Electronic Journal of Differential Equations, Vol. **1999**(1999), No. 10, pp. 1–33.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Implicit quasilinear differential systems: a geometrical approach *

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Abstract

This work is devoted to the study of systems of implicit quasilinear differential equations. In general, no set of initial conditions is admissible for the system. It is shown how to obtain a vector field whose integral curves are the solution of the system, thus reducing the system to one that is ordinary. Using geometrical techniques, we give an algorithmic procedure in order to solve these problems for systems of the form $A(\mathbf{x})\dot{\mathbf{x}} = \alpha(\mathbf{x})$ with $A(\mathbf{x})$ being a singular matrix. As particular cases, we recover some results of Hamiltonian and Lagrangian Mechanics. In addition, a detailed study of the symmetries of these systems is carried out. This algorithm is applied to several examples arising from technical applications related to control theory.

1 Introduction

Implicit systems of differential equations appear in many theoretical developments in Physics (such as analytical mechanics, relativistic models or gauge theories), as well as in technical applications (for instance, circuit theory, network analysis, large-scale interconnected systems or social-economic systems). Systems of this kind appear for two reasons:

- There exist some constraints relating the variables and their derivatives up to a certain order. Thus the number of degrees of freedom of the system is often less than the number of variables.

- The system has some kind of physical symmetry: there is a group of symmetries acting on the phase space of the system and some functions (the constraints) are invariant.

* *1991 Mathematics Subject Classifications:* 34C40, 57R25, 58A10, 58F99, 70Q05.
Key words and phrases: Implicit differential equations, constrained systems, vector fields, differentiable manifolds.

Submitted November 30, 1998. Published April 1, 1999.

The outstanding feature that distinguishes this kind of differential system is that the equations are not written in normal form. That is to say, all the highest order derivatives of the variables are not isolated as functions of the lower ones. Hence the general form of such a system (of order r) is $\mathbf{F}(t,\mathbf{x},\mathbf{x}',\dots,\mathbf{x}^{(r)})=0$. In this paper we only deal with systems of the form

$$\sum_j A_{ij}(\mathbf{x},\mathbf{x}',\dots,\mathbf{x}^{(r-1)})x^{j\,(r)}=\alpha_i(\mathbf{x},\mathbf{x}',\dots,\mathbf{x}^{(r-1)})$$

where A_{ij} is a matrix. Taking into account that every system of order r can be transformed into another of first order, only systems of the form

$$\sum_j A_{ij}(\mathbf{x})\dot{x}^j=\alpha_i(\mathbf{x}) \quad \text{or simply} \quad A(\mathbf{x})\dot{\mathbf{x}}=\alpha(\mathbf{x}) \tag{1}$$

need to be considered. The question then is to find the functions $x^j(t)$ which give the evolution of the variables.

If the matrix A is regular, the system is reduced to the normal form by multiplying both members of the equation by the inverse matrix A^{-1}. In this case, the system is given by a vector field; that is, an ordinary system of differential equations in the normal form, and the solutions are the integral curves of this vector field.

In this paper, we assume that the matrix A is singular and has constant rank. Therefore, there is no vector field representing the system, and it cannot be integrated analytically or numerically. This fact creates a new set of problems. In fact, to obtain an appropriate geometric solution to the problem, it is stated in terms of a differentiable manifold M acting as the phase space of the system. Then the above equations represent the coordinate form of the general problem.

As a consequence of the singularity of the matrix A, the system is possible incompatible: not every set of values of the variables is admissible as initial conditions for the system. Therefore, in order to solve the system, the first step of the procedure must be to identify the set of admissible initial conditions; these are termed “primary” constraints. This is not the end of the problem however, since another consistency condition is required. In fact, the evolution of the variables is restricted within the set of initial conditions. In other words, for every value of the evolution parameter t, the functions $\mathbf{x}(t)$ have to take on values in that set. Hence, the problem is not solved until a set of admissible initial conditions verifying this consistency condition is found. We will call this the final constraint submanifold. In order to transform the system into an ordinary one, a vector field tangent to this submanifold must be found such that its integral curves are the solutions of the initial system. This is the geometrical interpretation of the consistency condition. Furthermore, a problem of non-uniqueness of the solution (which is closely related to the identification of the true physical states of the system) can also appear.

In summary, there are two different problems. The first consists in finding the final constraints submanifold, and the second in transforming the system into a vector field tangent to this submanifold.

Solutions to these problems have been found by various means. In theoretical mechanics, these kinds of equations are called "singular", and the development of Analytical Mechanics provides better means of obtaining equations of motion of mechanical systems in which the true degrees of freedom are clearly identified, and the corresponding system of differential equations are non-singular (i.e., regular) (see, for instance, [12]). In theoretical physics, mechanical systems described by means of singular equations are both usual and important. Dirac and Bergmann [11], [2], [10] were the first to make an analytical study of such systems, although the greatest advances were made when the techniques of Differential Geometry were applied for describing mechanical systems [1]. The works of Gotay *et al* [15], [16], [17] and many others (see, for instance, [5], [19], [31] and all the references quoted therein) have been devoted to solving the problem of the compatibility and consistency of the equations of motion for singular mechanical systems, both in the Hamiltonian and the Lagrangian formulation, including systems of a higher order.

Nevertheless, equations of motion for mechanical systems are particular cases of differential equations, since they are obtained from a variational principle and start from a selected function (the *Lagrangian* or the *Hamiltonian*) which contains all the dynamical information of the system. As a consequence, the functions α_i and the matrix A are closely related to that dynamical function. In geometrical terms, A is the matrix representation of a closed 2-form and α_i are the component functions of a closed 1-form.

The most general case consists in having any matrix of functions and any 1-form (not necessarily closed). This situation arises in many technical applications, as mentioned above. Initial attempts at solving the problems posed by these systems were aimed at finding a method of "inversion" for the singular matrix of the system using analytical and algebraic methods. Thus in [4] equations like (1) with A and α constant matrices are studied, while in [37] a generalization of this case is considered, where these matrices depend on a small parameter and are singular for some values of this parameter. Questions on singular systems in control theory are also analyzed in [9],[40], [7] and [39]. Numerical methods relating to the numerical treatment of ordinary differential equations have likewise been developed (see, for instance, [22]).

The results thus obtained however are not completely satisfactory. Many questions concerning the method and the solution remain unclear. It is for this reason that some authors have investigated the geometrical treatment of these problems. So, [38] is a work where a general geometric framework for certain kinds of implicit equations is given, and a more precise formulation is developed in [21] for these systems, as well as a reduction algorithm for these systems. Questions related to the existence and uniqueness of solutions are treated in [33] (which is a more geometric treatment of previous works of the authors, such as [34] and [32]). The existence of symmetries and constants of motion, as well as integrability conditions, for these singular systems are analyzed in [25], [29] and [26]. A different geometric framework is given in [6] by studying normal canonical forms for singular systems. In [20], Gràcia and Pons have pioneered the study of these systems, in the most general situation, from a geometric point of view. Using fiber bundles techniques and the concept of *sections along a map*, they have developed a geometrical framework which enables us to describe these generalized singular systems, and to obtain therefrom the theoretical mechanical models as a particular case (even for higher order theories). Finally, [18] is a brief review of some of these problems and techniques.

In this paper, our goal is to use geometrical techniques to study implicit systems of the kind mentioned above, in addition to some of their applications. Within this context, our objectives could be stated more precisely as:

1. To give a simple geometrical description of these systems, including the study of symmetries and the analysis of the mechanical systems as a particular case ("simple", in the sense that the geometrical objects involved are as easy as possible). Matrix A is understood as the representation of a bilinear form.

2. To clarify and solve the problems of the compatibility and consistency of implicit differential equations dealt with.

3. To develop a detailed algorithm which gives the solution to the problem. This algorithm is the generalization of a previous one used for Hamiltonian and Lagrangian systems.

4. To apply the obtained results to systems arising from particular problems.

In general, the algorithm we develop is easier to implement than the analytical or algebraic methods, and our geometric treatment is simpler than those referred to above.

The paper is organized as follows: In section 2, we establish some preliminary algebraic results which are then used in section 3, which in turn is devoted to the study of the differentiable theory, and where we describe the algorithmic procedure which solves the problems of compatibility and consistency, and which constitutes the main part of the work. In section 4, we study symmetries for implicit differential equations. Section 5 is devoted to explaining specific cases of first and second-order implicit differential equations recovering as particular cases, some results of the mechanics of singular systems. The final section contains a detailed analysis of some examples of *Control theory*.

All the manifolds and maps are assumed to be real, and C^∞ and the notation is the usual one (for example, see [1]). In particular, $\mathfrak{X}(M)$ and $\Omega^p(M)$ are respectively the $\mathrm{C}^\infty(M)$-modules of differentiable vector fields and p-forms on M.

2 Algebraic theory

The general case

Let E be a finite dimensional real vector space with $\dim\ E = n$ and T a covariant 2-tensor on E (that is a bilinear form). Contraction with this tensor defines the following maps, which we call *left* and *right contractions* respectively:

$$\begin{array}{rcccl} \Re_T & : & E & \to & E^* \\ & & v & \mapsto & \Re_T(v) \quad : u \mapsto T(v,u) \\ \pounds_T & : & E & \to & E^* \\ & & v & \mapsto & \pounds_T(v) \quad : u \mapsto T(u,v) \end{array}$$

Both are $\mathbb{R}$-linear. If T is a symmetric (or antisymmetric) tensor then $\Re_T = \pounds_T$ (or $\Re_T = -\mathrm{l}_T$). We will write $\Re(v)T$ and $\pounds(v)T$ for the images of v by $\Re_T$ and $\pounds_T$ respectively. It is easy to prove that $\ker \Re_T = (\mathrm{Im}\, \pounds_T)^{\vdash}$ and $\ker \pounds_T = (\mathrm{Im}\, \Re_T)^{\vdash}$, then $\dim\ (\ker \Re_T) = \dim\ (\ker \pounds_T)$. (Let us recall that if S is a subspace of E, then $S^{\vdash} := \{\alpha \in E^* \mid \alpha(v) = 0,\ \forall v \in E\}$ is the incident subspace of S. The same holds for a subspace V of E^*). If both r_T and $\pounds_T$ are injective then we say that T has no radical.

Let $\{e_1, \ldots, e_n\}$ be a basis of E and $\{\omega^1, \ldots, \omega^n\}$ its dual basis. If the expression of T is $T = \sum_{i,j} a_{ij}\omega^i \otimes \omega^j$ and A is the matrix (a_{ij}), then the matrices of $\Re_T$ and l_T in those bases are ${}^tA$ and A respectively.

Let $\alpha \in E^*$ be a linear 1-form. We are interested in the study of the following equations:

$$\pounds(v)T = \alpha \qquad (2)$$

$$\Re(v)T = \alpha \qquad (3)$$

That is: is there any vector $v \in E$ such that $\Re_T(v) = \alpha$ (or $\pounds_T(v) = \alpha$ for the second)?.

Obviously the necessary and sufficient condition for these equations to have any solution is that $\alpha \in \mathrm{Im}\, \pounds_T$ and $\alpha \in \mathrm{Im}\, \Re_T$, respectively.

The following proposition transforms this obvious condition into a more suitable one:

Proposition 1 *1. The necessary and sufficient condition for $\pounds(v)T = \alpha$ to have any solution is that* $\ker \pounds_T \subset \ker \alpha$.

2. The necessary and sufficient condition for $\Re(v)T = \alpha$ to have any solution is that $\ker \Re_T \subset \ker \alpha$.

Proof.

1. Condition is necessary. In fact, suppose $u \in E$ verifies $\Re_T(u) = \alpha$, then for any $e \in \ker\, \pounds_T$ we have:
$$\alpha(e) = \Re_T(u)(e) = T(e,u) = 0$$
Condition is sufficient. If $\ker\, T^\sharp \subset \ker\, \alpha$ then $\alpha \in (\ker\, \pounds_T)^\vdash = \mathrm{Im}\, \Re_T$ and the equation has a solution.

2. The same as above. $\diamondsuit$

Comments.

- In the given basis of E and E^*, the matrix expressions of (2) and (3) are
$$A \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad ; \quad {}^tA \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$
respectively, where $v = \sum_i \lambda^i e_i$, $\alpha = \sum_i \alpha_i \omega^i$ and ${}^tA$ is the transpose of A.
- If $\dim\, (\ker\, \Re_T) = \dim\, (\ker\, \pounds_T) = \mathit{0}$, then the equations (2) and (3) have a single solution for every α.
- If $\dim\, (\ker\, \Re_T) = \dim\, (\ker\, \pounds_T) > \mathit{0}$, and the equations (2) and (3) have a solution, then it is not single. The difference of two solutions of (2) is an element of $\ker\, \Re_T$ and two solutions of (3) differ in an element of $\ker\, \pounds_T$.
- If T is symmetric or antisymmetric then both equations are the same (except for a sign in the second case).
- The tensor T defines another T' by means of the formula $T'(u,v) = T(v,u)$. If A is the matrix of T in a basis, then ${}^tA$ is the matrix of T' in the same basis. Hence, the problem $\Re(v)T = \alpha$ is the same as $\pounds(v)T' = \alpha$. From now on we will consider the “left” problem.

Restriction to subspaces

A problem we need to solve is the following: Let H be a subspace of E, is there any $v \in H$ such that
$$\pounds(v)T = \alpha \tag{4}$$
where $\alpha \in E^*$?.

In order to solve this we require the following

Lemma 1 *If H is a subspace of E then*

1. $\Re_T(H) = (H^\perp)^\vdash$

2. $\pounds_T(H) = ({}^{\perp}H)^{\vdash}$

where $H^{\perp} := \{u \in E \mid T(v,u) = 0 \ , \ \forall v \in H\}$ *is the right orthogonal subspace of* H *and* ${}^{\perp}H := \{u \in E \mid T(u,v) = 0 \ , \ \forall v \in H\}$ *is the left orthogonal subspace of* H*.*

Proof.

1. We have that $u \in (\Re_T(H))^{\vdash}$ if and only if $T(v,u) = 0$ for every $v \in H$ and this is equivalent to saying that $u \in H^{\perp}$. This shows that $(\Re_T(H))^{\vdash} = (H^{\perp})$ and, since we are working with finite dimensional spaces, this proves the assertion.

2. The proof is much the same as the last one. $\diamondsuit$

Now, in a way similar to the above proposition, we can prove that:

Proposition 2 *The necessary and sufficient condition for the existence of any* $v \in H$ *such that equation (4) holds is that* $\alpha \in (H^{\perp})^{\vdash}$*.*

3 Differentiable theory

Statement of the problem and compatibility conditions

Let M be a n-dimensional differentiable manifold and T a 2-covariant tensor field on M; that is, a section of the bundle $\mathrm{T}^*M \otimes \mathrm{T}^*M \to M$.

Contraction with this tensor defines the following $\mathrm{C}^{\infty}(M)$-module homomorphisms:

$$\begin{array}{rcccll} \Re_T & : & \mathfrak{X}(M) & \to & \Omega^1(M) & \\ & & X & \mapsto & \Re_T(X) & : Y \mapsto T(X,Y) \\ \pounds_T & : & \mathfrak{X}(M) & \to & \Omega^1(M) & \\ & & X & \mapsto & \pounds_T(X) & : Y \mapsto T(Y,X) \end{array}$$

We will write $\pounds(X)T$ and $\Re(X)T$ for the images of X by $\Re_T$ and $\pounds_T$ respectively.

Let (U, x^i) be a local system in M. Then, in the open set U we have

$$T = \sum_{i,j} a_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where $a_{ij} = T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. If we denote by $A = (a_{ij})$ the matrix of the coordinates of T in this local system, then

$$\Re_T = \sum_{i,j} a_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j \ , \ \pounds_T = \sum_{ij} a_{ji} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

Notice that $\ker \Re_T = (\mathrm{Im}\, \pounds_T)^{\vdash}$, and $\ker \pounds_T = (\mathrm{Im}\, \Re_T)^{\vdash}$. Consequently, for any $m \in M$, $\dim\, \ker (\Re_T)_m = \dim\, \ker (\mathrm{l}_T)_m$. If $\ker (\Re_T)_m = \ker (\pounds_T)_m = 0$ for every $m \in M$, we say that T has no radical.

$\dim\, \ker (\Re_T)_m$ is a locally constant function on M. We suppose it is constant everywhere in M.

Assumption 1 $\dim\, \ker (\Re_T)_m$ *does not depend on the point* $m \in M$*.*

Given a 2-covariant tensor field T on M and a differentiable 1-form $\alpha \in \Omega^1(M)$, we can consider the following equations for a vector field $X \in \mathfrak{X}(M)$:

$$\pounds(X)\, T = \alpha\,, \quad \Re(X)\, T = \alpha$$

If T has no radical, then both equations have a single solution, since $(\Re_T)_m$ and $(\pounds_T)_m$ are isomorphisms, for any $m \in M$. Observe that, as in the algebraic case, given T, we can define another T' by means of $T'(X, Y) := T(Y, X)$. Hence the equation $\Re(X)T = \alpha$ is the same as $\pounds(X)\, T' = \alpha$.

Following from the above, henceforth we only consider the equation $\pounds(X)\, T = \alpha$, and we call it a *linear singular differential system*; that is, a LSDS.

Given the LSDS $\pounds(X)\, T = \alpha$ on the manifold M, the problem we try to solve is the following: to find a submanifold $P_f \subset M$ and a vector field $X \in \mathfrak{X}(M)$ such that

1. $\pounds(X)\, T \underset{P_f}{=} \alpha$ (the symbol $\underset{N}{=}$ means an equality which holds on the points of N for any closed submanifold $N \subset M$).
2. X is tangent to P_f.
3. P_f is maximal with respect to conditions (1) and (2).

Observe that if (P_f, X) is the solution of the problem, then the integral curves of X are tangent to P_f, and the tangent vectors to these curves verify the equation at any point of the curves. Notice that when we obtain a solution to the problem, we can calculate the integral curves of the vector field by analytical or numerical methods. In general the vector field X is not unique.

As we have seen in section 2, the system has no solution anywhere in M, but only on the points $m \in M$ satisfying $\alpha_m \in \mathrm{Im}\, (\Re_T)_m$. So we have:

Definition 1 *The set*

$$P_1 = \{m \in M \ \mid\ \alpha_m \in \mathrm{Im}\, (\Re_T)_m\}$$

is called the primary constraint submanifold *associated to the equation* $\pounds(X)\, T = \alpha$*.*

It is clear that P_1 is the maximal subset of M in which this equation has solution. In other words, only the points of M belonging to P_1 are admissible as initial conditions of the equations. For this reason, P_1 is sometimes called the *submanifold of initial conditions*.

Assumption 2 *P_1 is a closed submanifold of M.*

(Let us recall that a *closed submanifold* of M is a closed subset that is itself a manifold. For instance, if $\xi^1, \dots, \xi^h$ are independent differentiable functions on M, then the set $\{m \in M \; ; \; \xi_i(m) = 0 \; , \; i = 1, \dots, h\}$ is a closed submanifold of M).

Proposition 3 $P_1 = \{m \in M \;\; | \;\; \ker (\pounds_T)_m \subset \ker \alpha_m\}$

Proof. ($\Longrightarrow$) If $m \in P_1$, then $\alpha_m \in \mathrm{Im}\,(\Re_T)_m$, then, according to proposition 1, we have that $\ker (\pounds_T)_m \subset \ker \alpha_m$.

($\Longleftarrow$) If $m \in M$ and $\ker (\mathrm{l}_T)_m \subset \ker \alpha_m$ then $\alpha_m \subset (\ker (\pounds_T)_m)^\vdash = \mathrm{Im}\,(\Re_T)_m$, and therefore the equation has a solution. $\diamondsuit$

Comments.

- If (U, x^i) is a local system on M, $T = \sum a_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$ and $\alpha = \sum \alpha_i \mathrm{d}x^i$, then the local expression of the system $\pounds(X)T = \alpha$ is
$$A \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$
where $X = \sum \lambda_i \frac{\partial}{\partial x^i}$ is the vector field we are looking for and A is the matrix (a_{ij}).

- If T has no radical, that is $\dim\, \ker (\Re_T)_m = 0$ for every $m \in M$, then the system is compatible everywhere in M, and the solution is unique. This means that $P_1 = M$ and there is only one vector field X which is a solution at every point of M.

- If T has radical, that is $\dim\, \ker (\Re_T)_m > 0$ for every $m \in M$, then the system has a solution only on the points of P_1. Moreover, the solution is not unique and the difference between two solutions is in $\ker \Re_T := \{X \in \mathfrak{X}(M) \;\; | \;\; T(X,Y) = 0 \; , \; \forall Y \in \mathfrak{X}(M)\}$. Hence, if X_0 is a solution of the system, then $X_0 + \ker \Re_T$ is the set of solutions at this level of the study.

- Sometimes we have an LSDS on a manifold M, but we are only interested in its restriction to a submanifold $M_0 \subset M$ (for instance, when dealing with singular Hamiltonian systems). In this case, the primary constraints submanifold is $P_1 \cap M_0$. Then, we suppose that M_0 is the zero set of a finite family of functions on M, and the study continues in the same way.

- Since we have assumed that P_1 is a closed submanifold of M, we can suppose that the vector field solutions of the system are defined everywhere in M, although they are solutions only on the points of P_1.

It is worth pointing out that the submanifold P_1 can be defined as the zero set of a family of functions. In fact, we have:

Corollary 1 $P_1 = \{m \in M \mid \alpha(Z)_m = 0 \ \forall Z \in \ker \pounds_T\}$.
If $Z \in \ker \pounds_T$, then the function $\xi^{(1)} := \alpha(Z)$ is the primary constraint *associated to Z.*

Stability conditions

Suppose that dim $P_1 \neq 0$ (if this is not the case, P_1 is a set of isolated points and the differential system has no solution) and let $X_0 + \ker \Re_T$ be the set of solutions of the system on the submanifold P_1.

The vector field solutions are solutions of the system only on the points of P_1. Then the integral curves of these vector fields must be contained in P_1. That is, these vector fields must be tangent to P_1. As a consequence of this fact, new conditions on the solutions arise. Hence we define:

Definition 2 *The set*

$$P_2 := \{m \in P_1 \mid \exists Y \in \ker \Re_T \ , \ (X_0 + Y)(m) \in \mathrm{T}_m P_1\}$$

is called the first generation secondary constraints submanifold *associated to the equation $\pounds(X)T = \alpha$.*

But the submanifold P_1 is defined as the zero set of a family of functions, the primary constrains. Hence, the condition for a vector field to be tangent to P_1 is that its action on this family of functions goes to zero on the points of P_1. That is:

$$P_2 = \{m \in P_1 \mid \exists Y \in \ker \Re_T \ , \ (X_0 + Y)_m \alpha(Z) = 0 \ , \ \forall Z \in \ker \pounds_T\}$$

Now, in order to give another characterization of P_2, we need to introduce some notation. We denote by $\underline{\mathfrak{X}(N)}$ the subset of $\mathfrak{X}(M)$ made of the vector fields tangent to N. Then

$$\mathfrak{X}(N)^\perp := \{X \in \mathfrak{X}(M) \mid T(Y,X) \underset{N}{=} 0 \ , \ \forall Y \in \underline{\mathfrak{X}(N)}\}$$

is the right orthogonal submodule of $\mathfrak{X}(N)$ with respect to T. More precisely, for any $m \in N$ we write

$$\mathrm{T}_m N^\perp := \{u \in \mathrm{T}_m M \mid T_m(v,u) = 0 \ , \ \forall v \in \mathrm{T}_m M \mid_N\}$$

then the vector fields of $\mathfrak{X}(N)^{\perp}$ are the sections of the subbundle of $\mathrm{T}M\mid_N$ whose fibers are the subspaces $\mathrm{T}_m N^{\perp}$. In the same way, we can define

$$^{\perp}\mathcal{X}(N) := \{X \in \mathcal{X}(M) \mid T(X,Y) \underset{N}{=} 0 \ , \ \forall Y \in \underline{\mathfrak{X}(N)}\}$$

which is called the *left orthogonal submodule* of $\mathfrak{X}(N)$ with respect to T.

The following theorem gives a characterization of P_2.

Theorem 1

$$P_2 = \{m \in P_1 \mid \alpha(Z)_m = 0 \ , \ \forall Z \in \mathfrak{X}(P_1)^{\perp}\}$$

Proof. Let $C = \{m \in P_1 \mid \alpha(Z)_m = 0 \ , \ \forall Z \in \mathfrak{X}(P_1)^{\perp}\}$. We have

1. $P_2 \subset C$.

   If $m \in P_2$, there exists $Y \in \ker \Re_T$ such that $(X_0 + Y)_m \in \mathrm{T}_m P_1$. If $Z \in \mathfrak{X}(P_1)^{\perp}$, we have:

   $$\alpha(Z)_m = (\Re_T(X_0 + Y)(Z))_m = (T(X_0 + Y, Z))_m = 0$$

   then $m \in C$.

2. $C \subset P_2$.

   If $m \in C$ and $Z \in \mathfrak{X}(P_1)^{\perp}$, then $\alpha(Z)_m = 0$, hence we have that $\alpha_m \in (\mathrm{T}_m P_1^{\perp})^{\vdash} = \Re_T(\mathrm{T}_m P_1)$. Hence, the system $\pounds(v)T_m = \alpha_m$ has solution in $\mathrm{T}P_1$ and therefore $m \in P_2$. $\diamondsuit$

Definition 3 *If $Z \in \mathfrak{X}(P_1)^{\perp}$, the function $\xi^{(2)} := \alpha(Z)$ is the* secondary constraint *associated to Z.*

At this point we find ourselves in the same situation as at the end of the last subsection. Hence the stability condition must be imposed again. The procedure is iterative and we are going to analyze the general situation.

Consider the following sequence of subsets of M defined inductively:

$$\begin{aligned} P_0 &:= M \\ P_1 &:= \{m \in M \mid \alpha(Z)_m = 0 \ , \ \forall Z \in \ker \pounds_T\} \\ P_{i+1} &:= \{m \in P_i \mid \alpha(Z)_m = 0 \ , \ \forall Z \in \mathfrak{X}(P_i)^{\perp}\} \quad (i \geq 1) \end{aligned}$$

and the following

Assumption 3 *The subsets P_i (for all $i \geq 1$) are closed submanifolds of M.*

Theorem 2 *The equation $\pounds(X)T = \alpha$ has solution tangent to P_k if and only if*

$$\langle \mathfrak{X}(P_k)^{\perp}, \alpha \rangle_m = 0 \ , \ \forall m \in P_k$$

Proof. ($\Longrightarrow$) Let $X \in \mathfrak{X}(P_k)$ be a solution of the equation. If $Z \in \mathfrak{X}(P_k)^{\perp}$ and $m \in P_k$, we have:

$$\alpha(Z)_m = (\pounds(X)T)(Z)_m = T(X, Z))_m = 0$$

($\Longleftarrow$) If the condition holds, and $m \in P_k$, then we have that $\alpha_m \in (\mathrm{T}_m P_k^{\perp})^{\vdash} = \Re_T(\mathrm{T}_m P_k)$. Hence the equation has solution at m. $\diamondsuit$

Comments.

- The sequence $\{P_i\}$ stabilizes necessarily. That is, there exists $k \geq 0$ such that $P_k = P_{k+1}$, because dim M is finite. This condition implies that theorem 2 holds for P_k.
- If $P_k = \emptyset$ or dim $P_k = 0$, then the system has no solution.
- If dim $P_k > 0$, then the system is compatible and the equation $\pounds(X)T = \alpha$ has solution tangent to P_k. In this case we call P_k the *final constraints submanifold* and denote it by P_f. If X_0 is a solution, then the set of solutions is $X_0 + \ker \mathrm{r}_{T_f}$, where

$$\ker \mathrm{r}_{T_f} := \{Z \in \underline{\mathfrak{X}(P_f)} \ \mid \ \Re_T(Z) \underset{P_f}{=} 0\}$$

  In some cases (classical analytical mechanics, linear circuits,...), we have that $\ker \mathrm{r}_{T_f} = \{0\}$. Then the solution is unique.

  In other cases (relativistic mechanics, electromagnetism,...), the solution is not unique. The non uniqueness of the solution is called (in Physics) *gauge freedom*, the elements of $\ker \mathrm{r}_{T_f}$ are the *gauge fields* and two different solutions are called *gauge equivalent vector fields* (see, for instance, [3] for more details).

Notice that the last theorem characterizes P_f, but does not give the tangent vector field X. In order to apply the algorithm for obtaining the couple (P_f, X) we will proceed as follows:

- We calculate a local basis for $\ker \Re_T$. Let $\{Z_1, \ldots, Z_h\}$ be this basis. Then P_1 is defined by $\alpha(Z_i) = 0$, for $I = 1, \ldots, h$. These are the compatibility conditions.
- If X_0 is a solution of the system on P_1, then the general solution on P_1 is $X = X_0 + \sum_j f_j Z_j$, with f_j arbitrary functions.
- In order to obtain P_2, we apply X to the functions $\alpha(Z_i)$ and we obtain the system

$$X_0(\alpha(Z_i)) + \sum_j f_j Z_j(\alpha(Z_i)) = 0 \quad , \quad i = 1, \ldots, h$$

The equations with $Z_j(\alpha(Z_i)) = 0$ for every j leads to new constraints given by $X_0(\alpha(Z_i)) = 0$. They are the stability conditions defining P_2. The other equations fix conditions on the functions $\{f_j\}$. That is, they define a submodule of ker $\Re_T$.

- Now, we proceed in the same way on P_2, with the remaining functions f_j in the general solution and the new constraints. The algorithm stops when we do not obtain new constraints.

This procedure is a generalization of the one developed in references [31] and [30], where the particular cases of Hamiltonian and Lagrangian mechanics is studied; that is, when the tensor T is antisymmetric and closed under the action of the exterior differential operator (see also Sections 5 and 5). We call it the *Stabilization General Algorithm*, or, in abbreviation, SGA.

Remark. Observe that in the first step of the algorithm (compatibility conditions), we transform the initial implicit differential equation $A\dot{\mathbf{x}} - \alpha = 0$ into a differential algebraic equation (DAE), which is given by the vector field $X_0 + \sum_{j=1}^{h} f_j Z_j$ and the constraints $\langle Z_j, \alpha \rangle = 0$ defining P_1 ($Z^1, \ldots, Z^h$ is a basis of ker $\Re_T$ and $f^1, \ldots, f^h$ are free). Then we apply the algorithm (stabilization condition) to this DAE to get P_f and the reduced equation on it.

The so-called *index* of this DAE (see, for instance, [23] or [13], and references quoted therein) coincides with the number of steps needed to stabilize the system, and to obtain P_f and the reduced equation, and hence it is equal to $f - 1$.

4 Symmetries

Symmetries of a singular differential system

Let $\pounds(X)T = \alpha$ be a singular differential system on M, which is assumed to be compatible, P_f its final constraint submanifold, and $\mathcal{S} := \{X_0 + \ker \mathrm{r}_{T_f}\}$ the set of solutions.

Definition 4 *A* symmetry *of the system is a diffeomorphism $\varphi: M \to M$ such that:*

1. $\varphi(P_f) = P_f$.
2. *If $X \in \mathcal{S}$, then $\varphi_* X \in \mathcal{S}$.*

(The symbols φ_* and φ^* denote the *push-forward* and the *pull-back* defined by the diffeomorphism φ [1]).

In the classical study of symmetries of a regular Hamiltonian system (M, Ω, h) (where Ω is a symplectic form and h is a smooth function called the *Hamiltonian* of the system), the symmetries are considered to be symplectomorphisms of (M, Ω); that is, diffeomorphisms $\varphi: M \to M$ such that $\varphi^*\Omega = \Omega$. A distinction is then made between *symmetries of the system*, which verify that $\varphi_* X_h = X_h$, (where X_h is the Hamiltonian vector field associated to h), and *symmetries of the Hamiltonian*, satisfying $\varphi^* h = h$. It is clear that every symmetry of the Hamiltonian is a symmetry of the system, but not the converse, and a simple example is a symplectomorphism φ such that $\varphi^* h = h + c$ (with $c \in \mathbb{R}$), which is not a symmetry of the Hamiltonian but a symmetry of the system (see, for instance, [1] for more information on these topics).

In the present case, the situation is different because now M has no prescribed geometric structure. Thus, we have to study methods in order to find symmetries which preserve the elements defining our system; that is, T and α. The results thus obtained are:

Theorem 3 *Let $\varphi: M \to M$ be a diffeomorphism such that $\varphi(P_f) = P_f$.*

1. *If $\varphi^* T \mid_{P_f} = T \mid_{P_f}$ and $\varphi^*\alpha \mid_{P_f} = \alpha \mid_{P_f}$, then φ is a symmetry of the system.*
2. *If φ is a symmetry of the system and $\varphi^* T \mid_{P_f} = T \mid_{P_f}$, then $\varphi^*\alpha \mid_{P_f} = \alpha \mid_{P_f}$*
3. *If φ is a symmetry of the system and $\varphi^*\alpha \mid_{P_f} = \alpha \mid_{P_f}$ then T and $\varphi^* T$ give equivalent systems (in the sense that they have the same solutions).*

In order to prove these results, we require the following:

Lemma 2 *If $X \in \mathfrak{X}(M)$ and $\varphi: M \to M$ is a diffeomorphism, then*

$$(\varphi^* \circ \pounds(\varphi_* X))T = (\pounds(X) \circ \varphi^*)T$$

Proof. Let $Y \in \mathfrak{X}(M)$, we have:

$$\begin{aligned}
[(\varphi^* \circ \pounds(\varphi_* X))T]Y &= (\varphi^*(\pounds(\varphi_* X)T))Y = (\pounds(\varphi_* X)T))\varphi_* Y \\
&= T(\varphi_* X, \varphi_* Y) = (\varphi^* T)(X, Y) \\
&= [\pounds(X)(\varphi^* T)]Y = [(\pounds(X) \circ \varphi^*)T]Y
\end{aligned}$$

Proof of the theorem.

1. Consider $X \in \mathcal{S}$, then

$$\pounds(\varphi_* X)T \mid_{P_f} = (\varphi^{*-1} \circ \pounds(X)) \circ \varphi^*)T \mid_{P_f} = \varphi^{*-1}\alpha \mid_{P_f} = \alpha \mid_{P_f}$$

2. If $X \in \mathcal{S}$, then $\pounds(X)T\,|_{P_f} = \alpha\,|_{P_f}$, therefrom
$$\varphi^*(\pounds(X)T\,|_{P_f}) = \varphi^*\alpha\,|_{P_f}$$
and, furthermore
$$\varphi^*(\pounds(X)T\,|_{P_f}) = \pounds(\varphi_*^{-1}X)\circ\varphi^*T\,|_{P_f} = \pounds(\varphi_*^{-1}X)T\,|_{P_f} = \alpha\,|_{P_f}$$

3. Let X be a solution of the system defined by T, that is, $\pounds(X)T\,|_{P_f} = \alpha\,|_{P_f}$. Then, we have:
$$\pounds(X)\varphi^*T\,|_{P_f} = \varphi^*\pounds(\varphi_*X)T\,|_{P_f} = \varphi^*\alpha\,|_{P_f} = \alpha\,|_{P_f}$$
Therefore, X is also a solution of the system defined by φ^*T. But φ^{-1} also verifies the same properties, hence the converse is also true. $\diamondsuit$

Infinitesimal symmetries

Next we see how the above concepts can be interpreted in terms of vector fields, which is to say the infinitesimal version of the concept of symmetry.

Definition 5 *A vector field $D \in \mathfrak{X}(M)$ is an* infinitesimal symmetry *of the system iff*

1. *D is tangent to P_f.*
2. *If $X \in \mathcal{S}$ then $\mathrm{L}(D)X \in \mathcal{S}$; that is, $\mathrm{L}(D)X - X \in \ker \Re_T$ ($\mathrm{L}(D)$ denotes the Lie derivative with respect to the vector field D).*

Observe that this definition is equivalent to saying that, if φ_t is a local uniparametric group associated to D, then φ_t are symmetries in the sense of definition 4.

In a way analogous to the above subsection, we first have the following results:

Proposition 4 *1. If $X \in \mathcal{S}$ and $D \in \mathfrak{X}(M)$ is a vector field tangent to P_f such that $\mathrm{L}(D)T = 0$ and $\mathrm{L}(D)\alpha = 0$, then $\mathrm{L}(D)X\,|_{P_f} \in \ker \Re_T\,|_{P_f}$.*

2. If $D \in \mathfrak{X}(M)$ is an infinitesimal symmetry and $\mathrm{L}(D)T = 0$, then $\mathrm{L}(D)\alpha = 0$.

Proof.

1. If $X \in \mathcal{S}$ then $\mathrm{L}(X)T\,|_{P_f} = \alpha\,|_{P_f}$. Therefore
$$\begin{aligned} 0 &= \mathrm{L}(D)\alpha\,|_{P_f} = \mathrm{L}(D)(\pounds(X)T)\,|_{P_f} \\ &= \pounds(\mathrm{L}(D)X)T\,|_{P_f} + \pounds(X)\,\mathrm{L}(D)T\,|_{P_f} = \pounds(\mathrm{L}(D)X)T\,|_{P_f} \end{aligned}$$

2. If $X \in \mathcal{S}$ then $\mathrm{L}(X)T\mid_{P_f} = \alpha\mid_{P_f}$. Therefore

$$\begin{aligned} 0 &= \mathrm{L}(D)\alpha\mid_{P_f} = \mathrm{L}(D)(\pounds(X)T)\mid_{P_f} \\ &= \pounds(\mathrm{L}(D)X)T\mid_{P_f} + \pounds(X)\,\mathrm{L}(D)T\mid_{P_f} \end{aligned}$$

$\diamondsuit$

The main result in relation to this topic is the following:

Proposition 5 *Let $D \in \mathfrak{X}(M)$ be a vector field tangent to P_f satisfying that $\mathrm{L}(D)T\mid_{P_f} = 0$ and $\mathrm{L}(D)\alpha\mid_{P_f} = 0$. Then the local uniparametric groups $\{\varphi_t\}$ of D transform integral curves of vector field solutions of the system into integral curves of vector field solutions.*

Proof. Let $m \in M$ and $\gamma\colon [0,s] \to M$ be an integral curve of a vector field solution, that is, $\gamma(t) \in P_f$, $\forall t \in [0,s]$ and $\pounds(\dot\gamma(t))T\mid_{\gamma(t)} = \alpha\mid_{\gamma(t)}$.

Let $\{\varphi_t\}$ be the local uniparametric group of D defined in a neighborhood of m. Consider now the curve $\sigma(t) := (\varphi_t \circ \gamma)(t)$. We see that

$$\pounds(\dot\sigma(t))T\mid_{\sigma(t)} = \alpha\mid_{\sigma(t)}$$

We have that $\dot\sigma(t) = \varphi_{t*}\dot\gamma(t)$, and using lemma 2 we obtain

$$\begin{aligned} [\varphi_t^* \circ \pounds(\dot\sigma(t))]T &= [\varphi_t^* \circ \pounds(\varphi_{t*} \circ \pounds(\varphi_{t*}\dot\gamma(t))]T = (\pounds(\dot\gamma(t))(\varphi_t^* T) \\ &= \pounds(\dot\gamma(t))T = \alpha\mid_{\gamma(t)} \end{aligned}$$

because T is invariant by φ_t. Then

$$(\pounds(\dot\sigma(t))T = \varphi_t^*\alpha\mid_{\gamma(t)} = \alpha\mid_{\sigma(t)}$$

because $\mathrm{L}(D)\alpha = 0$, hence α is invariant by φ_t. $\diamondsuit$

Comments.

- The vector fields $D \in \mathfrak{X}(M)$ tangent to P_f, which transform integral curve solutions of the system into integral curve solutions of the system, are called *gauge vector fields* of the system.

- Observe that, if $D \in \ker \Re_T\mid_{P_f}$, then $\alpha(D)\mid_{P_f} = 0$, since P_f is a submanifold such that $\langle \mathfrak{X}(P_f)^\perp, \alpha\rangle = 0$ and $\ker \mathrm{r}_{T_f}\mid_{P_f} \subset \mathfrak{X}(P_f)^\perp$. Therefrom, if α is a closed form and $D \in \ker \mathrm{r}_{T_f}\mid_{P_f}$, then $\mathrm{L}(D)\alpha = 0$. Hence, if $D \in \ker \mathrm{r}_{T_f}\mid_{P_f}$, $\mathrm{L}(D)\alpha = 0$ and α is closed, then D is a gauge vector field.

- Furthermore, if $T \in \Omega^2(M)$; that is, an antisymmetric tensor; and α is closed, then $D \in \ker \mathrm{r}_{T_f}\mid_{P_f}$ if and only if $\mathrm{L}(D)T\mid_{P_f} = 0$, and all the vector fields belonging to $\ker \mathrm{r}_{T_f}\mid_{P_f}$ are gauge vector fields.

- In this last case, $\ker \mathrm{r}_{T_f} |_{P_f}$ parametrizes the set of vector fields which are solutions of the system, and the local uniparametric groups of these fields parametrize the set of integral curve solutions passing through a point. This is the case of Hamiltonian mechanics.

5 Particular cases

Presymplectic problems

In these cases, the general situation is the following: M is a n-dimensional manifold, $\alpha \in \Omega^1(M)$ (is a 1-form) and T is a closed 2-form (i.e., $T \in \Omega^2(M)$ with $\mathrm{d}T = 0$) such that $\mathrm{rank}\, T < n$; that is, T is not regular.

These kind of systems have been studied in recent years. The problem of the compatibility and consistency of the equations of motion is solved by applying different but equivalent methods, such as the *Presymplectic Constraint Algorithm* [15], [16], the algorithmic procedure of [30] and [31], or the generalized algorithm of [20].

Hamiltonian problems

There are two cases of interest:

1. **Singular Hamiltonian systems**.

   They can be treated as a particular case of the above, where now $\dim\, M = n = 2k$ and $\alpha = \mathrm{d}h$ (at least locally) for some $h \in \Omega^0(M)$ called the *Hamiltonian function*.

2. **Regular Hamiltonian systems**.

   Once again, $\dim\, M = n = 2k$ and $\alpha = \mathrm{d}h$ (at least locally) for some $h \in \Omega^0(M)$. The difference with the singular case is that $\mathrm{rank}\, T = 2k$ (i.e., it is regular). The consequence of this fact is that $\Re_T$ and l_T are linear isomorphisms, hence the equations of motion are compatible everywhere in M, and determined. This implies that $P_f = M$ and the solution $X_h \in \mathfrak{X}(M)$ is unique [1]. In this case, T is called a *symplectic form* and (M, T) is a *symplectic manifold*.

   In addition, we have that every symmetry of the Hamiltonian is a symmetry of the system.

Second Order Differential Equations problems: Mechanical systems

Next we consider the particular case in which $M = \mathrm{T}Q$ is the tangent bundle of some n-dimensional manifold Q. Then, let $\pi\colon \mathrm{T}Q \to Q$ be the natural projection,

J the vertical endomorphism and $\Delta \in \mathcal{X}(\mathrm{T}Q)$ the Liouville vector field (i.e., the vector field of dilatations along the fibers of $\mathrm{T}Q$). If (q^i, v^i) are natural coordinates in $\mathrm{T}Q$, the local expression of these elements are

$$J = \sum_i \mathrm{d}q^i \otimes \frac{\partial}{\partial v^i} \quad , \quad \Delta = \sum_i v^i \frac{\partial}{\partial v^i}$$

(see, for instance, [24] and [8] for more details about these geometric concepts).

The most interesting cases are the Lagrangian systems in which T is a 2-form; that is, an antisymmetric 2-covariant tensor field. The standard formulation of these models deals with a dynamical function $\mathcal{L} \in \Omega^0(\mathrm{T}Q)$, called the *Lagrangian function*, from which an exact 2-form $\omega \in \Omega^2(\mathrm{T}Q)$ can be constructed, given by $\omega := -\mathrm{d}(\mathrm{d}\mathcal{L} \circ J)$, and another function $E \in \Omega^0(\mathrm{T}Q)$ (the *energy function*), defined as $E := \mathcal{L} - \Delta(\mathcal{L})$ (see, for instance, [24] and [5] for more details about all these concepts). In this situation, the problem is the same as in the above cases, where ω plays the role of T and $\alpha = \mathrm{d}E$; that is, the equation to be solved is

$$i(X)\omega = \mathrm{d}E \tag{5}$$

Nevertheless, physical and variational considerations lead us to introduce an additional feature: we must search for solutions satisfying the so-called *second order condition*:

$$\mathrm{T}\pi \circ X = \mathrm{Id}_{\mathrm{T}Q} \tag{6}$$

which can be set equivalently as

$$J(X) = \Delta$$

A vector field satisfying the second order condition is said to be a *holonomic vector field* or a *second order differential equation* (SODE), and is characterized by the fact that its integral curves are canonical lifting of curves in Q.

If the system is regular (i.e., ω is a non-degenerate form) then the system (5) is compatible and determined everywhere in $\mathrm{T}Q$, and its unique solution automatically satisfies the SODE condition (6) (see [5]).

If the system is singular (ω is a degenerate form), then to the problem of the incompatibility and inconsistency of equations (5) we must add condition (6) for the solutions. This fact introduces new constraints to those obtained in the stabilization algorithm applied to the equations (5), and restricts the gauge freedom (for a detailed discussion on this problem, see [31]).

6 Examples

Example 1: Control systems

A general situation

First of all we analyze an example in *Control theory* that shows how the algorithm behaves, depending on different situations which can arise.

Consider a *control system* described by the equations

$$A\dot{\mathbf{x}} = B\mathbf{x} + C\mathbf{u}$$

where $\mathbf{u} = \{u_i\}$ $(i = 1, \ldots, m)$ are the control parameters or *inputs*, $\mathbf{x} = \{x_i\} \in \mathbb{R}^n$ are the *outputs*, and A, B, C are constant matrices. In some cases the inputs have the form $\mathbf{u} = D\dot{\mathbf{x}}$ (D being another constant matrix), and hence the equations of the system take the form

$$T\dot{\mathbf{x}} = B\mathbf{x} \tag{7}$$

where $T = A - CD$ is sometimes a singular matrix.

In this case, the manifold M is $\mathbb{R}^n$, (with $\{x_i\}$ as coordinates), the 1-form α is $\alpha = \sum_{i,j} b^{ij} x_j \mathrm{d}x_i$ (where $B = (b^{ij})$) and T represents a constant 2-tensor in M. Then we have that $X = \sum_i f_i(x_j) \frac{\partial}{\partial x_i}$ and the integral curves of the vector field satisfying the equality are the solution of the system; that is, $\dot{x}_i = f_i(\mathbf{x})$.

For simplicity, we only consider the case $n \geq 4$ and $\dim(\ker T) = 2$. Then, let

$$Y^1 = \sum_i w_i^1 \frac{\partial}{\partial x_i} \quad , \quad Y^2 = \sum_i w_i^2 \frac{\partial}{\partial x_i}$$

be a basis of $\ker T$ and

$$Z_1 = \sum_i v_i^1 \frac{\partial}{\partial x_i} \quad , \quad Z_2 = \sum_i v_i^2 \frac{\partial}{\partial x_i}$$

a basis of $\ker T^t$. Thus, the compatibility conditions are

$$\alpha(Z_1) \equiv \sum_{i,j} v_i^1 b^{ij} x_j = 0 \quad , \quad \alpha(Z_2) \equiv \sum_{i,j} v_i^2 b^{ij} x_j = 0 \tag{8}$$

and, if X_0 is a particular solution of the system, the general one is

$$X = X_0 + g_1 Y^1 + g_2 Y^2 \tag{9}$$

where, at first, g_1, g_2 are arbitrary functions.

We have the following possible situations:

1. Both equalities (8) hold identically.

   Then the system of equations (7) *is compatible everywhere in M but undetermined* because its solution is (9), which *depends on two arbitrary functions.*

2. The first equality does not hold identically, but the second does (or conversely). Then
$$\xi_1^{(1)} := \alpha(Z_1) = \sum_{i,j} v_i^1 b^{ij} x_j$$
is the only primary constraint which defines the submanifold P_1, where the system is compatible and has the vector field (9) as solution.

   Now the stability condition of the SGA algorithm must be imposed, which leads to
$$X(\xi_1^{(1)}) \equiv X_0(\xi_1^{(1)}) + g_1 Y^1(\xi_1^{(1)}) + g_2 Y^2(\xi_1^{(1)}) = 0 \quad (\text{on } P_1)$$
   Notice that the coefficients $Y^\alpha(\xi_\alpha^{(1)})$ are constant, since Y^α are constant vector fields. Then we have the following possibilities:

   (a) $Y^1(\xi_1^{(1)}) \neq 0$ and $Y^2(\xi_1^{(1)}) \neq 0$.
   In this case, one function g can be expressed in terms of the other. For instance,
$$\begin{aligned} g_2|_{P_1} &= -g_1 \frac{Y^1(\xi_1^{(1)})}{Y^2(\xi_1^{(1)})} - \frac{X_0(\xi_1^{(1)})}{Y^2(\xi_1^{(1)})} \\ &= -\frac{1}{\sum_{i,j} v_i^1 b^{ij} w_j^2} \left(g_1 \sum_{i,j} v_i^1 b^{ij} w_j^1 + X_0(\sum_{i,j} v_i^1 b^{ij} x_j) \right) \end{aligned}$$
   P_1 is the final constraint submanifold, and *the solution is not unique* since *it depends on one arbitrary function*
$$\begin{aligned} X|_{P_1} &= X_0 + g_1 Y^1 \\ &\quad - \frac{1}{\sum_{i,j} v_i^1 b^{ij} w_j^2} \left(g_1 \sum_{i,j} v_i^1 b^{ij} w_j^1 + X_0(\sum_{i,j} v_i^1 b^{ij} x_j) \right) Y^2 \\ &\equiv X'_0 + g_1 \left(Y^1 - \frac{\sum_{i,j} v_i^1 b^{ij} w_j^1}{\sum_{i,j} v_i^1 b^{ij} w_j^2} Y^2 \right) \end{aligned}$$

   (b) $Y^1(\xi_1^{(1)}) = 0$ and $Y^2(\xi_1^{(1)}) \neq 0$ (or conversely).
   In this case, one function g can be completely determined
$$g_2|_{P_1} = -\frac{X_0(\xi_1^{(1)})}{Y^2(\xi_1^{(1)})} = -\frac{X_0(\sum_{i,j} v_i^1 b^{ij} x_j)}{\sum_{i,j} v_i^1 b^{ij} w_j^2}$$
   P_1 is the final constraint submanifold, and *the solution is not unique: it depends on one arbitrary function*
$$X|_{P_1} = X_0 - \frac{X_0(\sum_{i,j} v_i^1 b^{ij} x_j)}{\sum_{i,j} v_i^1 b^{ij} w_j^2} Y^2 + g_1 Y^1 \equiv X'_0 + g_1 Y^1$$

(c) $Y^1(\xi_1^{(1)}) = 0$, $Y^2(\xi_1^{(1)}) = 0$.

We have two possible situations:

i. $X_0(\xi_1^{(1)})|_{P_1} = 0$.

No arbitrary function can be determined or expressed in terms of others. P_1 is the final constraint submanifold, and *the solution is not unique: it depends on two arbitrary functions*

$$X|_{P_1} = X_0 + g_1 Y^1 + g_2 Y^2$$

ii. $X_0(\xi_1^{(1)})|_{P_1} \neq 0$.

We have a new constraint

$$\xi_1^{(2)} := X_0(\xi_1^{(1)}) = X_0(\sum_{i,j} v_i^1 b^{ij} x_j)$$

which, together with $\xi_1^{(1)}$, defines the submanifold P_2. Now we are again in the same situation as at the beginning of item 2, and the procedure continues in the same way.

3. Neither equality holds identically, and the functions $\xi_\alpha^{(1)}$ are linearly independent. Then

$$\xi_1^{(1)} := \alpha(Z_1) = \sum_{i,j} v_i^1 b^{ij} x_j \quad , \quad \xi_2^{(1)} := \alpha(Z_2) = v_i^2 b^{ij} x_j$$

are the primary constraints defining the submanifold P_1, where the system is compatible and has the vector field (9) as solution.

Now the stability condition of the SGA algorithm must be imposed, which leads to the system of equations

$$\begin{aligned} X(\xi_1^{(1)}) &\equiv X_0(\xi_1^{(1)}) + g_1 Y^1(\xi_1^{(1)}) + g_2 Y^2(\xi_1^{(1)}) = 0 \qquad \text{(on } P_1) \\ X(\xi_2^{(1)}) &\equiv X_0(\xi_2^{(1)}) + g_1 Y^1(\xi_2^{(1)}) + g_2 Y^2(\xi_2^{(1)}) = 0 \qquad \text{(on } P_1) \end{aligned}$$

This system can be written in matrix form as

$$E \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \equiv \begin{pmatrix} Y^1(\xi_1^{(1)}) & Y^2(\xi_1^{(1)}) \\ Y^1(\xi_2^{(1)}) & Y^2(\xi_2^{(1)}) \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = - \begin{pmatrix} X_0(\xi_1^{(1)}) \\ X_0(\xi_2^{(1)}) \end{pmatrix} \qquad \text{(on } P_1)$$

and we have the following possibilities:

(a) rank $E = 2$.

Both arbitrary functions can be determined by solving the last linear system. P_1 is the final constraint submanifold, and *the solution is unique.*

(b) rank $E = 1$.

One function g can be completely determined or expressed in terms of the other, and, in general, a new compatibility condition appears (a function which must vanish on P_1). If this condition holds on P_1, then P_1 is the final constraint submanifold, and *the solution is not unique: it depends on one arbitrary function*. Otherwise, a new constraint $\xi_1^{(2)}$, defining the new submanifold P_2, is obtained, and the tangency condition must be applied to this constraint, as above.

(c) rank $E = 0$.

In this case, both functions are the compatibility conditions for the system, and we are in the same situation as at the beginning of the procedure, except that now the vanishing of these functions must be studied on the submanifold P_1.

Remarks:

- All the submanifolds and constraints appearing are linear.
- All the steps in the algorithm can be implemented using the Gauss method, since all the systems of equations that appear are linear.
- The algorithm ends when we arrive at one of the situations marked in the items 1, 2a, 2b, 2c(i), 3a, and in one of the situations of 3b, or when dim $P_i = 0$. Only in this last case does the problem have no dynamically consistent solution.

A particular case

As a particular case of this kind of situation, consider the following system of equations:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + u_1 \\ 0 &= A_{21}x_1 + A_{22}x_2 + u_2 \end{aligned}$$

where the A_{ij} and u_i are constants. These kind of system appears in many technical applications. For instance, when subsystems with widely separated natural frequencies are coupled (such as in the modeling of parasitic elements in a system, or when an electrical generator is connected to an electrical transmission network). Actually, this is a simplified model of a more general case in which x_i are vectors and A_{ij} matrices [40].

The manifold M is $\mathbb{R}^2$ coordinated by $\{x_1, x_2\}$. The general form of the vector field solution will be

$$X = f_1(x)\frac{\partial}{\partial x_1} + f_2(x)\frac{\partial}{\partial x_2}$$

The tensor T is symmetric, again:

$$T = \mathrm{d}x_1 \otimes \mathrm{d}x_1$$

that is, its associated matrix is

$$\Re_T = \pounds_T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the 1-form α is

$$\alpha = (A_{11}x_1 + A_{12}x_2 + u_1)\mathrm{d}x_1 + (A_{21}x_1 + A_{22}x_2 + u_2)\mathrm{d}x_2$$

A basis of $\ker T$ is made up by the vector field

$$Z_1 = \frac{\partial}{\partial x_2}$$

Now, the primary constraint defining P_1 is

$$\xi_1^{(1)} := \alpha(Z_1) = A_{21}x_1 + A_{22}x_2 + u_2$$

and the solution on the points of P_1 is

$$X|_{P_1} = (A_{11}x_1 + A_{12}x_2 + u_1)\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2}$$

Using the SGA, we obtain

$$0 = X(\xi_1^{(1)})|_{P_1} = A_{21}(A_{11}x_1 + A_{12}x_2 + u_1) + A_{22}f_2 \tag{10}$$

and the final solution depends on the value of the coefficients A_{ij}. So we have the following options:

1. $A_{22} \neq 0$: then equation (10) enables us to determine the arbitrary function:

   $$f_2|_{P_1} = \frac{A_{21}}{A_{22}}(A_{11}x_1 + A_{12}x_2 + u_1)$$

   and the final solution is

   $$X|_{P_1} = (A_{11}x_1 + A_{12}x_2 + u_1)(\frac{\partial}{\partial x_1} + \frac{A_{21}}{A_{22}}\frac{\partial}{\partial x_2})$$

   Summarizing, the system to be solved has been reduced to

   $$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + u_1 \\ \dot{x}_2 &= \frac{A_{21}}{A_{22}}(A_{11}x_1 + A_{12}x_2 + u_1) \end{aligned}$$

   on the submanifold defined by

   $$0 = A_{21}x_1 + A_{22}x_2 + u_2$$

   The solution of this system is unique (there is no gauge freedom).

2. $A_{22} = 0$: in this case we have:

   (a) If $A_{21} = 0$ (and this implies $u_2 = 0$ in order the system to be compatible): then equation $\alpha(Z_1) = 0$ is satisfied everywhere in M and the solution is

   $$X = (A_{11}x_1 + A_{12}x_2 + u_1)\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2}$$

   In other words, there is gauge freedom. From the physical point of view, this means that the coordinate x_2 is an ignorable degree of freedom.

   (b) If $A_{21} \neq 0$: then equation (10) gives the secondary constraint

   $$\xi_2^{(2)} = A_{11}x_1 + A_{12}x_2 + u_1$$

   Once again, we have two possible cases:

   i. $A_{12} = 0$: then the constraints $\xi_1^{(1)}$ and $\xi_2^{(2)}$ define two parallel lines in M, that is, $P_f = \emptyset$.

   ii. $A_{12} \neq 0$: then the constraints $\xi_1^{(1)}$ and $\xi_2^{(2)}$ define P_2, which is a single point. Therefore, this is another case with no solution.

Example 2: Sliding control

Single input systems

Once again, in the framework of *Control theory*, a particular problem which often arises is the following: let

$$\dot{\mathbf{x}} = F + Gu$$

be a system of differential equations in $U \subset \mathbb{R}^n$, where $\{x_i\}$ $(i = 1, \dots n)$ are the coordinates, $u\colon U \to \mathbb{R}$ is the *input* and

$$\dot{\mathbf{x}} = \sum_i \dot{x}_i \frac{\partial}{\partial x_i} \quad , \quad F = \sum_i f_i(\mathbf{x})\frac{\partial}{\partial x_i} \quad , \quad G = \sum_i g_i(\mathbf{x})\frac{\partial}{\partial x_i}$$

are vector fields in U. Then the question is to seek an input u such that the evolution of the system is constrained to be in a submanifold

$$S \equiv \{\mathbf{x} \in U \mid \xi^{(1)}(\mathbf{x}) = 0\}$$

where $\xi^{(1)}\colon U \to \mathbb{R}$ is a differentiable function satisfying $\nabla\xi^{(1)}(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in U$.

The study of this problem is equivalent to solving the following singular system in $M \subset \mathbb{R}^{n+1}$ (with $\{x_i, u\}$ as coordinates)

$$\begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} F + Gu \\ \xi^{(1)} \end{pmatrix}$$

Now we apply the SGA as indicated in the above example. We first have that the solution on S is

$$X = \sum_i (f_i + g_i u) \frac{\partial}{\partial x_i} + \gamma \frac{\partial}{\partial u} \equiv F + uG + \gamma \frac{\partial}{\partial u}$$

Then the stability condition $X(\xi^{(1)}) = 0$ (on S) means that the evolution of the system must be in S. Thus there are two options:

1. If $G(\xi^{(1)}) \neq 0$ (on S), then it is said that the system verifies the *transversality condition*. In this case

   $$F(\xi^{(1)}) + uG(\xi^{(1)}) = 0 \quad \Leftrightarrow \quad u = -\frac{F(\xi^{(1)})}{G(\xi^{(1)})} \quad \text{(on S)}$$

   and we obtain the so-called *equivalent control method* in the study of the *sliding control*.

2. If $G(\xi^{(1)}) = 0$ (on S) then $\xi^{(2)} \equiv F(\xi^{(1)})$ is a new constraint, and the stabilization algorithm leads either to the new condition

   $$\begin{aligned} (F + uG)(F(\xi^{(1)})) = (F(F(\xi^{(1)})) + u[G, F](\xi^{(1)}) = 0 \\ \Leftrightarrow \quad u = -\frac{F(F(\xi^{(1)}))}{[G, F](\xi^{(1)})} \qquad \text{(on S)} \end{aligned}$$

   if $[G, F](\xi^{(1)}) \neq 0$ (on S), or to another constraint $\xi^{(2)} \equiv F(F(\xi^{(1)}))$, and so on.

Observe that, with this method, the problem can be solved even though the transversality condition does not hold. Compare with [36] and the method of *equivalent control* for the solution of this kind of problem.

Multiple input systems

We now suppose that the control system has more than one input. In this case the system of differential equations is

$$\dot{\mathbf{x}} = F + G^j u_j$$

defined in $U \subset \mathbb{R}^n$. As above, $\{x_i\}$ $(i = 1, \dots n)$ are the coordinates, $u_j : U \to \mathbb{R}$ $(j = 1, \dots m)$ are the inputs, and

$$\dot{\mathbf{x}} = \sum_i \dot{x}_i \frac{\partial}{\partial x_i} \quad , \quad F = \sum_i f_i(\mathbf{x}) \frac{\partial}{\partial x_i} \quad , \quad G^j = \sum_i g_i^j(\mathbf{x}) \frac{\partial}{\partial x_i}$$

are vector fields in U. Inputs u_j must now be found such that the evolution of the system is constrained to a $(n-m)$-dimensional submanifold

$$S \equiv \{\mathbf{x} \in U \mid \xi_j^{(1)}(\mathbf{x}) = 0\}$$

where $\xi_j^{(1)}\colon U \to \mathbb{R}$ are independent differentiable functions for every $\mathbf{x} \in U$.

Once again, the study of the problem is equivalent to solving the following singular system in $M \subset \mathbb{R}^{n+m}$ (with $\{x_i, u_j\}$ as coordinates)

$$\begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & (0)_m \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} F + \sum_j G^j u_j \\ \xi^{(1)} \end{pmatrix}$$

where $\mathbf{u} = (u_1, \dots, u_j)$ and $\xi^{(1)} = (\xi_1^{(1)}, \dots, \xi_j^{(1)})$ The solution on S is

$$X = \sum_i (f_i + \sum_j g_i^j u_j) \frac{\partial}{\partial x_i} + \sum_j \gamma_j \frac{\partial}{\partial u_j} \equiv F + \sum_j u_j G^j + \sum_j \gamma_j \frac{\partial}{\partial u_j}$$

Now the stability conditions are $X(\xi_j^{(1)}) = 0$ (on S) and several options exist:

1. If the matrix $G^j(\xi_k^{(1)})$ $(j, k = 1, \dots, m)$ has rank equal to m, then all the functions u_j can be determined, no new constraints appear and the procedure ends.
2. If $0 \leq \operatorname{rank} G^j(\xi_k^{(1)}) < m$, then only some (or none) of the functions u_j can be determined; in addition, new constraints arise. Therefore, the procedure follows in an iterative way.

A particular case

As a particular case, consider the following control system: a *telescopic robotic arm* in an horizontal plane which moves following a determined trajectory, that is, a *tracking problem* (see [28] for more details on the study of this system).

The robot is made of two arms of length l: the first, with mass m_1, has one fixed end, and the other, with mass m_2, slides inside of the first. There is a motor in the fixed end which makes the robot turn around this point, and another motor, with mass m and a rotor of radius R, at the other end of the first arm, which makes the second arm slide. The robot must carry a mass m_0, placed on the outer end of the second arm, from one point to another one along a fixed curve. The problem consists in determining the torques τ_1, τ_2 of both motors.

We take the origin at the fixed end. Then φ denotes the angle swept by the robot and x the length of the second arm which emerges from the first one. Then, the dynamical equations are

$$\ddot{\varphi} = \frac{\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\dot{\varphi}\dot{x}}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx}$$

$$\ddot{x} = \frac{1}{m_0+m_2}\left(\frac{\tau_2}{R}+\dot{\varphi}^2(m_2(x+\frac{l}{2})+m_0(x+l))\right)$$

(where $I = \left(m+m_0+\frac{m_1+m_2}{3}\right)l^2$). This system of second order differential equations can be transformed into one of first order by adding the variables ω, v, and the equations

$$\dot{\varphi} = \omega \quad , \quad \dot{x} = v$$

As a trajectory, we take an arc of spiral whose implicit equation is

$$l + x - \varphi = 0 \tag{11}$$

The new system can be expressed in the form (6), (in $U \subset \mathbb{R}^4$), where φ, ω, x, v are the variables, $\tau_1\tau_2$ are the inputs, and

$$\begin{aligned}
F &= \omega\frac{\partial}{\partial\varphi} - \frac{(2(m_0+m_2)x+(2m_0+m_2)l)\omega v}{I+(m_0+m_2)x^2+(2m_0+m_2)lx}\frac{\partial}{\partial\omega} \\
&\quad + v\frac{\partial}{\partial x} + \frac{\omega^2(m_2(x+\frac{l}{2})+m_0(x+l))}{m_0+m_2}\frac{\partial}{\partial v} \\
G^1 &= \frac{1}{I+(m_0+m_2)x^2+(2m_0+m_2)lx}\frac{\partial}{\partial\omega} \\
G^2 &= \frac{1}{R(m_0+m_2)}\frac{\partial}{\partial v}
\end{aligned}$$

and subjected to the constraint (11).

Now we write the system in the form of equation $\pounds(X)T = \alpha$, in $M \subset \mathbb{R}^6$, which has $\{\varphi, \omega, x, v, \tau_1, \tau_2\}$ as coordinates. Hence, the tensor T and the 1-form α are

$$\begin{aligned}
T &= \mathrm{d}\varphi\otimes\mathrm{d}\varphi + \mathrm{d}\omega\otimes\mathrm{d}\omega + \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}v\otimes\mathrm{d}v \\
\alpha &= \omega\mathrm{d}\varphi + \frac{\tau_1-(2(m_0+m_2)x+(2m_0+m_2)l)\omega v}{I+(m_0+m_2)x^2+(2m_0+m_2)lx}\mathrm{d}\omega + v\mathrm{d}x \\
&\quad + \frac{1}{m_0+m_2}\left(\frac{\tau_2}{R}+\omega^2(m_2(x+\frac{l}{2})+m_0(x+l))\right)\mathrm{d}v + (l+x-\varphi)\mathrm{d}\tau_1
\end{aligned}$$

that is, the matrix form of the system is

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix}\dot{\varphi}\\ \dot{\omega}\\ \dot{x}\\ \dot{v}\\ \dot{\tau}_1\\ \dot{\tau}_2\end{pmatrix} = \begin{pmatrix} \omega \\ \frac{\tau_1-(2(m_0+m_2)x+(2m_0+m_2)l)\omega v}{I+(m_0+m_2)x^2+(2m_0+m_2)lx} \\ v \\ \frac{\frac{\tau_2}{R}+\omega^2(m_2(x+\frac{l}{2})+m_0(x+l))}{m_0+m_2} \\ l+x-\varphi \\ 0 \end{pmatrix}$$

As $\ker T$ is spanned by the vector fields $\frac{\partial}{\partial\tau_1}, \frac{\partial}{\partial\tau_2}$, the compatibility condition gives the constraint

$$\xi^{(1)} := l + x - \varphi = 0$$

which defines $P_1 \hookrightarrow M$. Then, the vector field solution on P_1 is

$$\begin{aligned} X &= \omega\frac{\partial}{\partial\varphi} + \frac{\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx}\frac{\partial}{\partial\omega} + v\frac{\partial}{\partial x} \\ &+ \frac{1}{m_0+m_2}\left(\frac{\tau_2}{R} + \omega^2(m_2(x+\frac{l}{2}) + m_0(x+l))\right)\frac{\partial}{\partial v} \\ &+ f_1\frac{\partial}{\partial\tau_1} + f_2\frac{\partial}{\partial\tau_2} \end{aligned} \tag{12}$$

Next, we have to impose (in an iterative way) the stability condition which enables us to obtain a sequence of submanifolds $P_1 \hookleftarrow P_2 \hookleftarrow P_3$ defined by the constraints

$$\begin{aligned} \xi^{(2)} &:= X(\xi^{(1)}) = v - \omega \\ \xi^{(3)} &:= X(\xi^{(2)}) = \frac{\frac{\tau_2}{R} + \omega^2[m_2(x+\frac{l}{2}) + m_0(x+l)]}{m_0+m_2} - \\ &\quad \frac{\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx} \end{aligned}$$

and a relation between the arbitrary functions f_1, f_2

$$\begin{aligned} 0 &= X(\xi^{(3)}) \\ &= \frac{2\omega[m_2(x+\frac{l}{2}) + m_0(x+l)][\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v]}{(m_0+m_2)[I + (m_0+m_2)x^2 + (2m_0+m_2)lx]} + \\ &\quad \frac{[\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v][2(m_0+m_2)x + (2m_0+m_2)l]v}{(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)^2} \\ &\quad + v\omega^2 + \frac{2\omega v^2(m_0+m_2)}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx} + \\ &\quad \frac{v[2x(m_0+m_2) + (2m_0+m_2)l][\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v]}{(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)^2} \\ &\quad + \frac{[\frac{\tau_2}{R} + \omega^2(m_2(x+\frac{l}{2}) + m_0(x+l))][2(m_0+m_2)x + (2m_0+m_2)l]\omega}{(m_0+m_2)(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)} \\ &\quad - \frac{f_1}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx} + \frac{f_2}{R(m_0+m_2)} \end{aligned}$$

Thus, the vector field solution on P_3 is (12), where the variables and the arbitrary functions are related by all the above relations. So we have the (regular) system of differential equations

$$\begin{aligned} \dot\varphi &= \omega \\ \dot\omega &= \frac{\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx} \end{aligned}$$

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= \frac{1}{m_0+m_2}\left(\frac{\tau_2}{R} + \dot{\varphi}^2(m_2(x+\frac{l}{2}) + m_0(x+l))\right) \\
\dot{\tau}_1 &= f_1 \\
\dot{\tau}_2 &= \left(\frac{2\omega[m_2(x+\frac{l}{2}) + m_0(x+l)][\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v]}{(m_0+m_2)[I + (m_0+m_2)x^2 + (2m_0+m_2)lx]} + \right. \\
& \frac{[\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v][2(m_0+m_2)x + (2m_0+m_2)l]v}{(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)^2} \\
& + v\omega^2 + \frac{2\omega v^2(m_0+m_2)}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx} + \\
& \frac{v[2x(m_0+m_2) + (2m_0+m_2)l][\tau_1 - (2(m_0+m_2)x + (2m_0+m_2)l)\omega v]}{(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)^2} \\
& + \frac{[\omega^2(m_2(x+\frac{l}{2}) + m_0(x+l))][2(m_0+m_2)x + (2m_0+m_2)l]\omega}{(m_0+m_2)(I + (m_0+m_2)x^2 + (2m_0+m_2)lx)} \\
& \left. - \frac{f_1}{I + (m_0+m_2)x^2 + (2m_0+m_2)lx}\right)(-R(m_0+m_2))
\end{aligned}
$$

Observe that the inputs τ_1, τ_2 are not determined (their evolution depends on an arbitrary function). At this point some criteria can be imposed for tracking the trajectory in a predefined way, for instance, minimizing the cost (for going from one point to another in a given time). Of course, the integral curves of X are on the surface P_3, and, hence, on P_1.

7 Conclusions

The goal of this work is to give a relatively simple geometric framework which allows us to describe systems of differential equations

$$A(\mathbf{x})\dot{\mathbf{x}} = \alpha(\mathbf{x})$$

(where A is a singular matrix which represents a 2-covariant tensor), as well as solve the problems of incompatibility and inconsistency of these equations.

Our treatment enables us to overcome the difficulties arising from other analytic and algebraic procedures previously developed for these systems. It is important to point out that the geometric framework here developed is not as general as the one in reference [20], but it is simpler. Both of them are equivalent when they are used for describing the same system. In addition, in our treatment we pay special attention to the study of the symmetries of these systems and we give an accurate description of the algorithm.

It also represents an improvement on the other geometric treatments cited in the introduction, since we have developed an algorithm which solves the above mentioned problems of incompatibility and inconsistency. In the most interesting cases, the final result of the algorithm is a submanifold (of the space of states where the system is defined) and a vector field solution tangent to this submanifold, whose integral curves are the trajectories of the system. Consequently, the restriction of the system of differential equations to the submanifold found can be integrated by analytic or numerical methods.

In general, the vector field solution is not single. In fact, there are two possibilities :

1. If the singularity of the initial system of differential equations arises from a non suitable choice of the variables (that is, the initial phase space is too large in order to describe the real degrees of freedom of the problem), then in the final constraint submanifold the system has a single solution.

2. If the singularity of the system is a consequence of the existence of a certain kind of internal symmetry, then the final system of equations can be undetermined; that is, the solution of the system is not single. This means that for every point in the final constraint submanifold which is taken as an initial condition, the evolution of the system is not determined because a multiplicity of integral curves (of different vector fields solution) pass through it. This is known as the *gauge freedom* (in the physical literature). The question of removing this ambiguity has already been studied for some special cases (see, for instance, [27]).

Another essential point to which we pay special attention is the study of the symmetries of these systems. This is a subject which has not previously been treated (at least geometrically), and we believe our analysis is enlightening.

An interesting subject might be the study of *non autonomous singular differential equations*; that is, those of the form:

$$A(\mathbf{x}, t)\dot{\mathbf{x}} = \alpha(\mathbf{x}, t)$$

As is known, these systems can be considered as autonomous by adding the equation $t' = 1$. It is obvious that this equation remains unchanged throughout the algorithmic procedure, so the method is directly applicable to this kind of system. Observe that the constraints may depend on time.

We further believe that the analysis of *second order singular differential equations* is a subject of interest. Finally, as we have remarked in the examples, problems on *Control theory* (such as those related to *sliding control* and others) could be analyzed in this way.

Acknowledgments We thank the financial support of the CICYT TAP97-0969-C03-01. We thank Mr. Jeff Palmer for the correction of the English version of the manuscript.

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