Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 99, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS TO BIHARMONIC EQUATIONS WITH SIGN-CHANGING COEFFICIENTS

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Communicated by Vicentiu D. Radulescu.

Abstract. In this article, we study the existence of solutions for the semilinear elliptic equation

$$\Delta^2 u - a(x)\Delta u = b(x)|u|^{p-2}u$$

with Navier boundary condition $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a bounded domain with smooth boundary and $2 < p < 2^*$. We consider two different assumptions on the potentials a and b, including the case of sign-changing weights. The approach is based on the Nehari manifold with variational arguments about the corresponding fibering map, which ensures the multiple results.

1. Introduction and preliminary results

The literature concerning the existence of solution of the elliptic PDEs is very extensive, (for instance see [4, 6, 11, 18] and the references therein). Since fourth-order PDEs have been appeared in various models such as micro-electro-mechanical systems, phase field models of multiphase systems (see [7, 9, 16]), a number of articles have been devoted to the fourth-order elliptic PDEs; we refer the interested readers to [2, 5, 12, 13, 15, 20, 21, 22].

In particular, the biharmonic equation $\Delta^2 u + c\Delta u = d[(u+1)^+ - 1]$, in which $u^+ = \max\{u, 0\}$, have attracted the attention of the mathematicians. This type of elliptic equation furnishes a model to study the traveling waves in suspension bridges, which is first developed by Lazer and Mckenna [14]. For $u = u(x_1, \dots, x_N)$ the bi-Laplacian operator is defined by

$$\Delta^2 u = \sum_{i=1}^{N} \frac{\partial^4 u}{\partial x_i^4} + \sum_{i,j=1; i\neq j}^{N} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2}.$$

The fourth-order equations, which are studied in the most papers, has the form $\Delta^2 u + c\Delta u = f(x, u)$, in which f satisfied certain conditions, $c < \mu_1$ and sometimes $c > \mu_1$; where μ_1 is the first eigenvalue of $-\Delta u = \lambda u$ with Dirichlet boundary condition.

Micheletti and Pistoia [15] provided a geometrical structure of the equation $\Delta^2 u + c\Delta u = bg(x, u)$ similar to the linking theorem, by supposing $2G(x, s) \le s^2$,

2010 *Mathematics Subject Classification.* 35A01, 35J35, 35D30, 35J91.
Key words and phrases. Bi-Laplacian operator; weak solution; Nehari manifold; fibering map.

Submitted July 17, 2017. Published Aapril 28, 2018.

$\limsup_{s\to-\infty} G(x,s)/s^2 \le 0$ and $\liminf_{s\to 0} G(x,s)/s^2 = l(x)$; where $G(x,u) = \int_0^u g(x,s)ds$, and consequently they derived the multiplicity existence results.

In [21], based on the mountain pass theorem, the existence of positive solutions for the problem $\Delta^2 u + c\Delta u = f(x,u)$ is studied in which f satisfies the local superlinearity or sublinearity conditions and $c < \mu_1$. The similar problem in [10] is studied under the conditions $\liminf_{|u|\to\infty} G(x,u)/|u|^2 = \infty$ and $ug(x,u) - 2G(x,u) \ge d|u|^\sigma$ where $\sigma > \frac{2N}{N+4}$ and by using the variant fountain theorem the existence of multiple solutions is derived. In [22] by using the least action principle, the Ekeland variational principle and the mountain pass theorem, the multiplicity of solutions for the problem $\Delta^2 u + c\Delta u = a(x)|u|^{s-2}u + f(x,u)$ with the combined nonlinearity on f is studied.

In [20] the equation $\Delta^2 u + c\Delta u = \lambda u + f(u)$ was studied in which f has subcritical growth condition, i.e., $|f(s)| \le d_1|s| + d_2|s|^{p-1}$ for some $p \in [2, 2^*)$ and $d_1, d_2 > 0$, under Navier boundary condition by applying the topological degree theory.

In this paper, we consider the problem

$$\begin{gathered}\Delta^2 u - a(x)\Delta u = b(x)|u|^{p-2}u, \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$ with $N > 4$ and $2 < p < 2^* = \frac{2N}{N-2}$. Moreover, one of the following assumptions is satisfied.

- (A1) $a, b \in L^\infty(\Omega)$ and $a(x), b(x) \ge 0$ a.e. in Ω, or
- (A2) $a, b \in L^\infty(\Omega)$ and a, b may change sign.

The main results of the article are in two subsections. In the first one, we consider problem (1.1) by assuming condition $(A1)$ and so we seek the solutions through providing a minimizer sequence.

In the second subsection, where condition (A2) is satisfied, we study the existence results due to the behavior of the corresponding fibering map, while $a^+ < \mu_1$ or $\mu_1 < a^- < a^+ < \mu_1 + \sigma$ for some appropriate σ which is introduced later, $a^+ = \operatorname{ess\,sup}\{a(x), x \in \Omega\}$ and $a^- = \operatorname{ess\,inf}\{a(x), x \in \Omega\}$.

It is known that if $I(u)$ denotes the energy functional corresponding to an equation, all of the critical points of I must lie on the manifold $\{u; \langle I'(u), u\rangle = 0\}$, which is known as the Nehari manifold (see [17, 19]). Moreover, the fibering map $(\varphi_u : t \to I(tu))$ which is closely linked to the Nehari manifold is an interesting approach for describing of the energy functional's behavior on the Nehari manifold (see [3, 8]).

Consider the Sobolev space

$$H^1(\Omega) := \{u \in L^2(\Omega) : u_{x_i} \in L^2(\Omega), 1 \le i \le N\}.$$

It is known that $H^1(\Omega)$ with the inner product $\langle u, v\rangle := \int_\Omega |\nabla u \nabla v| dx$ is a Hilbert space. Moreover, let

$$\begin{gathered}H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}, \\ H^2(\Omega) := \{u \in L^2(\Omega) : u_{x_i}, u_{x_i x_j} \in L^2(\Omega),\ 1 \le i, j \le N\}.\end{gathered}$$

We recall that $H^2(\Omega)$ with the inner products $\langle u, v\rangle = \int_\Omega |\triangle u \triangle v| dx$ or

$$\langle u, v\rangle = \int_\Omega |\triangle u \triangle v| dx - c\int_\Omega |\nabla u \nabla v| dx,$$

with $c < \mu_1$ and $\mu_1 = \inf\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} : 0 \neq u \in H^1(\Omega)\}$ is a Hilbert space. We remark that all of the derivatives in the above definitions are in the weak sense; for more details see [1].

The compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ is known, thus there exists a positive constant e such that $\|u\|_2 \leq e\|\nabla u\|_2$; in which $\|\cdot\|_2$ is the usual norm on $L^2(\Omega)$. Indeed, the sharp constant e is equal to $\frac{1}{\sqrt{\mu_1}}$. Hence

$$\|u\|_2 \leq \frac{1}{\sqrt{\mu_1}}\|\nabla u\|_2; \quad \forall u \in H^1(\Omega). \tag{1.2}$$

Also we have, $H^2(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$. Let

$$\mu_1^2 = \inf\{\frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |u|^2 dx} : 0 \neq u \in H^2(\Omega)\}. \tag{1.3}$$

By the natural continuous map, $H^2(\Omega)$ is embedded into $H^1(\Omega)$, so for some positive constant k, we insert that $\|\nabla u\|_2 \leq k\|\Delta u\|_2$. By considering (1.2) and (1.3), the sharp constant k would be $\frac{1}{\sqrt{\mu_1}}$, i.e.,

$$\mu_1 = \inf\{\frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |\nabla u|^2 dx} : 0 \neq u \in H^2(\Omega)\}. \tag{1.4}$$

We assume throughout this paper, φ_1 as a unit vector in $H^2(\Omega)$, which $\mu_1 = \frac{\int_\Omega |\Delta\varphi_1|^2 dx}{\int_\Omega |\nabla\varphi_1|^2 dx}$ and let $X = H^2(\Omega) \cap H_0^1(\Omega)$, which is a Hilbert space equipped under the inner product

$$\langle u, v\rangle = \int_\Omega (\triangle u \triangle v + a(x)\nabla u \nabla v)dx.$$

2. Main results

From the basic variational arguments we insert that the weak solutions of (1.1) are corresponded to the local minimizer of

$$I(u) = \frac{1}{2}\int_\Omega (|\triangle u|^2 + a(x)|\nabla u|^2)dx - \frac{1}{p}\int_\Omega b(x)|u|^p dx.$$

Since $p > 2$, for every $u \neq 0$, $I(tu)$ tends to $-\infty$ as t tends to $+\infty$. Thus, I is not bounded below and so the minimizing approach in X may fail.

2.1. **Case of nonnegative coefficients.**

For every $\alpha \in \mathbb{R}$, let

$$S_\alpha := \{u \in X : \int_\Omega b(x)|u|^p = \alpha\}.$$

Then for every $u \in S_\alpha$, $I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\alpha$. Thus $I|_{S_\alpha}$ is certainly bounded below and the process of minimizing I on S_α is equivalent to the process of minimizing $\|u\|$ or $\|u\|^2$ on S_α. Set $\inf_{u\in S_\alpha}\|u\|^2 =: m_\alpha$, we will show that m_α is achieved by a function, and a multiple of this function is a minimizer of I on X and so a weak solution of (1.1).

Lemma 2.1. *For every $\alpha > 0$, there exists a nonnegative function $u_\alpha \in S_\alpha$ such that $\|u_\alpha\|^2 = m_\alpha$.*

Proof. By the coercivity of I on S_α (i.e., $\lim_{\|u\|\to\infty, u\in S_\alpha} I(u)=\infty$), there exists a bounded minimizer sequence $\{u_n^{(\alpha)}\}$ for $f(u):=\|u\|^2$ on S_α. Obviously, since $\{|u_n^{(\alpha)}|\}$ is still a minimizer sequence in S_α, we can suppose that $u_n^{(\alpha)}(x)\geq 0$ a.e. in Ω. By reflexivity of X, there exists a subsequence of $u_n^{(\alpha)}$ (still denote it by $u_n^{(\alpha)}$), which is weakly convergent to $u_\alpha \in X$ ($u_n^{(\alpha)} \rightharpoonup u_\alpha$) and therefore the Sobolev compact embedding ensures that $u_n^{(\alpha)}$ is strongly convergent in $L^p(\Omega)$. Hence

$$lim_{n\to\infty}\int_\Omega b(x)|u_n^{(\alpha)}|^p dx = \int_\Omega b(x)|u_\alpha|^p,$$

which means $u_\alpha \in S_\alpha$. If $u_n^{(\alpha)} \not\rightarrow u_\alpha$ in X, we have that $\|u_\alpha\|^2 < \liminf \|u_n^{(\alpha)}\|^2 = m_\alpha$, which is a contradiction, since $u_\alpha\in S_\alpha$. Hence $u_n \to u_\alpha$ in X and since $u_\alpha \in S_\alpha$, u does not vanish identically. □

Theorem 2.2. *Suppose that a, b satisfy condition* (A1)*, then problem* (1.1) *admits at least one weak solution in X.*

Proof. Let $g(u) := \int_\Omega b(x)|u|^p dx$ and $f(u) := \|u\|^2$. Relying on the Lagrange multiplier theorem, if u_α is a minimizer of f under the condition $g(u)=\alpha$, then there exists $\lambda\in\mathbb{R}$ such that $f'(u_\alpha)=\lambda g'(u_\alpha)$; that is

$$\langle u_\alpha, v\rangle = \frac{p\lambda}{2}\int_\Omega b(x)|\nabla u_\alpha|^{p-2}\nabla u_\alpha \nabla v dx, \tag{2.1}$$

for every $v\in X$. By taking $u_\alpha = Cw_\alpha$ for an appropriate constant C, which will be introduced in the sequel, it yields

$$C\langle w_\alpha, v\rangle = \frac{p\lambda}{2}C^{p-1}\int_\Omega b(x)|\nabla w_\alpha|^{p-2}\nabla w_\alpha\nabla v dx.$$

Now, by considering $C=(\frac{2}{p\lambda})^{\frac{1}{p-2}}$ we have $\langle w_\alpha, v\rangle = \int_\Omega b(x)|\nabla w_\alpha|^{p-2}\nabla w_\alpha \nabla v dx$, namely w_α is a weak solution of (1.1). □

Remark 2.3. For $\alpha\neq\beta$ the minimizers of f on S_α and S_β give the same weak solution of (1.1).

Proof. For $\alpha\neq\beta$, one can readily check that $m_\alpha = \big(\frac{\alpha}{\beta}\big)^{2/p} m_\beta$. Indeed,

$$S_\alpha = \big\{u\in X : \int_\Omega b(x)|u|^p = \alpha\big\} = \big\{\big(\frac{\alpha}{\beta}\big)^{1/p} v : v\in X, \int_\Omega b(x)|v|^p = \beta\big\}.$$

Thus

$$m_\alpha = \inf_{u\in S_\alpha}\|u\|^2 = \big(\frac{\alpha}{\beta}\big)^{2/p} m_\beta. \tag{2.2}$$

So u_α minimizes $\|u\|^2$ on S_α if and only if $(\frac{\beta}{\alpha})^{1/p}u_\alpha$ minimizes $\|u\|^2$ on S_β. Moreover, it is easy to see that $\lambda_\alpha = \frac{2m_\alpha}{p_\alpha}$ and $C_\alpha = (\frac{\alpha}{m_\alpha})^{\frac{1}{p-2}}$; indeed, it is sufficient to rewrite (2.1) by substituting $v=u_\alpha$. Therefore

$$\begin{aligned} w_\alpha &= \frac{1}{C_\alpha}u_\alpha = (\frac{m_\alpha}{\alpha})^{\frac{1}{p-2}}\big(\frac{\alpha}{\beta}\big)^{1/p}u_\beta \\ &= (\frac{m_\beta}{\beta})^{\frac{1}{p-2}}u_\beta = \frac{u_\beta}{c_\beta} = w_\beta. \end{aligned}$$

□

Corollary 2.4. *Let $a \in L^\infty(\Omega)$ which is a.e. nonnegative. Every $\mu > 0$ is an eigenvalue of problem (2.3) where*

$$\begin{gathered} \Delta^2 u - a(x)\Delta u = \mu|u|^{p-2}u, quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega; \end{gathered} \tag{2.3}$$

2.2. **Case of sign-changing coefficients.** Now we consider problem (1.1) in which a, b meet the condition (A2). The fibering map corresponding to the Euler-Lagrange functional of problem (1.1) is defined as a map $\varphi : [0, \infty) \to \mathbb{R}$ with $\varphi_u(t) = I(tu)$. Hence,

$$\begin{gathered} \varphi_u(t) = \frac{t^2}{2} \int_\Omega (|\triangle u|^2 - a(x)|\nabla u|^2)dx - \frac{t^p}{p} \int_\Omega b(x)|u|^p dx, \\ \varphi_u'(t) = t \int_\Omega (|\triangle u|^2 - a(x)|\nabla u|^2)dx - t^{p-1} \int_\Omega b(x)|u|^p dx. \end{gathered}$$

Obviously, $\varphi_u'(1) = 0$ if and only if $u \in N := \{u \in X; \langle I'(u), u\rangle = 0$. It is natural to divide the critical points of $\varphi_u'(t)$ into three subsets containing local minimuma, local maximuma and inflection points and so we define $N^+ := \{u \in N, \varphi_u''(1) > 0\}$, $N^- := \{u \in N, \varphi_u''(1) < 0\}$ and $N^0 := \{u \in N, \varphi_u''(1) = 0\}$.

In this section, we consider X with the norm $\|u\| = (\int_\Omega |\triangle u|^2 dx)^{1/2}$; moreover $A(u) := \int_\Omega (|\triangle u|^2 - a(x)|\nabla u|^2)dx$ and $B(u) := \int_\Omega b(x)|u|^p dx$. Hence for each $u \in X$ we have $\varphi_u'(t) = 0$ if and only if $A(u) = t^{p-2}B(u)$. Moreover, if $A(u)B(u) > 0$ then there exists $t_0 > 0$ such that $\varphi_u(t_0) = 0$, i.e. $t_0 u \in N$ and otherwise no multiple of u belongs to N. Finally, if $t_0 u \in N$, then

$$\varphi_{t_0 u}''(1) = (2-p)A(t_0 u) = (2-p)t_0^2 A(u).$$

Hence, for $p > 2$, if $A(u) > 0$ we derive $t_0 u \in N^-$ and if $A(u) < 0$ we conclude $t_0 u \in N^+$.

Lemma 2.5. *If $a^+ < \mu_1$, then there exists $\delta > 0$ such that for every $u \in X$, $A(u) \geq \delta\|u\|^2$.*

Proof. If $\int_\Omega a(x)|\nabla u|^2 dx \leq 0$ then the assertion is obvious. Let us suppose that $\int_\Omega a(x)|\nabla u|^2 dx > 0$ and argue by contradiction. If for each $\delta > 0$ there exists $u \in X$ such that $A(u) < \delta\|u\|^2$, we derive that

$$\|u\|^2 < \frac{\int_\Omega a(x)|\nabla u|^2 dx}{1-\delta} < \frac{a^+ \int_\Omega |\nabla u|^2 dx}{1-\delta}. \tag{2.4}$$

Now, by considering $\delta < 1 - \frac{a^+}{\mu_1}$ we have $\frac{a^+}{1-\delta} < \mu_1$ and thus (2.4) leads to a contradiction with (1.4). $\square$

Theorem 2.6. *If $a^+ < \mu_1$, then I admits a minimizer on N.*

Proof. Since $a^+ < \mu_1$, we deduce that $N^+ = \emptyset$; thus $\inf_{u\in N} I(u) = \inf_{u\in N^-} I(u)$. We will show that $\inf_{u\in N^-} I(u) > 0$. For $u \in N$, $A(u) = B(u)$ and hence $\|u\|^2 = (\frac{A(v)}{B(v)})^{\frac{2}{p-2}}$ where $v = \frac{u}{\|u\|}$. Consequently, for $u \in N$ we have

$$I(u) = (\frac{1}{2} - \frac{1}{p})A(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2 A(v) = (\frac{1}{2} - \frac{1}{p})\frac{A(v)^{\frac{p}{p-2}}}{B(v)^{\frac{2}{p-2}}}.$$

Lemma 2.5 ensures that $A(v) \geq \delta$ for some $\delta > 0$. Moreover, by Sobolev embedding $X \hookrightarrow L^p(\Omega)$, for a positive constant C we have, $\int_\Omega |v|^p dx < C$. Hence

$$I(u) \geq (\frac{1}{2} - \frac{1}{p}) \frac{\delta^{\frac{p}{p-2}}}{(b^+C)^{\frac{2}{p-2}}};$$

and thus $\inf_{u\in N^-} I(u) > 0$. Set $m := \inf_{u\in N^-} I(u)$ and let us consider $\{u_n\} \subset N^-$, which $\lim_{n\to\infty} I(u_n) = m$. In this cae, the coercivity of I on N^-, $\{u_n\}$ would be bounded and so by reflexivity of X, up to subsequence, there exists $u_0 \in X$ such that u_n is weakly convergent to u_0, ($u_n \rightharpoonup u_0$). Since $u_n \to u_0$ in $L^p(\Omega)$ and $u_n \in N$, then

$$m = \lim_{n\to\infty} I(u_n) = (\frac{1}{2} - \frac{1}{p}) \lim_{n\to\infty} B(u_n) = (\frac{1}{2} - \frac{1}{p}) B(u_0).$$

Thus $B(u_0) > 0$ and hence $u_0 \neq 0$. Moreover, since $a^+ < \mu_1$ we have $A(u_0) > 0$. Therefore, a multiple of u_0 (t_0u_0; $t_0^{p-2} = \frac{A(u_0)}{B(u_0)}$) belongs to N^-. If $u_n \not\to u_0$ in X then $\|u_0\| < \liminf_{n\to\infty} \|u_n\|$ and so

$$A(u_0) - B(u_0) < \liminf_{n\to\infty} (A(u_n) - B(u_n)) = 0.$$

Consequently, $t_0 < 1$ and $\varphi'_{u_0}(1) < 0$. Therefore

$$I(t_0u_0) < \liminf_{n\to\infty} I(t_0u_n) = \liminf_{n\to\infty} \varphi_{u_n}(t_0) < \liminf_{n\to\infty} \varphi_{u_n}(1) = \liminf_{n\to\infty} I(u_n) = m,$$

which is in contrast with $t_0u_0 \in N^-$. Hence, $u_n \to u_0$ in X and $u_0 \in N$, since $A(u_0) = B(u_0)$. □

Lemma 2.7. *There exists $\sigma > 0$ in a way that for every $\mu \in (\mu_1, \mu_1 + \sigma)$ if $\int_\Omega (|\triangle u|^2 - \mu|\nabla u|^2) dx \leq 0$ then $u = k\varphi_1$ for some $k \in \mathbb{R}$.*

Proof. Suppose the sequences $\{\mu_n\}$ and $\{u_n\}$ are such that $\mu_n \to \mu_1^+$ (i.e., $\mu_n \to \mu_1$ and $\mu_n > \mu_1$) and $\int_\Omega (|\triangle u_n|^2 - \mu_n|\nabla u_n|^2) dx \leq 0$. Without loss of generality, let $\|u_n\| = 1$. Since $\{u_n\}$ is bounded, there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. If this convergence is not strong in X then

$$\int_\Omega (|\triangle u_0|^2 - \mu_1|\nabla u_0|^2) dx < \liminf \int_\Omega (|\triangle u_n|^2 - \mu_n|\nabla u_n|^2) dx \leq 0$$

which is impossible. Hence $u_n \to u_0$ and so $\|u_0\| = 1$. Moreover, we deduce that $\int_\Omega (|\triangle u_0|^2 - \mu_1|\nabla u_0|^2) dx \leq 0$ which holds if and only if $u_0 = k\varphi_1$, for some constant k. □

Theorem 2.8. *Suppose that $B(\varphi_1) \neq 0$ and let $\sigma > 0$ as introduced in lemma 2.7. If $\mu_1 < a^- \leq a^+ < \mu_1 + \sigma$ then I admits a minimizer on N^+.*

Proof. Firstly, we show that N^+ is bounded. Let us argue by contradiction, so assume that there exists an unbounded sequence $\{u_n\} \subseteq N^+$, which $\|u_n\| \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|}$, thus by boundedness of v_n, up to a subsequence, it would be weakly convergent to some $v_0 \in X$. We have $A(u_n) = B(u_n)$ then

$$A(v_n) = \|u_n\|^{p-2} B(v_n). \tag{2.5}$$

Moreover, $|A(v_n)| \leq 1 + |\int_\Omega a(x)|\nabla v_n|^2 dx| < 1 + C^2a^+$, so $A(v_n)$ is uniformly bounded and this in conjunction with (2.5) ensures that $\lim_{n\to\infty} B(v_n) = 0$ and since $v_n \to v_0$ in $L^p(\Omega)$, we get $B(v_0) = 0$. If $v_n \not\to v_0$ in X we have

$$A(v_0) < \liminf A(v_n) \leq 0; \tag{2.6}$$

therefore by regarding to the lemma 2.7 we deduce $v_0 = k\varphi_1$. Since $B(v_0) = 0$, while $B(\varphi_1) \neq 0$, we insert that $k = 0$, which contradicts (2.6). Hence $v_n \to v_0$ in X and so $\|v_0\| = 1$ and further $A(v_0) = \liminf A(v_n) \leq 0$. Due to the lemma 2.7 and since $B(v_0) = 0$, we get $v_0 = 0$, which contradicts $\|v_0\| = 1$, hence N^+ is bounded.

Hence, let us suppose $\{u_n\}$ as a bounded minimizer sequence of I on N^+ and set

$$m := \inf I(u)_{u\in N^+} = \lim_{n\to\infty} I(u_n).$$

Then, up to a subsequence, there exists $u_0 \in X$ in a way that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^p(\Omega)$. Hence

$$B(u_0) = \lim_{n\to\infty} B(u_n) = (\frac{2p}{p-2})m < 0,$$
$$A(u_0) \leq \liminf A(u_n) = (\frac{2p}{p-2})m < 0.$$

Consequently, a multiple of u_0 (t_0u_0; $t_0^{p-2} = \frac{A(u_0)}{B(u_0)}$) belongs to N and since

$$\varphi''_{t_0u_0}(1) = (2-p)t_0^2 A(u_0) > 0,$$

then $t_0u_0 \in N^+$. If $u_0 \not\to u_0$ in X, we have

$$A(u_0) < \liminf A(u_n) = \liminf B(u_n) = B(u_0)$$

and thus $t_0 < 1$. Therefore

$$I(t_0u_0) = (\frac{1}{2}-\frac{1}{p})t_0^2A(u_0) < (\frac{1}{2}-\frac{1}{p})A(u_0) < (\frac{1}{2}-\frac{1}{p})\liminf A(u_n) = (\frac{1}{2}-\frac{1}{p})m < m;$$

which contradicts $t_0u_0 \in N^+$. Hence, we deduce that u_n converge strongly to u_0 in X and $A(u_0) = B(u_0)$, i.e., $u_0 \in N$ and since $B(u_0) < 0$ we derive that $u_0 \in N^+$. □

Theorem 2.9. *Suppose that $B(\varphi_1) < 0$ and let $\sigma > 0$ as introduced in lemma 2.7. If $\mu_1 < a^- \leq a^+ < \mu_1 + \sigma$ then I admits a minimizer on N^-.*

Proof. In the first step, by an argument similar to the proof of theorem 2.8, we deduce that every minimizer sequence of I on N^- is bounded. In what follows, we will show that $\inf_{u\in N^-} I(u) \neq 0$. Let us argue by contradiction. Suppose that, for a bounded minimizer sequence $\{u_n\} \subset N^-$, $A(u_n) = B(u_n) \to 0$ as $n \to \infty$.

Let $v_n = \frac{u_n}{\|u_n\|}$, then $A(v_n) \to 0$ as $n \to \infty$, and for some $v_0 \in X$, up to a subsequence, $v_n \rightharpoonup v_0$ in X. If $v_n \not\to v_0$ then

$$A(v_0) < \liminf A(v_n) = 0. \tag{2.7}$$

Thus, by lemma 2.7, v_0 is a multiple of φ_1 such as $v_0 = k\varphi_1$.

Further, $B(v_n) \to B(v_0)$ which $B(v_n) = \|u_n\|^{-p}B(u_n) > 0$, thus $B(v_0) \geq 0$. But since $v_0 = k\varphi_1$ and $B(\varphi_1) < 0$ we derive that $k = 0$ and so $v_0 = 0$, which gives a contradiction with (2.7), hence $v_n \to v_0$ in X and $\|v_0\| = 1$.

In addition, if $A(v_0) \leq 0$, by applying lemma 2.7 we deduce $v_0 = 0$, which contradicts $\|v_0\| = 1$, hence $\inf_{u\in N^-} I(u) > 0$. In the sequel we will show that, I achieves its minimum on N^-. We insert that $u_n \rightharpoonup u_0$ for some $u_0 \in X$. One can derive that $A(u_0) \leq 0$; indeed, if $A(u_0) > 0$ by lemma 2.7, $u_0 = k\varphi_1$ and yields

$$|k|^pB(\varphi_1) = B(u_0) = (\frac{2p}{p-2})\inf_{u\in N^-} I(u) > 0,$$

which is not compatible with the assumption $B(\varphi_1) < 0$.

Hence $A(u_0) > 0$ and so a multiple of u_0 $(t_0u_0; t_0^{p-2} = \frac{A(u_0)}{B(u_0)})$ belongs to N^-. If $u_n \not\to u_0$ then $A(u_0) < B(u_0)$ and thus $t_0 < 1$, which leads to

$$\begin{aligned} I(t_0u_0) &< \liminf I(t_0u_n) = \liminf \varphi_{u_n}(t_0) \\ &\le \liminf \varphi_{u_n}(1) = \liminf E(u_n) = \inf_{u\in N^-} I(u). \end{aligned}$$

This is in contrast with $t_0u_0 \in N^-$, hence u_0 is a nontrivial weak solution of the problem, which belongs to N^-. □

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