m \longrightarrow u$ in $\Omega\times]0,T[$ almost everywhere. Hence,

$$|u_m|^\rho u_m \longrightarrow |u|^\rho u \quad \text{in} \quad \Omega\times]0,T[\quad \text{almost everywhere.} \tag{2.17}$$

From (2.11) we also have that

$$(u_m) \quad \text{is bounded in} \quad L^\infty\left(0,T;L^{\rho+2}(\Omega)\right). \tag{2.18}$$

From (2.17), (2.18) and Lions' lemma (cf. Lions [4], lemma 1.3) we obtain

$$|u_m|^\rho u_m \longrightarrow |u|^\rho u \quad \text{weak star in} \quad L^\infty(0,T;L^{p'}(\Omega)), \tag{2.19}$$

where $p=\rho+2$. Since T any positive real number then the convergence (2.19) is also held for all $t\in[0,\infty[$.

We still can obtain from (2.11) and (2.16) the following convergences

$$\begin{aligned}
u_m &\longrightarrow u && \text{weak star in} && L^\infty(0,\infty;H_0^1(\Omega)),\\
u'_m &\longrightarrow u' && \text{weak star in} && L^\infty(0,\infty;H_0^1(\Omega)),\\
u''_m &\longrightarrow u'' && \text{weak star in} && L^\infty(0,\infty;L^2(\Omega)),\\
\theta_m &\longrightarrow \theta && \text{weak in} && L^2(0,\infty;H_0^1(\Omega)),\\
\theta'_m &\longrightarrow \theta' && \text{weak in} && L^2(0,\infty;H_0^1(\Omega)).
\end{aligned} \tag{2.20}$$

By using (2.19) and (2.20), we can take the limit in the system (2.6) (2.7), and the statement (2.4) and (2.5) are given by these convergences.

The regularity (2.3) is guaranteed by results on elliptic regularity (see, for instance, Medeiros & Milla Miranda [7] or Nirenberg [8]) in view of

$$\Delta u = \frac{1}{\mu}\left\{u'' + \sum_{i=1}^n \frac{\partial\theta}{\partial x_i} + \gamma|u|^\rho u\right\} \quad \text{belongs to} \quad L^2(0,\infty;L^2(\Omega))$$

and

$$\Delta\theta = \theta + \sum_{i=1}^n \frac{\partial u'}{\partial x_i} \quad \text{belongs to} \quad L^2(0,\infty;L^2(\Omega)).$$

From the two previous results we can conclude that equations (1.1) and (1.2) are given in the sense of $L^\infty(0,\infty;L^2(\Omega))$ As a consequence of (2.3)-(2.5), the functions u, u' and θ are continuous. (see Lions & Magenes [5] Vol. I, Cap. 1). Therefore, the initial conditions (1.3) are well defined. $\diamondsuit$

Uniqueness. If $\{u,\theta\}$ and $\{\hat{u},\hat{\theta}\}$ are solutions of (1.1)-(1.4), then for all φ and ξ belong to $L^2\left(0,\infty;L^2(\Omega)\right)$ the functions $v=u-\hat{u}$ and $w=\theta-\hat{\theta}$ satisfy

$$\begin{gathered}
(v''(t),\varphi)+\mu(t)((v(t),\varphi))+\sum_{i=1}^{n}\left(\frac{\partial w}{\partial x_i}(t),\varphi\right)\\
+\gamma\left(|u(t)|^\rho u(t)-|\hat{u}(t)|^\rho\hat{u}(t),\varphi\right)=0,\\
(w'(t),\xi)+((w(t),\xi))+\sum_{i=1}^{n}\left(\frac{\partial v'}{\partial x_i}(t),\xi\right)=0,\\
v(0)=0,\quad v'(0)=0\quad\text{and}\quad w(0)=0\,.
\end{gathered}$$

When we replace φ for $2v_t$ and ξ for $2w$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{|v'(t)|^2+\mu(t)\|v(t)\|^2+|w(t)|^2\right\}+2\|w(t)\|^2\\
&\qquad+2\gamma\int_\Omega\left(|u(t)|^\rho u(t)-|\hat{u}(t)|^\rho\hat{u}(t)\right)v'(t)dx \;=\; 2\mu'(t)\|v(t)\|^2.
\end{aligned}$$

Using the hypothesis (2.1) we can write

$$\begin{aligned}
&\frac{d}{dt}\left\{|v'(t)|^2+\mu(t)\|v(t)\|^2+|w(t)|^2\right\}+2\|w(t)\|^2\\
&\quad\leq\; 2\gamma\int_\Omega\left||u(t)|^\rho u(t)-|\hat{u}(t)|^\rho\hat{u}(t)\right||v'(t)|dx\,.
\end{aligned}\tag{2.21}$$

Applying the mean value theorem we get

$$\begin{aligned}
&\int_\Omega\left||u(t)|^\rho u(t)-|\hat{u}(t)|^\rho\hat{u}(t)\right||v'(t)|dx\\
&\quad\leq\;(\rho+1)\int_\Omega\sup_{0\leq t\leq T}\left\{|u(t)|^\rho,|\hat{u}(t)|^\rho\right\}|v(t)||v'(t)|dx.
\end{aligned}$$

By using Hölder's generalized inequality for the case $\frac{1}{q}+\frac{1}{n}+\frac{1}{2}=1$, there exists a constant $c>0$ such that

$$\begin{aligned}
&\int_\Omega\left||u(t)|^\rho u(t)-|\hat{u}(t)|^\rho\hat{u}(t)\right||v'(t)|dx\\
&\quad\leq\; c\left\{\||u|^\rho\|_{L^n(\Omega)}+\||\hat{u}|^\rho\|_{L^n(\Omega)}\right\}\|v(t)\|_{L^q(\Omega)}\|v'(t)\|_{L^2(\Omega)}.
\end{aligned}$$

As $\|\,|z|^\rho\,\|_{L^n(\Omega)}=\|z\|^\rho_{L^{n\rho}(\Omega)}$ and $n\rho\leq q$, then using the same argument of the estimate II we obtain $\|\,|z|^\rho\,\|_{L^n(\Omega)}\leq c\|z\|^\rho_{H_0^1(\Omega)}$ for all $z\in H_0^1(\Omega)$.

On the other hand, the estimate (2.11) implies that $\|z(t)\|^{\rho}_{H^1_0(\Omega)}$ is bounded for all $t \geq 0$. Therefore, a positive real constant c exists such that

$$\int_{\Omega} ||u(t)|^{\rho}u(t) - |\hat{u}(t)|^{\rho}\hat{u}(t)|\,|v'(t)|dx \leq c\|v\|_{H^1_0(\Omega)}|v'|_{L^2(\Omega)}. \tag{2.22}$$

Substituting (2.22) in (2.21) and integrating from 0 to t we get

$$\begin{aligned} &|v'(t)|^2 + \mu(t)\|v(t)\|^2 + |w(t)|^2 + 2\int_0^t \|w(s)\|^2 ds \\ &\leq \gamma \int_0^t \left\{|v'(s)|^2 + c^2\|v(s)\|^2\right\} ds. \end{aligned}$$

As $\frac{\mu(t)}{\mu_o} \geq 1$ then choosing $k = \max\left\{\gamma,\ \frac{c^2}{\mu_0}\right\}$ yields

$$\begin{aligned} &|v'(t)|^2 + \mu(t)\|v(t)\|^2 + |w(t)|^2 + 2\int_0^t \|w(s)\|^2 ds \\ &\leq k \int_0^t \left\{|v'(s)|^2 + \mu(s)\|v(s)\|^2\right\} ds. \end{aligned}$$

Finally, by Gronwall's inequality it follows that $v(t) = w(t) = 0$ on $[0,\infty[$. Hence, Theorem 2.1 is established $\diamondsuit$

Now, we shall present a result of the global existence and uniqueness of weak solution for the system (1.1)-(1.4).

Theorem 2.2 (Weak solution) *Let the assumptions of Theorem 2.1 about μ, ρ, and γ be satisfied. If $p = \rho + 2$ and*

$$u_0 \in H^1_0(\Omega) \cap L^p(\Omega), \quad u_1 \in L^2(\Omega) \quad \textit{and} \quad \theta_0 \in H^1_0(\Omega),$$

then the system (1.1)-(1.4) has a unique solution $\{u, \theta\}$ such that

$$u \in L^{\infty}\left(0,\infty; H^1_0(\Omega) \cap L^p(\Omega)\right), \quad u' \in L^{\infty}\left(0,\infty; L^2(\Omega)\right), \tag{2.23}$$

$$u'' \in L^2\left(0,\infty; H^{-1}(\Omega) + L^{p'}(\Omega)\right), \tag{2.24}$$

$$\theta \in L^{\infty}\left(0,\infty; L^2(\Omega)\right) \cap L^2\left(0,\infty; H^1_0(\Omega)\right), \tag{2.25}$$

$$\theta' \in L^2\left(0,\infty; H^{-1}(\Omega)\right), \tag{2.26}$$

and the equation (1.1) is given in the sense of $L^2\left(0,\infty; H^{-1}(\Omega) + L^{p'}(\Omega)\right)$ and (1.2) is given in the sense of $L^2\left(0,\infty; H^{-1}(\Omega)\right)$.

Proof. *Existence.* The demonstration of existence of a weak solution is made by the same method utilized to get the strong solution. However, we now consider the basis $(w_j)_{j\in\mathbb{N}}$ in the space $H_0^1(\Omega)\cap L^p(\Omega)$, and the initial conditions sastisfying

$$u_{0m} \longrightarrow u_0 \quad \text{strongly} \quad \text{in} \quad H_0^1(\Omega)\cap L^p(\Omega), \tag{2.27}$$
$$\theta_{0m} \longrightarrow \theta_0 \quad \text{strongly} \quad \text{in} \quad H_0^1(\Omega), \tag{2.28}$$
$$u_{1m} \longrightarrow u_1 \quad \text{strongly} \quad \text{in} \quad L^2(\Omega). \tag{2.29}$$

Under these conditions, we have that the system (2.6) (2.7) has a local solution $\{u_m(t), \theta_m(t)\}$ on the interval $[0, t_m[$, and this solution will be extended over the whole interval $[0, \infty[$ independent of m and t thanks to estimate (2.30) above.

Using the same procedure used to get estimate I of Theorem 2.1, we get

$$\begin{aligned} &|u_m'(t)|^2 + |\theta_m(t)|^2 + \mu(t)\|u_m(t)\|^2 \\ &+\frac{2\gamma}{p}\int_\Omega |u_m(t)|^p dx + 2\int_0^t \|\theta_m(s)\|^2 ds \quad \leq \quad k \quad \text{for all} \quad t\geq 0. \end{aligned} \tag{2.30}$$

Hence, there exist subsequences of $(u_m)_{m\in\mathbb{N}}$ and $(\theta_m)_{m\in\mathbb{N}}$, which we denote as the original sequences, and functions $u : \Omega\times]0,\infty[\to \mathbb{R}$ and $\theta : \Omega\times]0,\infty[\to \mathbb{R}$ such that

$$u_m \longrightarrow u \quad \text{weak star in} \quad L^\infty(0,\infty; H_0^1(\Omega)\cap L^p(\Omega)), \tag{2.31}$$
$$u_m' \longrightarrow u' \quad \text{weak star in} \quad L^\infty(0,\infty; L^2(\Omega)), \tag{2.32}$$
$$\theta_m \longrightarrow \theta \quad \text{weakly in} \quad L^2(0,\infty; H_0^1(\Omega)). \tag{2.33}$$

From (2.31)-(2.33) we can take the limit in (2.6) (2.7) yielding

$$\begin{aligned} &-\int_0^\infty (u'(t), v)\phi'(t)dt + \int_0^\infty \mu(t)((u(t), v))\phi(t) \\ &+\sum_{i=1}^n \int_0^\infty \left(\frac{\partial\theta}{\partial x_i}(t), v\right)\phi(t)dt + \gamma\int_0^\infty (|u(t)|^\rho u(t), v)\,\phi(t)dt \quad = \quad 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &-\int_0^\infty (\theta(t), w)\phi'(t)dt + \int_0^\infty ((\theta(t), w))\phi(t)dt \\ &\qquad +\sum_{i=1}^n \int_0^\infty \left(\frac{\partial u'}{\partial x_i}(t), w\right)\phi(t)dt \quad = \quad 0, \end{aligned} \tag{2.35}$$

where v and w belong to $H_0^1(\Omega)$ and ϕ belongs to $\mathcal{D}(0,\infty)$.

Hence, following the same procedure of Clark at al [1] (Theorem 4.1) we can conclude

$$\begin{aligned} &u'' - \mu\Delta u + \sum_{i=1}^n \frac{\partial\theta}{\partial x_i} + \gamma|u|^\rho u = 0 \quad \text{in} \quad L^2\left(0,\infty; H^{-1}(\Omega) + L^{p'}(\Omega)\right), \\ &\theta' - \Delta\theta + \sum_{i=1}^n \frac{\partial u'}{\partial x_i} = 0 \quad \text{in} \quad L^2\left(0,\infty; H^{-1}(\Omega)\right). \end{aligned} \tag{2.36}$$

The statements (2.23)-(2.26) are consequences of (2.36) and of convergences (2.31)-(2.33).

The functions u, u' and θ are continuous, thanks to (2.23)-(2.26). Thus, the initial conditions (1.3) are verified.

Uniqueness. From $(2.23)_2$ and (2.24) the duality $\langle u'', u' \rangle$ does not make sense. Thus, it is necessary to utilize the regularization's method of Ladyzhenskaya-Visik [3], see also Lions [4] (Theorem 1.2, pp. 14), and we proceed such as in the demonstration of uniqueness of Theorem 2.1 Therefore, Theorem 2.2 has been proven ♢

3 Asymptotic behavior

The aim of this section is to prove that the total energy associated to solutions of (1.1)-(1.4) has exponential decay when the time t goes to $+\infty$. To reach our goal we will utilize the method of Komornik-Zuazua [2].

The total energy of the system (1.1)-(1.4) is given by

$$E(t) = |u'(t)|^2 + \mu(t)\|u(t)\|^2 + |\theta(t)|^2 + \frac{2\gamma}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx.$$

To obtain the asymptotic behavior of the energy $E(t)$ it is necessary to consider the following hypothesis about the constant γ,

$$\gamma \leq \frac{\mu_0^{\rho+1}}{2ck^\rho}\,, \tag{3.1}$$

where μ_0 and k are fixed in (2.1) and (2.11) respectively, and c is the constant of continuous injection of $H_0^1(\Omega)$ into $L^{\rho+2}(\Omega)$. From estimate I and hypothesis (2.1) it is easy to see that the energy $E(t)$ is not increasing in view of

$$\frac{d}{dt}E(t) \leq -\|\theta(t)\|^2 \quad \text{for all} \quad t \geq 0. \tag{3.2}$$

The asymptotic behavior of the energy of our system is given by

Theorem 3.1 *If the constant γ satisfies the hypothesis (3.1) then the energy $E(t)$ associated to weak solution of the system (1.1)-(1.4), guaranteed by Theorem 2.2, satisfies*

$$E(t) \leq E(0)\exp(-\omega t) \quad \textit{for all} \quad t \geq 0, \tag{3.3}$$

where ω is a real-positive number defined in (3.23).

Proof. First we get (3.3) for the energy $E(t)$ given by the strong solutions of (1.1)-(1.4) guaranteed by Theorem 2.1. Thus, the stability result for the energy associated to the weak solution is established without difficulty by density arguments. Let us consider the auxiliary real function

$$\psi(t) = 2\left(u'(t), (x\cdot\nabla)\,u(t)\right) + (n-1)\left(u'(t), u(t)\right) \quad \text{for all} \quad t \geq 0, \tag{3.4}$$

where $x \in \mathbb{R}^n$ is nonzero, "$\cdot$" denotes the usual scalar product in $\mathbb{R}^n$, and $n \in \mathbb{N}$.

Differentiating $\psi(t)$ with respect to t we obtain

$$\begin{aligned} \psi'(t) &= 2\left(u''(t), (x\cdot\nabla)\,u(t)\right) + 2\left(u'(t), (x\cdot\nabla)\,u'(t)\right) \\ &\quad +(n-1)\left(u''(t), u(t)\right) + (n-1)\left|u'(t)\right|^2. \end{aligned}$$

Replacing $u''(t)$ by $\mu(t)\Delta u(t) - \sum_{i=1}^n \frac{\partial\theta}{\partial x_i}(t) - \gamma|u(t)|^\rho u(t)$ in the identity above it implies

$$\begin{aligned} \psi'(t) &= 2\mu(t)\int_\Omega \Delta u(t)\,(x\cdot\nabla)\,u(t)dx - 2\sum_{i=1}^n \int_\Omega \frac{\partial\theta}{\partial x_i}(t)\,(x\cdot\nabla)\,u(t)dx \\ &\quad -2\gamma\int_\Omega |u(t)|^\rho u(t)\,(x\cdot\nabla)\,u(t)dx + 2\int_\Omega u'(t)\,(x\cdot\nabla)\,u'(t)dx \\ &\quad +(n-1)|u'(t)|^2 + (n-1)\mu(t)\int_\Omega \Delta u(t)u(t)dx \\ &\quad -(n-1)\sum_{i=1}^n\int_\Omega \frac{\partial\theta}{\partial x_i}(t)u(t)dx - \gamma(n-1)\int_\Omega |u(t)|^{\rho+2}dx. \end{aligned} \tag{3.5}$$

Our next objective is to limit each term of the right-hand side of (3.5). In the following steps we will use the Green's formula and the boundary conditions (1.4) several times.

Step 1: First term of (3.5). Since $u(t) \in H_0^1(\Omega)\cap H^2(\Omega)$, the Green's formula we have

$$\begin{aligned} 2\int_\Omega \Delta u\,(x\cdot\nabla)\,udx &= 2\sum_{i,j=1}^n \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)x_j\frac{\partial u}{\partial x_j}dx \\ &= -2\sum_{i,j=1}^n\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)dx \\ &= -2\sum_{i=1}^n\int_\Omega\left(\frac{\partial u}{\partial x_i}\right)^2dx - 2\sum_{i,j=1}^n\int_\Omega \frac{\partial u}{\partial x_i}x_j\frac{\partial^2 u}{\partial x_i\partial x_j}dx. \end{aligned} \tag{3.6}$$

However,

$$\begin{aligned} &-2\sum_{i,j=1}^n\int_\Omega \frac{\partial u}{\partial x_i}x_j\frac{\partial^2 u}{\partial x_i\partial x_j}dx \\ &= 2\sum_{i,j=1}^n\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\left(x_j\frac{\partial u}{\partial x_i}\right)dx \end{aligned}$$

$$= 2\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial x_j}{\partial x_j}\frac{\partial u}{\partial x_i}dx + 2\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}x_j\frac{\partial^2 u}{\partial x_j \partial x_i}dx.$$

Thus,

$$\begin{aligned} -2\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}x_j\frac{\partial^2 u}{\partial x_i \partial x_j}dx &= \sum_{i,j=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial x_j}{\partial x_j}\frac{\partial u}{\partial x_i}dx \\ &= n\sum_{i=1}^{n}\int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 dx. \end{aligned} \tag{3.7}$$

Substituting (3.7) in (3.6) yields

$$2\mu(t)\int_\Omega \Delta u(t)\,(x\cdot\nabla)\,u(t)dx = -(2-n)\mu(t)\|u(t)\|^2. \tag{3.8}$$

Step 2 Second term of (3.5).

$$\begin{aligned} -2\sum_{i=1}^{n}\int_\Omega \frac{\partial \theta}{\partial x_i}(t)\,(x\cdot\nabla)\,u(t)dx &\leq \frac{nr_\Omega^2}{\mu_o}\|\theta(t)\|^2 + \sum_{i=1}^{n}\frac{\mu_o}{4n}\|u(t)\|^2 \\ &= \frac{nr_\Omega^2}{\mu_o}\|\theta(t)\|^2 + \frac{\mu(t)}{4}\|u(t)\|^2, \end{aligned} \tag{3.9}$$

where $r_\Omega = \|x\|_{L^\infty(\Omega)}$.

Step 3 Third and eighth terms of (3.5).

$$\begin{aligned} &-2\gamma\int_\Omega |u(t)|^\rho u(t)\,(x\cdot\nabla)\,u(t)dx \\ &= -2\gamma\sum_{i=1}^{n}\int_\Omega |u(t)|^\rho u(t)x_i\frac{\partial u}{\partial x_i}(t)dx \\ &= 2\gamma\sum_{i=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\left(|u(t)|^\rho u(t)x_i\right)u(t)dx \\ &= 2\gamma\rho\sum_{i=1}^{n}\int_\Omega |u(t)|^{\rho+1}x_i\frac{\partial}{\partial x_i}|u(t)|dx \\ &\quad +2\gamma\sum_{i=1}^{n}\int_\Omega |u(t)|^\rho u(t)x_i\frac{\partial u}{\partial x_i}(t)dx + 2\gamma\sum_{i=1}^{n}\int_\Omega |u(t)|^{\rho+2}dx. \end{aligned}$$

Observe that

$$\begin{aligned} 2\gamma\rho\sum_{i=1}^{n}\int_\Omega |u(t)|^{\rho+1}x_i\frac{\partial}{\partial x_i}|u(t)|dx &= \frac{2\gamma\rho}{\rho+2}\sum_{i=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\left(|u(t)|^{\rho+2}\right)x_i\,dx \\ &= -\frac{2n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx\,. \end{aligned}$$

Thus,

$$-2\gamma \int_\Omega |u(t)|^\rho u(t)\,(x\cdot\nabla)\,u(t)dx = -\frac{n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx + n\gamma\int_\Omega |u(t)|^{\rho+2}dx.$$

Hence, adding the third and eighth terms of (3.5) yields

$$\begin{aligned}&-2\gamma \int_\Omega |u(t)|^\rho u(t)\,(x\cdot\nabla)\,u(t)dx - \gamma(n-1)\int_\Omega |u(t)|^{\rho+2}dx \\ &= -\frac{n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx + \gamma\int_\Omega |u(t)|^{\rho+2}dx.\end{aligned} \tag{3.10}$$

As $H_0^1(\Omega)$ is continuously embedded into $L^{\rho+2}(\Omega)$ then there exists a real positive constant c such that

$$\gamma\int_\Omega |u(t)|^{\rho+2}dx \le \gamma c\|u(t)\|^{\rho+2} = \gamma c\|u(t)\|^\rho\|u(t)\|^2.$$

From (2.11) we have $\|u(t)\|^\rho \le \frac{k^\rho}{\mu_0^\rho}$ for all $t\ge 0$. Therefore,

$$\gamma\int_\Omega |u(t)|^{\rho+2}dx \le \frac{\gamma c k^\rho}{\mu_0^\rho}\|u(t)\|^2.$$

As $\frac{\mu(t)}{\mu_o}\ge 1$ then

$$\gamma\int_\Omega |u(t)|^{\rho+2}dx \le \frac{\gamma c k^\rho}{\mu_0^{\rho+1}}\mu(t)\|u(t)\|^2.$$

Substituting this expression in (3.10), we obtain

$$\begin{aligned}&-2\gamma \int_\Omega |u(t)|^\rho u(t)\,(x\cdot\nabla)\,u(t)dx - \gamma(n-1)\int_\Omega |u(t)|^{\rho+2}dx \\ &\le -\frac{n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx + \frac{\gamma c k^\rho}{\mu_0^{\rho+1}}\mu(t)\|u(t)\|^2.\end{aligned} \tag{3.11}$$

Step 4: Fourth term of (3.5).

$$2\int_\Omega u'(t)\,(x\cdot\nabla)\,u'(t)dx = \sum_{i=1}^n\int_\Omega x_i\frac{\partial}{\partial x_i}[u'(t)]^2dx = -\sum_{i=1}^n\int_\Omega [u'(t)]^2dx.$$

Hence,

$$2\int_\Omega u'(t)\,(x\cdot\nabla)\,u'(t)dx = -n|u'(t)|^2\,.$$

Step 5: Sixth term of (3.5).

$$(n-1)\mu(t)\int_\Omega \Delta u(t)u(t)dx = (1-n)\mu(t)\|u(t)\|^2. \tag{3.12}$$

Step 6: Seventh term of (3.5).

$$
\begin{aligned}
-(n-1)\sum_{i=1}^{n}\int_\Omega \frac{\partial\theta}{\partial x_i}(t)u(t)dx &\le \frac{\lambda_1 n(n-1)^2}{\mu_o}\|\theta(t)\|^2+\sum_{i=1}^{n}\frac{\mu_o}{4n\lambda_1}|u(t)|^2\\
&\le \frac{\lambda_1 n(n-1)^2}{\mu_o}\|\theta(t)\|^2+\frac{\mu(t)}{4}\|u(t)\|^2, \qquad (3.13)
\end{aligned}
$$

where λ_1 is defined by $|v|^2 \le \lambda_1\|v\|^2$ for all $v \in H_0^1(\Omega)$. Substituting (3.8)-(3.13) in (3.5) yields

$$
\begin{aligned}
\psi'(t) &\le -|u'(t)|^2-\left(\frac{1}{2}-\frac{\gamma ck^\rho}{\mu_0^{\rho+1}}\right)\mu(t)\|u(t)\|^2 \qquad (3.14)\\
&\quad +\left(\frac{nR_\Omega^2+\lambda_1 n(n-1)^2}{\mu_o}\right)\|\theta(t)\|^2-\frac{n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx.
\end{aligned}
$$

The constant $\frac{1}{2}-\frac{\gamma ck^\rho}{\mu_0^{\rho+1}}$ is positive thanks to the hypothesis (3.1). Now, multiplying (3.14) for a suitable ϵ and taking into account that $-\|\theta\|^2 \le -\frac{1}{\lambda_1}|\theta|^2$, for all $\theta \in H_0^1(\Omega)$ then we have from (3.2) and (3.14) that

$$
\begin{aligned}
E'(t)+\epsilon\psi'(t) &\le -\epsilon|u'(t)|^2-\epsilon\left(\frac{1}{2}-\frac{\gamma ck^\rho}{\mu_0^{\rho+1}}\right)\mu(t)\|u(t)\|^2\\
&\quad -\frac{1}{\lambda_1}\left[1-\epsilon\left(\frac{nr_\Omega^2+\lambda_1 n(n-1)^2}{\mu_o}\right)\right]|\theta(t)|^2 \qquad (3.15)\\
&\quad -\epsilon\frac{n\gamma\rho}{\rho+2}\int_\Omega |u(t)|^{\rho+2}dx.
\end{aligned}
$$

Let ϵ and τ be real positive numbers such that

$$
\epsilon=\min\left\{\frac{\mu_0}{nr_\Omega^2+\lambda_1 n(n-1)^2},\quad \frac{1}{\tau_0}\right\}, \qquad (3.16)
$$

$$
\tau=\min\left\{\frac{\mu_0^{\rho+1}-2\gamma ck^\rho}{2\mu_0^{\rho+1}},\ \frac{\mu_0-\epsilon\left[nr_\Omega^2+\lambda_1 n(n-1)^2\right]}{\mu_o\lambda_1\epsilon},\ \frac{n\rho}{2}\right\}, \qquad (3.17)
$$

where τ_0 is defined in (3.19). Taking into account (3.16) and (3.17) in (3.15) we get

$$
E'_\epsilon(t) \le -\epsilon\tau E(t) \quad \text{for all} \quad t \ge 0, \qquad (3.18)
$$

where $E_\epsilon(t)=E(t)+\epsilon\psi(t)$.

On the other hand,

$$
\begin{aligned}
|\epsilon\psi(t)| &\le \epsilon\left\{\frac{1}{\mu_o}|u'(t)|^2+r_\Omega^2\mu_o\|u(t)\|^2+\frac{n-1}{2\mu_o}|u'(t)|^2+\frac{(n-1)\mu_o}{2}|u(t)|^2\right\}\\
&\le \epsilon\left\{\frac{n+1}{2\mu_o}|u'(t)|^2+\left[\frac{2r_\Omega^2+\lambda_1(n-1)}{2}\right]\mu(t)\|u(t)\|^2\right\}.
\end{aligned}
$$

Choosing

$$\tau_0 = \max\left\{\frac{2\mu_o}{n+1}, \quad \frac{2}{2r_\Omega^2 + \lambda_1(n-1)}\right\}, \tag{3.19}$$

it follows

$$|\epsilon\psi(t)| \le \epsilon\tau_0 E(t) \quad \text{for all} \quad t \ge 0. \tag{3.20}$$

Note that

$$|E_\epsilon(t) - E(t)| \le \epsilon|\psi(t)| \quad \text{for all} \quad t \ge 0. \tag{3.21}$$

From (3.20) and (3.21) we have

$$(1-\epsilon\tau_0)E(t) \le E_\epsilon(t) \le (1+\epsilon\tau_0)E(t) \quad \text{for all} \quad t \ge 0. \tag{3.22}$$

From (3.18) and (3.22) we obtain

$$E'(t) \le -\omega E(t) \quad \text{or all } t \ge 0\,, \tag{3.23}$$

where $\omega = \epsilon\tau/(1-\epsilon\tau_0)$. From (3.16) it follows that $1-\epsilon\tau_0$ is a real positive number. Thus, there exists a real positive number ω satisfying (3.3). Therefore, the proof of Theorem 3.1 is complete ♢

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