

DESCRIBING STUDENTS' ANALOGICAL REASONING WHILE CREATING  
STRUCTURES IN ABSTRACT ALGEBRA

by

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A dissertation submitted to the Graduate Council of  
Texas State University in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
with a Major in Mathematics Education  
May 2021

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## **DEDICATION**

I dedicate this work to my parents and to my sister, Elizabeth, who have given me the love and support needed to continue moving forward to finish this project.

## **ACKNOWLEDGEMENTS**

First, I would like to thank my co-chairs, Kate Melhuish and Jessica Bishop, for their unending support in developing and ultimately completing this project. I could not have asked for better role models, not just for conducting mathematics education research, but also for learning a bit more about how to navigate life in general. Kate, thank you for your dedication to shaping me as a scholar and helping me improve my work over the last three years. Jessica, thank you for introducing me to the world of mathematics education research and helping me cement my decision to continue pursuing a PhD in this field. I very much look forward to continue collaborating with the both of you in the future!

I thank Jennifer Czoher for helping me to jumpstart my dissertation as a part of a course and later pushing me to try different methods of qualitative research and ways of thinking from what I had found myself accustomed to. At one point, a group of students began calling me “the male Dr. Czoher”. I took this as a compliment. I also thank Elise Lockwood, Maria Acosta, and Kristen Lew for serving on my dissertation committee and providing me with valuable insight to make my work stronger.

I thank my research participants for their commitment, patience, and endurance through several hours of interviews doing mathematics that they may (or may not) have enjoyed. I could not have hoped to have completed this work without them.

I thank the faculty, lecturers, and staff in the mathematics department at Texas State. When I first came here in 2010, I didn’t have much direction in life aside from the

vague goal of finishing college and perhaps becoming a teacher someday. I thank many of them for assisting with my development as a mathematician and finding direction. Susan Morey was the first mathematics instructor I had at the college level and I owe much of my proof writing skill to her. Enrolling in Analysis I and meeting Terry McCabe was a life changing experience for me as he helped me to solidify my decision to eventually attend graduate school. (It was also a joy to meet someone who shared so many of my interests in music!) I also thank Max Warshauer and Thomas Keller, who greatly helped expand my interests in abstract algebra which later became a main topic of this dissertation.

I thank the graduate students of Texas State's Mathematics Education program that I have come to know in my time here, including those who have been with me through everything and those who chose to eventually pursue different paths after all. It has been an honor to work alongside so many great people and I am going to sorely miss being able to call myself a graduate student in this department. I especially thank Suby for her support in both professional and personal life, for being my travel partner to so many conferences, and for putting up with me as her officemate for so long as she has. I also thank Christina for being such an excellent research partner on the AAA team.

Finally, I would like to thank my parents Anne and Duane, my sisters Elizabeth and Erin, my grandparents, and all of my family for their patience and understanding of why I couldn't be around as much as I perhaps could have had I chosen a different path in life.

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## LIST OF ABBREVIATIONS

| Abbreviation | Description                         |
|--------------|-------------------------------------|
| AO           | Actor-Oriented                      |
| ARM          | Analogical Reasoning in Mathematics |
| DBA          | Design-by-Analogy                   |
| IMS          | Internal Mathematical Structure     |
| OCA          | Open-Classical Analogy              |
| SMT          | Structure-Mapping Theory            |

## ABSTRACT

Analogical reasoning has played a significant role in the development of modern mathematical concepts. Although some perspectives in mathematics education have argued against the use of analogies and analogical reasoning in instructional contexts, some attempts have been made to leverage the pedagogical power of analogies. I assert that with a greater understanding of how students develop analogies and reason by analogy, analogies can indeed be used productively for the teaching and learning of mathematics. Using abstract algebra as the primary context, I propose three papers: (1) a theoretical paper orienting analogical reasoning as a *way of thinking* in mathematics (and thus learnable by students), (2) an empirical paper contributing the Analogical Reasoning in Mathematics (ARM) framework for interpreting students' activity during analogical reasoning and (3) a practitioner paper detailing a full lesson incorporating analogical reasoning as a tool for exploratory structure creation in abstract algebra.

Paper #1 identifies analogical reasoning as a way of thinking in the context of advanced mathematics. There has been critique of the use of analogies for the purpose of students learning new content because students may fail to appropriately recognize the analogical connections developed by instructors. I counter that students can productively reason by analogy to understand new mathematics when provided with settings to develop this way of thinking. In this paper, I use examples from the work of mathematicians to argue for the important role of analogy for the purpose of mathematical discovery. I then provide an illustration of an undergraduate student

engaged in similar productive analogical reasoning as they develop analogs between structures in group and ring theory. Through this process, the student showed increasing awareness of how and why they were engaging with such reasoning. This observation evidences the potential for students to reason by analogy for mathematical discovery.

Paper #2 establishes the Analogical Reasoning in Mathematics (ARM) framework for describing students' analogical activity in mathematics contexts. I first outline a definition of analogy and contrast it with the concept of metaphor. I then introduce ARM, which categorizes analogical reasoning activity that is unique to the context of doing mathematic and explicates features of analogical reasoning that are largely implicit in existing models. Constructed from an analysis of interviews with four students engaged with analogical tasks in abstract algebra using basic qualitative methods related to grounded theory, ARM includes three dimensions of analogical activity: mapping/non-mapping across domains (MAD), attending to similarity and difference (SAD), and foregrounding a domain (FAD). Built upon these dimensions, analogical activities are identified and explicated for the purpose of analyzing student analogical reasoning. I provide examples of several of these activities in the context of abstract algebra.

Finally, Paper #3 proposes a novel lesson for introducing structures in ring theory by reasoning analogically about structures already known in group theory. In this way, students come to creatively establish new structures that they may take ownership of while providing opportunities for rich discussion about the purpose of these structures. The lesson consists of four key components: (a) introducing the definition of ring, (b)

introducing the idea of analogy and analogical reasoning between groups and rings, (c) developing structures (i.e., subrings, ring homomorphisms, and quotient rings) through analogical reasoning with known structures, and (d) developing theorems/proofs through analogical reasoning. Throughout this paper, I provide thoughts and insights from previous implementations and conclude by reflecting on what has (and has not) worked well in my experience with implementing these tasks.

Taken together, these papers offer insight into understanding how students reason by analogy and suggests implications for productively incorporating analogical reasoning into instruction. Directions for future research involving analogical reasoning in mathematics are outlined based on these contributions.

## I. INTRODUCTION

Throughout history, analogical reasoning has played a significant role in the discovery of scientific and mathematical concepts. Analogy has been recognized as playing an equally important role in the way we act and behave in everyday life and has even been shown to be present in children's development of mathematical thinking (English, 2004). A powerful example of analogy in history can be found within Descartes' recognition of similarities between algebraic and geometric concepts and the attempt to unify arithmetic, algebra, and geometry with a theory of proportions. On this topic, Crippa (2017) states:

Euclid's theory of proportions is a general theory thanks to a logic of analogy, which allows one to circumvent the prohibition of transferring proofs from one kind to the other... In other words, in virtue of analogy the proof that certain properties hold for a certain subject matter, i.e. geometry, can extend its validity and hold also within a different subject matter, for instance arithmetic. (p. 1244-1245)

The analogies between algebra and geometry allowed for a translation between problems in algebra and problems in geometry in such a way that the insights gained from one context could be used to extend insights into the other. Furthermore, the union of algebra and geometry laid the foundation for further studies in mathematics, most notably the field of algebraic geometry in modern algebra (Kendig, 1983).

From examples such as these, it is evident that the creation of analogies can be beneficial in establishing new ways of thinking and reasoning in mathematics. However, it has been found that instructors may produce most analogies within instructional contexts, and it is unclear to what extent students are understanding the analogies for themselves (Richland, Holyoak, & Stigler, 2004). As a result, students may be missing opportunities to productively reason by analogy in mathematics classrooms.

Polya (1954) documented the importance of analogy in mathematics in volume 1 of his book *Induction and Analogy in Mathematics*. He proposed a framework of mathematical reasoning consisting of generalizing, specializing, and analogy. Polya argued that these three forms of inductive reasoning often co-occur when approaching problems in mathematics. In the mathematics education literature, generalizing has been acknowledged as being critical to engaging in mathematical reasoning (e.g. Kaput, 1999; Mason, 1996; Ellis, 2007)). Although analogy has been widely identified as crucial to learning in other content areas such as general science education (e.g. Hesse, 2000) and computer science (e.g. Gentner, 1983), analogy and analogical reasoning have not received the same level of attention in the mathematics education literature (English & Sharry, 1996). In order to provide a more complete picture of how students across the K-20 spectrum might reason mathematically, it is necessary to characterize students' mathematical activity as they engage in analogical reasoning as well as understand how to promote students to reason by analogy.

Because analogies are commonly leveraged in undergraduate abstract algebra, the subject is an excellent one for investigating how students reason by analogy in mathematics. The similarity between the structures of the mathematical objects in group theory and ring theory is no accident; historically, there was purposeful intent in unifying the structures between group theory and ring theory (Hausberger, 2018). As a result, abstract algebra provides a rich environment for students to establish clear connections across different mathematical domains and thus provides ample opportunity for accessible reasoning by analogy.

Furthermore, abstract algebra has been characterized as being an enormously

difficult undertaking for undergraduate mathematics students (Dubinsky, Dautermann, Leron & Zazkis, 1994). Given that abstract algebra is a foundational course for future mathematics teachers as well as future mathematicians, the persistently high difficulty in teaching abstract algebra remains to be a great concern. Researching how students reason by analogy in abstract algebra also opens up possibilities for investigating students' thinking about topics in ring theory. In contrast to student thinking about groups and associated content in group theory, there is little research on student thinking in ring theory. The majority of existing research on student thinking in ring theory currently focuses on how students might reinvent the ring and come to understand basic aspects of rings. For example, Simpson and Stehlíková (2006) attended to student development of structural understanding by exploring how one student came to apprehend the structure of a commutative ring, and Cook (2012) has explored the ways in which students might reinvent the concepts of ring, integral domain, and field. However, there is a lack of research on students' understanding of topics beyond these basic elements of ring theory. Given that future teachers and mathematicians alike are expected to take courses in abstract algebra, there is a need to contribute to the literature in mathematics education on the topic of ring theory so that we may teach the subject more efficiently and effectively. I hypothesize that by investigating how students reason about topics in ring theory by analogy with structures in group theory, several results pertaining to students' understanding of structures in ring theory will be in reach soon thereafter, and informed efforts to develop curriculum for teaching ring theory leveraging analogy and analogical reasoning can begin.

This dissertation takes as its premise that analogical reasoning can play a

significant role in assisting students to learn about topics in ring theory by analogy with topics in group theory by providing students an opportunity to develop structures in ring theory through inventive reasoning by analogy in a manner similar to that of mathematicians doing research (e.g., Ouvrier-Buffet, 2015). The following questions guided the research presented in this dissertation:

RQ1: How do students reason by analogy about rings, subrings, ring homomorphisms, and quotient rings in abstract algebra?

RQ2: How might students come to productively reason by analogy in abstract algebra?

RQ3: How might analogy and analogical reasoning be effectively incorporated into abstract algebra curriculum focused on introducing ring theory after group theory?

In Chapter II, I review literature pertaining to analogy and analogical reasoning within and without mathematics education. In Chapter III, I provide an overview of the methods used to answer the questions posed above. I then propose three papers which make contributions to research on understanding how students reason by analogy in mathematics and make suggestions for productively incorporating analogical reasoning into instruction in mathematics classrooms. Specifically, Chapter IV proposes a theoretical paper orienting analogical reasoning as a *way of thinking* in mathematics and thus argues that reasoning by analogy is learnable by students. Chapter V proposes an empirical paper contributing the Analogical Reasoning in Mathematics (ARM) framework for interpreting students' activity during analogical reasoning. Chapter VI proposes a practitioner paper detailing a full lesson incorporating analogical reasoning as

a tool for exploratory structure creation in abstract algebra. Finally, Chapter VII summarizes the contributions made by these papers and proposes several directions for future research

## **II. REVIEW OF LITERATURE**

I begin this chapter by describing several conceptualizations of analogy, giving the most attention to the conceptualizations of analogy that use some variation of mapping theory. In particular, the most influential and widely recognized of these theories, Structure-Mapping Theory (Gentner, 1983), is introduced here. I then describe the ways in which analogy has been thought of as method of formulating inferences and creating new knowledge in mathematics. I give special attention to research on the generation of mathematical structure through reasoning by analogy, a subsection of research on analogy in mathematics education that has not been strongly developed.

### **Conceptualizations of Analogy**

The concept of analogy has been conceived of in various ways within the literature. In this section, I describe historical and modern conceptualizations of analogy in the literature and the ways in which analogy has been modeled and operationalized. I begin with a description of several definitions of analogy before narrowing the scope to modern theories of studying analogy through mappings and constraint satisfaction. I then describe the literature on analogy within mathematics education.

### **Defining Analogy**

Several definitions and representations of analogy have been proposed in an attempt to capture the intricacies of what is meant by analogy. Before reviewing the literature on the more complex modern conceptualizations of analogy, it is helpful to look back on the historical uses of analogy in order to gain some intuition about the general definitions and uses of analogy.

**A Historical Account of Analogy.** Earlier conceptualizations of analogy fall under the umbrella of *classical analogy theory*, initially put forth by Aristotle, Thomas Aquinas, and Cardinal Cajetan (Haaparanta, 1992). Haaparanta distinguished between three ways of discussing classical analogy: the analogy of things (*analogia entis*), the analogy of terms (*analogia nominum*), and the analogical argument (*analogia rationis*).

Classical analogy theory primarily relates to the analogy of things and the analogy of terms. The analogy of things refers to those analogies that are created by noting similarities between two objects, events, or systems. This type of analogy would include statements such as “the car is as fast as a cheetah”, and “an atom is like our solar system.” Polya (1954) defined analogy using a similar language of objects, arguing that analogy is “similarity on a more definite and more conceptual level... Similar objects agree with each other in some aspect.” (p. 13) While the analogy of things and Polya’s definition of analogy both rely on comparing two or more objects, Polya’s definition explicitly points to similarities that are more conceptual in nature.

The analogy of terms refers to those analogies that do not explicitly draw comparisons between two objects, events, or systems, but instead suggest comparisons between meanings of terms that are similar in given contexts, but not necessarily the same. For example, the meaning of the word *save* is similar in the contexts of *saving time* and *saving money* even though the word is not being used in exactly the same way in each context. The difference between the analogy of things and the analogy of terms is subtle at first glance: whereas the analogy of things refers explicitly to analogies that make comparisons between characteristics or relations of two or more things, the analogy of terms is comparing the meaning of a single term as it might be used in various

contexts. It is important to note that there are implicit systems suggested by the analogy of terms, although the systems are never explicitly compared as being similar.

| <b>Table 1. Historic Conceptualizations of Analogy</b> |  |   |   |
|--|--|---|---|
|  | <b>Structures under Consideration</b>                                  | <b>Purpose of Category</b>  | <b>Example</b>  |
| <b>Analogy of Things</b>                               | Objects, events, or systems  | Note similarities between objects, events, or systems.              | “An atom is like our solar system.”   |
| <b>Analogy of Terms</b>                                | Linguistic or conversational analogies. Meaning of terms is the focus. | Describe analogies between meanings of terms in different contexts. | The analogy between the word <i>save</i> in the phrases <i>save money</i> and <i>save time</i> .    |
| <b>Analogical Argument</b>                             | Arguments that utilize analogical reasoning.                           | Analogy as inference. Used to create knowledge.                     | Describing heat transfer from a warm house in winter by arguing that <i>heat flows like water</i> . |

Finally, the category of analogical argument focuses on the actual process of reasoning by analogy rather than characterizing a type of analogy. Analogical arguments often involve conjecturing about similarities between two structures where there is information that is known about a given structure and information that is wished to be known about another structure. Gentner (1983) provides an example of an analogical argument based on the analogy *heat is like water* in the context of explaining heat transfer from a warm house in cold weather. An analogy is then introduced wherein a house and its roof is like a container with a lid. The fact that puncturing the lid of the container can cause water to leak out is then used to further the analogy and establish a new conjecture: heat may “leak” from a house if there is a “puncture” in the roof.

A summary of the differences among the categories of classical analogies can be found in Table 1. In this study, I primarily seek to study the analogy of things and analogical arguments in the context of abstract algebra. However, I make use of modern theories to operationalize analogy in this study. I discuss some aspects of modern conceptualizations of analogy in the next section.

**Modern Conceptualizations of Analogy.** The brief historical account of analogy makes it clear that observing similarities between unlike things is a generic way to view what is meant by analogy. Moving beyond this generic perspective, modern conceptualizations of analogy have made large strides in creating more precise notions of the meaning of analogy while also expanding the contexts in which they may apply. In the midst of the cognitive revolution, there was a surge in representing analogies as models of mental constructs and mappings between those constructs (e.g. Gentner, 1983; Holyoak & Thagard, 1989). I discuss this perspective on analogy situated within cognitive science in this section.

I also briefly note here that a popular way of describing analogies has been through the theoretical lens of cognitive transfer. According to Day and Goldstone (2012) a characteristic trait of this traditional perspective on transfer is that knowledge is considered to be “represented in terms of systems of discrete symbols, each of which corresponds to a meaningful concept,” (p. 154) and that transfer is viewed as “the recruitment of previously known, structured symbolic representations in the service of understanding and making inferences about new, structurally similar cases.” (p. 154) However, I do not focus this section on transfer and instead focus primarily on the theory of mapping as a way of conceptualizing analogy. This choice is made from a desire to describe analogy and analogical reasoning as being their own constructs rather than situating them within broader theories of transfer.

***Aspects of Mapping Theories.*** Modern conceptualizations of analogy have largely focused on analogical mapping in the context of problem solving (Carbonell, 1983; Greer & Harel, 1998), representations (Reed, 2012), explanation (Gentner, 1983;

Holyoak, Gentner, & Kokinov, 2001), and theory formation in science (Hesse, 2000). In an effort to construct a model of the relational nature of analogy, several theories of analogy have adopted a representation of analogy through some model incorporating mappings. In general, these models include a form of matching between one structure to another.

In her seminal paper on analogical reasoning, Gentner (1983) introduced one of the most prominent theories of analogy to incorporate mapping, the Structure-Mapping Theory (SMT). Gentner asserts that a theory of analogy must describe how the meaning of an analogy is derived from the meaning of its parts. Within the SMT framework, analogies are created by comparing across structures that are dissimilar in some respect. Holyoak and Thagard (1989) also make use of mapping to represent analogies and analogical arguments in their contribution of a theoretical framework for analogical mapping based on constraints. A commonality among these approaches using mappings is the recognition of a *source* and a *target* when describing the structures to be compared. During an episode of analogical reasoning, the source refers to the structure from which an analogy originates, while the target refers to the structure that is being mapped onto. The language of *domains* has been widely adopted to refer to structures under consideration. According to Gentner (1983), domains are psychologically viewed as systems of objects, object-attributes and relations between objects, all of which together constitute a structure in the mind. Domains consist of not only the objects under consideration when forming an analogy, but also the underlying relational properties that the object may possess. Domains can be representative of various levels of structure, and domains can be imbedded within other domains. For example, one can consider the

domain of objects such as groups, but one can also consider the domain of group theory as a whole. Extending further, one can consider the domain of abstract algebra, the domain of advanced mathematics, the domain of mathematics, and so on.

| <b>Table 2. Summary of Themes in Modern Analogy Theories</b> |  |  |   |
|--|--|--|---|
|  | <b>Description</b>   | <b>Purpose</b>   | <b>Example</b>  |
| <b>Domains</b>   | Refers to the objects/structures under consideration.                        | Constitutes the system of nodes, attributes, and relations to be mapped. | The domain of ring theory, consisting of objects such as rings, subrings, ring isomorphisms, etc.                                       |
| <b>Mappings</b>  | The formation or recognition of similarities between domains.                | The underlying mechanism through which analogies are created.            | Mapping the property of subgroups being a set to the property of subrings being a set.  |
| <b>Constraints</b>   | Underlying restrictions that dictate or influence the creation of analogies. | Studied to determine the reasons for the creation of analogies.          | An individual proposes the existence of ring homomorphisms because of the important role that group homomorphisms play in group theory. |

In contrast to the work of Gentner, Holyoak and Thagard (1989) introduced a theory of analogical reasoning based on the idea that there are certain constraints that individuals adhere to when forming analogies and creating mappings between domains. Rather than create analogies that are dependent on structure-mapping as proposed by Gentner, Holyoak and Thagard argued that analogy creation is goal-driven. This theory is known as *multiconstraint theory*. A constraint is some criterion that individuals subconsciously attempt to meet when engaging in analogical reasoning. There are three broad classes of constraints when reasoning analogically: similarity, structural consistency, and pragmatic centrality. When forming an analogy, people are thought to only generate analogies when there is recognized similarity (similarity), and a need for the generation of such an analogy (pragmatic centrality). During the formation of an analogy, there it is argued that there is a tendency to construct, as best as possible, a structural isomorphism between the two domains (structural consistency). A summary of

the themes found in modern theories of analogy can be found in Table 2.

As I will discuss in Paper #2, I make heavy use of the language of domains and mappings in operationalizing analogy. However, I do not make assumptions that students adhere to any constraints when formulating analogies to avoid biasing my interpretations of an analogy.

***Processes of Reasoning Analogically.*** How analogies are initially formed is described as *analogical access* (Hummel & Holyoak, 1997). In general, similarity of concepts at any level of abstraction is thought to contribute to analogical thinking, particularly during initial access. However, it has been shown individuals may not necessarily attend to the analogous features of two situations unless explicitly told to search for an analogy. This was famously exhibited within a study by Gick and Holyoak (1980) in which participants were asked to generate solutions to a famous medical problem involving how to distribute focused radiation to destroy cancer cells after having been given a solution to a military problem involving how to appropriately distribute troops to take siege of a fort. Gick and Holyoak found that while there was a clear analogy between the two problems (from the expert's perspective), the participants of the study did not generate analogies between the two problems spontaneously.

The example described by Gick and Holyoak points to the potential for differences in the ways that novices and experts might reason by analogy. Attempts to describe a generic process of analogical reasoning have been made in the modern conceptualizations of analogy through mapping. Here I present two conceptions of the steps involved in the process of analogical reasoning.

Keane, Ledgeway, and Duff (1994) identified five stages involved with analogical

reasoning: representation, retrieval, mapping, adaptation, and induction. Similarly, in the language of mapping theory, Holyoak and Thagard (1989) characterized four steps involved with analogical reasoning: selecting an appropriate source, generating a mapping between the source and target, generating analogical inferences, and subsequent learning and understanding of the target. Keane et al.'s conception of representation and retrieval maps onto the first of Holyoak and Thagard's four steps. In particular, representation refers to the reorganization of information about a domain in such a way that an individual can make sense of the domain, while retrieval refers to the act of searching for and retrieving information about an appropriate source domain from which the analogy to the target domain can be established. Mapping, being the ubiquitous step in modern theories of analogy, is found in both of Keane et al.'s and Holyoak and Thagard's framework. Adaptation relates to the step of generating analogical inferences. These steps refer to the necessity of conjecture and refinement during the process of analogical reasoning, as it is not always possible for analogies to be immediately recognizable. Finally, the inductive stage maps onto the final step of developing an understanding of the target, in which new knowledge about the target domain based on the source domain is created.

While the processes of reasoning by analogy described above are reasonable summaries of a generic process, they may not fully capture the range of analogical activities exhibited by students in mathematics. For example, both processes refer to some variety of adaptation appearing during analogical reasoning. However, as I will discuss throughout the proposed papers, students may not choose to adapt at all. In addition, these processes do not expound upon *how* each stage of the process is

performed. I attend to this point in Paper #2. In the next section, I discuss research on analogy specific to mathematics education.

### **Analogy in Mathematics Education**

The literature in mathematics education pertaining to analogy has been largely dominated by investigating how children reason by analogy (Alexander, White, & Daugherty, 1997; English, 2004; Goswami, 2013). In this section, I present the current state of literature on analogy in mathematics education. I divide this section based on three prominent ways of reasoning by analogy identified by English (2004): problem analogy, pedagogical analogy and classical analogy.

#### **Problem Analogies**

One approach to the study of analogy in mathematics education has been through the investigation of students' analogical reasoning between similar problem types.

Carbonell (1983) defines problem solving by analogy as follows:

Analogical problem solving consists of transferring knowledge from past problem-solving episodes to new problems that share significant aspects with corresponding past experience -- and using the transferred knowledge to construct solutions to the new problems. (p. 3)

In general, problem analogies are exclusively concerned with the creation of isomorphisms between problem types (Greer & Harel, 1998). Reed (2012) expanded on the notion of mapping across problem types by introducing a taxonomy of types of mapping and structures to map between. This taxonomy acknowledged the existence of one-to-many and partial mappings in addition to isomorphic mappings, and acknowledged the existence of mapping between situations, representations, solutions, and sociocultural contexts. English (1998) investigated the ways in which children would reason analogically between problems with similar structures, one of which was less

complex than the other. She found that the children in the study required assistance in applying analogical reasoning to problems that were more complex in nature, although they were able to exhibit analogical reasoning in everyday contexts.

Problem analogies have been implicitly studied in the literature involving the generation of connections between similar problems. For example, Lockwood (2011) investigated ways in which post-secondary students generated connections while solving counting problems. She presented a case study in which she examined student-generated connections among counting problems and addressed possible implications of such connections for students and for researchers. In this way, problem analogies have been considered as a useful tool for investigating the ways that students call upon previous knowledge when encountering new problems or situations.

### **Pedagogical Analogies**

The second type of analogy that has received attention in the literature are pedagogical analogies. Pedagogical analogies are those that appear in the context of instruction as an aid in explaining a particular concept. The usual goal of the analogy creation is to extend knowledge of a domain that is easily accessible (relative to the learner) to a domain that is not understood as well, namely to reduce the abstraction of a concept by creating an analogy to a more concrete setting. Several examples of pedagogical analogy exist in the realm of science education as a result (Amin, 2015; Close & Scherr, 2015; Jeppsson, Haglund, & Amin, 2015; Niebert & Gropengiesser, 2015).

Pedagogical analogies move away from examining specific problems or problem types and look toward the potential for benefiting instructional contexts, usually to aid in

explaining a particular concept. Richland et al. (2004) analyzed data from 8th grade classrooms drawn from the Third International Mathematics and Science Study (Stigler, Gonzales, & Kawanaka, 1999) and found that teachers developed the majority of analogies within those classes. Pedagogical analogies can be found in a variety of contexts. For example, Dawkins and Roh (2016) outline three analogies used by an instructor to assist students with understanding concepts in real analysis, such as uniqueness being similar to finding a white tiger in a forest. Peled (2007) discusses tasks designed to elicit analogical thinking for undergraduate pre-service teachers in order to assist them with understanding cognitive processes involved with learning mathematics. These examples are claimed to be effective tools for teaching in their respective contexts. However, there are mixed reports on the effectiveness of pedagogical analogies in the classroom. Specifically, a potential danger of pedagogical analogies is that students may not be able to fully understand the analogy generated by the teacher because the student is not the one creating the analogy (Cobb, Yackel & Wood 1992). The teacher, with their greater grasp of the content knowledge, may be able to appreciate the analogy they generate, but the message may not be appropriately conveyed to students. For example, Greer and Harel (1998) found that students did not always understand the common analogy between physical manipulatives and abstract arithmetic. Although Thompson (1994) argued that the concretization of mathematical concepts (such as physical manipulatives for arithmetic) can be an effective aid in teaching, he noted that it should only be done after carefully considering what it is that students are meant to gain from it and utilizing the concrete representations in a well thought out manner. Thus, it may be that the negative reports on the use of pedagogical analogies is tied to a lack of

understanding of how to leverage them appropriately.

### Classical Analogies

Classical analogies in mathematics education typically take on a form of reasoning as one would reason with proportions. That is, the analogies have the form  $A:B::C:D$ , read as “A is to B as C is to D”. An example in advanced mathematics is created by allowing A and C to be normal subgroups and ideals respectively and allowing B and D to be kernels of group homomorphisms and kernels of ring homomorphisms respectively. A common classical analogy problem is to provide a partial analogy and ask for the completion of the analogy by determining the missing piece (Piaget & Cook, 1952).

| Table 3. Types of Analogy in Mathematics Education |   |   |
|--|---|---|
|  | Description   | Example   |
| <b>Problem Analogy</b>                             | Analogies formed between similar problem types.                 | Comparing the following problems:<br>(a) <i>Jim has 14 goldfish. Peter has 6 more goldfish than Jim. How many goldfish does Peter have?</i> (English, 1998)<br>(b) <i>The Year 4 and 5 classes at Mayfield School had a street stall. The Year 5 class raised \$652. The Year 4 class raised \$155 more than the Year 5 class. How much money did the Year 4 class raise?</i> (English, 1998) |
| <b>Pedagogical Analogy</b>                         | Analogies leveraged during instruction to aid in understanding. | Leveraging similarities between groups and rings when introducing the definition of rings during lecture.   |
| <b>Classical Analogy</b>                           | Analogies of the classic form “A is to B as C is to D”          | “Consider a tetrahedron is similar to a triangle. Conjecture a property of a tetrahedron analogous to the property of a triangle you already know. Explain your answer.” (Lee & Sriraman, 2011, p. 126)   |

Lee and Sriraman (2011) argued that classical analogies were too restrictive due to the nature of the sources and targets already being predetermined by an observer and proposed an expanded version of classical analogy called Open Classical Analogy (OCA). Within the OCA, analogies are not considered to be predetermined by an observer but are rather left open to the student for generating conjectures. The use of analogies in generating conjectures is discussed further in the following section. Unlike

problem and pedagogical analogies, classical analogies have not received as much attention in the literature in mathematics education. A summary of the different classifications of analogy in mathematics education can be found in Table 3.

### **Analogy as a Heuristic for Mathematical Discovery**

Analogical reasoning is recognized by psychologists and mathematics educators alike as a process that allows for the generation of conjectures and hypotheses about concepts or ideas that are initially unfamiliar. Spearman (1923) once argued that all intellectual acts involve some form of analogical reasoning. Although this extreme perspective has not become mainstream, analogy has long been regarded as an important tool in the process of discovering new mathematical concepts and ideas. Polya (1954) points out that analogy plays a role in all discoveries and that, in some cases, analogy plays such a significant role that it can be viewed as the originator of the discovery.

According to Holyoak and Hummel (2001):

Analogy provides an important example of what appears to be highly general cognitive mechanism that takes specific inputs from essentially any domain that can be represented in explicit propositional form, and operates on them to produce inferences specific to the target domain. (p. 162)

Cañadas, Deulofeu, Figueiras, Reid, and Yevdokimov (2007) argued that analogies played a central role on the development of conjecture in mathematics in the context of problem solving. These descriptions of analogy elucidate one perspective on the power that analogical reasoning can have in the formation of new inferences.

Individuals can spontaneously generate analogies that may instantly make intuitive sense (Kapon & diSessa, 2012). Other times, an analogy may be formed as the result of a long process of reasoning in which the analogy is purposefully constructed. The process through which analogy is purposefully considered for the formation of

inferences about some domain of interest has been termed Design-by-Analogy (DBA) (Singh & Casakin, 2018). In this section, I elaborate on the existing literature in which analogy is considered as the motivator for the creation of a new mathematical structure.

### **Analogy for the Invention of New Concepts in Math Education**

Although discovery by analogy has been identified as an important tool for discovering mathematics, there is very little research on the role that analogical reasoning can play in the invention of new concepts. Even in the context of DBA, it is typically assumed that the target domain is known to the learner and the goal is to reveal further information about the target domain based on what is already known about the domain. So far, the described literature has primarily focused on the generation of analogies in two situations: those in which the analogy is formed between an abstract concept and a concrete concept, and those in which the analogy is formed between two domains that already existed within the mind of the learner. The aim of this section is to briefly summarize the literature that deviates from these constraints.

**Pure Carryover.** Although outside of the mathematics education literature, the process of creating analogies between a known source domain and an unknown target domain has been considered within the SMT framework as pure carryover (Gentner, 1983). Pure carryover occurs when information from a source domain is extracted and transplanted into a target domain, usually with no consideration of whether it makes sense to do so. In the language of mapping, pure carryover occurs when knowledge in the source domain is mapped to the unknown or poorly understood target domain. Research on pure carryover itself is scarce and is usually seen in instances of students failing to produce analogies in “correct” ways (often seen in research adopting an expert-novice

paradigm). This was typically seen when students failed to adapt relations in the source domain to relations in the target once the mapping was made (Novick & Holyoak, 1991). For example, a student may establish a subring test to be the exact same as the subgroup test, making no modifications to account for the new setting of ring theory.

**Conjecturing by Analogy.** While pure carryover is concerned with establishing inferences in a given target by means of directly leveraging knowledge of the source, conjecturing by analogy opens the possibility for a student to select the target themselves. In their paper, Lee and Sriraman (2011) acknowledge analogies for purposes of invention and attempt to reformulate classical analogy in such a way as to capture those instances of analogical reasoning that can result in conjecturing about new mathematical objects. This reconceptualization of classical analogy is referred to as Open Classical Analogy (OCA). In the classical analogy framework, analogies are described as making connections between object X with property A and object Y with property B. However, as Lee and Sriraman point out, the property A and the object Y are each known to the individual from the beginning, leaving no possibility for situations where the target object may be unknown to the learner from the outset.

To explore the notion of OCA, Lee and Sriraman conducted a 3-hour lesson in which 8th grade students were asked to engage with OCA problem types in the context of geometry. Of the students in the class, 3 of them participated in 1-hour clinical interviews before and after the lesson. An example of the kind of question asked was as follows: “Three median lines of a triangle meet in a single point. Select other lines and conjecture a property that is analogous to the given property. Explain your answer.” (Lee & Sriraman, 2011, p.126). Analysis of the 3 students’ activity revealed that students

attended to surface similarities when searching for a target domain while attention to relational similarities led to conjecturing about the target domain.

**Creating Structure by Analogy.** Finally, Stehlíková and Jirotková (2002) and Hejný (2002) recognized the possibility of constructing mathematical structures through analogical reasoning, with Hejný describing it as “transmission of an already known structure to a new context.” (p. 23) In their work, structure creation is theoretically described with a model of Internal Mathematical Structure (IMS), a representation of mathematical structures that are within an individual’s mind. In particular, Hejny (2003) asserts that knowledge is gained by connecting experiences that are disconnected until a new experience causes a sudden recognition of connections between the new and previous experiences. The connections formed between all of these experiences result in an intricate network of knowledge about a structure. Hejný claims that the IMS is the key to organizing these networks and binding them together.

| <b>Table 4. Summary of Research on Analogy as a Tool for Invention</b> |  |   |
|--|--|---|
|  | <b>Description</b>   | <b>Example</b>  |
| <b>Pure Carryover</b>  | The process of carrying over knowledge to a target that is virtually unknown by the student. | Carrying over the idea of group homomorphism to rings to create ring homomorphisms.   |
| <b>Conjecturing by Analogy</b>   | Analogies in which the target domain is open to choice by the student.                       | “Three median lines of a triangle meet in a single point. Select other lines and conjecture a property that is analogous to the given property. Explain your answer.” (Lee & Sriraman, 2011, p.126) |
| <b>Creating Structure by Analogy</b>                                   | Building a structure by comparing two structures by analogy.                                 | Conjecturing about properties of the structure of quotient rings using knowledge of quotient groups.  |

The theory of IMSs has produced some empirical results about analogical structure creation. Stehlíková and Jirotková (2002) investigated the processes of building IMSs in the specific context of reasoning analogically between two arithmetic structures. They claimed that “when building a new structure as an analogy to an existing structure,

the new structure is usually richer in concepts and ideas and enriches the original structure.” (p. 105) Stehlíková and Jirotková described five phenomena that they claimed were specific to the building of a structure as an analogy: regularities, anomalies, broadening intuition, obstacles, and the development of new strategies.

Although analogy was not an explicit motivator for the study, Hausberger (2017) presented a classroom experiment in which knowledge of group homomorphisms was leveraged for teaching ring homomorphisms. This study was an attempt to reveal how students might better understand one structure on the basis of another. Thus, this study is implicitly related to processes of creating or understanding structure through analogical reasoning. However, because the students possessed some knowledge of ring homomorphisms beforehand, the structures were not created on the basis of group homomorphisms. According to Kvasz (2005), whether it is possible for an individual to spontaneously construct one structure on the basis of another remains an open question for experimental research. A table summarizing the review of research on how analogy has been used to consider the invention of new concepts can be found in Table 4.

There has been some pushback against using analogy as a heuristic for discovery in mathematics education. Cobb, Yackel, and Wood (1992) argued that a mapping between two domains can be easily explained by an instructor because they have already developed a rather sophisticated understanding of the concepts involved. Thus, students who have not developed a sophisticated understanding may not find mappings very useful, meaning that the use of analogies may be difficult for students to incorporate when learning new concepts. Greer and Harel (1998) acknowledged that an important implication of this argument is that analogy should only be used in order to reinforce

existing conceptions rather than attempt to build new ones. However, it should be noted that there is very little research on the role that analogy can play in the formation of new mathematical concepts. Although the five phenomena described by Stehlíková and Jirotková (2002) is a foundation for describing how students might approach the invention of a new mathematical structure through analogy, there is still much to be discovered on how students engage in these activities and the nature of other types of phenomena that could exist. In particular, the analogical activity specific to students' analogical reasoning has not been well developed. For example, in what ways do students leverage regularities and anomalies when generating structures? I attend to questions such as these within Paper #1 and Paper #2.

### **III. OVERVIEW OF METHODS**

This interpretive study incorporated qualitative methods of research related to grounded theory (Corbin & Strauss, 2014). In this section, I provide an overview of the methods used to answer the guiding questions posed at the beginning. I first describe a pilot study that investigated the ways in which students generated connections between topics in group theory and ring theory and explain how the current study was informed by the pilot. I then describe the recruitment of participants, design of tasks, and analysis of data for the current study.

#### **Brief Overview of the Pilot Study**

In the summer of 2018, I initiated a pilot study with the intention of exploring how students generated connections between several structures in group theory and ring theory. At the outset of the pilot, observation of the phenomenon of analogical reasoning was yet to be made an explicit goal of the study. Instead, the goal of this pilot study was to: (1) test the viability of the hypothesis that connections between such structures were indeed accessible to students, and that knowledge of group theory could provide a suitable foundation for beginning the study of ring theory; and (2) test out tasks designed for the purpose of eliciting connections between structures in group theory and ring theory. Two students were recruited for the pilot: an undergraduate majoring in mathematics, and a graduate student pursuing a PhD in mathematics education.

The first student was administered a set of tasks over the course of five interviews with three different goals: (a) an interview designed to assess content knowledge of group theory, (b) an interview introducing the definition of ring, and (c) three interviews focused on conjecturing definitions for structures in ring theory. The general structure of

this interview process was extended into the interviews with the second pilot student, and the interviews collected for the dissertation as a whole.

Informed by the pilot interview process, modifications were made to these original pilot tasks and new tasks were introduced to focus on analogical reasoning. First, the analogizing task (described below) was created and was designed to allow for students to have flexibility in naming the structure. In contrast, the original tasks provided names to the students (e.g., subring) when asking students to define the structure. Second, the pilot process revealed that students might need time to engage with the new concepts in ring theory in order to more effectively engage with the construction of new structures. Thus, each interview related to structures in ring theory contained tasks that were meant to give the student an opportunity to work with the new structure (such as checking properties of ring) without necessarily having to reason by analogy. In the following sections, I describe the final task design used for data collection.

### **Participants and Setting**

In order to investigate students' analogical reasoning in mathematics, I acquired rosters from recently-taught introductory undergraduate abstract algebra courses at a large public 4-year university and emailed an invitation to students within the rosters. From a pool of approximately 40 students total, four students accepted the invitation to be interviewed. Three of the recruited students were undergraduate mathematics majors (1 woman and 2 men), while one was a male graduate student in mathematics education. The graduate student had taken the undergraduate abstract algebra course as a leveling course since he had not needed any abstract algebra for his particular undergraduate

degree in applied mathematics. With each participant, I conducted five 60-90-minute-long semi-structured interviews designed to elicit analogical reasoning.

The first three students were interviewed in person and tasks were administered on paper. One camera was positioned to record the students' writing and a lapel microphone was provided to record audio synchronously with the video recording. Due to complications with COVID-19, the fourth student was interviewed using a video-conferencing tool for all five interviews. For this student, tasks and questions were administered both verbally and through a chat interface within the tool, while the student wrote and recorded their work with the use of an electronic tablet and writing instrument. The tablet screen was shared through the tool and was visible to myself during the whole interview. A real-time recording of the work was collected from the student's tablet, while synchronous audio was recorded from the video-conference tool.

### **Interview Tasks**

The context of abstract algebra was chosen because of the existence of several naturally occurring structural similarities between group theory and ring theory: subgroups are similar to subrings, group homomorphisms are similar to ring homomorphisms, and normal subgroups are similar to ideals. In this section, I outline the design of the tasks and the interviews utilized in this study.

### **Assessment of Knowledge of Group Theory**

The initial interview helped to assess the participant's content knowledge of abstract algebra before beginning to explore topics in ring theory. The purpose of this initial interview was to determine what students recalled about groups, subgroups, group homomorphisms, and quotient groups, but also to ascertain what students knew about

ring theory, if anything, before further interviews were conducted. The students were found to have varying degrees of knowledge of rings from their previous classes, but none of the students had any significant knowledge of topics beyond basic structures of ring, integral domain, and field. After the final interview, one student revealed that they had indeed seen the definition of ideal in their previous course. However, they did not recall any information about the definition of ideal, and did not make any use of it during the interview process.

Tasks in this initial interview consisted of asking students to provide definitions of various structures in group theory: group, subgroup, group homomorphism, quotient group, and normal subgroup. In addition, along with each definition were prompts to provide an example of each structure and check given examples. An example of the interview protocol for this initial interview can be found in Appendix A.

### **Introducing the Definition of Ring**

The second interview provided participants with the definition of ring and various tasks designed to acclimate the student to working with rings. These tasks focused primarily on getting student acquainted with basic properties of rings to prepare them for establishing new structures by analogy in the later interviews.

First, the participant was provided with a definition of ring and asked to talk-aloud as they studied the definition. During this portion, it was common for the participant to immediately begin making comparisons between the structure of group and ring. Next, the participant was asked to elaborate on similarities and differences they noted during the talk-aloud exploration of the definition, as well as if there were any other similarities or differences worth mentioning. The remainder of the ring interview

consisted of example generation, example checking, as well as exposure to brief proofs/computations involving the new setting of ring theory. An example of the interview protocol for the ring interview can be found in Appendix B.

### **Constructing Mathematical Structures by Analogy**

The three subsequent interviews constituted the heart of the data collection related to analogical reasoning: each interview focused on reconstructing one of subrings, ring homomorphisms, or quotient rings by analogy with a structure in group theory. The focal task of each of these interviews was the *analogizing task*. These are tasks of the form: Make a conjecture for the definition of a structure in ring theory analogous to  $X$  in group theory. In this research,  $X$  was taken to be one of subgroup, homomorphism, or quotient group. Further tasks were constructed around three basic types: (1) elicitation of comparisons between structures when spontaneous comparisons tapered off (i.e., “In what other ways are the structures of group and ring the same?”), (2) example generation and checking (i.e., “give an example of a subring.”), and (3) proof-writing (i.e., “Is the homomorphic image of a commutative ring commutative? Does this statement compare to any theorem/proof you know from group theory?”).

As the interviews progressed, I allowed the students to refer back to their work from previous rounds of interviews. This not only gave the students an opportunity to recall information if they were stuck, but also allowed me the opportunity to see exactly what information they wanted to refer back to while they reasoned by analogy. An

example of the interview protocol for the interviews surrounding the construction of a structure by analogy can be seen in Appendix C.

### **Data Analysis**

In order to analyze the data, I made use of techniques outlined by Corbin and Strauss (2014) and the constant comparative method. First, immediate reflections were written after each interview was completed. This process of reflecting allowed me to record my thoughts about what I witnessed during the interview. After an interview was completed, the audio was sent for transcription. Beginning in order of collection with the five interviews from each of the first and second student, initial coding consisted of searching for evidence of analogical activity as defined by the Structure-Mapping Theory (SMT) framework. That is, I searched for any evidence that the student was making comparisons and mapping across domains. This coding produced a rough categorization of dimensions of analogical activity that expanded beyond the basic tenets of mapping and similarity: (1) mapping and non-mapping activity, (2) attending to similarity and differences between domains, and (3) foregrounding a domain. These are discussed in detail in Paper #2.

#### **Unit of Analysis: Instances**

The unit of analysis was an *instance* of analogizing. Instances comprised a student's analogy over the course of an interview with the analogous object under consideration depending upon the interview. While smaller analogies may be contained within a single instance (i.e., observing a similarity between binary operations in groups and rings), the analogies investigated in this study were the ring theory analogues to groups, subgroups, group homomorphisms, and quotient groups. Instances were typically

1-3 sentences in length and were identified by two criteria: presence of analogical reasoning and shifts in mathematical focus. Presence of analogical reasoning was determined by the students' language contextualized within the broader transcript indicating that a comparison was presently being made, indicating that a comparison was going to be made, or indicating a discussion of a newly formed structure just after completing a comparison. Here, comparisons were identified by the student making connections between content in the source and target domain. Shifts in mathematical focus were determined by the mathematical content being attended to. For example, consider the following excerpt:

So, now we're going to call this normal subring. We give this one a name. First condition is that  $S$  is a subring of  $R$ . Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it.

This brief excerpt was subdivided into three instances:

- "So, now we're going to call this normal subring. We give this one a name."
- "First condition is that  $S$  is a subring of  $R$ ."
- "Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it."

Within all three instances, evidence of analogical reasoning was determined by language such as "we give *this one* a name", "*conditions*", and "*just to copy it*." In the first instance, the student is attending to *naming conventions* as the mathematical content. In the second and third instance, the student is attending to definitional properties of normal subrings. I discuss the list of mathematical content codes identified in my analysis in Paper #2.

I identified instances across all transcripts. To each instance, I initially assigned codes of whether there was mapping or non-mapping activity, whether the source or

target was being foregrounded, and whether the focus of attention was on similarity or difference. As more data was collected, I engaged in multiple rounds of axial coding and over time the coding shifted away from assigning the broader codes to assigning codes explicating specific analogical activity. The final round of coding included assigning codes to identify the analogical activity involved, and the mathematical content under consideration. For example, consider the instances from above:

- “So, now we're going to call this normal subring. We give this one a name.”
- “First condition is that  $S$  is a subring of  $R$ .”
- “Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it.”

Rather than assign the group of codes of (a) mapping, (b) attending to similarity, and (c) foregrounding source domain to each of these instances, I assigned codes that described the students’ analogical activity. In this case, all instances were examples of exporting. The codes assigned to the instances above were *exporting naming convention* from the source of normal subgroups to the target of normal subrings for the first instance and *exporting definitional property* from the source of normal subgroups to the target of normal subrings for the second and third instance. In order to make this decision, I attended to both the language within the segments, as well as the physical work of the students as they made their connections. In particular, for this trio of instances, the student had their definition of normal subgroup at their side and was glancing at this definition while generating their definition of normal subring. In addition, I noted the students’ use of language such as “just to copy it”, as this signified that he was making decisions based solely on his older definition without thinking about what might change.

Along with the shift to assigning the activity code of exporting, the previous

codes of mapping, attending to similarity, and foregrounding the source became known as the dimensions of the activity of exporting. I explicate further analogical activities identified through my analysis in Paper #2 and share their dimensions.

### **Microanalysis**

As a part of ongoing analysis, I routinely engaged with three general techniques: microanalysis, diagramming, and memoing (Corbin & Strauss, 2014).

Microanalysis involves close examination of one (or several) segment(s) of data with the goal of deeply understanding the phenomenon found within. I intermittently performed microanalysis on segments when the nature of the analogical activity was especially unclear. Several novel interpretations of students' analogical activity were produced from these analyses.

To explicate the process of microanalysis, I briefly present an example here of what microanalysis typically entailed in this study. Consider the following excerpt of a transcript with a student, Trixie (a pseudonym). This excerpt was chosen for microanalysis in the study due to a lack of clarity of whether or not analogical reasoning was occurring:

- Trixie: Yeah, there's not a ring. Zero, one, negative one. That's 'cause it doesn't have closedness. So I guess really I'm showing a group, and then applying an extra property on it. Or an extra...
- Interviewer: Is that always the case?
- Trixie: Well, yeah. 'Cause you gotta check that  $R$  plus is a group on its own, right? 'Cause it has all the ... Like I said earlier, it has all the properties it needs, plus extras. Then you gotta check it on  $R$  star. And it could, just minus the inverses. And then plus the extra fun properties of associativity.
- Interviewer: Okay. This is making sense to me how you write it here. Are you thinking of this as a group too?
- Trixie: It's kind of like it wants to be a group.
- Interviewer: It wants to be a group?
- Trixie: But it isn't. I know I'm writing its notation like a group.
- Interviewer: It's fine. Yeah.

Trixie: This isn't a group.

In this segment, microanalysis initially involved a close inspection of the meanings of phrases such as: “I guess really I’m showing a group”, “It’s kind of like it wants to be a group”, and “This isn’t a group.” This analysis revealed that a student may view an analogically created structure as being part of the old structure, but still maintain separation from the old structure. Further analysis involved detailing *how* Trixie was leveraging the older structure to create the newer one: she was “applying an extra property on it” and noting that “R plus is a group on its own.” From this microanalysis, two concepts emerged: (1) the hypothesis that analogical activity could be present even when a clear analogy is not explicated, and (2) a precursor to the analogical activity of *extending*.

### **Diagramming**

In addition, diagramming was incorporated to aid in making sense of how the concepts arising from the coding fit together in new (or otherwise difficult to detect) ways. Specifically, diagrams consisted of: (a) constructing visual drawings and representations of my participants’ analogical activity, making heavy use of the naturally visual nature of mapping between domains; or (b) creating tables/diagrams that connected and categorized codes with one another. Appendix D displays an example of one such diagram along with a description of categories within the diagram. This example in the appendix is an early example of the coding categories I used in the initial coding process.

## **Memos**

Finally, I regularly wrote memos that explicated my thinking about concepts and generating new hypotheses. These memos were written either with specific thoughts in mind that required critical thinking and elaboration to make sense of, or written in a stream of consciousness when only vague thoughts were present. Memos varied in length from a paragraph, to several page long writing sessions. Memo writing was always performed after a round of microanalysis in order to collect and organize my thoughts, but was also performed whenever any idea I considered relevant would arise. Each memo was dated at the time of writing and served as a record of my thinking over time so that old observations could be rediscovered and potentially reintegrated into new coding schemes when appropriate. Oftentimes, the contents of these memos included ideas that were not immediately available for investigation or researchable, or were theoretical in nature. As such, the memos also serve as a record of my thinking of potential ideas and research avenues to pursue in more depth in the future. The following is a full example of one such memo in which the concept of a naming convention was being explicated:

**Naming Conventions:** The naming of an analogous structure appears to have some implications in the process of reasoning by analogy. First, the name can help draw immediate connections between structures. For example, Student A was able to immediately generate a new structure based on subgroup after having been given the name “subring” to work with. Beforehand, she was generating a structure that was a subgroup adorned with properties to make it a ring (without necessarily calling it a subring.) Student B would clearly make his own connections which was visible by his immediate naming of a structure based off

the source structure. In the context of abstract algebra, this naming convention idea is a bit “boring” because the true naming really is as simple as sub-(group/ring) or (group/ring) homomorphism, etc. However, I think this naming convention idea could be generalized to other cases where the naming is not set in stone beforehand. For example, a child reasoning by analogy between integers and blocks (let’s say blue and red blocks) might begin to call the red blocks: *negative blocks*. I believe this reveals aspects of the child’s reasoning between these domains.”

An example of a more elaborate memo can be found in Appendix E. The results microanalysis, diagramming, and memoing were regularly shared with colleagues through conversation and presentation to assist in ensuring the viability of my interpretations. This process of collecting and analyzing data continued until no new examples of analogical activity were found within new rounds of collected interview student with the last student, and the new data only provided further evidence of previously identified activities. At this point, saturation was achieved. The coding categories produced a classification of dimensions of analogical activities, mathematical content codes, and several analogical activities. I describe the coding categories and analogical activities in Paper #2.

## **IV. PAPER #1: ANALOGICAL REASONING AS A WAY OF THINKING IN ADVANCED MATHEMATICS**

### **Abstract**

Analogies and analogical reasoning have played significant roles in the development of modern mathematics. However, there has been critique of the use of analogies for the purpose of students learning new content because students may fail to appropriately recognize the analogical connections developed by instructors. I counter that students can productively reason by analogy to understand new mathematics when provided with settings to develop this way of thinking. In this paper, I use examples from the work of mathematicians' to argue for the important role of analogy for the purpose of mathematical discovery. I then provide an illustration of an undergraduate student engaged in similar productive analogical reasoning as they develop analogs between structures in group and ring theory. Through this process, the student showed increasing awareness of how and why they were engaging with such reasoning. This observation evidences the potential for students to reason by analogy for mathematical discovery. Further, the documented activity provides a blueprint for how analogies can be engendered and how analogical reasoning for discovery can be developed amongst students.

### **Introduction**

The following excerpt displays an undergraduate student, Nathan, working toward developing the structure of factor rings in abstract algebra:

So, what a factor group is, is this group where you get two different groups, and there's this... It's a set of these members where they undergo this operation. That's the condition for them. So, I think for... I don't know what we call it, haven't given it a name yet. I'm going to give it a name. Maybe we'll just call them factor

rings. Oops. Factor rings. I was thinking just to copy it, but it's missing an operation. And last time I did that, I did need both operations because otherwise, what's the difference? So, I say, well, maybe this factor ring is going to be the set of all the [cosets] under both operations.

How did Nathan come to this point in constructing the structure of factor ring? What precisely is Nathan doing, and what are his goals in establishing the structure? As I will discuss in this paper, Nathan is reasoning by analogy and had developed analogical reasoning as a way of thinking.

Similarity lies at the heart of analogical reasoning. There are multiple ways in which two mathematical entities can be construed as similar or different (Melhuish & Czocher, 2020). For example, metaphor is a kind of similarity that is often leveraged for understanding mathematical concepts with less abstraction (e.g., the collapsing metaphor for limits in calculus identified by Oehrtman [2009]). Pimm (1981) explicated one class of metaphor that was based in analogy: structural metaphors. In contrast, analogies may represent a broader kind of similarity. Consider the following quote by Polya:

The essential difference between analogy and other kinds of similarity lies, it seems to me, in the intentions of the thinker. Similar objects agree with each other in some aspect. If you intend to reduce the aspect in which they agree to definite concepts, you regard those similar objects as analogous. (1954, p. 13)

Thus, analogies are formed when similarities are observed between two objects and are discriminately established relative to what is important to the thinker. When solving problems in mathematics, Polya argues that analogies can play a vital role in developing new and creative solutions. Such claims echo the work of modern mathematicians, where analogical reasoning can be a valuable tool for developing new mathematics (e.g., Ouvrier-Buffet, 2015). In addition, both mathematical and non-mathematical analogies have been found to be used in curriculum (e.g., Harel, 1987). For example, in abstract algebra, textbook authors often present topics in ring theory as analogous to those in

group theory (e.g., Gallian, 2013).

While analogy has been documented to play a role in pedagogy, researchers have cast doubt on whether analogies can support student development of new concepts. Cobb, Yackel, and Wood (1992) argued that attempting to engender a mapping between two domains in students' reasoning may be inadequate for designing effective lessons because teachers have the advantage of already possessing a deep understanding of the content prior to creating the mapping. Naturally, students cannot be expected to understand deep similarities between two domains when investigating content for the first time and are perhaps less likely to understand or appreciate the effectiveness of an instructive analogy designed by the teacher. Building off the argument of Cobb et al., Greer and Harel (1998) suggest that analogies are best used for reinforcing pre-existing conceptions rather than constructing new ones.

These positions suggest a divide between the productive use of analogies for the purpose of doing mathematical research and analogies for the purpose of teaching and learning mathematics. On the one hand, analogy is a tool for creation and discovery of new mathematics. On the other hand, analogy is considered inadequate for establishing new content and only thought of as useful for reinforcement. In this paper, I seek to bridge the divide by proposing that the pedagogical use of analogies can be implemented productively for the purpose of constructing new knowledge given an appropriate set of circumstances. For instance, it is interesting to consider the accessibility of analogs in such a way that the analogy is not entirely predetermined by the instructor from the outset of the lesson. If students are given the opportunity to develop their own skill in reasoning by analogy, then they may be able to construct new knowledge. I contend that analogical

reasoning for mathematical discovery can be a *way of thinking* (Harel, 2008a), and this way of thinking can be fostered within student's reasoning.

I use the context of abstract algebra to explore this conjecture. I consider the questions: is it feasible that a student who possesses basic knowledge of only group theory may leverage pertinent aspects of that knowledge when coming to learn about rings? How might students come to recognize analogies such as those between group theory and ring theory on their own? Finally, how might students learn to productively engage with analogical reasoning to assist with the construction of new concepts in ring theory? In the sections that follow, I describe the conceptualization of analogy used in this paper. I then share examples of analogy and analogical reasoning being used for the purpose of mathematical discovery. Finally, I offer evidence of a student developing analogy for discovery as a way of thinking during their inquiry into the connections between group and ring theory. Although there are likely several strategies that students may develop when given the opportunity to purposefully reason by analogy, I focus on one strategy that I observed to naturally and spontaneously occur within my students' reasoning. In particular, I explore the subtle ways in which one student's reasoning evolved as they became more aware of their process of analogical reasoning.

### **Conceptualizing Analogy**

In order to situate a discussion of analogy and analogical reasoning in the context of mathematics, I briefly introduce the overall conceptualization of analogy used in this paper. Reasoning by analogy is a mental process that involves the comparison of two cognitive domains. Researchers tend to operationalize individuals' analogies via a process of organizational matching between one structure and another (e.g., Gentner,

1983). A commonality among modern approaches to describing analogies and analogical reasoning is to consider mappings between a *source* and a *target* domains. During an episode of analogical reasoning, the source refers to the structure from which an analogy originates while the target refers to the structure that is being mapped onto. One domain is typically already known to the individual while the other is a lesser-known domain that is under investigation. For example, the situation of electricity flowing in a circuit is commonly compared analogically to the situation of water flowing through a river. The power of the analogy, however, does not necessarily lie in any surface level comparisons between the two. Instead, the pertinent similarities are underlying relations identified between the two domains. In this way, analogies may be leveraged to make comparisons between two structures in order to reveal new information about one or the other.

The language of domains has been widely adopted to refer to the origin of structures under consideration. I use this language throughout this paper to explicitly identify which domains are being compared in the context of reasoning by analogy about mathematical structures. Domains can be representative of various levels of structure and can be imbedded within other domains. For example, one can consider the domain of structures such as groups, but one can also describe the domain of group theory as a whole. Extending further, one can consider the domain of abstract algebra, the domain of advanced mathematics, or the domain of mathematics.

### **Analogy for the Purpose of Mathematical Discovery**

Analogies and analogical reasoning have been prominent in the development of modern mathematics. In particular, analogies have played a significant role in the discovery of new mathematics. Recognizing the importance of analogy in mathematical

reasoning, Polya (1954) named reasoning by analogy along with generalizing and specializing as key ways of thinking. Polya claimed that these three forms of reasoning often co-occur when approaching problems in mathematics and are each ubiquitous to the development of mathematical thinking and reasoning. With respect to analogy, the key suggestion provided by Polya was to consistently make connections and comparisons of new problems back to those problems that were previously solved. With Polya's observation of the importance of analogizing in mathematics in mind, I share three cases to illustrate the important role of analogy in mathematical discovery.

### **Constructing New Information**

History was made when Descartes established a formal connection between concepts in algebra and geometry with the creation of analytic geometry. Discussed in Descartes' treatise, *La Géométrie* (1637), this revelation changed the way that mathematicians viewed both of algebra and geometry and ushered in a new paradigm for discovering relatively simple solutions to previously difficult and seemingly impossible problems. One underlying role in Descartes' development of this connection was reasoning by analogy (Crippa, 2017), specifically the act of recognizing and constructing parallels between the domains of algebra and geometry. Descartes created what was essentially a codex for translating between arithmetic and geometric relations. One of the most influential discoveries made by constructing this analogical relation between algebra and geometry was the Cartesian plane. The recognition of the deep analogies between algebra and geometry proved to be useful for establishing new mathematical information. For example, Lützen (2010) noted that, among several other accomplishments, Descartes "translated the problem of solution of a quadratic equation

into a construction using ruler and compass, and claimed that conversely no other equation could be solved by these means” (p. 27).

### **Developing New Techniques**

To illustrate the key role of analogy in the process of inductive mathematical reasoning, Polya (1954) shared an example from Euler’s work on finding the sum of reciprocals of squares:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

The core of Euler’s analogy was to adapt methods verified for the source domain of finite cases of algebraic equations to a target domain of a case that was infinite. Euler took advantage of the fact that an  $n^{\text{th}}$  degree equation with  $n$  roots could be decomposed into a product of linear factors and proceeded to decomposed an “infinite degree” equation with “infinite roots” into a product of infinite linear factors. By mapping a known strategy from a finite domain to an infinite one, Euler uncovered a new method for finding the value of certain infinite sums. Although it was quickly pointed out by Euler’s contemporaries that the analogically conceived conjecture could not immediately be seen to be true, it did produce the successful and now famous result: the sum of the reciprocals of squares is  $\pi^2/6$ .

### **Analogy in the Defining Process of Modern Mathematics Research**

Finally, I exemplify the use of analogy in formulating new mathematical concepts and definitions through the adaptation of previously known definitions. Ouvrier-Bufferet (2015) investigated the ways in which mathematicians define mathematical objects and developed a model of mathematicians’ process for generating definitions. Ouvrier-Bufferet found that analogies play a role at various stages of the defining model: during initial

exploration when generating a definition, and throughout defining. No matter the stage at which analogies are being used, the purpose remains consistent: new concepts and theories pertaining to the novel definition are generated through comparison to existing concepts and theories.

Using components of Ouvrier-Buffet's defining model Martín-Molina, González-Regaña & Gavilán-Izquierdo (2018) shared an example of a research mathematician describing her work in generating a definition by establishing an appropriately generalized version of an existing definition. In particular, the mathematician in their study uses analogies to compare between the source domain of already known definitions, and the target domain of generalized versions of the source definitions.

Martin-Molina et al. elaborate on this matter, stating:

Alice noticed the great similarities between the introduction of 'generalized complex space forms' and 'generalized Sasakian space forms'. She described how similarly the new definitions appeared, how they were checked to be valid and how new examples of the newly defined manifolds were found. (p. 1077)

Analogical reasoning was purposefully invoked by these mathematicians, specifically in the way that the research mathematician knowingly creates a new domain to which similarities and differences can be observed. A summary of the relevant aspects of analogy for the purpose of mathematical discovery across each of these three cases can be found in Table 5.

| Table 5. Analogy for Mathematical Discovery |  |  |
|---|--|--|
|   | Description  | Significance   |
| <b>Descartes' Use of Analogy</b>            | Formalization of the connection between algebraic and geometric concepts with the help of analogy.                     | Several new mathematical problems became accessible in the years that followed.                            |
| <b>Euler's Use of Analogy</b>               | Introduction of a technique to find the sum of the reciprocals of squares by analogizing with finite degree equations. | Euler produced a now famous mathematical result and a new technique for solving similar types of problems. |
| <b>Analogy in Modern Research</b>           | Creation of new definitions via analogy with previous definitions.   | Analogical reasoning is seen to be closely related to the establishment of new mathematical content.       |

### Examining a Student's Analogical Reasoning in Abstract Algebra

Given the role of analogy in mathematics history and modern mathematicians' work, it is reasonable to suggest that students could leverage similar strategies while learning new mathematical content. I situate this work in the context of ring theory because there are several natural analogies between the structures in group theory and ring theory. To name a few, groups are analogous to rings, subgroups are analogous to subrings, and normal subgroups are analogous to ideals. Because of these naturally occurring analogs, the context of abstract algebra is an accessible one for investigating how students might productively reason by analogy.

In this section, I present an example of an undergraduate student, Nathan (a pseudonym), engaging with analogical reasoning in this context. My main objective in presenting these examples is to illustrate my earlier claim: students can develop new mathematics through analogy. More substantially, this case illustrates analogical reasoning becoming a way of thinking as the student leveraged their previous analogical reasoning as they progressed through the tasks.

In order to situate the examples that follow, I briefly describe here the focal task for the interviews that I call the *analogizing task*. This task involved a providing a

statement of the form: Make a conjecture for a structure in ring theory analogous to \_\_\_\_\_ in group theory. The goal of this task was to provide students an opportunity to reason by analogy between structures in groups and potential ring structures. The task sought to allow students to purposefully reason by analogy while maintaining enough freedom for students to generate their own mathematics and generate spontaneous analogies as they thought about what the analogous ring structure would be. In the following sections, I display examples of Nathan reasoning by analogy and for each, I (1) describe Nathan's activity with respect to the algebra, (2) describe the analogical activity present in his reasoning, and (3) situate his analogical reasoning within known phenomenon found within research on analogy and analogical reasoning.

### **The Creation of an Analogy**

What are the first aspects that come to mind when beginning to compare two domains? It is natural that one would begin with what is available on the surface rather than immediately formulating deep rooted connections. From there, additional connections can be made. In this section, I briefly describe some aspects of Nathan's process in generating an initial analogy between subrings and subgroups.

**Analogical Access: A Name.** In the following excerpt, Nathan is presented the first analogizing task:

Interviewer: Okay. First question here is that I want you to make a conjecture for a structure in ring theory that is analogous to subgroups in group theory.

Nathan: So I'm just making it up? Like a subring? I guess we'll call it that.

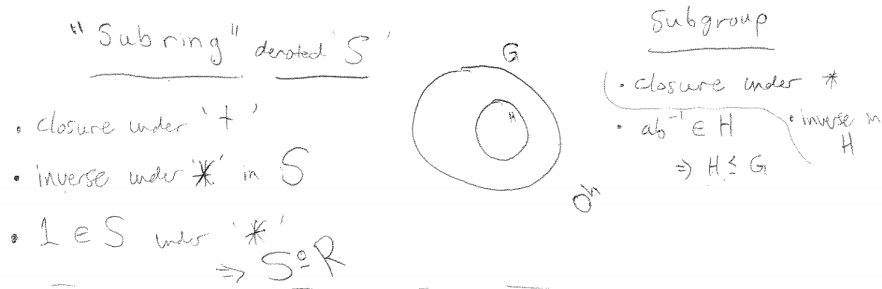
Nathan has begun the construction of what he has termed a "subring". Nathan begins with a simple naming of the new structure under consideration. Reasoning by analogy is

visible in his chosen name for the new structure wherein Nathan has conformed to the naming convention established by subgroups. While it may appear obvious or uninteresting to establish this name on the surface, the naming convention provides potential foreshadowing into what Nathan believes will characterize a structure in ring theory analogous to that of subgroups. His decision to name the new structure a “subring” indicates the first sign of a connection to the structure of subgroups in group theory.

The establishment of the new structure as being a “subring” is a marker for Nathan’s entry point into his reasoning about what a structure in ring theory analogous to subgroups should be. A common term for this initial step in the process of reasoning by analogy is *analogical access* (e.g., Hummel & Holyoak, 1997). In Nathan’s case, he was skeptical about what it was he was trying to accomplish (i.e., “I’m just making this up?”). However, by using the name of the structure as a point of access, Nathan was able to generate a definition for his new structure as discussed in the next section.

**Equipping the Analog with Properties.** After establishing the name of the structure, Nathan provided a description of the newly conceived structure as it would be in the context of ring theory. In the following example, Nathan is summarizing his process for generating properties of the structure he had created:

I just compared it to subgroup. We know that subgroup has a closure under its binary operation and it contains the inverse. So I said maybe it has closure under just one of them, one of the binary operations and then maybe the inverse is under the other operation. And then I said, well since ring had more properties than the group, then maybe a subring has more properties than the subgroup. Then I just added a unity or what do you call it, identity, under one of the operations. Because it's a subring, maybe it doesn't exhibit all the properties, just like subgroup doesn't. So, I said maybe it just has one of them.



**Figure 1:** Nathan's Reasoning Between Subgroups and "Subrings"

Nathan's focus on describing the structure of subring was to make a direct comparison to the structure of subgroup and decide what aspects of the two would remain the same.

Nathan's goal in this excerpt was to generate properties that could then be used to define a subring. From this excerpt, I infer that Nathan was indeed making explicit connections to his domain knowledge of subgroups to assist in formulating a description of the structure in the domain of rings. In particular, Nathan heavily emphasizes the properties of subgroups that he was aware of and mapped them over to the context of subring with the assumption that it would possess similar properties. The initial written process of Nathan's reasoning between the structures of subgroups and subrings, in which he considered the properties needed for each structure, can be seen in Figure 5.

Nathan's analogical reasoning in this example relied heavily on assuming that certain properties of the subring structure would in fact remain consistent with what he knew about subgroups. For example, Nathan was initially considering the need for subrings to require a multiplicative inverse for all elements, despite the fact that multiplicative inverses are not guaranteed in rings. This analogical process is known as *carry over* (Gentner, 1983), wherein an individual maps aspects of the source domain over to the target domain without giving much justification for why the aspect should be mapped (other than that it is consistent with what they knew to be true about the source.)

In particular, Nathan does not appear to give much attention to his process of mapping aside from believing that certain necessary properties, decided as such because of what he knows to be true about subgroups, are requirements for the new structure of subring. In Nathan's case, the initial unnecessary properties carried over for the definition of subring were often resolved through deeper inquiry into the structure. Furthermore, as the interviewer, I asked probing questions to assist Nathan in refining his selection of properties to map, especially by attending to differences between the structures of group and ring in order to facilitate changes in the structure of subring as compared to subgroup.

### **The Evolution of Nathan's Analogical Reasoning**

After creating the subring structure by analogy with subgroups, Nathan engaged with two more analogizing tasks (beginning with group homomorphisms and factor groups) in subsequent interviews. These tasks revealed progressive differences in how Nathan approached reasoning by analogy stemming from his previous experience with reasoning by analogy between subgroups and subrings.

**A More Pointed Inquiry.** When presented with the analogizing task for subgroups, Nathan initially responded with hesitation on how to proceed. After initial exposure to the analogizing task, however, Nathan was more purposeful in initiating the construction of structures by analogy in the interviews that followed. In the following excerpt, Nathan has recalled what he knows about group homomorphisms and is now beginning to reason about a structure he has termed a "*ring homomorphism*". Within the first analogizing task concerning subgroups, Nathan focused more narrowly on recreating properties of subgroups that would be similar for his newly created structure of subrings.

However, during the construction of ring homomorphisms, Nathan paused to consider the fact that subrings were different from subgroups and that differences were to be expected.

So, now let's talk about ring homomorphism. Homomorphism. I don't know, I feel like it would be the same thing. There exists, let's just say  $\psi$  or something from one ring to another ring such that  $\psi$  is one-to-one. Or, I don't know what you say, onto... The same thing kind of goes here. So, I don't know, I feel like it ... I don't know, I hate saying it's the same thing, because last time subring was different because the ring is different.

Nathan expressed concern with describing the newly conceived structure as “the same thing” as a group homomorphism. By invoking his experience with the first analogizing task, in which he found subrings to be different from subgroups because of the fact that rings are inherently different from groups, Nathan shifted his attention away from simply carrying over aspects from the source into the target. Nathan was more aware of differences when constructing an analog to group homomorphisms. As such, Nathan asserted a distinction from his thinking about group homomorphisms. Nathan’s awareness and anticipations of differences is also evident within the analogizing task for factor groups.

Factor rings. I was thinking just to copy it, but it's missing an operation. And last time I did that, I did need both operations because otherwise what's the difference? So... maybe this factor ring is going to be the set of all the  $Sa$  [cosets] under both operations.

In this example, Nathan briefly alluded to there being a purpose for considering more than just similarities when formulating his analogy when he asked rhetorically “otherwise, what’s the difference?” Initially, Nathan considered only the need for one operation as is the case for a factor group. However, his observation based on his previous analogical reasoning results in a new hypothesis in which he considers a factor ring as being a set of cosets with both of the operations required to form a ring.

**Abstracting Conditions.** In the following excerpt, Nathan has been given the analogizing task for factor (quotient) groups. His first instinct was to consider his strategies for reasoning by analogy in the context of subrings and ring homomorphisms. In the following excerpt, Nathan is reasoning about what constitutes a factor ring by analogy with factor groups while leveraging his previous reasoning about the analogizing task for group homomorphisms:

Oh, it's the fact that these [conditions] have to be true. That's what I was thinking. I was thinking there was multiple phis, but it's multiple conditions that have to be true. So, here, I don't know, it doesn't feel like we have conditions. I understand that this is a condition.  $G$  over  $H$  is a set of cosets under this operation. So,  $R$  over  $S$ , I'm guessing is going to be cosets. It's just confusing now. I want to just copy it, but what's the operation between here, and why is it only one operation?

Providing further evidence of an increasing awareness of how he wished to approach the process of analogical reasoning, Nathan developed a notion of “conditions” to which he wanted to attend to. These conditions were representative of specific properties of a given structure that could be abstracted between the source and target structures, such as a group homomorphism requiring the homomorphism property:

$$\varphi(a *_1 b) = \varphi(a) *_2 \varphi(b)$$

In particular, Nathan was purposefully attending to his reasoning by analogy to ensure that what resulted was more than just a comparison of surface level features. Maintaining consistency with his recognition that merely copying the structure directly from group theory was insufficient, Nathan identified aspects of group homomorphism that could be appropriately abstracted and translated to the context of ring homomorphisms. In the excerpt provided above, Nathan was beginning to recognize that the same could be true for the analogizing task for factor groups as well identified the notion of a factor ring as being a set of cosets as being one such condition to attend to. As discussed in the

previous excerpt, Nathan would eventually hypothesize that the factor ring structure would be defined with two operations to form a ring, whereas he initially was considering the need for only one operation as was the case for factor groups.

### **Discussion**

By investigating Nathan's activity while reasoning by analogy, we can see that it is indeed possible for students to engage in productive reasoning by analogy when learning about new mathematics. Analogical reasoning is a *way of thinking* that students can develop while investigating structures in abstract algebra. Specifically, Nathan spontaneously began to invoke his previous analogical reasoning strategies as he progressed through the interviews. The increasing awareness of his reasoning by analogy suggests that the process of analogizing is learnable and can be used to assist students to independently reason by analogy. Although Cobb et al. (1992) made an argument against the pedagogical use of analogies that are designed by the teacher beforehand, I argue that analogies may be pedagogically effective when students themselves are allowed the opportunity to produce the analogies in the moment and, perhaps more importantly, engage in analogical reasoning that is meaningful to them. In this way, the promotion of analogical reasoning in students' mathematical thinking can provide an effective way of leveraging previous knowledge to both initiate and extend inquiry into new topics. Much like the use of analogy as wielded by research mathematicians discussed by Ouvrier-Bufferet (2015), Nathan began to utilize analogy in a purposeful and productive manner. For example, Nathan searched for underlying properties in order to advance his understanding of the newly created structure. This process of searching for conditions suggests that Nathan's analogical reasoning had progressed from attempting to simply

carry over properties from the source domain into the target to identifying which aspects of the domains were meaningful enough to be mapped.

The focus of this paper has been on analogical reasoning in the context of advanced mathematics, specifically abstract algebra. While there is significant literature to be found detailing the process of analogical reasoning observed in children (e.g., Goswami, 2001), the focus is rarely on analogical reasoning for the purpose of discovery. I conclude this paper by arguing that productive and complex forms of analogical reasoning, such as those exhibited by mathematicians, can be found outside of advanced mathematics as well. Consider the following statement in which a young child, Lynn, is solving the problem “ $-8 - 3 = \square$ ” (Bishop, Lamb, Phillip, Whitacre, Schappelle & Lewis, 2014).

I'll just start counting by, start counting down at 8 because it's negative. Eight, 7 (holds up one finger), 6 (holds up two fingers), 5 (holds up three fingers). Negative 5. (Pause.) Wait, I think it's switched. I don't know. At first I was thinking since it was a minus [the subtraction symbol in  $-8 - 3$ ] so it would have to be a minus. But now I'm thinking since this one was a plus (she points to the previous problem  $-3 + 6 = \square$ ) and I had to do minusing, that this one [ $-8 - 3 = \square$ ] is plus on the negative numbers. (Lynn correctly counted up 6 units from -3 for the previous problem but because the absolute values of the numbers decreased, she said it was “like minusing” to her.) I want to change my answer and count up now. Eight, 9 (holds up one finger), 10 (holds up two fingers), 11 (holds up three fingers). Negative 11.

In this example, the child is mapping between the source domain of positive integers and the target domain of negative integers. Not only does Lynn reason about negative integers by analogy with positive integers, but she also reasons by analogy about the operations of addition and “minusing.” In particular, Lynn’s reasoning echoes the use of analogy employed by Euler when making a conjecture across domains to make a discovery: Lynn makes a conjecture about how to handle the case of subtraction from a negative number

by analogy with a previous case of addition to a negative number. Further inquiry into the process of analogical reasoning, as well as careful development of tasks that promote explicit analogical reasoning, such as variants of the analogizing task presented in this paper, could help to explicate what student analogical reasoning looks like across several content domains and grade bands in mathematics. By attending to the development of such tasks and promoting analogical reasoning as a way of thinking in mathematics, we can better equip students with reasoning skills in mathematics that prepare them to autonomously generate rich connections by analogy across a variety of content domains.

## **V. PAPER #2: THE ANALOGICAL REASONING IN MATHEMATICS**

### **FRAMEWORK**

#### **Abstract**

This paper establishes the Analogical Reasoning in Mathematics (ARM) framework for describing students' analogical activity in mathematics contexts. I first outline a definition of analogy and contrast it with the concept of metaphor. I then introduce ARM, which categorizes analogical reasoning activity that is unique to the context of doing mathematics and explicates features of analogical reasoning that are largely implicit in existing models. Constructed from an analysis of interviews with four students engaged with analogical tasks in abstract algebra, ARM identifies three dimensions of analogical activity: mapping/non-mapping across domains (MAD), attending to similarity and difference (SAD), and foregrounding a domain (FAD). Built upon these dimensions, analogical activities are identified and explicated for the purpose of analyzing student analogical reasoning. I provide examples of several of these activities in the context of abstract algebra.

#### **Introduction**

Reasoning by analogy offers students a way to develop creative insight into new problems and situations by leveraging their knowledge about previously known information. Analogical reasoning has even been conjectured to be the basis for all reasoning (Spearman, 1923). While this claim perhaps overgeneralizes the power of analogies, reasoning by analogy is indeed recognized as a valuable tool for the teaching and learning of mathematics (English, 2004), has ties to mathematical processes such as abstraction (English & Sherry, 1996), and inspires mathematics research ventures

(Ouvrier-Buffet, 2015). Several existing models of analogical reasoning are available (e.g., Gentner, 1983; Gelernter, 1985; Holyoak & Thagard, 1989), and these models provide useful ways to operationalize analogical reasoning in a variety of content areas. However, a description of how students reason by analogy in the context of mathematics has not been explicated in detail. In particular, existing models often do not explicitly account for the ways that students may reason by analogy that are unique to mathematics. Furthermore, existing models are often concerned with describing perfect representations of analogy generation and analogical reasoning; this need not be the case in exploratory or instructional contexts. Because of the importance of analogy for mathematical thinking and reasoning (Polya, 1954), there is a need to describe how students themselves reason by analogy. By doing so, we can begin to better understand how to promote productive analogical reasoning in mathematics classrooms.

While research in mathematics education has leveraged analogy as a tool for investigating student mathematical thinking, analogy is often conflated with a related concept: metaphor. The lack of a rigorous distinction between analogy and metaphor obfuscates the utility of each construct for research on student thinking and learning in mathematics. Although some scholars have contributed to the discourse establishing differences between the analogy and metaphor (e.g., Pimm, 1981), and there is theoretical literature outside of mathematics education that ponders the ways in which analogy and metaphor are the same and different (e.g., Bailer-Jones, 2002), a formal distinction between analogy and metaphor is not clear within mathematics education. Thus, in order to assist in describing students' processes of analogical reasoning, there is a need to clarify key differences between analogy and metaphor.

The goal of this paper is to establish a framework for interpreting how students reason by analogy in mathematics. I propose a theoretical framing for operationalizing and interpreting students' analogical reasoning, and argue for the differences between analogy and metaphor with this framing in mind. I then introduce the Analogical Reasoning in Mathematics (ARM) framework constructed from an analysis of students' reasoning by analogy in the context of abstract algebra, a content area rich in a variety of accessible analogies to be created and studied. Informed by literature within and without mathematics education, this framework establishes a tool for interpreting students' reasoning by analogy in mathematics. The following question guided this qualitative research:

RQ1. What is the nature of students' analogical reasoning as they create analogies between known mathematical objects and new mathematical objects in a different context?

RQ2. What are students' analogical activities as they reason by analogy about objects between two mathematics contexts?

The specific context of this study is research on students' analogical reasoning in abstract algebra. I chose this context in part due to the existence of several natural similarities between the topics of group theory and ring theory: several structures associated with rings can be thought of as direct analogues to structures associated with groups. As such, I seek to answer the following sub questions specific to the context of abstract algebra in order to gain insight into analogical reasoning more generally:

SQ1. What is the nature of students' analogical reasoning as they create analogies between groups and rings, subgroups and subrings, group homomorphisms and ring homomorphisms, and quotient groups and quotient rings?

SQ2. What are students' analogical activities as they reason by analogy about groups and rings, subgroups and subrings, group homomorphisms and ring homomorphisms, and quotient groups and quotient rings?

### **Theoretical Framing**

In this section, I describe my stance toward interpreting students' mathematical activity with respect to their reasoning by analogy and the underlying foundations of the Analogical Reasoning in Mathematics (ARM) framework.

#### **The Actor-Oriented Perspective**

I adopt the Actor-Oriented (AO) perspective (Lobato, 2012) to foreground students' own thinking and activity. Much of the literature on analogy has examined participants' processes of analogical reasoning based on whether they were able to recreate predetermined analogies (e.g. Gick & Holyoak, 1980). Because observers tend to be well-versed in the contexts which they are studying, observers may bring along with them a bias concerning the analogies expected to be generated by the participant. This is especially pertinent to structural analogies in abstract algebra, where several potential analogies are indeed predetermined to adhere to convention. It is often the case that if a participant does not generate the analogies expected by the expert, then the participant is thought to have failed in creating any analogy at all. The AO perspective mitigates this bias by ensuring that student activity is examined from the perspective of the individual. The AO perspective situates the role of the researcher as an interpreter of students'

activity while backgrounding expectations for what is thought to be correct. The activities enacted by the students while engaging with analogical reasoning are, within reason, interpreted from the perspective of the student. In this way, there are no “incorrect” activities or connections that may be generated by a participant while reasoning by analogy. This orientation is key to developing a framework for how students reason by analogy in mathematics rather than modeling a perfect representation of analogy and analogical reasoning.

### **Theoretical Foundations**

In order to operationalize analogy, I adopt Structure-Mapping Theory (SMT) (Gentner, 1983). Structure-Mapping Theory proposes a model for describing analogies and the process of reasoning by analogy through mental mappings across *domains*. ARM makes use of the following three features of SMT:

- Analogies compare between a source and target domain.
- Analogies consist of mappings between the source and target.
- Mappings consist of linking content between the source and target.

Not unlike the notion of concept image (Tall & Vinner, 1981), a domain is a collection of knowledge that one possesses about a given topic or idea. Domains may vary in size and scope depending on the context in which one is reasoning by analogy. For instance, one may hold the singular concept of a group (in group theory) as constituting a domain, or one may take as a domain the entirety of group theory, including anything relevant to groups within the domain (even concepts that they have only just discovered or are not well understood can reside within the domain.) To operationalize the act of creating an analogy, Structure-Mapping Theory introduced a formal description of the activity of

mapping. During the process of reasoning by analogy, an individual generates connections by mapping between one domain typically already known (the source domain) and another domain that is being investigated (the target domain.) When mapping between domains, students map some form of content between the domains. The contents of domains within the SMT are described as objects, attributes, and relations. However, as I will describe later, I do not use the content outlined by Gentner in this study and instead describe content that is specific to the practice of mathematics.

### **Analogy in Mathematics Education**

While Structure-Mapping Theory introduced several key characteristics for identifying and describing analogies, there is further need to distinguish between closely related types of similarity: metaphor and analogy. In this section, I further expand on the assumptions I make about analogy by contrasting with metaphor. I then discuss literature pertaining to analogy in mathematics education.

#### **Differentiating Analogy and Metaphor**

Whether a comparison is an analogy or a metaphor is not dependent upon the structure or the content of the comparison. Instead, I argue that it is dependent upon the individual's orientation toward the comparison. In this section, I make three claims: (1) a conscious separation of domains is vital to analogy; (2) analogies often involve bidirectional mappings whereas metaphors typically do not; and (3) analogies can involve comparisons between vaguely understood domains.

**Analogies Separate Domains.** To reason metaphorically is to assert that one entity is another in some sense. Of course, it is rarely the case that a metaphor establishes two domains as being entirely equivalent. Instead, an assertion of similarity is made

within some bounds of the domains; the bound of one domain is the bound of the compared domain. For instance, the basic metaphor for infinity is a conceptually finite process that “goes on and on” (Lakoff & Nunez, 2000), and function is traveling (Zandieh et al., 2017). Analogies do not make such extreme assertions of similarity and instead, analogically compared domains remain separate.

Let us now clarify this distinction between metaphor and analogy with an example that is common in topology: representations of abstract topological spaces drawn on a chalkboard. Because the space is represented on a flat surface, it is never truly representative of any space that is say, nonmetrizable. However, nonmetrizable spaces are often represented as pictures drawn on a flat surface. Metaphorical reasoning between the two constructs occurs in one of two ways: either (a) it is understood what the bounds of the metaphor are (that the chalkboard more accurately represents a subspace of  $\mathbb{R}^2$ , for example) and thus one can freely reason about the space while being wary of its limitations, or (b) the perceived bounds of the metaphor extend beyond what is intended, and key differences remain unnoticed. In the latter case, a student might implicitly believe that all topological spaces behave like subspaces of  $\mathbb{R}^2$ . Thus, metaphors may unintentionally pose threats to mathematical understanding (see Pimm, 1981; 1988).

On the other hand, to reason analogically between the representation on the board and the abstract notion of a topological space is to maintain awareness that there is a separation between them. The drawing on the board is never meant to represent an abstract space in its entirety. To be clear, an individual may not be fully aware of all the differences that are key to distinguishing the two in the intended way, nor is the individual required to fully explicate every difference between the two. Instead,

awareness of the fact that the constructs are different is what is key to thinking analogically.

**Analogies are Bidirectional.** The direction of mapping is a key defining feature of analogy. Consider the difference in meanings between the following metaphors: “some surgeons are butchers”, and “some butchers are surgeons” (Ortony, 1979). The first statement has a markedly different meaning than the second. The asymmetry of metaphors is found in the mathematics education literature, such as the metaphor cluster documented by Olsen and colleagues (2020): doing mathematics is a journey. The language surrounding this metaphor often involves references to learning mathematics as consisting of a series of landmarks or covering a certain amount of “ground.” What does the symmetric statement entail? Reversing the direction establishes the following: a journey is doing mathematics. Not only is the original meaning changed, but an unusual metaphor is produced: doing mathematics would be an esoteric way of understanding a journey.

In contrast, when an individual reasons by analogy, the comparison can typically be made in both directions: understanding of either domain can assist in making inferences in the other. Refer back to the example of the abstract topological space represented on a flat surface. When reasoning by analogy between the two domains, no extreme assertion of similarity is necessarily made (i.e., the domains are separated.) Instead, some effort is made to compare and contrast the two domains. For instance, one may reason analogically about the definition of limit of a sequence in the usual topology on  $\mathbb{R}^2$ , versus the definition of limit of a sequence in a general topological space being represented on the flat surface. If a sequence converges in  $\mathbb{R}^2$  (with the usual topology),

then the limit is unique. This need not be true in general: there are infinitely many points to which the sequence  $\{1/n\}$  converges within  $T$ , the finite complement topology on  $\mathbb{R}$ . A comparison can be achieved in either direction. It is just as viable to state that “limits in  $\mathbb{R}^2$  are like/unlike limits in  $T$ ” as it is to state that “limits in  $T$  are like/unlike limits in  $\mathbb{R}^2$ .”

**Analogies Can be Vague or Imprecise.** Metaphors are often interpreted as comparisons between the known and the unknown. Consider Zandieh et al.’s (2017) notion of traveling as a metaphor for function: students are perceived as having an understanding of the concept of “traveling” which they then use to gain an intuitive understanding of the abstract notion of “function”. Analogies are sometimes thought to only allow for comparisons between two already known domains. However, this does not allow for vague or imprecise analogies, which exist in the literature. A prominent example outside of mathematics education is known as pure carryover (Gentner, 1983) which involves mapping known content into a completely foreign domain. Within mathematics education literature, analogy has been leveraged for the purpose of creation or invention (e.g., Lee & Sriraman, 2011; Stehlíková and Jirotková, 2002), and mathematicians establish new lines of research into unknown topics by analogy (Ouvrier-Buffet, 2015). Thus, it is indeed possible to generate analogies in which one domain remains vague and imprecise. I present further evidence for this claim within the results of this paper: students will be shown to construct unknown (to them) objects in ring theory by analogy with known structures in group theory.

### **Mathematics Education Literature Pertaining to Analogy**

Having argued a distinction between analogy and metaphor, I now discuss existing literature related to analogy in mathematics education. I incorporate three

categories introduced by English (2004) to organize this section: problem analogy, pedagogical analogy, and classical analogy.

There has been great interest in investigating how people leverage their knowledge of a solution to one problem to determine a solution to another; these are known as problem analogies (Carbonell, 1983). In general, research on problem analogies is concerned with the creation of isomorphisms between problem types (e.g., Greer & Harel, 1998). English (1998) investigated the ways in which children would reason analogically between problems with similar structures, one of which was less complex than the other. She found that the children in the study required assistance in applying analogical reasoning to problems that were more complex in nature, although they were able to exhibit analogical reasoning in everyday contexts. Some literature has actively looked beyond the basic notion of structural isomorphism. Reed (2012) introduced a taxonomy of types of mapping and structures to map between. This taxonomy acknowledged the existence of one-to-many and partial mappings in addition to isomorphic mappings.

Pedagogical analogies move away from examining specific problems or problem types and look toward the potential for benefiting instructional contexts, usually to assist with explaining a particular concept. While pedagogical analogies may be leveraged by either students or the instructor, some evidence suggests that students are not typically the creator of such analogies (Richland et al., 2004). A common goal of pedagogical analogies is to help reduce the abstraction of a concept. For example, Dawkins and Roh (2016) outline three analogies used by an instructor to assist students with understanding concepts in real analysis, such as uniqueness being similar to finding a white tiger in a

forest. Peled (2007) discusses tasks designed to elicit analogical thinking for undergraduate pre-service teachers in order to assist them with understanding cognitive processes involved with learning mathematics. However, there are mixed reports on the effectiveness of pedagogical analogies in the classroom. One potential danger is that students may not fully understand the intended analogy (Cobb, Yackel & Wood 1992). For instance, Greer and Harel (1998) found that students may not grasp the analogy between physical manipulatives and abstract arithmetic.

Finally, classical analogies take on a simple form of proportional reasoning (e.g. Modestou & Gagatsis, 2010). For example, comparing normal subgroups and ideals respectively with kernels of group homomorphisms and kernels of ring homomorphisms respectively produces a classical analogy. Unlike problem analogies, classical analogies in educational contexts typically provide a partial analogy to a student (such as removing  $D$  from the analogy above) and then the students are asked to complete the analogy (Piaget & Cook, 1952). Lee and Sriraman (2011) argued that classical analogies were too restrictive due to their predetermined nature. They proposed an expanded version of classical analogy called Open Classical Analogy (OCA), within which analogies are left open to the student for generating conjectures. The introduction of the OCA provides insight into designing tasks that may elicit productive student analogical reasoning and was a partial inspiration for the design of the analogizing task described in the next section. Lee and Sriraman investigated the notion of OCA in the context of an 8th grade geometry classroom wherein students made conjectures about geometrical properties. In this paper, I will expand on the types of activities to which OCA problems can be used:

making conjectures to construct and define new structures by analogy in advanced mathematics.

### **Research Methods**

To investigate students' analogical activity in mathematics, I chose the context of abstract algebra because of the existence of several naturally occurring structural similarities between group theory and ring theory. The well-behaved mapping between these structures is intentional: algebraic structures were synthesized in the early 20th century to unify mathematics research of the era (Hausberger, 2018). In this section, I outline the research methods implemented to access students' analogical reasoning in abstract algebra and the process for developing the ARM framework.

#### **Participants and Setting**

I acquired rosters from recently taught introductory undergraduate abstract algebra courses at a large public research university and emailed an invitation to all students on the rosters. From a pool of approximately 40 students total, four students accepted the invitation to be interviewed. Three of the recruited students were undergraduates in mathematics, while one was a graduate in mathematics education. With each participant, I conducted five 60-90-minute-long semi-structured interviews designed to elicit analogical reasoning.

The first three students were interviewed in person and tasks were administered on paper. One camera was positioned to record the students' writing and a lapel microphone recorded audio synchronously with the video recording. Due to complications with COVID-19, the fourth student was interviewed using a video-conferencing tool for all five interviews. For this student, tasks and questions were

administered both verbally and through a chat interface within the tool, while the student wrote and recorded their work with the use of an electronic tablet and writing instrument. The tablet screen was visible to myself during the whole interview and a real-time recording of the tablet screen and synchronous audio was made available from the video-conference tool.

### **Interview Tasks**

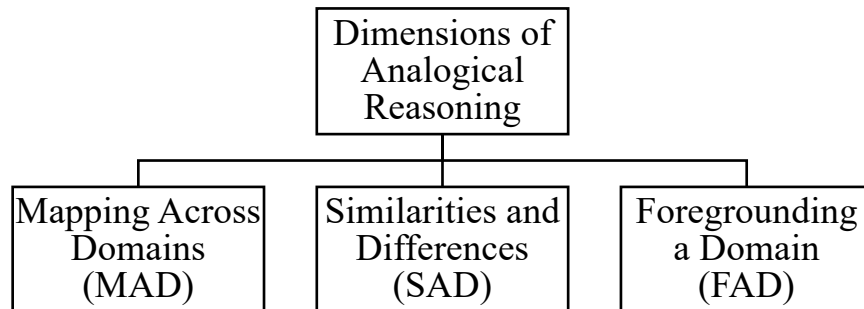
The initial interview was meant to assess the participant's content knowledge of abstract algebra before beginning to explore rings. The purpose of this initial interview was to determine what students recalled about group theory as well as ascertain what students knew about ring theory, if anything, before further interviews were conducted. The second interview provided participants with the definition of ring and various tasks designed to acclimate the student to working with rings. This interview focused primarily on getting student acquainted with basic properties of rings, but included opportunities for the students to compare rings to groups analogically.

The three subsequent interviews constituted the heart of the data collection: each interview focused on reconstructing one of subrings, ring homomorphisms, or quotient rings by analogy with a structure in group theory. The focal task of each of these interviews was the analogizing task. These are tasks of the form: Make a conjecture for the definition of a structure in ring theory analogous to  $X$  in group theory. In this research,  $X$  was taken to be one of subgroup, homomorphism, or quotient group. Further tasks were constructed around three basic types: (1) elicitation of comparisons between structures when spontaneous comparisons tapered off (i.e., "In what other ways are the structures of group and ring the same?"), (2) example generation and checking (i.e., "give

an example of a subring.”), and (3) proof-writing (i.e., “Is the homomorphic image of a commutative ring commutative?”) As the interviews progressed, I allowed students to refer back to their work from previous interviews. This not only gave the students an opportunity to recall information if they were stuck, but also allowed me to see exactly what information they wanted to refer back to while they reasoned by analogy.

### **Data Analysis**

I used techniques outlined by Corbin and Strauss (2014) and the constant comparative method to analyze the data. First, immediate reflections were written after each interview was completed and the audio was transcribed. The process of analyzing data occurred in two main parts: (1) establishment of an initial coding scheme based largely on the known concepts of mapping activity and attention to similarity, and (2) refinement of the initial coding scheme to begin capturing unique aspects of analogical activity not yet made explicit.



**Figure 2:** Primary Dimensions of Analogical Reasoning

| <b>Table 6. Dimensions of Analogical Activity</b> |  |  |
|---|--|--|
| <b>Name</b>                                       | <b>Description</b>   | <b>Significance</b>  |
| <b>Mapping Across Domains (MAD)</b>               | Includes the ubiquitous component of mapping across domains (inter-domain activity) as well as activity where mapping does not occur (intra-domain activity).                          | The introduction of the negative case of inter-domain activity, intra-domain activity, allows for the observation of activity that assists with analogical reasoning, but is not necessarily tied to analogical reasoning. |
| <b>Similarity and Difference (SAD)</b>            | While reasoning by analogy, attention can be given to what is either the same or different across domains. Attention may also be given to neither.                                     | Explicating attention to differences allows for a broader classification of types of analogical activity beyond what attention to similarity allows for.   |
| <b>Foregrounding a Domain (FAD)</b>               | Placing emphasis on a domain during analogical reasoning. Typical candidates include the source or target domain, although both domains, and even a third domain, may be foregrounded. | Foregrounding provides insight into what the focus of analogical reasoning at a given point in the process of reasoning by analogy.  |

Beginning in order of collection with the five interviews from each student, initial coding consisted of searching for evidence of analogical activity as defined by the Structure-Mapping Theory framework. That is, I searched for any evidence that the student was making comparisons and mapping across domains. This initial coding produced a categorization of dimensions of analogical activity that expanded beyond the basic tenets of mapping and similarity: (1) mapping and non-mapping activity, (2) attending to similarity and differences between domains, and (3) foregrounding a domain. Figure 2 displays a visual of this categorization, and a description of these dimensions can be found in Table 6. The unit of analysis was an *instance* of analogizing and evolved as I continued the coding process. Instances were identified by two criteria: presence of analogical reasoning and shifts in mathematical focus. Presence of analogical reasoning was determined by the students' language contextualized within the broader transcript indicating that a comparison was presently being made, indicating that a comparison was going to be made, or indicating a discussion of a newly formed structure just after completing a comparison. Shifts in mathematical focus were determined by the

mathematical content being attended to. For example, consider the following excerpt:

So, now we're going to call this normal subring. We give this one a name. First condition is that  $S$  is a subring of  $R$ . Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it.

This brief excerpt was subdivided into three instances:

- “So, now we're going to call this normal subring. We give this one a name.”
- “First condition is that  $S$  is a subring of  $R$ .”
- “Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it.”

Within all three instances, evidence of analogical reasoning was determined by language such as “we give *this one* a name”, “*conditions*”, and “*just to copy it.*” In the first instance, the student is attending to *naming conventions* as the mathematical content. In the second and third instance, the student is attending to definitional properties of normal subrings. I further discuss the different types of mathematical content below.

To each instance, I assigned a code of whether there was mapping or non-mapping activity, whether the source or target was being foregrounded, and whether the focus of attention was on similarity or difference. I initially coded for the type of content being attended to beginning with the content outlined by the SMT: object, attribute, or relation. However, these codes were often unsatisfying for describing the students' focus. As such, a categorization of types of content was developed alongside the types of analogical activity.

As more data was collected, I engaged in multiple rounds of axial coding that related the dimensions together to form types of analogical activities and over time the coding shifted away from assigning the broader codes to assigning codes explicating specific analogical activity. For example, rather than assign the codes of (a) mapping, (b)

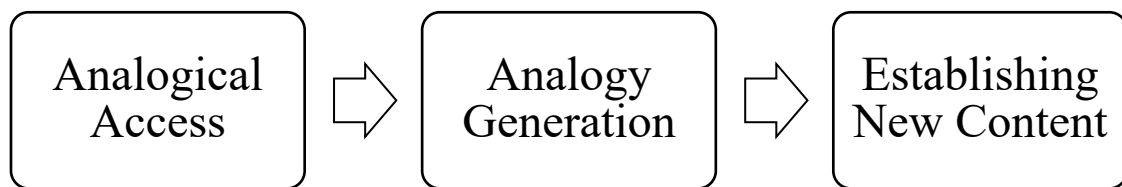
attending to similarity, and (c) foregrounding source domain to an instance, I assigned a code that integrated these three to describe the students' analogical activity, such as exporting. Mapping, attending to similarity, and foregrounding the source became known as the dimensions of the activity of exporting. I present examples of analogical activities and their dimensions in the results. The categorization of content types can be found in Table 7.

| <b>Table 7. Types of Content</b>           |  |  |
|--|--|--|
| <b>Name</b>                                | <b>Attends to:</b>   | <b>Example</b>   |
| <b>Structural Property</b>                 | A specific property of a structure                                       | Recognizing that groups and rings both require closure.  |
| <b>Definition or Definitional Property</b> | A mathematical definition, either taken as a whole or just one part.     | Looking back at the definition of ring when developing subrings.   |
| <b>Naming Convention</b>                   | The name of a structure or property in the source or target.             | Defining a structure in ring theory analogous to normal subgroups in group theory as "normal subrings."                    |
| <b>Relation</b>                            | A connection within a domain between two or more other forms of content. | Recognizing that subrings are a subset of rings (whether or not the same relation is explicated for groups and subgroups.) |
| <b>Example</b>                             | An example, usually of a mathematical structure                          | Calling on the example of integers from groups to initiate exploration of an example in rings.                             |
| <b>Process</b>                             | An algorithm or set of step-by-step rules.                               | The subgroup lemma viewed as a process of checking for closure and inverses.   |
| <b>Theorem</b>                             | Statements of theorems, lemmas, etc.                                     | The subgroup lemma viewed as a theorem.  |
| <b>Proof</b>                               | A mathematical proof, either taken as a whole or just one part.          | Recalling the proof of the subgroup lemma to assist with conjecturing the subring lemma.                                   |

As a part of ongoing analysis, I routinely leveraged three techniques: microanalysis, diagramming, and memoing (Corbin & Strauss, 2014). Microanalysis involves close examination of portions of data with the goal of deeply understanding phenomenon contained within. I intermittently performed microanalysis on segments where the nature of the analogical activity was especially unclear. Several novel interpretations of analogical activity were produced from these analyses. In addition, diagramming was incorporated to make sense of how concepts arising from the coding fit

together in new (or otherwise difficult to detect) ways. Diagrams consisted of: (a) constructing visual representations of my participants' analogical activity, making heavy use of the naturally visual nature of mapping between domains, or (b) creating tables that connected and categorized codes with one another. Finally, I regularly wrote memos that explicated my thinking about concepts and generating new hypotheses. These memos acted as a record of my thinking over time and allowed older thoughts and observations to be rediscovered and reintegrated when appropriate. The results of microanalysis, diagramming, and memoing were regularly shared with colleagues through conversation and presentation to assist in ensuring the viability of my interpretations.

The process of collecting and analyzing data continued until saturation was achieved: the final round of interviews did not produce any novel analogical activities and only corroborated the existing activities. The end result of this process was the ARM framework: a classification of dimensions of analogical activities and mathematical content, and several identified analogical activities described with the help of the dimensions. I present the analogical activities in the next section.



**Figure 3:** An Overarching Process of Analogical Reasoning

| <b>Table 8. Overview of Analogical Activities</b> |                       |  |   |
|---|-----------------------|--|---|
|   | <b>Name</b>           | <b>Description</b>   | <b>Dimensions</b>   |
| <b>Access</b>                                     | <b>Recalling</b>      | Recalling or remembering content about one domain, usually the source.   | <ul style="list-style-type: none"> <li>• Intra-domain</li> <li>• Neither similarity/difference</li> <li>• Foregrounding source</li> </ul>     |
|   | <b>Distinguishing</b> | Identifying differences between the source and target domain.  | <ul style="list-style-type: none"> <li>• Inter-domain</li> <li>• Difference</li> <li>• Foregrounding source</li> </ul>                        |
|   | <b>Associating</b>    | Identifying similarities between the source and target domain.   | <ul style="list-style-type: none"> <li>• Inter-domain</li> <li>• Similarity</li> <li>• Foregrounding source</li> </ul>                        |
| <b>Generation</b>                                 | <b>Exporting</b>      | Mapping exact content from the source to the target; often associated with assuming that domains are completely similar with respect to some content.                    | <ul style="list-style-type: none"> <li>• Inter-domain</li> <li>• Similarity</li> <li>• Foregrounding source</li> </ul>                        |
|   | <b>Importing</b>      | Purposefully selecting content from the source to map to the target domain; associated with discriminately forming similarities rather than assuming content is similar. | <ul style="list-style-type: none"> <li>• Inter-domain</li> <li>• Similarity</li> <li>• Foregrounding target</li> </ul>                        |
|   | <b>Extending</b>      | Viewing one structure as being grounded within another and establishing the new structure by “decorating” the old.   | <ul style="list-style-type: none"> <li>• Intra-domain</li> <li>• Similarity</li> <li>• Foregrounding source</li> </ul>                        |
| <b>Establishment</b>                              | <b>Adapting</b>       | Making changes to content to account for differences found between domains.  | <ul style="list-style-type: none"> <li>• Inter-domain</li> <li>• Difference</li> <li>• Foregrounding source</li> </ul>                        |
|   | <b>Elaborating</b>    | Expanding on what is known about a domain, usually the target.   | <ul style="list-style-type: none"> <li>• Intra-domain</li> <li>• Neither similarity or difference</li> <li>• Foregrounding target.</li> </ul> |

## Findings

In this section, I present the results of my inquiry into how students reason by analogy in mathematics. I use ARM to characterize several analogical activities that contribute to the process of analogical reasoning. The goal of this section is to elucidate the power of ARM for describing analogical activities that were previously implicit or absent in the literature by situating activities within a process for analogical reasoning: access, generation, and establishing new content (see Figure 3). Table 8 provides a brief overview of the activities characterized and described in this section. I note here that the described process is not intended to capture every case of analogical reasoning and is not

meant to be interpreted as a list of steps that must occur. Rather, it is an exemplar of a common process that students may partake in while reasoning by analogy.

### **Access: Organizing Content**

Rather than spontaneously begin formulating analogies, there is typically a point of entry from which analogical reasoning begins. This phenomenon is known as analogical access (Hummel & Holyoak, 1997). In this section, I characterize three activities associated with analogical access: recalling, associating and distinguishing activity.

**Recalling Source Content.** By expanding the component of mapping across to domains to include intra-domain activity, a greater range of mathematical activities during analogical reasoning become observable. One such activity is that of recalling source content. Consider the following example of a student recalling attributes after being asked to make a conjecture for a structure in ring theory analogous to group homomorphisms:

So, let me just try to recall that... So, group homomorphism. So, there exists a  $\phi$  that maps from  $A$  to  $B$ . So,  $A \rightarrow B$  or, I guess maybe it'd be easier to say  $\phi$  maps from  $(A, *)$  to  $(B, *)$ . So,  $\phi(a)$  equals some  $b$ .

In this case, the student is foregrounding the source and is neither attending to similarity or difference. Although the student is not explicitly engaging in reasoning by analogy in this example, the student is recalling content about the source domain with the intent of using the information for the purpose of analogical reasoning.

**Associating and Distinguishing Content.** While recalling focuses on just one domain, associating and distinguishing domains are other potential form of access that compare across domains. Associating occurs when a student observes a similarity

between two domains. In contrast, distinguishing occurs when a student recognizes an anomaly between the source and target domain. Consider the following example in which a student is making observations about the definition of ring:

So it has ... what I noticed immediately is that, it has two binary operations addition and multiplication. Whereas with group theory we're only dealing with one binary operation at a time.

In this example the student is distinguishing structural properties, specifically the property of rings having two binary operations as opposed to the one operation defined on groups. As I will illustrate in the next section, this example of distinguishing activity later contributed to this student's analogical construction of other structures in ring theory.

### **Generating Analogies**

Once there is access, an informed creation of analogies can begin. In this section, I illustrate three types of activities that are tied to the generation of analogies: exporting, importing, and extending activity. Taken together, these represent a range of analogical activity describing how students generate analogies. The first two activities, exporting and importing, are representative of the well-known form of analogy generation by mapping similarity from the source to the target. However, the dimension of foregrounding a domain reveals a distinction based on what domain is emphasized.

Normal Subring

i)  $S \subseteq R$   
 ii)  $rSr^{-1} \subseteq S \quad \forall r \in R$   
 w.t.s:  $(S, +, x) \cong (R, +, x)$

Normal Subgroup

i)  $H \subseteq G$   
 ii)  $gHg^{-1} \subseteq H \quad \forall g \in G$   
 w.t.s:  $(H, *) \cong (G, *)$

**Figure 4:** Student Exporting Properties from Source to Target

**Export/Import Activity.** Here, I share two activities that agree along the dimensions of similarity and inter-domain activity but differ for which domain is foregrounded: exporting and importing activity. Exporting across domains occurs when a student projects content of the source domain over to the target domain. In contrast, importing occurs when a student selectively pulls content over from the source domain into the target domain. Consider the following example of a student recalling definitional properties in the source, followed by exporting a definition and naming convention from the source domain of groups to the target domain of rings:

So, normal subgroup... First condition is that  $H$  is a subgroup of  $G$ , and the second condition is that  $gHg^{-1}$  is a part of  $H$ , and then you can say, therefore,  $H$  is normal to  $G$ . So, now we're going to call this normal subring. We give this one a name. First condition is that  $S$  is a subring of  $R$ . Second condition, I don't know. Maybe we say  $rSr^{-1}$  is in  $S$ , just to copy it.

A visual of this student's work is seen in Figure 4 above. In this example, the student is constructing a definition for what they call "normal subrings" by copying over known aspects of normal subgroups into the context of ring theory. To contrast with the activity of exporting, consider the following example in which a student is making a conjecture about what comes next in the study of ring theory after having developed the concept of

subring:

Like Abelian rings, or like... giving them that type of thing where you give them special names... Special types of rings, "This is the golden ring." So these, you gave me these properties on the last page. But I'm sure if you have all these properties, it's probably a special type of ring.

In this example, the student is importing the notion of structural properties associated with “special” groups, such as Abelian groups, from the source domain into the target domain. The emphasis is on the target, and thus the student does not immediately assume that the naming convention of Abelian carries over to rings. Rather, this student has discriminately chosen which aspect of group theory to map into the target while also maintaining a level of vagueness as to what the properties may even be (i.e., commutativity itself is not what is mapped, only that a “special” property might exist.)

**Extending Activity.** It is not always the case that a mapping must occur to generate analogies. For example, there may be cases where one domain is temporarily viewed as being grounded within another. I refer to this activity as extending. In other words, one domain is taken as a foundation and “decorated” with properties in order to recognize or produce the new domain. In the following example, a student is establishing a structure in ring theory analogous to subgroups in group theory:

I'm trying to see how I can say that if something is a sub-group, and there's going to be some qualifying sentence that relates to how it could be a ring. So, like with sub-group and with something else, which is what I'm trying to figure out, then it will be a ring.

Unlike recalling activity, more than one domain is recognized in this example of intra-domain activity. However, because the source domain is being extended to form the target, no explicit mapping across domains is achieved, although extending may work in conjunction with a previous mapping. Consider the following example of extending

activity in the next quote in which a student is justifying why the additive identity of a ring is unique following an exportation of a structural property:

Because the additive identity of a group is unique. And I just kind of extended it, because the  $R$  plus together forms our group. So, if the addition somehow didn't have a unique identity, then what I stripped out the multiplication and just looked at  $R$  plus that would also not have a unique identity, but that contradicts what I know about groups.

In this example, the student is arguing that the additive identity of a ring is unique because it is unique for groups as well. This argument constitutes an exportation of a structural property. The student then justifies his argument by suggesting that, if one considers only the additive operation on the set  $R$ , then you are left with a group. As such, removal of the uniqueness of the additive identity would create a contradiction if one thought of the structure as a group. The examination of the structure as being a group constitutes another example of extending activity.

### **Establishing New Content**

During or after the formation of analogies, students may establish new content in the target. In this section, I describe two activities: adapting and elaborating activity. Each of these activities contributes to the formation of new content in the target and signifies a release from potential constraints that may exist when reasoning by analogy (such as that exhibited with exporting activity.)

$$\phi(a + b * c) = \phi(a) + \phi(b) * \phi(c)$$

$$\boxed{\phi(a * b) = \phi(a) * \phi(b)}$$

$$\boxed{\phi(a + b) = \phi(a) + \phi(b)}$$

$$\phi(a + b) = \phi(a) * \phi(b)$$

$$\phi(a * b) = \phi(a) + \phi(b)$$

**Figure 5:** Student Adapting a Structure to the Target

**Adapting Activity.** As previously noted, attention to differences often impacted reasoning by analogy at a later time. One example of this was through adapting. Adapting occurs when a student modifies the target to accommodate a distinction between the source and target. In contrast to extending activity, which can also involve modifications to the target, adapting is focused on reconciling an observed difference. In the previous example of distinguishing, the student noted the existence of two binary operations for rings as opposed to one for groups. With this in mind, consider the following statement from the student as they conjectured about the definition of ring homomorphism:

What would be one for.... We have two [operations] here. Start with phi going from G to H. There's two operations here so I'm like, I don't exactly know if it should just be one of them, or both of them, or how I would do that here. Could I do like three elements, like a, b, and c, and then have like the addition and multiplication?

A visual of this student's work is seen in Figure 5 above. In this instance, the student is keying in on the difference she identified between the domains two interviews prior, and adapts the homomorphism property she learned in group theory to the context of rings.

**Elaborating Activity.** The final analogical activity I share is not explicitly analogical reasoning. In the case of adapting activity, students are still acknowledging a difference between the source and target when establishing new content about the target.

However, students may disengage from the source when establishing new content, signifying a transition to reasoning about the target that is independent of the source. I refer to this activity as elaboration of the target. Consider the following example in which a student is thinking of how to prove whether a homomorphic image of a commutative ring is commutative:

Yes, [it is commutative]. Just because you can ... You can move it around before you take phi of it. Right? Yeah. How do I show commutativity in rings? Is it just you've got to ... I'm assuming you can combine the left- and right-hand distribution laws to get there.

Rather than recall or adapt previous knowledge about groups to approach the task, the student is thinking solely about what he knows about the target alone. In particular, he is wondering about how the distributive properties could be used to show commutativity in the image.

### **Discussion**

In this paper, I illustrated an analytic framing for exploring students' analogical reasoning activity in mathematics. I established an argument that clarifies analogy and metaphor, and in doing so, offered a more precise distinction between them. Having established a more robust definition of analogy, I outlined the Analogical Reasoning in Mathematics framework: a collection of analogical activities and their underlying dimensions that allow for a more precise description of students' analogical reasoning in mathematics.

This research is intended to complement work prior work related to both of analogy and metaphor in mathematics education. In particular, this paper introduced a framework for interpreting analogical reasoning that makes explicit features of analogical reasoning that were either previously unidentified or remained implicit: non-mapping

activity, attention to difference, and foregrounding activity. In particular, foregrounding is a novel dimension introduced in this framework. By attending to the foregrounding of domains, the distinction of certain activities was made possible. The most prominent example of this distinction lies between the export and import activities: each activity focuses on mapping activity and similarity, but the foregrounded domain differs between them. As a result, how students make use of similar content between domains can be parsed in a way that was not previously available in other frameworks for analogical reasoning.

The results of this study suggest that students can indeed construct mathematical structures on the basis of analogy with others. In particular, the analogical activities described in this paper assisted the students in this study in accomplishing the task of conjecturing and then constructing structures in ring theory by analogy with what they knew from group theory. Taken individually, the analogical activities allow for the dissection of analogical reasoning at various stages in a process of reasoning by analogy such as access, generation, and establishment of new content. However, this process need not be the only way to reason by analogy. For instance, while one student leveraged extending activity after analogical access (as described in the second extending example in the previous section) to assist with generating an analogy, another student had leveraged extending activity to establish analogical access. Therefore, dissection of analogical reasoning with ARM followed by reintegration of the activities suggests possibilities for identifying and describing other processes of reasoning by analogy that are perhaps invisible without the close examination made available by ARM.

As I noted in the review of the literature, evidence suggests that teachers

formulate the majority of analogies in mathematics classroom (Richland et al., 2004). Furthermore, some scholars have expressed skepticism about the productivity of pedagogical analogies for learning new mathematics (Cobb et al., 1992). Of particular note in this research is that the tasks used in the interviews show promise as tools for promoting student-generated analogies: the students successfully engaged in analogically reconstructing structures as they were asked to do. As such, another contribution of this paper is the analogizing task itself. Much of the previous research on analogy in mathematics education involved closed forms of analogical reasoning with predetermined answers. In contrast, this research has focused on inventive, student-driven analogy, similar to the notion of open-classical analogy (OCA) (Lee & Sriraman, 2011). The present research expanded on the scope of mathematical activity that can be expected by implementing an OCA problem: students engaged in the act of defining mathematical structures that were made accessible to them through analogical reasoning.

### **Using ARM to Interpret Student Analogical Reasoning: An Example**

Having displayed the analogical activities associated with ARM, I now turn briefly to detailing how the ARM framework can be used to explore and interpret students' analogical reasoning in mathematics. To accomplish this, I share and contrast two excerpts of students' initial constructions of objects in ring theory analogous to subgroups in group theory. By sharing these excerpts and using ARM to analyze them, I illustrate that students may productively reason by analogy in a variety of ways. I refer to different approaches to analogizing as *pathways of analogical reasoning* (Hicks, 2020b). In these excerpts, instances are denoted by [ ]. Consider the first case by Student A:

(1)[All right, so, we don't ever deal with both, like more than one operation interacting for groups of ...] (2)[We also don't have an inverse for multiplication,

which you need an inverse for a group, therefore you need an inverse for the subgroup.] (3)[So, I think that if we have a subgroup, then we have all of these <points to group definition on the desk>. We have the identity, the inverse. We have associativity and closure.] (4)[but what we don't have is commutativity, necessarily, or this use of multiple properties at once.] (5)[So, we could say that if we have a subgroup, then if this subgroup is also ... so this was just kind of me brainstorming up here, but if a subgroup  $H$  is commutative and distributes ... Right hand side and left hand side, then  $H$  is a ring ...]

In this case, Student A begins by making several observations differences between groups and rings. Within the first two instances, this student *distinguishes* between *structural properties* in groups and rings. In the third instance, the student *recalls* what she knows about subgroups: they possess all the properties of a group, and then continues *distinguishing* in the fourth instance. In the fifth instance, Student A establishes a definition for her new object analogous to subgroups. In particular, Student A is *extending* the *mathematical structure* of subgroup by adding on the missing properties of ring she identified in the previous instances, thereby forming a ring using subgroup as a base.

In contrast, let us now turn attention to a description of a similar initial creation by Student B:

(1)[So I just compared it to subgroup.] (2)[So we know that subgroup has a closure under its binary operation and it contains the inverse.] (3)[So I said maybe it has closure under just one of them, one of the binary operations and then maybe the inverse is under the other operation.] (4)[And then I said, well since ring had more properties than the group, then maybe a subring has more properties than the subgroup. So then I just added a unity or what do you call it, identity, under one of the operations.] (5)[Because it's a subring, maybe it doesn't exhibit all the properties, just like subgroup doesn't. So I said maybe it just has one of them.]

In this excerpt, Student B begins in the first instance by explicitly *associating* subgroups to the new analogous object in ring theory. In the second instance, Student B *recalls* *properties* about the subgroup structure, and then simultaneously *exports* and *adapts*

properties in instance 3. In particular, he exports the specific properties of closure and inverse from the source domain of subgroups, but then makes a small modification about where those properties are needed for the subring structure (closure for just one operation, and inverses for the other.) In instance 4, Student B *adapts* again to form a new structural property: an identity is needed for one of the operations in the subring because rings have more properties than groups. Finally, Student B *exports* his definition of subgroup (which relies on the subgroup lemma) to argue that subrings also don't require all properties.

From this brief analysis of these two excerpts, we are able to describe how these students were thinking about the objects in ring theory analogous to subgroups in group theory as well as their pathways for initiating the construction of the analogous object. In particular, this analysis has revealed two types of pathways that students might take to begin constructing a ring theory analogue to subgroups in group theory. First, Student A's construction was centered around an eventual extension of the subgroup structure to form a ring. In other words, Student A's activity relied on thinking about what properties of rings were missing from her definition of subgroup, and then added properties to the subgroup structure based on her analysis of what was different. It is unclear if she ever fully recognized "subrings" as a structure on their own during the excerpt; instead, the structure a subgroup that was being transformed into a ring and the name of "subring" did not make an appearance. In contrast, Student B readily developed the notion of "subring" and proceeded to construct the structure by comparing directly to subgroups, namely by exporting exact content and including minor adaptations to properties of subgroup in the context of subrings. For Student B, the notion that rings may be groups never made an

appearance; the structures were always completely separated from the beginning.

Previous models of analogical reasoning focus holistically upon the mapping process to describe the formation of analogies, but do not describe the individual activities that bring about the mappings. By using ARM to analyze and contrast the excerpts above, we are able to parse analogical reasoning in such a way that two cases of analogical reasoning that appear similar in terms of the content being mapped, but are actually quite different when investigating the activities involved. In particular, both students were focused on the notion of subgroup during their analogical reasoning, and the mathematical content of the two excerpts is concerned with the various properties of subgroups. However, an analysis of the analogical activities contained within the two excerpts allows for insight into how the students understood their analogical creations: Student A's conception of an analogue for subgroups heavily leveraged the fact that rings can be obtained by adorning groups with further properties; Student B's conception of an analogue for subgroups was to generate a notion of subring and export his knowledge of subgroups into the ring context.

### **Directions for Future Research**

The results in this paper have only analyzed students' analogical reasoning in an interview setting. Thus, there is much to be learned about how students might productively leverage analogical reasoning over an extended period of time so that analogical reasoning may be integrated within a structured curriculum. A unique characteristic of the present research was attention to how students might come to construct mathematical structures by analogy with known structures in abstract algebra. Naturally, this process was similar to that of guided reinvention (Gravemeijer &

Doorman, 1999). The task design implemented for data collection in this research suggests the potential for a new heuristic: analogical reinvention. While inroads have been made into the guided reinvention of ring, integral domain, and field (Cook, 2014; 2017), there has yet to be an established curriculum developing further structures, such as ideals, as for structures in group theory (e.g., Larsen, 2013). Previous inquiries into the matter have proven difficult, likely due to the unintuitive foundations of the structure. One approach to achieving reinvention of ideal could be to design a curriculum leveraging analogical reinvention with a known structure: normal subgroups. In particular, the structure of ideal arose naturally during my participants' development of the quotient ring structure by analogy with quotient groups: they referred to the structure as a "normal subring." Further research can explicate what such a reinvention process would entail by exploring the usefulness of analogical reinvention in various contexts and generating models of the analogical reinvention process.

## **Limitations**

This paper presented a qualitative analysis of the analogical reasoning of four students. As such, no claims of generalizability are made. It is possible that idiosyncratic forms of analogical reasoning may have been present within the participants in part due to their previous exposures to topics in abstract algebra. Thus, I note that variability was present among the participants' previous learning and exposure to abstract algebra. Of the interviewed students, one claimed to have never seen any topics in ring theory, two had seen only the definition of ring, and one had seen the definition of ring, integral domain, and field before. In addition, three of the students had studied group theory in a traditional lecture-based course, while one had studied groups in an inquiry-oriented

course.

I also acknowledge that the analogical reasoning elicited from students may be context dependent. Although abstract algebra proved to be fertile ground for eliciting analogical reasoning, the analogies between structures in group theory and ring theory may not have captured the full range of dimensions within other cases of analogical reasoning. The existence of several surface similarities between the structures in group theory and ring theory may have hindered the diversity of the participants' analogical reasoning. However, robust examples of analogical reasoning were indeed elicited by the participants and no participant focused solely on surface level features. The novel setting of defining mathematical structures by analogy may have strengthened the diversity of their analogical reasoning beyond attending to surface features.

Finally, I was the sole researcher making decisions about interpreting my participants' analogical activity. As such, bias may be present in the ARM framework in terms of the selected dimensions, activities, and mathematical content. Steps were taken to curb this bias by sharing my interpretations with colleagues at various stages in the data analysis. In addition, I recognize that bias may have been introduced into the interpretation of my participants' analogical activity by my own personal background with abstract algebra; it is possible my interpretations of students' activity was influenced by what I perceived as being either typical or nonstandard. As such, other activities or dimensions of analogical may have become available had I investigated analogical reasoning in a different context. In order to expand on the contexts in which ARM can be applied, future research can explore the nature of students' analogical reasoning in other content areas.

## **VI. PAPER #3: EXPLORATORY STRUCTURE CREATION THROUGH REASONING BY ANALOGY IN ABSTRACT ALGEBRA**

### **Abstract**

Analogy has played an important role in developing modern mathematics and continues to play a role in modern mathematics research. Although analogy does make casual appearances in several mathematics textbooks, it is unclear to what extent students are granted opportunities to reason by analogy in productive ways in their undergraduate courses. This paper proposes a novel lesson for introducing structures in ring theory by reasoning analogically about structures already known in group theory. In this way, students come to creatively establish new structures that they may take ownership of while providing opportunities for rich discussion about the purpose of these structures. The lesson consists of four key components: (a) introducing the definition of ring, (b) introducing the idea of analogy and analogical reasoning between groups and rings, (c) developing structures (i.e., subrings, ring homomorphisms, and quotient rings) through analogical reasoning with known structures, and (d) developing theorems/proofs through analogical reasoning. Throughout, I provide thoughts and insights from previous implementations and conclude by reflecting on what has (and has not) worked well in my experience with implementing these tasks.

### **Introduction**

Analogies have played a significant role in the development of several mathematical concepts. (Polya, 1954). Reasoning by analogy is also a tool that is purposefully leveraged by mathematicians when creating and conjecturing new structures and results in modern mathematics research (Ouvrier-Buffet, 2015). Recently, I have

been investigating the ways in which students reason by analogy in the context of abstract algebra (Hicks, 2020a; 2020b). Specifically, I have been exploring the ways in which students reason analogically between groups and rings, subgroups and subrings, group and ring homomorphisms, and finally, quotient groups and quotient rings. As is evident in several abstract algebra textbooks, these analogies are frequently observed when introducing these topics in ring theory after exposing students to the analogous topics in group theory. Consider the following quote from Gilbert and Gilbert (2015):

In this chapter, we develop some theory of rings that parallels the theory of groups presented in Chapters 3 and 4. We shall see that the concept of an *ideal* in a ring is analogous to that of a *normal subgroup* in a group.

However, it is unclear what students take away from casually observing these analogies. While they may appear simple on the surface, what do students understand about the analogy when it is merely provided to them by the instructor? Perhaps more importantly, simply providing students with these analogies eliminates the opportunity for students to reason by analogy themselves. This raises a pertinent question: How might undergraduate students productively reason by analogy in abstract algebra?

In this article, I describe a lesson focused on productive student analogical reasoning and share examples from implementing tasks focused on developing structures in ring theory through reasoning by analogy about structures in group theory. By having students develop ring theoretic structures through analogy, they are able to take ownership of the newly formulated structures in ring theory rather than being given the connections by the instructor. In addition, this lesson acts as a solid review of the details of group theoretic structures that students may need a refresher of or may have forgotten over a period of time. In the following sections, I share the details of the tasks and the

lesson plan, describe the implementation of the lesson (including thoughts and insights from both the implementation of a version of this lesson in a classroom, as well as my observations from engaging students in one-on-one interviews), and conclude with a reflection on implementing the tasks in these settings.

### **The Analogizing Task**

The focal task provided to students is deemed the *analogizing task*. These are tasks of the form: Make a conjecture for the name and definition of a structure in ring theory that is analogous to  $X$  in group theory. In the context of this paper,  $X$  is to be one of: subgroups, homomorphisms, and quotient groups. The focus of the lesson revolves around the analogizing task and explorations of the structures developed from the task. This task was created for two reasons: (1) it encourages students to reason by analogy in such a way that they could reasonably innovate and generate their independent thoughts about the new structures, and (2) the task is bounded enough as to be productive for the overall goal of introducing topics in ring theory.

The analogizing task sets the stage for students to engage with the creation of mathematical structures in ring theory by analogy with structures that they know from group theory. In my personal experience, I have found that this task provides students with a creative outlet unavailable to them when instructors assume that the analogies between these structures are obvious. In addition, the conjectured names and definitions developed by students provides an opportunity for rich discussion of the purpose of these structures while establishing new information about rings as well as reviewing old information about groups.

## Outline of the Lesson

The following is an outline for structuring the lesson in one 80-minute session and will be used to describe the lesson in this paper. Further options for implementation can be found in the Appendix: an extended iteration that covers two days, and a “bitesize” iteration that spreads the activity over several days. However, all implementations cover the full range of analogical reasoning activity: (1) Developing a structure by analogy, (2) comparing newly created structures to previous structures, and (3) conjecturing and proving theorem statements by analogy.

### 1. Students familiarize themselves with the definition of Ring (15 min)

- (a) Introduce the definition of ring.
- (b) Ask students to identify similarities/differences between groups and rings.
- (c) Ask students to determine examples of rings using what they know about groups.
- (d) Introduce further examples of rings as appropriate.

### 2. Establishing Rationale for Analogy and Analogical Reasoning (5min)

- (a) Briefly review structures in group theory
- (b) Leverage student’s observations of similarity/differences from before to argue that these observations can be extended to other structures.
- (c) If desired, draw diagram developing a mapping between group/ring structures.

### 3. Students do Analogizing Task for Subgroups in small groups (20 min)

- (a) Students generate a name and definition.
- (b) If time permits, ask students to generate a test analogous to the subgroup test.

### 4. Discuss Subring as Class (10 min)

- (a) Establish subring name and definition.
  - (b) Ask students to identify similarities and differences between subgroups and subrings.
  - (c) If time permits, discuss subring test.
5. Students do Analogizing Task for Group Homomorphisms in small groups (15 min)
- (a) Students generate a name and definition for analogous structure.
6. Discuss Ring Homomorphism as Class (10 min)
- (a) Establish ring homomorphism name and definition.
  - (b) Ask students to identify similarities and differences between group and ring homomorphisms
7. Wrap up and next steps (5 min)

Although the structure of this lesson plan was developed with group and ring theory topics in mind, I note that the structure and time of the lesson plan may be appropriately adapted to other mathematical settings in which analogies may play a role (with varying degrees of time, difficulty and complexity for the expected analogy students are to generate depending on their maturity). For, an instructor in an introductory analysis course might encourage students to independently generate analogous definitions and theorems for monotonically decreasing sequences to monotonically increasing sequences, or generate an analogous definition for an open ball in  $\mathbb{R}^n$  by analogy with an open ball in  $\mathbb{R}$ .

### **Goal of the Lesson**

The overarching goal of this lesson plan is to provide students with an opportunity to develop structures in ring theory. Rather than replace a full unit on each of subrings, ring

homomorphisms, and quotient rings, this lesson is meant to orient students toward recognizing that many of the concepts in ring theory can be thought of as analogous with concepts in group theory. In particular, this lesson is intended as a way to launch a unit on rings while allowing students ample opportunity to generate connections, structures, and theorems that they may take ownership of. As such, I suggest that this lesson is best used in the following ways:

- Launching a course/unit on ring theory in which students have already learned basic group theory.
- Wrapping up a course on group theory in which a quick detour into rings is done at the end.

In either case, the lesson can serve double duty in introducing new concepts to students while also allowing for a broad review of some topics in group theory. In addition, this lesson provides an opportunity for students to become familiar with the act of reasoning by analogy in mathematics, a skill that can be useful to them in other mathematical contexts and research.

### **Implementing the Lesson**

This lesson plan was developed for an introductory undergraduate abstract algebra class intended for mathematics majors at a large university in the Southwestern United States. It was planned to be implemented at the beginning of a unit on ring theory after having been through a unit on the basics of group theory. However, this lesson plan can be suitably adapted for any level of abstract algebra in which the instructor wishes for their students to engage with constructing mathematical structures through conjecturing, defining, and analogizing activity rather than introducing definitions through more

traditional means.

### **Familiarizing Students with the Definition of Ring**

The first part of this lesson involves introducing students to the definition of ring. Depending on the wishes of the course instructor, the definition may either be introduced with or without unity with little variation in the rest of the lesson. This part of the lesson launches the unit on ring theory and assists students in getting to know the basics of the new mathematics they will be investigating. Students are provided with a sheet consisting of the definition of ring and then asked to analyze the definition in small groups and make comments on what stands out to them. During this activity, students are expected to independently begin formulating connections between the structure of group and ring, such as noticing what is the same (e.g., identity elements exist) and what is different (e.g., one operation defined on a set vs. two operations.) The recognition of similarities and differences can be leveraged as a tool for comparing other structures later on in the lesson.

After students have had time to analyze the definition, discuss understandings and noticings of the definition as a class. One key point of discussion that can be had here is how the structure of ring relates to the structure of a group. I have seen students take one of several stances on this:

- Rings are groups that are “decorated” with properties and are viewed as extensions of groups (i.e., all rings are constructed from a group and are thus groups themselves.)

- Rings and groups are distinct structures, but group makes up part of the definition of ring (i.e., recognizing the set with the additive operation forms an abelian group, but the ring itself is not seen as a group.)
- Rings and groups may have similarities, but are distinct standalone structures (i.e., groups are never rings, rings are never groups.)

During this time, you may also ask students to try generating examples of rings and provide them with examples of rings as you see fit. The generation of examples is another period where you can expect students to leverage their knowledge of groups to establish an initial pool of examples.

### **Introducing Analogical Reasoning**

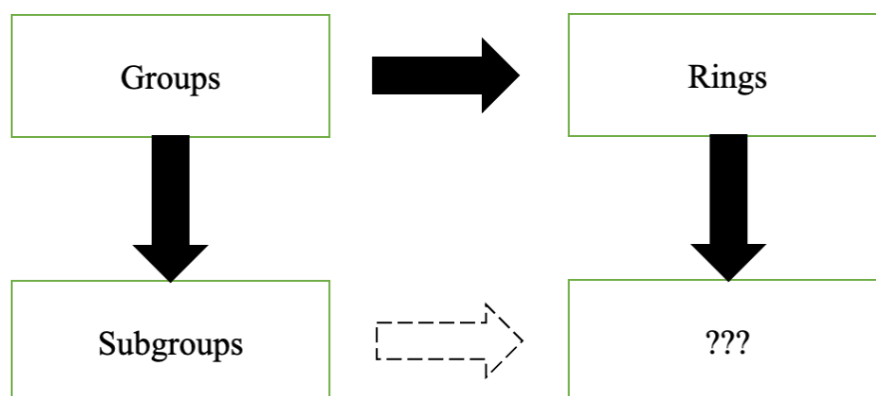
Before jumping into the analogizing tasks, a brief introduction to the idea of analogy and analogical reasoning is in order. The goal of this introduction is to situate the students' goal in the rest of the lesson and contextualize their analogical activity as helping to develop new structures in ring theory. More importantly, this introduction can help to clarify what is meant by analogy in the tasks that follow. First, discuss as a class what structures you have studied in group theory before having begun the study of rings. This typically involves at minimum structures such as subgroups, group homomorphisms, and quotient groups. During the first part of this lesson, students will likely have drawn several connections between the structure of group and ring themselves. Leverage these observations as a way to extend the possibility of the existence of analogous for rings as well. For example, you may ask the students to conjecture what other concepts might be similar and different between all of group theory and ring theory aside from just similarities and differences of groups and rings themselves.

## Implementing the Analogizing Task

Following the introduction of the definition of ring and the idea of analogy, the analogizing task can be provided to students to work on in small groups. For this explanation, I will discuss the implementation of the analogizing task for subgroups, although this explanation can be appropriately adapted to the other structures listed in this lesson as well (i.e., group homomorphisms, and quotient groups).

**Launching the Analogizing Task.** After students are placed into their groups, provide them with the analogizing task for subgroups as follows: Make a conjecture for the name and definition of a structure in ring theory that is analogous to *subgroups* in group theory.

Students may sometimes express confusion over the meaning of analogy or analogous structure when they are doing this task for the first time. In these cases, you may wish to make use of a diagram that further clarifies the intent of the task. For example, to launch the exploration of a structure in ring theory analogous to subgroups in group theory, drawing a diagram as in Figure 6 can be a helpful to clarify the task for students:



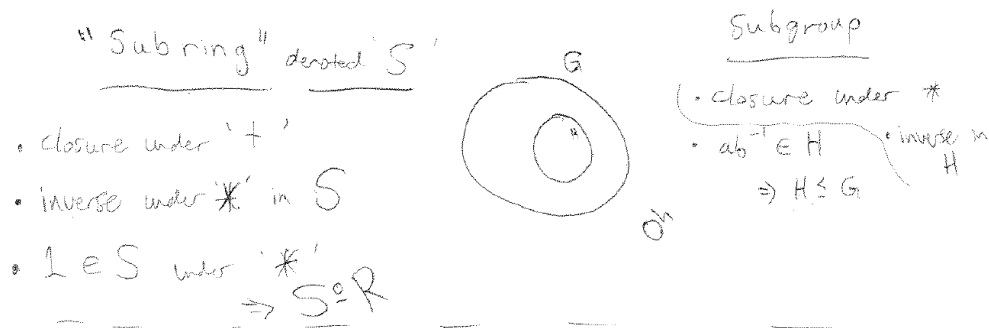
**Figure 6:** Subgroup Analogy Diagram

**Naming the Structure.** In my experience, virtually all students quickly establish the name of *subring* for their newly generated structure. Although establishing the name of subring may seem trivial on the surface, the naming of the structure possesses high importance for the students' understanding of the structure they are generating. Describing the structure as a *substructure* indicates that they are cognizant of the relationship between groups and subgroups and are preparing for mapping the relation of a *substructure* over to the context of rings.

**What to Expect.** Once students have had time to establish names and definitions in their small groups, you may regroup the students to discuss their findings, conjectures, and observations as a class. It is very common for students to generate the definition of subring by attending to the subgroup test rather than the structure of subgroup itself. The following was an example of what one student stated about their newfound structure:

So I just compared it to subgroup. So we know that subgroup has a closure under its binary operation and it contains the inverse. So I said maybe [the subring] has closure under just one of... the binary operations and then maybe the inverse is under the other operation. And then I said, well since ring had more properties than the group, then maybe a subring has more properties than the subgroup. So then I just added an... identity, under one of the operations. Because it's a subring, maybe it doesn't exhibit all the properties, just like subgroup doesn't. So I said maybe it just has one of them.

As can be seen in this example, the student was primarily attending to the need for closure and inverses to generate a subring. Further evidence of this is seen in their written work as they considered how to approach this task (see Figure 7).



**Figure 7: Comparing Subgroup and Subring**

In addition, due to the simplicity of the structure of subrings, students may also generate definitions of subring that simply take their known definition of subgroup and replace all instances of “group” with “ring.” In this case, a brief review of the difference between the definition of subgroup and the subgroup test can assist in establishing the definition of subring as well as a more productive discussion of the subring test later on. Details about observations of what students have generated when creating other structures by analogy can be found in Paper #3 Appendix B.

After the class comes to an agreement on the definition of subring, you may ask the students to make observations about the similarities and differences between subgroup and subring. The goal of this subtask is to keep the students focused on questioning the ways in which the developed structure is similar and different from the structure they knew before.

### **Conjecturing Theorem Statements and Developing Proofs by Analogy**

After students have generated the analogous structure in ring theory, then can engage with the new structure in a proof setting. For subrings, we asked students to establish a process for determining whether a subset of a ring was a subring by analogy with the process for subgroups: the one-step subgroup test. (Examples of theorems provided for other structures can be found in Appendix B.) This task is begun by giving

students the following prompt: *Let  $G$  be a group and let  $H$  be a nonempty subset of  $G$ . If for all  $a$  and  $b$  in  $H$ ,  $ab^{-1}$  is in  $H$ , then  $H$  is a subgroup of  $G$ . Construct an analogous statement of this theorem in the context of ring theory. Afterward, attempt a proof of your statement.* In addition, you may wish to provide students with a proof of the subgroup statement. When approaching this task, students are expected to begin making appropriate adaptations to the theorem rather than simply replacing the word “group” with “ring.” Figure 8 displays an example of what one participant of mine produced for their subring test.

Given that  $(R, +, *)$  is a ring,  
 and  $H \subseteq R$  : H satisfies inverse and  
closure properties  
 if  $H$  is a subgroup of  $R$  under  $+$ ,  
 and  $H$  is closed under  $*$ ,  
 then  $(H, +, *)$  is a "subring" of  $(R, +, *)$ .

**Figure 8:** Example of Subring Test Developed by Analogy

### Wrapping Up

At the end of the lesson, you may want to provide a brief recap of the structures that were generated from the analogizing task. If more than one structure was developed, you can refer back to the diagram from the beginning of the lesson and describe. An interesting subtask to engage with as a whole class is to ask students to make inferences of what they think might come next in the study of rings. Doing so can prompt the students to think further about what topics in ring theory may be in the future, and quite often, students generate inferences that easily set the stage for discussion about other

structures or theorems in the future. One student of mine suggested the possibility for different “types” of rings:

There's a lot more stuff going on with rings, like [with groups] we had Sylow groups we had permutations, we had dihedrals, alternating, all that kind of stuff. You had a lot of different groups. Same thing is going to go for the rings. You're going to have different types of rings... I bet there's just a ton of them.

Depending on the amount of material covered in the lesson, there are various homework tasks that can be assigned following this lesson that leverage analogical reasoning. Some examples of tasks are:

- As groups under addition, we know that  $2\mathbb{Z}$  and  $\mathbb{Z}$  are isomorphic. Are  $2\mathbb{Z}$  and  $\mathbb{Z}$  isomorphic as rings under multiplication and addition? Why or why not?
- For any group homomorphism  $\varphi: G \rightarrow H$ ,  $\varphi(e_g) = e_h$ . Is there an analogous property for ring homomorphisms?
- Provide students with a complete proof that, if  $G$  is a group and  $H, K$  are subgroups of  $G$ , then  $H \cap K$  is a subgroup of  $G$ . Ask students to conjecture an analogous theorem statement for rings, and then ask them how they could leverage the proof of the statement in group theory to develop the proof of the conjectured statement in ring theory. What aspects of the proof are the same? What aspects are different?

In addition, if it can't be fit in to class time, students can explore the analogizing task for quotient groups at home to begin the process of thinking about what a quotient ring looks like and what it is conceptually.

## **Reflections on Implementation and Concluding Thoughts**

I end this paper with a brief discussion of aspects of the lesson I have found to be the most motivating as well as comments about the task from my interviewed students. I also discuss a task that I tried out in my interview setting, but did not explicitly write into the lesson plan because it did not seem to work as intended (although I encourage instructors to keep an open mind and try things out for themselves!)

### **What Worked Well**

Much like the statements found in several abstract algebra textbooks I have seen, I used to believe that analogous structures between group theory and ring theory were rather obvious to generate and define. When you already possess a deep knowledge of structures such as subgroups, it can be difficult to envision establishing an analogous structure in ring theory that is anything but the traditional definition of subring. However, my implementation of the analogizing tasks has led to me witnessing a lot of creative mathematical thinking about the definitions and purposes of structures in ring theory from students. It is clear from my experience with implementing this task that the generation of analogous structures is far from a trivial task, and thus there is great potential for rich discussion in the classroom. I also believe that this process of developing structures by analogy (among others) is an excellent way to engage students with more genuine mathematical activity. In particular, I have personally found that implementing the analogizing task to be quite powerful in getting students to engage with the act of defining as one might expect from the work of a research mathematician. The implemented tasks have also been well-received by my participants: One of my interviewed participants explained that they greatly enjoyed the tasks and that it

motivated them to want to continue the study of ring theory beyond what was introduced to them with the tasks.

### **What Might be Improved**

While constructing tasks for the student interviews, I explicitly included a section wherein students generated examples on their own and then checked examples provided to them. The goal of this subtask was to engage the students with the newly formed structure in a concrete way. My hope was that it might also elicit more spontaneous reasoning by analogy. Although the task was not a failure by any means, it did not appear to elicit anything interesting in the way of analogical reasoning. As such, while I think that example generation is still an important part of learning and understanding new structures and definitions, I don't currently think that analogical reasoning necessarily assisted with the construction of examples in any exciting way within my implementation. As such, example generation and checking can be done within this lesson plan without the aid of analogical reasoning, or modifications can be tried out instructors that can improve this task.

### **Lesson Plan Modifications (Paper #3 Appendix A)**

#### **Two-Session Modification**

Understandably, two class sessions may be too much time to devote to keep a class on track, especially if the course syllabus demands both group and ring theory in the same term or semester. However, devoting two sessions to the analogizing tasks can allow for deeper independent exploration of the analogous structures during class time and more time for class discussion as a result. The following schedule outlines an expanded version of the lesson outlined in the paper. This expanded version places more

focus on not just developing structures, but also more opportunity to engage with conjecturing theorem statements by analogy, establishing proofs of those statements, and tackling the analogizing task for quotient rings (and likely by extension, the structure of ideals.)

**Day One.** Day one lesson is as follows:

- 1a. Students familiarize themselves with the definition of Ring (15 min)
- 2a. Students do Analogizing Task for Subgroups in small groups (15 min)
- 3a. Students Develop Subring by analogy with subgroups as a class (15 min)
- 4a. Students Conjecture Theorem Statement(s) and Proof by Analogy (15 min)
  - (a) Example: Conjecture shortcut test for subrings
- 5a. Discuss Conjectured Statement(s) (and Share Proofs) as Class (15 min)
- 6a. End of class wrap up (5min)

**Day Two.** Day two lesson is as follows:

- 1b. Review the Definition of Ring (5 min)
- 2b. Students do Analogizing Task for Group Homomorphism in small groups (10min)
  - (a) See Appendix B for information on implementing this task.
- 3b. Discuss Ring Homomorphism Name and Definition as Class (10 min)
- 4b. Students Conjecture Theorem Statement(s) and Proof by Analogy (15 min)
- 5b. Discuss Conjectured Statement(s) (and Share Proofs) as Class (15 min)
- 6b. Students do Analogizing Task for Quotient Groups in small groups (20 min)
  - (a) See Appendix B for information on implementing this task.
- 7b. End of class wrap up (5 min)

### **“Bitesize” Modification**

Some instructors may not wish to devote an entire day to the analogizing tasks but may still wish to use the analogizing task to introduce structures on their own schedule, or simply have fuller autonomy in picking and choosing which structures they wish to introduce in this way. For example, an instructor may wish to gloss over the definition of subring and only conduct this lesson for ring homomorphisms.

For these instructors, the “bitesize” lesson plan outlines a condensed version of the lesson in the paper in which the analogizing task occurs only on the day that the structure is intended to be introduced to students and only partially accounts for the full day’s activities. This plan outlines a condensed iteration of the lesson for ring homomorphisms and accounts for half of an 80-minute session. This outline would be intended for use on the class day that homomorphisms are to be introduced in the curriculum.

1. Students do Analogizing Task for Group Homomorphism in small groups (10min)
  - (a) If needed, explain what is meant by analogy.
  - (a) See Appendix B for information on implementing this task.
2. Discuss Ring Homomorphism Name and Definition as Class (10 min)
3. Students Conjecture Theorem Statement(s) and Proof by Analogy (15 min)
4. Discuss Conjectured Statement(s) as Class (5 min)

### **Analogies with Other Structures (Paper #3 Appendix B)**

The lesson outlined in this article was confined to only displaying the tasks related to generating and exploring the structure of subring. However, the full lesson (including homework) incorporates further generation and exploration of two other

structures in ring theory: ring homomorphisms and quotient rings. In this appendix, I provide some thoughts, insights, and examples of theorems related to analogical reasoning surrounding these structures.

## Ring Homomorphisms

A slight step up in difficulty from the analogizing task for subgroups, the analogizing task for group homomorphisms prompts more creative results from students. In particular, students are confronted with how to adapt the homomorphism concept to *two* operations rather than just one. Common observations that arise are:

- Students may question whether the naming convention of “homomorphism” is appropriate in the ring context, or if a new name is needed.
- Students may wish to mix the preservation of operations (e.g.,  $\varphi(x+y) = \varphi(x)*\varphi(y)$ )
- Students may argue about whether one or both operations requires the homomorphism property (e.g.,  $\varphi(x+y) = \varphi(x) + \varphi(y)$ , but not necessarily that  $\varphi(x*y) = \varphi(x)*\varphi(y)$ ).
- Students may assert that two separate functions are required to define the analogous structure (e.g.,  $\varphi(x+y) = \varphi(x) + \varphi(y)$ , and  $\psi(x*y) = \psi(x) * \psi(y)$ )
- Students may argue the need for 1-1 and onto.

These student observations can lend themselves to a review of the purpose of the homomorphism property and homomorphisms for groups and a rich discussion of the analogous purpose for ring homomorphisms as functions that relate two rings together. Ideally, students can abstract the notion of homomorphism from this discussion as being a function that relates two structures. In addition, this activity can lead to a review and

discussion of isomorphisms being related to the concept of “sameness” of two groups or rings.

Once the structure of ring homomorphism is established, you may provide the students with a theorem from group theory to be translated into a theorem for ring homomorphisms. In my implementation, I used the following statement:

*Suppose  $G$  and  $H$  are groups and  $\varphi: G \rightarrow H$  is a group homomorphism. Then the image of a subgroup of  $G$  is a subgroup of  $H$ .*

### **Quotient Rings**

A significant step up in difficulty from subrings is the construction of the quotient ring by analogy with quotient groups. Because of the nature of the quotient ring structure, it is unlikely that students will develop a complete definition of the structure on their own. However, the goal of the activity is only to introduce the concept of quotient rings to students and have students recognize the existence of an analogous structure; the complete and correct definition can be provided either within this lesson or later on when appropriate depending on the wishes of the instructor. Common observations that arise during the analogizing task for quotient groups are:

- Students will wonder about the operation used for generating cosets (i.e.,  $a+H$  vs.  $a*H$ ).
- If they haven’t done so by this point, students may begin to recognize that rings are also examples of Abelian groups and thus (1) quotient rings are quotient groups by default, and (2) every subgroup is a normal subgroup.

- Students will conjecture the existence of a “normal subring” by analogy with normal subgroups. This is an excellent way to motivate the need for ideals in ring theory.
- When pursuing the “normal subring” concept, students may replace the word “group” in the usual definition of normal subgroup with the word “ring”. If this occurs, it can provide a review of the purpose of normal subgroups and a rich discussion to help motivate the need for the structure of ideals in the future.

In my implementation of this analogizing task in an interview setting, I allowed my student to revisit the definition of a quotient group and revisit the proof that a set of cosets  $G/H$  is a quotient group if and only if  $H$  is a normal subgroup of  $G$ . Letting them revisit this proof reminds them of the need to establish a well-defined operation on  $G/H$  to create a group while also reminding them the role normality plays. Rather than expecting the student to generate the definition of quotient ring or ideal from scratch, I expected my student to develop conceptual connections about the structures that were required to make a quotient ring “work”.

## VII. DISCUSSION

This dissertation has proposed three papers, each of which makes contributions to the research and instructional practice surrounding analogical reasoning in mathematics with specific attention to topics in abstract algebra. Together, these papers attend to the three research questions proposed in the beginning:

RQ1: How do students reason by analogy about rings, subrings, ring homomorphisms, and quotient rings in abstract algebra?

RQ2: How might students come to productively reason by analogy in abstract algebra?

RQ3: How might analogy and analogical reasoning be effectively incorporated into abstract algebra curriculum focused on introducing ring theory after group theory?

In particular, Paper #1 identified analogical reasoning as a *way of thinking* in mathematics that is learnable by students when given the opportunity to do so. Paper #2 introduced a framework for describing and interpreting students' analogical activity while doing mathematics. Finally, Paper #3 connected the findings of Paper #1 and Paper #2 to practice by outlining a full lesson in abstract algebra that features analogical reasoning as a tool for exploratory structure creation. In this final chapter, I connect my work back to previous literature and suggest future research possibilities that are made available based on my contributions, several of which are the product of writing memos recording my thoughts as part of data analysis.

## Connecting to Existing Literature

The primary contribution of this dissertation is the Analogical Reasoning in Mathematics (ARM) framework. The literature in mathematics education pertaining to analogical reasoning previously lacked a framework for operationalizing analogy for describing and interpreting students' analogical reasoning; ARM contributes one such way of doing so. Furthermore, much of the previous research on analogy in mathematics education involved closed forms of analogical reasoning with predetermined answers. In contrast, Lee and Sriraman (2011) introduced the notion of open-classical analogy (OCA) as a task for utilizing inventive analogy. OCA proposed three types of tasks relative to the amount of information given for a classical analogy (i.e., A is to B as C is to D):

- Provide the A and C terms of the analogy, leaving B, D to conjecture.
- Provide the A and B terms of the analogy, leaving the target to conjecture.
- Provide only A, leaving all other terms generated by the student through conjecture.

Lee and Sriraman investigated these problem types in the context of a middle school geometry classroom for gifted learners. The present research expands not only on the context and population studied, but also expands on the scope of mathematical activity that can be expected by implementing an OCA problem: students engaged in the act of defining and constructing mathematical structures that were made accessible to them through analogical reasoning. As I discuss below, the present research also observed students engage in analogical reasoning while constructing proofs, adding analogical proof activity to this expanded list.

**Abstraction and Generalization.** English and Sharry (1996) contend that abstraction is achievable through analogical reasoning and shared evidence of this phenomenon occurring within a group of grade 12 high school students who were asked to categorize a list of algebraic equations based on similarity. Looking beyond abstraction that students may achieve when given categorizing, I hypothesize that the phenomenon of analogical abstraction can play a significant role for inventive analogy. I suspect that the analogical activity a student enacts may be tied to their ability to abstract an underlying structure under consideration. For instance, the difference between exporting and importing the subgroup structure into the target domain of rings may depend upon the ability to first abstract the concept of a “sub-structure” from subgroup. In such cases, importation may even be conceptualized as the composition of an abstracting activity (which creates a third domain) together with an exportation of the abstracted structure from this third domain to the original target.

Polya (1954) further points to links between the activities of generalizing, specializing, and analogizing. Findings within this study provide evidence for this hypothesized connection between analogizing and generalizing: when analogically comparing between groups and rings, some students became aware of the possibility for more general structures, such as a set with three operations, or generalized their observations to similar ideas outside of abstract algebra, such as how the notion of homeomorphism relates to homomorphism and isomorphism. Connections can be made to the Relating-Forming-Extending framework developed by Ellis, Tillema, Lockwood and Moore (2017). In particular, this framework identifies inter-contextual and intra-contextual forms of generalizing to determine whether a student is perceiving similarity

as being established across contexts, or within the same context. The activities identified within ARM can be put into communication with the inter-contextual category of activity. In coordination with ARM, I hypothesize a more robust description of how students identify or generate similarities/differences across examples is possible.

**The Process of Analogical Structure Creation.** An original intent of this dissertation was to develop a model for how students create mathematical structures through analogy. As such, a key element of the analogizing task contributed by this dissertation is the construction of mathematical structure by way of analogy with a previously known structure. As I previously discussed, Stehlíková and Jirotková (2002) and Hejný (2002) recognized the possibility of constructing mathematical structures through analogical reasoning. Structure creation is described with a model of Internal Mathematical Structure (IMS), a representation of mathematical structures that lies within an individual's mind. Stehlíková and Jirotková (2002) investigated the processes of building IMSs in the specific context of reasoning analogically between two arithmetic structures, and described five phenomena that they claimed were specific to the building of a structure by analogy: regularities, anomalies, broadening intuition, obstacles, and the development of new strategies.

Much like the analogical activities presented in the ARM framework, these five phenomena relate to general aspects of analogical reasoning rather than describe a process of analogical reasoning as a whole. However, the ARM framework also identifies specific activities that can occur while reasoning by analogy. In particular, the ARM framework dissects the process of reasoning by analogy and interprets local analogical reasoning as opposed to the global process. The focus of this dissertation was on

students' analogical reasoning and analogical activity rather than attending specifically to the process of constructing a mathematical structure by analogy. Using ARM as a tool for describing student's mathematical activity while reasoning by analogy, I hypothesize that the process of generating structures can be abstracted and described across mathematics content areas outside of undergraduate algebra as well as within. This would synergize well with the heuristic of analogical reinvention: describing processes of analogical structure creation may produce generic templates for how students may approach analogical reinvention.

### **Directions for Future Research**

Several avenues of research become available from the line of inquiry into analogical reasoning in mathematics established in this dissertation. In this section, I outline directions for future research based both on the results coming from the proposed papers as well as my observations while working on this project. I situate these projects within three categories: (1) investigating what students understand and learn about structures in ring theory when reasoning by analogy about analogous structures in group theory; (2) developing curriculum incorporating analogy and analogical reasoning, specifically for the teaching of ring theory after group theory; and (3) investigating the process of analogical reasoning in various content areas of mathematics and its ties to other mathematical processes.

### **Students' Content Knowledge in Ring Theory**

The context of reasoning by analogy between structures in ring theory and group theory not only allows for the investigation of how students reason by analogy, but also establishes a way to investigate what students understand about concepts in ring theory.

In particular, there may be affordances to students' understanding of structures in ring theory provided by the novel context of constructing structures by analogy with previously known structures in group theory. Furthermore, students may come away with enriched understandings of concepts in group theory after developing structures in ring theory by analogy. Research on this type of interplay between knowledge of group theory and ring theory would be a step toward understanding how students can develop structuralist thinking in abstract algebra, such as structure sense (Simpson & Stehlíková, 2006).

I propose that students can build meaningful connections between the content of ring theory and group theory when reasoning by analogy in such a way that their understanding of both topics is strengthened. Some evidence of this phenomena exists within the results of the present study. For example, after discussing “normal subrings” and later being given the definition of ideal, one participant made the following observation about the structure:

That definitely did not look the same. I just tried to copy it and I had reasoning behind why I thought I would need the multiplication. It turns out no, we did addition. So, that was really confusing. And then I forgot I had learned ideal. I forgot I learned it because it didn't seem useful in any way. There was no use for me to have it at that time. But yeah, now when you start relating it to the group theory... If this was a class, I'll probably learn a lot more why I know this stuff, which is maybe how graduate school is; you start talking about the why's and stuff like that.

In particular, this student suggests that he had actually seen the definition for ideal in the past, but had forgotten about it because it was not “useful in any way”. However, because he had engaged with developing the structure through analogy and had made several connections back to factor groups and normal subgroups, he felt that he had a clearer grasp on the overall purpose for the structure. Future research can investigate how

students come to recognize these sorts of meaningful connections and determine which of those connections are most productive for assisting students with learning ring theory.

### **Developing Curriculum Incorporating Analogy**

I hypothesize that analogy and analogical reasoning can be incorporated into a variety of mathematics curriculum where exploration is a goal. In this dissertation, I displayed Paper #3 as a proof of existence for this hypothesis in the context of exploratory structure creation. My intuition suggests that this may be expanded into other domains as well, such as exploring and reinventing the definition of limit of a sequence in a general topology by analogy with the standard definition of limit of a sequence given in real analysis, or by analogy with the definition in a general metric space. In this section, I provide my thoughts on developing a curriculum incorporating analogy in the context of abstract algebra: analogical reinvention of ideal.

**Analogical Reinvention of Ideal.** Abstract algebra is a foundational course for future teachers of mathematics as well as future mathematicians. Being that research on student thinking in abstract algebra is overwhelmingly dominated by group theory, there is much to be learned about how students think about topics in ring theory. Some results exist investigating how students might reinvent rings, integral domains, and fields (Cook, 2012), as well as how students understand basic properties of rings (Cook, 2014; 2017). The results of this dissertation complement this work by offering a way to explore not only how students think about certain topics in ring theory, but also the interplay of knowledge between group and ring theory. Leveraging students' knowledge of group theoretic constructs to introduce topics in rings, in addition to further investigating how students understand ring theoretic structures when reasoning by analogy, provides a basis

for establishing a curriculum for the teaching of ring theory after introducing group theory.

The existing mathematics education literature related to ring theory has not fully approached the topic of ideal. This fact may be a reflection of the complexity associated with the structure of ideal (Jovignot, Hausberger & Durand-Guerrier, 2017), and the complexity of developing an effective curriculum for introducing the structure of ideal in an abstract algebra course. While the results of this dissertation do not offer insight into a stand-alone curriculum for the guided reinvention of ideal, this dissertation points to the possibility of a different approach: a heuristic for the exploration and discovery of ideals through analogy: analogical reinvention. In other words, guided reinvention of ideal could be achieved with the assistance of analogical reasoning with normal subgroups in group theory. The following outline suggests initial impressions of a possible learning trajectory for analogical reinvention of ideals:

- Students will understand that the group structure is embedded in rings (especially the subgroup structure being contained within subrings.)
- Students will understand that the ring structure contains an Abelian group, so every subgroup is normal.
- Students will develop the quotient ring structure by analogy with quotient group. A goal of this would be to recognize the need for ideals. Likely, they will be referred to as normal subrings. A subgoal would be to recognize that the normal subgroup structure dictates what the cosets of the factor ring would look like.
- Students will develop the notion of cosets during the creation of quotient ring as well as formulate initial thinking about ideals.

- Students will understand that the quotient ring structure should be a ring and need to properly define the operations on the cosets.
- Students will recall that the property of normality for normal subgroups allows operations on factor groups to be well defined. Students will abstract this concept and recognize that the operations of addition and multiplication must be well-defined to form a quotient ring.
- Students will establish the definition of ideal.

**Analogical Proof Activity.** An immediate observation coming out of the present study is that students' analogical proof activity is of particular interest. Examples of proof by analogy are ubiquitous in abstract algebra due to the existence of several commonalities between the structures of group and ring theory. For instance, consider the following quote from Gallian (2010): "The next three theorems parallel results we had for groups. The proofs are nearly identical to their group theory counterparts and are left as exercises" (p. 283). An underlying assumption is made in this quote: because the relevant analogous proofs are found in group theory, they are meant to be straightforward and do not require a proof written in the book. However, of the three theorems being referred to in this quote, one is the first homomorphism theorem for rings, a theorem that is hardly considered trivial in an introductory course in abstract algebra.

It is unclear whether students appreciate the apparent simplicity in suggesting that a theorem about a new context is obvious by analogy with a previously known theorem. This raises some questions: To what extent do students accept proofs by analogy? In the realm of science education, Kapon and diSessa (2012) found that while students accepted some analogical arguments outright, they also possess hesitations or reservations about

other analogically established arguments. Furthermore, how might students productively construct, comprehend, and validate proofs leveraging analogical reasoning?

## APPENDIX SECTION

### APPENDIX A

#### Group Theory Interview Protocol

##### Participant's History with Abstract Algebra

1. How many courses in modern algebra have you had? When did you last take a course in modern algebra?
2. Was there any coverage of ring theory in any of your classes? If so, what did you learn?

##### Definition of Group

3. What is the definition of group?

##### Student Generated Examples

4. Provide an example of a group.
  - *If no answer, ask them to recall examples from class.*
  - *If no answer, ask specifically about integers under addition*
5. Provide a nonexample of a group.
  - *If no answer, ask them if they can recall basic example from class.*
  - *If no answer, ask about integers under subtraction.*

##### Determining if given sets are Groups

6. Determine if the following sets are groups:
  - i. The set of integers modulo  $n$  under addition,  $\mathbb{Z}_n$ .
  - ii. The set of natural numbers under addition,  $\mathbb{N}$ .
7. Can the set  $\{0,2,4,6,8\}$  form a group? Why or why not?
8. Can the set  $\{a,b,c\}$  form a group? Why or why not?

##### Subgroups

9. What is the definition of subgroup?
10. Is  $\mathbb{Z}_3$  a subgroup of  $\mathbb{Z}_9$ ?
  - If no details given: How do you determine if a  $H$  is a subgroup of  $G$ ?

##### Group Homomorphisms

11. What is the definition of group homomorphism?
12. Is the function defined by  $f(x) = \frac{1}{2}x$  a homomorphism from  $2\mathbb{Z}$  to  $\mathbb{Z}$ ?

##### Definition of Normal Subgroups and Quotient Groups

13. What is the definition of a normal subgroup?
14. What is the definition of a quotient group?

##### Asking about Other Topics

15. Do any other topics from group theory come to mind that were not on this list? If so, please list them.

## APPENDIX B

### Ring Theory Interview Protocol

#### Introducing Definition of Ring

- Present the participant with the definition of ring. Allow them time to study the definition. Ask them to think aloud as they study the definition.

#### Working with Examples of Rings

1. Determine if the following sets are rings under their usual addition and multiplication:
  - iii. The set of integers modulo  $n$ ,  $\mathbb{Z}_n$ .
  - iv. The set of natural numbers,  $\mathbb{N}$ .
    - *If the student does not recognize the structures as groups, ask if and how knowledge of groups could have helped assess these examples.*
2. Can the set  $\{0,2,4,6,8\}$  form a ring? Why or why not?
  - *If answer is same/different from that given for groups, ask them to compare.*

#### Student Generated Examples of Ring

3. Construct an example of a ring.
  - *If no answer, ask them to recall examples from groups.*
  - *If no answer, ask specifically about integers under addition*
4. Construct a nonexample of a ring.
  - *If no answer, ask them if they can recall examples from groups.*
  - *If no answer, ask about integers under subtraction.*

#### Basic Proofs Involving Rings

5. Prove that the additive identity of a ring is unique.
  - *If spontaneous analogy is not present, ask about the analogous theorem in group theory.*
6. Let  $(R, +, *)$  be a ring. Prove that  $a^2 - b^2 = (a + b)(a - b)$  for all  $a, b \in R$  if and only if  $R$  is commutative.

#### Properties of Rings

7. Determine if the following properties are true for the ring  $\mathbb{Z}_6$ .
  - a.  $a^2 = a$  implies that  $a = 0$  or  $a = 1$ .
  - b.  $ab = 0$  implies  $a = 0$  or  $b = 0$ .
  - c.  $ab = ac$  and  $a \neq 0$  imply  $b = c$

#### Connections to Group Theory

8. What connections (if any) have you made to group theory during this session?
9. How might you expect the study of rings to proceed?

## APPENDIX C

### Example of Analogizing Task Interview

#### Constructing the Definition of Subring

1. Make a conjecture for a structure in ring theory that is analogous to subgroups in group theory.
  - a. *If the participant is unsure of the meaning of “analogous,” assist them with an example (i.e. analogy between atom and solar system) or a diagram.*
  - b. *When finished with spontaneous comparisons: Ask what similarities there are to subgroups.*
  - c. *Ask what differences there are to subgroups.*

#### Student Generated Examples of Subrings (given a Ring)

2. Is  $\mathbb{Z}$  (with usual addition and multiplication) a subring of any rings?
  - *If no answer, ask if  $\mathbb{Z}$  is a subgroup of any groups.*
3. Does  $\mathbb{Z}$  (with usual addition and multiplication) contain any proper subrings?

#### Conjecturing the Subring Test

4. If test did not show up during structure creation present the following task:  
Conjecture a way to test whether  $S$  is a subring of a ring  $R$ .
  - *If no answer, prompt them to recall the subgroup test.*

#### Determining Examples of Subrings

5. Is  $\mathbb{Z}_3$  a subring of  $\mathbb{Z}_9$ ?
  - *If no answer, ask to compare to response given for groups.*
6. Suppose that  $R$  is a ring and  $a$  is an element of  $R$  such that  $a^2 = 1$ .  
Let  $S = \{ara | r \in R\}$ . Prove that  $S$  is a subring of  $R$ .

#### Proofs Involving Subrings

7. Suppose  $R$  is a ring and  $S$  and  $T$  are subrings of  $R$ . Prove that  $S \cap T$  is a subring of  $R$ .

#### Connections to Group Theory

8. What connections (if any) have you made to group theory during this session?
9. How might you expect the study of rings to proceed?

## APPENDIX D

### Example of Diagram and Description of Codes

|              | Source Domain   | Target Domain                     |
|--------------|---|-----------------------------------|
| Similarities | Obstacles<br>Expanding?<br>Carry-over<br>Identifying regularities | Extending?                        |
| Differences  | Identifying anomalies   | Adapting<br>Broadening intuition? |

For this initial attempt at a diagram, I have decided to try and parse out different types of activities that may occur based on whether the student is foregrounding the source or target domain, and whether the focus is on identifying/generating(?) similarities or differences. I have attempted to map the five phenomena (regularities, anomalies, obstacles, broadening intuition, and adaption) into these categories as a starting point.

I describe the four resulting categories here:

#### ***Source Domain Foregrounded:***

***Similarities:*** This category describes those instances where a student is identifying or generating similarities between a source and target where the student's focus is on the source structure. This can occur when the student is "stuck" in their way of thinking about the old structure and is struggling to adapt to the new context of the target, or when the student is actively seeking similarities between the structures that originate from their understanding of the source.

Example: In this example, the student is conjecturing that Sylow theory will still be applicable in ring theory. The source domain is foregrounded because the conjecture is based entirely within knowledge of group theory with no clear sign of adapting to the new context of rings.

*Student:*        *P-Sylow subrings ...*

*Interviewer:*    *That Sylow theory will come back [crosstalk 01:07:37].*

*Student:*        *Yeah. I feel like it'll probably come back around. I assume something would still be applicable in ring theory.*

***Differences:*** This category describes those instances of analogical reasoning in which a student is identifying or generating differences while maintaining focus on then source structure. This can occur when the student is actively seeking what is different between

two analogous structures based on knowledge of the source structure.

Example: In this example, the student is examining the definition of ring and reflecting on the definition. He recognizes that identities are not necessary for multiplication and comments on how that is different from groups. The source is foregrounded because it is driving the recognition of the difference.

*Interviewer: Okay. So, our first question here is, just to reflect on the above definition. Is there anything that stands out to you about this definition?*

*Student: Yeah, I would say it's not necessary. No identity to the ... for multiplication.*

*Interviewer: Could you explain why that caught your attention?*

*Student: Because everything in group theory ... there's a very big part of it, is that there needs to be identity and there's also, even clear in this, an identity for addition it just seems interesting that when we throw in another binary operation it doesn't necessarily need to have an identity as well. Which is just interesting to think about.*

#### **Target Domain Foregrounded:**

**Similarities:** This category describes those instances of analogical reasoning in which a student is identifying or generating similarities while focusing on the target structure.

Example: In this example, the student is conjecturing the existence of ring homomorphisms. The target domain is foregrounded because although the conjecture is based within knowledge of group homomorphisms, the student is exhibiting signs of adapting to the new context of ring theory (sort of like recognizing or predicting that differences will eventually occur).

*Student: Well, we already talked about sub rings, so then after that we're probably gonna want to talk about functions on rings. So, homomorphisms but in terms of rings I don't know exactly what you would call them. Next so would be functions.*

*Interviewer: So some ideas similar to homomorphisms?*

*Student: Yeah. Functions on ring forming homomorphism-like structures. From there, once you're talking about homomorphisms that kind of just opens up a lot other areas.*

**Differences:** This category describes those instances of analogical reasoning in which a student is identifying or generating differences while focusing on the target structure.

Example: This example is a continuation of the source domain difference example above. In this example, the target is foregrounded because the student is now considering how the difference manifests itself in the context of rings.

*Student: Which also, I would say, opens up a lot more possibilities for what your ring is allowed for because I know if you're talking about ... if you're wanting to talk about multiplication, I can't have multiplication under ... like, if you do multiplication of positives or something like that when you're talking about integers to have a group and stuff. Or something like that, I don't remember exactly what it was, because the problem of the identity or something like that.*

*Student: But this'll probably open up a lot more options since you won't necessarily need the inverse part to work out or something like that, you know? Oh, that's right. It couldn't have zero because there's no inverse for zero. That's what it was. Under multiplication. But if you're talking about this same set were addition is the first property, then you possibly could still do multiplication because you don't need to have that inverse property under the multiplication.*

## APPENDIX E

### Example of Memo: Negative Analogical Reasoning

Date: May 7<sup>th</sup>, 2020

Title: Thoughts on Coding Quotient Ring A&B

In the quotient ring interview, Student A flat out admits that she can't recall anything about what a factor group is. I hadn't picked up on something like this just yet, which seems rather surprising given that it seems like a natural response. I suspect more of this happens in Student A's subring interview. I think that capturing this with some kind of code or flag could be useful for describing a student's overall process of reasoning by analogy. Struggling to recall information could be a sign for instructors to provide support in some way during instruction.

### Meta-Level Analogical Reasoning

Student B engaged with an interesting example of analogical reasoning about factor rings by first recalling what he did when constructing ring homomorphisms. This is the first instance of this that I have seen. In this case, the student is recalling a previous episode of reasoning by analogy and considers what was useful. He seems to abstract his previous episode by observing that he wanted *conditions* for ring homomorphisms (rather than multiple phis) and maps this concept of conditions to the factor group/ring context. What's interesting to me about this example is that it seems to suggest that he is learning about some form of reasoning by analogy in and of itself, rather than just engaging with the act of reasoning by analogy.

Student B: ... like the cosets. And then we'll put that mark, we'll call that coset. Factor groups. Now, what about rings? What would they even be called? How would they even work? Let's write it out. It would be G star and H

star versus here, we would have  $R$  plus and  $S$  plus. And why don't we say that  $S$  is a subring of  $R$ , and  $R$  over  $S$  would be the... Actually, these are using this binary operation. So, this would be the set of... Let's see. Last time we did subring, right?

Interviewer: The last interview?

Student B: Uh-huh (affirmative).

Interviewer: It was ring homomorphism. Subring was the previous one.

Student B: So, ring homomorphism was... We had two functions, right?

Interviewer: Can you explain?

Student B: Group homomorphism is, there's a  $\Phi$  defined describing the relationship from one group mapping to the next group, and then ring homomorphism is a function  $\Phi$  defined mapping... I thought it was a ring under one operation to the next ring under the first... so, first binary operations, and then there was another  $\Phi$  that mapped the other operation.

Interviewer: So, just one  $\Phi$ ?

Student B: There's one  $\Phi$ -

Interviewer: And it's actually mapping both operations?

Student B: Mapping the rings.

Interviewer: Yea. And both operations are involved in it, but just the one function?

Interviewer: (silence).

Interviewer: [inaudible 00:02:46].

Student B: Oh, it's the fact that these have to be true. That's what I was thinking. I was thinking there was multiple  $\Phi$ s, but it's multiple conditions that have to be true.

Interviewer: Right.

Student B: So, here, I don't know, it doesn't feel like we have conditions. I understand that this is a condition.  $G$  over  $H$  is a set of cosets under this operation. So,  $R$  over  $S$ , I'm guessing is going to be cosets. It's just confusing now. I want to just copy it, but what's the operation between here, and why is it only one operation? I'm thinking... Let's see, we'll put idea is that we have  $S$

times  $A$  for all  $A$  in  $R$ , we have  $S$  plus  $A$  for all  $A$  in  $R$ . And then it goes the other way too, because they're rings. Maybe it could be piecewise or something like that. Like this. Yeah, we'll do that.

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