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The Lazer Mckenna Conjecture for Radial Solutions in the R^N Ball *

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Abstract

When the range of the derivative of the nonlinearity contains the first k eigenvalues of the linear part and a certain parameter is large, we establish the existence of 2k radial solutions to a semilinear boundary value problem. This proves the Lazer McKenna conjecture for radial solutions. Our results supplement those in [5], where the existence of $k+1$ solutions was proven.

1 Introduction

Here we consider the boundary value problem

$$-\Delta u(x) = g(u(x)) + t\varphi(x) + q(x) \text{ for } x \in \Omega \tag{1.1}$$

$$u(x) = 0 \text{ for } x \in \partial\Omega\,, \tag{1.2}$$

where Δ denotes the Laplacean operator, Ω is a smooth bounded region in $R^N(N > 1)$, g is a differentiable function, q is a continuous function, and $\varphi > 0$ on Ω is an eigenfunction corresponding to the smallest eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. We will assume that

$$\lim_{u\to-\infty} \frac{g(u)}{u} = \alpha \quad \text{ and } \quad \lim_{u\to\infty} \frac{g(u)}{u} = \beta\,. \tag{1.3}$$

Motivated by the classical result of A. Ambrosetti and G. Prodi [1], equations of the form (1.1)–(1.2) have received a great deal of attention when the interval (α, β) contains one or more eigenvalues of $-\Delta$ with zero Dirichlet boundary data. In [1] it was shown that when (α, β) contains only the smallest eigenvalue then for $t < 0$ large enough the equation (1.1)–(1.2) has two solutions. Upon

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considerable research on extensions of this result, A. C. Lazer and P. J. McKenna conjectured that when (α, β) contains the first k eigenvalues then (1.1)–(1.2) has $2k$ solutions. Here we prove that such a conjecture is true if one restricts to radial solutions ($u(x) = u(y)$ if $||x|| = ||y||$) in a ball. This conjecture, however, is not true in general. In [7] E. N. Dancer gives an example where (α, β) contains more than two eigenvalues and yet (1.1)–(1.2) has only four solutions for $t < 0$ large. The reader is referred to [13] for an extensive review on problems with jumping nonlinearities and their applications to the modeling of suspension bridges.

Throughout this paper $[x]$ denotes the largest integer that is less than or equal to x. Our main result is stated as follows:

Theorem 1.1 *Let Ω be the unit ball in $R^N (N > 1)$ centered at the origin. Let $0 < \rho_1 < \rho_2 < \cdots < \rho_n < \cdots \to \infty$ denote the eigenvalues of $-\Delta$ acting on radial functions that satisfy (1.2). If*

$$\alpha < \rho_1([j/2] + 1)^2 < \rho_k < \beta < \rho_{k+1} \tag{1.4}$$

and q is radial function, then for t negative and of sufficiently large magnitude, problem (1.1)–(1.2) has at least $2(k - j)$ radial solutions, of which $k - j$ satisfy $u(0) > 0$.

This theorem with $j = 1$ proves the Lazer-McKenna conjecture in the class of radial functions. Theorem 1.1 extends the results of D. Costa and D. de Figueiredo (See [5]) since we do not require $\alpha < \rho_1$ and for any $N > 1$ we obtain k solutions with $u(0) > 0$. In [5] the authors proved, only for $N = 3$, that the equation (1.1)–(1.2) has k solutions with $u(0) > 0$. The reader is also referred to [14] for a study on the case $t > 0$. For other results on problems with jumping nonlinearities see [8], [11], [13] and references therein.

For the sake of simplicity we will assume that $\alpha > 0$. Minor modifications needed for the case $\alpha \leq 0$ are left to the reader.

2 Preliminaries

Since φ is a radial function, using polar coordinates ($r = \|x\|, \theta$) we see that finding radial solutions to (1.1)–(1.2) is equivalent to solving the two point boundary value problem

$$u'' + (\tfrac{N-1}{r})u' + g(u(r)) + t\varphi(r) + q(r) = 0 \quad r \in [0, 1]\,, \tag{2.1}$$

$$u'(0) = 0\,, \tag{2.2}$$

$$u(1) = 0\,, \tag{2.3}$$

where the symbol $'$ denotes differentiation with respect to $r = \|x\|$, $\varphi(r) \equiv \varphi(x)$, and $q(r) \equiv q(x)$.

Let $\tau(\varphi, q) = \tau$ be such that if $t < \tau$ then the problem (1.1)–(1.2) has a positive solution $U_t := U$ (See [5], [11]). Following the ideas in [14] we will seek solutions to (1.1)–(1.2) of the form $U+w$. It is easily seen that $U+w$ satisfies (1.1)–(1.2) if and only if w satisfies

$$w'' + \tfrac{N-1}{r}w' + \lambda[g(U(r)+w(r)) - g(U(r))] = 0\,, r \in [0,1] \tag{2.4}$$

$$w'(0) = 0\,, \tag{2.5}$$

$$w(1) = 0\,, \tag{2.6}$$

for $\lambda = 1$. We will denote by $w := w(\cdot, t, \lambda, d)$ the solution to (2.4)–(2.5) satisfying $w(0) = d$.

We prove Theorem 1.1 by studying the bifurcation curves for the equations (2.4)–(2.6). For future reference we note that, for fixed $t \in R$, the set

$$S \subset \{(\lambda, w) \in R \times (C(\Omega) - \{0\})\,;\ (\lambda, w) \text{ satisfies (2.4)–(2.6)}\}$$

is connected if and only if $\{(\lambda, w(0))\,;\ (\lambda, w) \in S\}$ is connected. This is an immediate consequence of the continuous dependence on initial conditions of the solutions to (2.4). In order to facilitate the proofs of the above theorems, we identify S with the latter subset of R^2. We consider solutions to (2.4)–(2.6) bifurcating from the set $\{(\lambda, 0);\ \lambda > 0\}$, which clearly is a set of solutions. Since the eigenvalues of the problem

$$z'' + \tfrac{N-1}{r}z' + \lambda g'(U)z = 0 \quad r \in [0,1] \tag{2.7}$$

$$z'(0) = 0\,, \tag{2.8}$$

$$z(1) = 0\,, \tag{2.9}$$

are simple, by general bifurcation theory (See [5]) it follows that if μ is an eigenvalue of (2.7)–(2.9) then near $(\mu, 0)$ there are solutions to (2.4)–(2.6) of the form $(\mu + o(s), s\psi + o(s))$ where $\psi \neq 0$ is an eigenfunction corresponding to the eigenvalue μ.

Given t, hence U, we will denote by $\mu_1 < \mu_2 < \cdots \to \infty$ the eigenvalues to (2.7)–(2.9). Now we are ready to establish the estimates on the points of bifurcation of (2.4)–(2.6).

Lemma 2.1 *If $\lim_{u\to+\infty} g(u)/u = \gamma$ then for any positive integer j and $\epsilon > 0$ there exists $T(j)$ such that if $t < T$ then $\mu_j < (\rho_j/\gamma - \epsilon)$*

Proof. Since U tends to ∞ uniformly on compact subsets of [0,1) as $t \to -\infty$, by the Courant-Weinstein minmax principle we have

$$\mu_j \le \sup_{u \in M - \{0\}} (\int_\Omega \nabla u \cdot \nabla u)/(\int_\Omega g'(U)u^2)\,, \tag{2.10}$$

where M is any j-dimensional linear subspace. On the other hand, letting M be the span of $\{\varphi_1, ..., \varphi_j\}$, where φ_i is an eigenfunction corresponding to the eigenvalue ρ_i we see that the numerator in the the right hand side of (2.10) is less than or equal to $\rho_j \int_\Omega u^2$. This implies that $\mu_j < (\rho_j/(\gamma-\epsilon))$ for $t \ll 0$, which proves the lemma.

Let $E(r,t,\lambda,d) := E(r) = ((w'(r,t,\lambda,d))^2/2) + \lambda\cdot(G(r,t,w(r,t,\lambda,d)))$, where $G(r,t,s) = \int_0^s (g(U(r)+x) - g(U(r)))\,dx$. Because of (1.3), arguing as in [2] (See also [4]), we see that for each t and λ in bounded sets

$$E(r,t,\lambda,d) \to +\infty \text{ uniformly on } [0,1] \text{ as } |d| \text{ tends to infinity.} \tag{2.11}$$

Remark 2.1 *By the uniqueness of solutions to the initial value problem (2.4)–(2.5), $w(0) = d$, we see that if $w(s) = w'(s) = 0$ for some $s \in [0,1]$ then $w(r) = 0$ for all $r \in [0,1]$.*

Lemma 2.2 *Let $t < \tau$ be given with α as in Theorem 1.1. If $\{(\lambda_n, w_n)\}$ is a sequence of solutions to (2.4)–(2.6) such that for each n w_n has exactly j zeros in (0,1), $\{\lambda_n\}$ converges to Λ, and $\{|w_n(0)|\}$ converges to infinity, then*

$$\alpha\Lambda \ge ([j/2]+1)^2 \rho_1\,.$$

Proof: Without loss of generality we can assume that $w_n(0) > 0$ for all n. Let $0 < r_{1,n} < \cdots < r_{k,n} < 1$ denote the zeros of w_n in (0,1]. For $i = 1, \cdots, k$, let $s_{i,n} \in (r_{i,n}, r_{i+1,n})$ be such that

$$|w_n(s_{i,n})| = \max\{|w_n(t)|; t \in [r_{i,n}, r_{i+1,n}]\}\,.$$

Since g is locally Lipschitzian, by the uniqueness of solutions to initial value problems we see that $|w_n(s_{i,n})| \neq 0$. Thus $w_n'(s_{i,n}) = 0$ By (2.11) we see that $\{w_n(s_{i,n})\}$ converges $to -\infty$ as n tends to infinity.

Now we analyze w_n on $[s_{i,n}, r_{i+1,n})$, for i odd. By the definition of α we see that $g(x) = \alpha x + h(x)$ with $\lim_{x\to-\infty} h(x)/x = 0$, for $x < 0$. Let s denote a limit point of $\{s_{i,n}\}$ and b a limit point of $\{r_{i.n}\}$. Thus $\{z_n := w_n/w_n(s_{i,n})\}$ converges, uniformly on $[s,b]$, to the solution to

$$z'' + \tfrac{N-1}{r} z' + \Lambda\alpha z = 0\,,\ r \in [s,b] \tag{2.12}$$

$$z(s) = 1\,,\ z'(s) = 0\,. \tag{2.13}$$

By the Sturm Comparison Theorem we know that $z > 0$ on $[s, s+(\rho_1/(\Lambda\alpha))]$. Hence for $\delta > 0$ sufficiently small there exists η such that if $n > \eta$ then $w_n < 0$ on $[s_{i,n}, s_{i,n} + (\rho_1/(\Lambda\alpha)) - \delta]$. Since this argument is valid for all i odd, we see that

$$m(\{x; w_n(x) < 0\}) > ([k/2]+1)((\frac{\rho_1}{\Lambda\alpha})^{1/2} - \delta)\,,$$

which proves the lemma.

Corollary 2.1 *Let $t < \tau$. If $\{(\lambda_n, w_n)\}$ is a sequence of solutions to (2.4)–(2.6), w_n has exactly k zeros in (0,1) for each n, $\{\lambda_n\}$ converges to Λ, and $\{|w_n(0)|\}$ converges to infinity, then $(\alpha+\beta)\Lambda \geq ([k/2]+1)^2\rho_1$, where $[x]$ denotes the largest integer less than or equal to x.*

Proof: Since $\beta \in R$ the arguments of the proof of Lemma 2.2 are also valid for the local maxima of w_n, which yields the Corollary.

3 Proof of Theorem 1.1

Let $m \leq k$ be a positive integer. By Lemma 2.1 there exists $T := T(m)$ such that if $t < T$ then $\mu_k < 1$. From general bifurcation theory for simple eigenvalues (see [6]) it follows that there exist two unbounded branches (connected components) of nontrivial solutions bifurcating from $(\mu_m, 0)$. We will denote these branches by $G_{m,+}$ and $G_{m,-}$ respectively. In addition, the branch $G_{m,+}$ (respect. $G_{m,-}$) is made up of elements of the form (λ, w), w has m zeros in (0,1], $w(0) > 0$ (respect. $w(0) < 0$), and contains elements of the form (λ, w) with λ near μ_m and $w(0)$ near zero. Hence

$$G_{j,\sigma} \cap G_{\kappa,s} = \Phi \text{ if } (j,\sigma) \neq (k,s). \tag{3.1}$$

Since $G_{m,s},\ s \in \{+,-\})$ is unbounded, and since there is no element of $G_{m,s}$ with $\lambda = 0$ (the only solution to (2.4)–(2.6) when $\lambda = 0$ is $w \equiv 0$), Lemma 2.2 implies that for $m \in \{j, ..., k\}$ the set $G_{m,s}$ contains an element of the form (λ, w) with $\lambda > 1$. By the connectedness of $G_{m,s}$ we see that it contains an element of the form $(1, w_{m,s})$ which proves that $U + w_{m,s}$ is a solution to $(1.1) - (1.2)$. Thus (1.1)–(1.2) has $2(k-j)$ solutions. In addition, since $U(0) > 0$ and $w_{m,+} > 0$ we see that $k - j$ of these solutions are positive at zero, which proves the Theorem.
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