

## EXISTENCE OF A SOLUTION AND ITS NUMERICAL APPROXIMATION FOR A STRONGLY NONLINEAR COUPLED SYSTEM IN ANISOTROPIC ORLICZ-SOBOLEV SPACES

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ABSTRACT. We study the existence of a capacity solution for a nonlinear elliptic coupled system in anisotropic Orlicz-Sobolev spaces. The unknowns are the temperature inside a semiconductor material, and the electric potential. This system may be considered as a generalization of the steady-state thermistor problem. The numerical solution is also analyzed by means of the least squares method in combination with a conjugate gradient technique.

### 1. INTRODUCTION

This work concerns a generalization of the steady-state thermistor problem. It consists of two coupled nonlinear elliptic equations governing the temperature,  $u$ , and the electric potential,  $\varphi$ , inside a semiconductor device, namely,

$$\begin{aligned} -A(u) &= \rho(u)|\nabla\varphi|^2 && \text{in } \Omega \\ \operatorname{div}(\rho(u)\nabla\varphi) &= 0 && \text{in } \Omega, \\ \varphi &= \varphi_0 && \text{on } \partial\Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Omega \subset \mathbb{R}^d$  (the thermistor geometry) is a bounded domain,  $d \geq 2$  is an integer, and the operator  $A$ , given by

$$A(u) = \sum_{i=1}^d \partial_i(a_i(x, u, \partial_i u)), \quad \partial_i = \frac{\partial}{\partial x_i},$$

which is assumed to be of the Leray-Lions type on certain Orlicz-Sobolev spaces. For each  $i = 1, \dots, d$ , the function  $a_i(x, s, \zeta) : \Omega \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  is a Carathéodory function, that is, measurable with respect to  $x$  in  $\Omega$  for all  $(s, \zeta) \in \mathbb{R}^2$ , and continuous with respect to  $(s, \zeta)$  for almost every  $x \in \Omega$ . The vector function  $\mathbf{a} = (a_1, \dots, a_d)$  satisfies certain monotonicity and coercivity conditions in the anisotropic Orlicz-Sobolev space

$$W^1L_{\mathcal{M}}(\Omega) = \{u \in L_{M_0}(\Omega) : \partial_i u \in L_{M_i}(\Omega), i = 1, \dots, d\},$$

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where  $\mathcal{M} = (M_1, \dots, M_d)$ ,  $M_1, \dots, M_d$  are  $N$ -functions which, in general, do not satisfy the  $\Delta_2$ -condition, and  $M_0 = \min_{1 \leq i \leq d} M_i$ . Also,  $\rho \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$  stands for the temperature dependent electric conductivity and  $\varphi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$  is given.

System (1.1) is a mathematical model which generalizes the so-called thermistor problem [2, 6, 10, 13]. In most practical cases, one has  $\rho(s) > 0$  for all  $s \in \mathbb{R}$  and  $\rho(s) \rightarrow 0$  as  $s \rightarrow +\infty$ . In particular, the equation for  $\varphi$  is nonuniformly elliptic, and consequently, no a priori estimates for  $\nabla\varphi$  will be available so that  $\varphi$  may not belong to a Sobolev space. This means that the search for weak solutions to (1.1) is not well-suited in this setting.

To deal with this difficulty, we consider the function  $\Phi = \rho(u)|\nabla\varphi|^2$  as a whole and then show that it belongs to  $L^2(\Omega)^d$ . A new formulation of system (1.1) will lead us to the introduction of the notion of capacity solution.

The concept of capacity solution was first introduced by Xu in [17] in the analysis of a modified version of the evolution thermistor problem. He also applied this concept to more general settings where weaker assumptions [18] or mixed boundary conditions [19] are considered. Later on, it has also been used by other authors in different situations [6, 12, 14]. For instance, in [6] the authors analyzed the existence of a capacity solution for the evolution thermistor problem in  $W^{1,p}$  for  $p \geq 2$ . Moussa et al. [12] studied this system in isotropic Orlicz-Sobolev spaces, whereas Talbi et al.

citeHajar considered the anisotropic case with polynomial growth with respect to the variable  $\zeta$ .

The goals of this paper are twofold. First, we analyze the existence of a capacity solution to (1.1) for arbitrary and different growths of the functions  $a_i(x, s, \zeta)$ ,  $i = 1, \dots, d$  (in particular, some or all of the functions  $M_i$ ,  $i = 1, \dots, d$ , do not need to satisfy the  $\Delta_2$ -condition). For instance, in  $d = 2$ , we may have  $a_1(x, s, \zeta) = |\zeta|^{p-2}\zeta$  and  $a_2(x, s, \zeta) = 2\beta \exp(\beta\zeta^2)\zeta$  where the corresponding  $N$ -functions are given by  $M_1(\zeta) = |\zeta|^p/p$  and  $M_2(\zeta) = \exp(\beta\zeta^2) - 1$ , respectively; in this case,  $M_1$  satisfies the  $\Delta_2$ -condition whereas  $M_2$  does not. Secondly, we describe a numerical algorithm for the approximation of the solution to problem (1.1) based on the least squares method combined with a conjugate gradient technique [4, 5]. Though some numerical simulations have led to good results by using this algorithm, the numerical resolution of (1.1) for arbitrary functions  $a_i$  remains a challenge; this is related to the machine precision. Indeed, since this algorithm generates a sequence of approximate solutions for the temperature, say  $(u_h^m)$ , by means of descent directions,  $(z_h^m)$ , that is  $u_h^{m+1} = u_h^m - \lambda_m z_h^m$ , where  $\lambda_m > 0$  is an optimal value, it may occur that  $\lambda_m$  becomes smaller than the machine precision which may result in an underflow situation. Consequently,  $\lambda_m$  is taken to be zero inside the machine, and the algorithm would produce  $u_h^{m+1} = u_h^m$  after that value of  $m$ .

Notice that the numerical resolution of system (1.1) is an important issue. These numerical simulations may yield a very useful information while designing a thermistor for certain specific purposes. In this sense, knowing the steady-state temperature distribution inside the thermistor for a given geometry  $\Omega$  and potential  $\varphi_0$  is crucial, although this is only a first step in this analysis (we also want to know the whole history of both, temperature and potential, from a known initial

temperature up to being close enough to the steady-state by solving the evolution problem).

This article is organized as follows. In Section 2 we introduce some notation, concepts and functional spaces together with certain technical results that will be needed along this paper. Section 3 states the assumptions on data and introduces the concept of a capacity solution to (1.1) in the framework given in the previous section. In Section 4 we present an existence theorem along with its proof. Finally, Section 5 is devoted to the description of a numerical algorithm for the approximate solution to problem (1.1) for certain choices of the functions  $a_i$ ,  $1 \leq i \leq d$ , including some numerical results obtained by the implementation of this algorithm.

## 2. PRELIMINARIES

We begin by recalling some definitions and properties of Orlicz spaces [1, 11] and then we introduce the anisotropic Orlicz-Sobolev spaces.

**2.1.  $N$ -functions.** The basic concept in an Orlicz normed space is that of  $N$ -function.

**Definition 2.1.** A function  $M: \mathbb{R} \rightarrow \mathbb{R}$  is called an  $N$ -function if it fulfills the following conditions:

- (i)  $M$  is convex in  $\mathbb{R}$ :  $M(\lambda s_1 + (1 - \lambda)s_2) \leq \lambda M(s_1) + (1 - \lambda)M(s_2)$ , for all  $s_1, s_2 \in \mathbb{R}$  and for all  $\lambda \in [0, 1]$ .
- (ii)  $M$  is an even function:  $M(s) = M(-s)$  for all  $s \in \mathbb{R}$ .
- (iii)  $M(0) = 0$  and  $M(s) > 0$  for all  $s \in \mathbb{R}$ .
- (iv)  $\frac{M(s)}{s} \rightarrow 0$  as  $s \rightarrow 0$  and  $\frac{M(s)}{s} \rightarrow +\infty$  as  $s \rightarrow +\infty$ .

An  $N$ -function  $M$  is said to satisfy the  $\Delta_2$ -condition for all  $s \in \mathbb{R}$  if, for some  $k > 0$ ,

$$M(2s) \leq k M(s) \quad \text{or all } s \in \mathbb{R}.$$

We say that  $M$  satisfies the  $\Delta_2$ -condition for  $s$  large if there exist  $s_0 \geq 0$  and  $k > 0$  such that

$$M(2s) \leq k M(s) \quad \text{for all } s \geq s_0.$$

An equivalent definition [11] of an  $N$ -function is a function  $M$  that admits the representation

$$M(s) = \int_0^{|s|} m(\sigma) \, d\sigma,$$

where  $m: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a non-decreasing and right-continuous function,  $m(s) > 0$  for all  $s > 0$  and  $m(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ . For an  $N$ -function  $M$ , the complementary or conjugate is defined by

$$\bar{M}(s) = \int_0^{|s|} \bar{m}(\sigma) \, d\sigma,$$

where  $\bar{m}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is given by  $\bar{m}(t) = \sup\{s : m(s) \leq t\}$ .

We have the Young's inequality

$$|ts| \leq M(t) + \bar{M}(s) \quad \text{for all } t, s \in \mathbb{R}.$$

Let  $\Omega$  be an open set in  $\mathbb{R}^d$  and  $d \in \mathbb{N}$ . The Orlicz class  $\mathcal{L}_M(\Omega)$  (resp. the Orlicz space  $L_M(\Omega)$ ) is defined as the set of (equivalence classes of) real-valued Lebesgue measurable functions  $u$  in  $\Omega$  such that

$$\int_{\Omega} M(u(x)) \, dx < +\infty \quad (\text{resp. } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0).$$

Notice that  $L_M(\Omega)$  is a Banach space under the so-called Luxemburg norm

$$\|u\|_M = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \right\},$$

and  $\mathcal{L}_M(\Omega)$  is a convex subset of  $L_M(\Omega)$ . Indeed,  $L_M(\Omega)$  is the linear hull of  $\mathcal{L}_M(\Omega)$ . The closure in  $L_M(\Omega)$  of the set of bounded measurable functions with compact support in  $\bar{\Omega}$  is denoted by  $E_M(\Omega)$ . The equality  $E_M(\Omega) = L_M(\Omega)$  holds if and only if  $M$  satisfies the  $\Delta_2$ -condition, for all  $s$  or for  $s$  large according to whether  $\Omega$  has infinite measure or not. The dual of  $E_M(\Omega)$  can be identified with  $L_{\bar{M}}(\Omega)$  by means of the duality pairing  $\int_{\Omega} u(x)v(x) \, dx$ , and the dual norm on  $L_{\bar{M}}(\Omega)$  is equivalent to  $\|\cdot\|_{\bar{M}}$ .

In  $L_M(\Omega)$  we define the Orlicz norm

$$\|u\|_{(M)} = \sup \int_{\Omega} u(x)v(x) \, dx \quad (2.1)$$

where the supremum is taken over all  $v \in E_{\bar{M}}(\Omega)$  such that  $\|v\|_{\bar{M}} \leq 1$ . It turns out that the norms  $\|\cdot\|_M$  and  $\|\cdot\|_{(M)}$  are equivalent. In fact, it can be shown that

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M \quad \text{for all } u \in L_M(\Omega). \quad (2.2)$$

Also, the Hölder inequality holds

$$\int_{\Omega} |u(x)v(x)| \, dx \leq \|u\|_M \|v\|_{(\bar{M})} \quad \text{for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega),$$

and by (2.2)

$$\int_{\Omega} |u(x)v(x)| \, dx \leq 2\|u\|_M \|v\|_{\bar{M}} \quad \text{for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega).$$

In particular, if  $\Omega$  has finite measure, Hölder's inequality yields the continuous inclusion  $L_M(\Omega) \subset L^1(\Omega)$ .

An important inequality in  $L_M(\Omega)$  is the following:

$$\int_{\Omega} M(u(x)) \, dx \leq \|u\|_{(M)} \quad \text{for all } u \in L_M(\Omega) \text{ such that } \|u\|_{(M)} \leq 1, \quad (2.3)$$

wherefrom we readily deduce

$$\int_{\Omega} M\left(\frac{u(x)}{\|u\|_{(M)}}\right) \, dx \leq 1 \quad \text{for all } u \in L_M(\Omega) \setminus \{0\}. \quad (2.4)$$

**Definition 2.2.** We say that  $(u_n) \subset L_M(\Omega)$  converges to  $u \in L_M(\Omega)$  for the modular convergence in  $L_M(\Omega)$  if, for some  $\lambda > 0$ , one has

$$\int_{\Omega} M\left(\frac{u_n(x) - u(x)}{\lambda}\right) \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Modular convergence is weaker than the convergence in the norm of  $L_M(\Omega)$ . However, it is enough to our purposes. The next result tells us that the modular convergence in  $L_M$  implies the convergence in the weak-\* topology  $\sigma(L_M, L_{\bar{M}})$ .

**Lemma 2.3** ([3, 7]). *Let  $(u_n) \subset L_M(\Omega)$ ,  $u \in L_M(\Omega)$  and  $v \in L_{\bar{M}}(\Omega)$  such that  $u_n \rightarrow u$  with respect to the modular convergence. Then*

- (1)  $u_n v \rightarrow uv$  strongly in  $L^1(\Omega)$ . In particular,  $\int_{\Omega} u_n v \rightarrow \int_{\Omega} uv$ .
- (2) Furthermore, if  $(v_n) \subset L_{\bar{M}}(\Omega)$  is such that  $v_n \rightarrow v$  with respect to the modular convergence, then  $u_n v_n \rightarrow uv$  strongly in  $L^1(\Omega)$ .

**2.2. Anisotropic Orlicz-Sobolev spaces.** Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ , and  $M_i$  be an  $N$ -function for each  $i = 1, \dots, d$ . We write  $\mathcal{M} = (M_1, \dots, M_d)$ ,  $\bar{\mathcal{M}} = (\bar{M}_1, \dots, \bar{M}_d)$ . The anisotropic Orlicz space  $L_{\mathcal{M}}(\Omega)$  (respectively,  $E_{\mathcal{M}}(\Omega)$ ) is defined by

$$L_{\mathcal{M}}(\Omega) = \prod_{i=1}^d L_{M_i}(\Omega) \quad (\text{respectively, } E_{\mathcal{M}}(\Omega) = \prod_{i=1}^d E_{M_i}(\Omega)),$$

endowed with the norm

$$\|u\| = \sum_{i=1}^d \|u_i\|_{M_i}, \tag{2.5}$$

To introduce the anisotropic Orlicz-Sobolev spaces it will be interesting to define the function

$$M_0(s) = \min_{1 \leq i \leq d} M_i(s). \tag{2.6}$$

**Remark 2.4.** *It is easy to check that:*

- (i) The function  $M_0$  is an  $N$ -function.
- (ii) The embedding  $L_{M_i}(\Omega) \hookrightarrow L_{M_0}(\Omega)$  is continuous for each  $i \in \{1, \dots, d\}$ .

The anisotropic Orlicz-Sobolev spaces are defined by

$$\begin{aligned} W^1 L_{\mathcal{M}}(\Omega) &= \{u \in L_{M_0}(\Omega) : \partial_i u \in L_{M_i}(\Omega), i = 1, \dots, d\}, \\ W^1 E_{\mathcal{M}}(\Omega) &= \{u \in E_{M_0}(\Omega) : \partial_i u \in E_{M_i}(\Omega), i = 1, \dots, d\}, \end{aligned}$$

which are Banach spaces under the norm

$$\|u\|_{1, \mathcal{M}} = \|u\|_{M_0} + \sum_{i=1}^d \|\partial_i u\|_{M_i}. \tag{2.7}$$

Both spaces,  $W^1 L_{\mathcal{M}}(\Omega)$  and  $W^1 E_{\mathcal{M}}(\Omega)$ , can be identified as subspaces of the product space  $\Pi = L_{M_0}(\Omega) \times L_{\mathcal{M}}(\Omega)$ . Then, the predual space of  $\Pi$ ,  $\hat{\Pi}$ , is  $\hat{\Pi} = E_{\bar{M}_0}(\Omega) \times E_{\bar{\mathcal{M}}}(\Omega)$ . We will use the weak-\* topology  $\sigma(\Pi, \hat{\Pi})$ . Let  $\mathcal{D}(\Omega)$  be the space of functions in  $\mathcal{C}^\infty(\Omega)$  with compact support in  $\Omega$ . The space  $W_0^1 E_{\mathcal{M}}(\Omega)$  is defined as the (norm) closure of the space  $\mathcal{D}(\Omega)$  in  $W^1 E_{\mathcal{M}}(\Omega)$ , and the space  $W_0^1 L_{\mathcal{M}}(\Omega)$  as the  $\sigma(\Pi, \hat{\Pi})$ -closure of  $\mathcal{D}(\Omega)$  in  $W^1 L_{\mathcal{M}}(\Omega)$ .

**Lemma 2.5** ([12]). *Let  $\Omega$  be a bounded and open set in  $\mathbb{R}^d$ . Assume that  $m_i(t) \geq t$  for all  $t \geq 0$  and all  $i = 1, \dots, d$ . Then the following continuous embeddings hold for  $i = 1, \dots, d$ :*

$$L_{M_i}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{M}_i}(\Omega).$$

*In particular,  $W_0^1 L_{\mathcal{M}}(\Omega) \hookrightarrow H_0^1(\Omega)$  and  $H^{-1}(\Omega) \hookrightarrow W^{-1} L_{\bar{\mathcal{M}}}(\Omega)$ .*

**Remark 2.6.** *Assume that, for each  $i = 1, \dots, d$ , one has  $m_i(t) \geq t$  for all  $t \geq 0$ . Then*

$$\int_{\Omega} v^2 \, dx \leq 2 \int_{\Omega} M_i(v) \, dx, \quad \text{for all } v \in \mathcal{L}_{M_i}(\Omega). \tag{2.8}$$

**Theorem 2.7** ([16]). *Let  $\Omega \subset \mathbb{R}^d$  be an open and bounded set with locally Lipschitz boundary. Then the embedding  $W^1L_{\mathcal{M}}(\Omega) \hookrightarrow E_{\mathcal{M}}(\Omega)$  is compact. Furthermore, the compact imbedding  $W_0^1L_{\mathcal{M}}(\Omega) \hookrightarrow E_{\mathcal{M}}(\Omega)$  holds without the locally Lipschitz boundary assumption.*

**Corollary 2.8.** *Let  $\Omega$  be an open and bounded set in  $\mathbb{R}^d$  and  $M_0$  the  $N$ -function defined in (2.6). Then, the embedding  $W_0^1L_{\mathcal{M}}(\Omega) \hookrightarrow E_{M_0}(\Omega)$  is compact.*

Poincaré's inequality in  $W_0^1L_{\mathcal{M}}(\Omega)$  also holds.

**Lemma 2.9** ([7]). *Let  $\Omega \subset \mathbb{R}^d$  be an open and bounded set. Then, there exist constants  $\kappa_0$  and  $\kappa_1 = \kappa_1(\Omega)$  such that*

$$\int_{\Omega} M_0(u) \, dx \leq \kappa_0 \sum_{i=1}^d \int_{\Omega} M_i(\kappa_1 \partial u_i) \, dx \quad \text{for all } u \in W_0^1L_{\mathcal{M}}(\Omega).$$

**Corollary 2.10.** *The seminorm  $u \in W^1L_{\mathcal{M}}(\Omega) \mapsto \sum_{i=1}^d \|\partial_i u\|_{M_i}$  is a norm in  $W_0^1L_{\mathcal{M}}(\Omega)$  and it is equivalent to the norm  $\|\cdot\|_{1,\mathcal{M}}$  given in 2.7.*

Since the elements of the space  $W_0^1L_{\mathcal{M}}(\Omega)$  have been defined as the weak-\* limit of convergent sequences in  $\mathcal{D}(\Omega)$ , the following result states that, for certain domains  $\Omega$ ,  $\mathcal{D}(\Omega)$  is 'dense' in  $W_0^1L_{\mathcal{M}}(\Omega)$  with respect to the modular convergence as well.

**Definition 2.11.** A bounded domain  $\Omega \subset \mathbb{R}^d$  is said to satisfy the segment property, if there exist a locally finite open covering  $\{U_i\}$  of  $\partial\Omega$  and corresponding vectors  $\{y_i\} \subset \mathbb{R}^d$  such that for all  $x \in \bar{\Omega} \cap U_i$  and any  $\mu \in (0, 1)$  one has  $x + \mu y_i \in \Omega$ .

**Lemma 2.12.** *Let  $\Omega \subset \mathbb{R}^d$  be an open and bounded set satisfying the segment property and  $u \in W_0^1L_{\mathcal{M}}(\Omega)$ . Then there exists a sequence  $(u_n) \subset \mathcal{D}(\Omega)$  such that  $u_n \rightarrow u$  with respect to the modular convergence in  $W^1L_{\mathcal{M}}(\Omega)$ ; that is, there exists  $\lambda > 0$  such that*

$$\int_{\Omega} M_0((u_n - u)/\lambda) + \sum_{i=1}^d \int_{\Omega} M_i((\partial_i u_n - \partial_i u)/\lambda) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The proof of the above lemma is a straightforward adaptation of [8, Theorem 4] for isotropic Orlicz-Sobolev spaces.

Finally, we introduce the following dual spaces

$$W^{-1}L_{\bar{\mathcal{M}}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = \sum_{i=1}^d \partial_i f_i \text{ with } f_i \in L_{\bar{M}_i}(\Omega), \text{ for all } i, 1 \leq i \leq d \right\}$$

$$W^{-1}E_{\bar{\mathcal{M}}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = \sum_{i=1}^d \partial_i f_i \text{ with } f_i \in E_{\bar{M}_i}(\Omega), \text{ for all } i, 1 \leq i \leq d \right\}$$

These spaces are equipped by their usual quotient norms.

### 3. ESSENTIAL ASSUMPTIONS AND MAIN RESULT

From this point on we will assume that  $\Omega \subset \mathbb{R}^d$  is an open and bounded set satisfying the segment property. We now state the assumptions on the differential operator in divergence form given by  $A: W_0^1L_{\mathcal{M}}(\Omega) \rightarrow W^{-1}L_{\bar{\mathcal{M}}}(\Omega)$

$$A(u) = \sum_{i=1}^d \partial_i (a_i(x, u, \partial_i u)).$$

- (A1) For each  $i = 1, \dots, d$  the function  $a_i: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $a_i = a_i(x, s, \zeta)$  is a Carathéodory function (measurable in  $x$  for all  $(s, \zeta)$  and continuous with respect to  $(s, \zeta)$  almost everywhere (a.e.) in  $\Omega$ ).
- (A2) There exist  $N$ -functions  $M_i$ ,  $i = 1, \dots, d$ , a function  $a_0 \in E_{\bar{M}_i}$  and positive constants  $k$  and  $\gamma$  such that a.e. in  $\Omega$ , for all  $(s, \zeta) \in \mathbb{R}^2$  and for  $i = 1, \dots, d$ .

$$|a_i(x, s, \zeta)| \leq \gamma (a_0(x) + \bar{M}_i^{-1}(M_i(k|\zeta|))). \quad (3.1)$$

- (A3) For a.e. in  $\Omega$ , for all  $s \in \mathbb{R}$ , and all  $\zeta, \zeta' \in \mathbb{R}^d$  with  $\zeta \neq \zeta'$ ,  $\zeta = (\zeta_1, \dots, \zeta_d)$ ,  $\zeta' = (\zeta'_1, \dots, \zeta'_d)$  we have

$$\sum_{i=1}^d [a_i(x, s, \zeta_i) - a_i(x, s, \zeta'_i)](\zeta_i - \zeta'_i) > 0 \quad (3.2)$$

- (A4) There exists  $\lambda_0 > 0$  such that a.e. in  $\Omega$ , for all  $s \in \mathbb{R}$ , and all  $(\zeta_1, \dots, \zeta_d) \in \mathbb{R}^d$  we have

$$\sum_{i=1}^d a_i(x, s, \zeta_i) \zeta_i \geq \alpha \sum_{i=1}^d M_i\left(\frac{\zeta_i}{\lambda_0}\right) \quad (3.3)$$

- (A5) Let  $B \subset \mathbb{R}$  be a bounded set. Then for a.e. in  $\Omega$ , for all  $s \in B$ , and  $\zeta' = (\zeta'_1, \dots, \zeta'_d) \in \mathbb{R}^d$

$$\sum_{i=1}^d [a_i(x, s, \zeta_i) - a_i(x, s, \zeta'_i)](\zeta_i - \zeta'_i) \rightarrow +\infty \quad (3.4)$$

as  $|\zeta| \rightarrow +\infty$ ,  $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{R}^d$ , uniformly in  $s$ .

- (A6) For all  $i = 1, \dots, d$ , we have

$$a_i(x, s, 0) = 0 \quad (3.5)$$

- (A7)  $\rho \in \mathcal{C}(\mathbb{R})$  and there exists  $\bar{\rho} \in \mathbb{R}$  such that for all  $s \in \mathbb{R}$ ,

$$0 < \rho(s) \leq \bar{\rho} \quad (3.6)$$

- (A8)  $\varphi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ .

Now we are ready to state the definition of a capacity solution to problem (1.1).

**Definition 3.1.** A triplet  $(u, \varphi, \Phi)$  is called a capacity solution to problem (1.1) if it satisfies the following conditions:

- (A9)  $u \in W_0^1 L_{\mathcal{M}}(\Omega)$ ,  $a_i(\cdot, u, \partial_i u) \in L_{\bar{M}_i}(\Omega)$  for  $1 \leq i \leq d$ ,  $\varphi \in L^\infty(\Omega)$  and  $\Phi \in L^2(\Omega)^d$ .
- (A10)  $(u, \varphi, \Phi)$  verifies the system of differential equations in the sense of distributions

$$-\sum_{i=1}^d \partial_i (a_i(x, u, \partial_i u)) = \operatorname{div}(\varphi \Phi) \quad \text{in } \Omega,$$

$$\operatorname{div} \Phi = 0 \quad \text{in } \Omega.$$

- (A11) For every  $S \in \mathcal{C}_0^1(\mathbb{R})$  (that is,  $S \in \mathcal{C}^1(\mathbb{R})$  and has compact support in  $\mathbb{R}$ ), one has  $S(u)\varphi - S(0)\varphi_0 \in H_0^1(\Omega)$ , and

$$S(u)\Phi = \rho(u)[\nabla(S(u)\varphi) - \varphi \nabla S(u)].$$

**Remark 3.2.** Notice that the concept of capacity solution involves a third component, namely,  $\Phi \in L^2(\Omega)^d$ . Where this vector field comes from? The original problem only has two unknowns so that there must be a relationship between  $\Phi$  and  $(u, \varphi)$ . Indeed, this is true and the relationship is given by the condition expressed in (A11). In fact, assume that  $u \in L^\infty(\Omega)$  and take  $S \in \mathcal{C}_0^1(\mathbb{R})$  such that  $S \equiv 1$  in the interval  $[-\|u\|_\infty, \|u\|_\infty]$ . Then, we deduce that  $\varphi \in H^1(\Omega)$ ,  $\Phi = \rho(u)\nabla\varphi$  and  $\operatorname{div}(\varphi\Phi) = \rho(u)|\nabla\varphi|^2$ . Consequently, bounded capacity solutions are weak solutions. In the general case where  $u \notin L^\infty(\Omega)$  the expression given in (A11) allows us to define the gradient of  $\varphi$  pointwise almost everywhere from the identity

$$\Phi\chi_{\{|u|<K\}} = \rho(u)\nabla\varphi\chi_{\{|u|<K\}} \quad \text{for any } K > 0.$$

#### 4. AN EXISTENCE RESULT

In this section we establish the main result of this article.

**Theorem 4.1.** *Under the assumptions (A1)–(A8), system (1.1) admits a capacity solution.*

To prove the main result, we will need to show the existence of a weak solution to a similar problem but under a more restrictive assumption, namely,

$$\begin{aligned} \rho \in \mathcal{C}(\mathbb{R}) \text{ and there exist } \rho_1 \text{ and } \rho_2 \in \mathbb{R} \text{ such that} \\ 0 < \rho_1 \leq \rho(s) \leq \rho_2, \text{ for all } s \in \mathbb{R}. \end{aligned} \quad (4.1)$$

**Theorem 4.2.** *Assume (A1)–(A6), (A8) and (4.1) hold. Then, there exists a weak solution  $(u, \varphi)$  to (1.1), that is  $u \in W_0^1L_{\mathcal{M}}(\Omega)$ ,  $a_i(\cdot, u, \partial_i u) \in L_{\bar{M}_i}(\Omega)$ , for all  $1 \leq i \leq d$ ,  $\varphi - \varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , and*

$$\begin{aligned} \sum_{i=1}^d \int_{\Omega} a_i(x, u, \partial_i u) \partial_i \phi &= \int_{\Omega} \rho(u) |\nabla \varphi|^2 \phi, \quad \text{for all } \phi \in W_0^1L_{\mathcal{M}}(\Omega), \\ \int_{\Omega} \rho(u) \nabla \varphi \nabla \psi &= 0, \quad \text{for all } \psi \in H_0^1(\Omega). \end{aligned}$$

*Proof.* To prove the existence of a weak solution, Schauder's fixed point theorem will be applied together with a result on the existence and uniqueness of a weak solution to an elliptic equation.

For a function  $\omega \in E_{M_0}(\Omega)$  we consider the elliptic problem

$$\begin{aligned} \operatorname{div}(\rho(\omega)\nabla\varphi) &= 0 \quad \text{in } \Omega, \\ \varphi &= \varphi_0 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.2)$$

Thanks to Lax-Milgram's theorem, (4.2) has a unique solution  $\varphi \in H^1(\Omega)$ . In this case, from the maximum principle we also have  $\varphi \in L^\infty(\Omega)$  and

$$\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi_0\|_{L^\infty(\Omega)}. \quad (4.3)$$

Using  $\varphi - \varphi_0 \in H_0^1(\Omega)$  as a test function in (4.2) we obtain

$$\int_{\Omega} \rho(\omega) |\nabla \varphi|^2 = \int_{\Omega} \rho(\omega) \nabla \varphi \nabla \varphi_0,$$

hence

$$\rho_1 \int_{\Omega} |\nabla \varphi|^2 \, dx \leq \int_{\Omega} \rho(\omega) |\nabla \varphi| |\nabla \varphi_0| \, dx \leq \rho_2 \int_{\Omega} |\nabla \varphi| |\nabla \varphi_0| \, dx.$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega} |\nabla\varphi|^2 dx \leq C(\rho_1, \rho_2, \varphi_0) = C. \quad (4.4)$$

Thanks to the elliptic equation for  $\varphi$ , the term  $\rho(\omega)|\nabla\varphi|^2$  also belongs to the space  $H^{-1}(\Omega)$ . Indeed, let  $\phi \in \mathcal{D}(\Omega)$  and take  $\psi = \phi\varphi$  as a test function in (4.2). We have

$$\int_{\Omega} \rho(\omega)\nabla\varphi\nabla(\phi\varphi) dx = 0,$$

that is

$$\int_{\Omega} \rho(\omega)|\nabla\varphi|^2\phi dx = - \int_{\Omega} \rho(\omega)\varphi\nabla\varphi\nabla\phi dx = \langle \operatorname{div}(\rho(\omega)\varphi\nabla\varphi), \phi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

This means that

$$\rho(\omega)|\nabla\varphi|^2 = \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \text{ in } \mathcal{D}'(\Omega). \quad (4.5)$$

Since  $\rho(\omega)\varphi\nabla\varphi \in L^2(\Omega)^N$  we finally deduce the regularity

$$\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega).$$

Now we consider the elliptic problem

$$\begin{aligned} - \sum_{i=1}^d \partial_i(a_i(x, \omega, \partial_i u)) &= \operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (4.6)$$

The variational formulation of this elliptic equation is

$$\begin{aligned} u &\in W_0^1 L_{\mathcal{M}}(\Omega), \quad a_i(\cdot, \omega, \partial_i u) \in L_{\bar{M}_i}(\Omega), \quad \text{for } i = 1, \dots, d, \\ \sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i \phi &= - \int_{\Omega} \rho(\omega)\varphi\nabla\varphi\nabla\phi, \quad \text{for all } \phi \in W_0^1 L_{\mathcal{M}}(\Omega). \end{aligned} \quad (4.7)$$

We have  $\operatorname{div}(\rho(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega) \leftrightarrow W^{-1} L_{\bar{\mathcal{M}}}(\Omega)$  because of Lemma 2.5. The proof of the existence and uniqueness of solution to (4.7) is a straightforward application of the result given in [15].

Now, we show that  $\partial_i u/\lambda_0 \in \mathcal{L}_{M_i}(\Omega)$  for  $i = 1, \dots, d$ , where  $\lambda_0$  is the constant appearing in (3.3), and the estimates

$$\sum_{i=1}^d \int_{\Omega} M_i(\partial_i u/\lambda_0) dx \leq C(\varphi_0, \alpha, \lambda_0, \rho_2) = C_0, \quad (4.8)$$

$$\sum_{i=1}^d \|a_i(\cdot, \omega, \partial_i u)\|_{\bar{M}_i} \leq C_1. \quad (4.9)$$

Indeed, let  $\lambda > 0$  such that  $|\partial_i u|/\lambda \in \mathcal{L}_{M_i}(\Omega)$  for  $i = 1, \dots, d$ . Since  $\varphi \in H^1(\Omega) \subset W^1 L_{\bar{\mathcal{M}}}(\Omega)$ , there exists  $\mu > 0$  such that  $\frac{1}{\mu}\rho_2\|\varphi_0\|_{L^\infty(\Omega)}\partial_i\varphi \in \mathcal{L}_{\bar{M}_i}(\Omega)$  for  $i = 1, \dots, d$ . By taking  $\phi = u$  as a test function in (4.7), from (3.3), (3.5), (4.1), (4.3) and Young's inequality, we obtain

$$\frac{\alpha}{\lambda\mu} \sum_{i=1}^d \int_{\Omega} M_i(\partial_i u/\lambda_0) dx$$

$$\begin{aligned} &\leq \frac{1}{\lambda\mu} \int_{\Omega} \sum_{i=1}^d a_i(x, \omega, \partial_i u) \partial_i u \, dx \\ &\leq \sum_{i=1}^d \int_{\Omega} \bar{M}_i(\rho_2 \|\varphi_0\|_{L^\infty(\Omega)} \partial_i \varphi / \mu) \, dx + \sum_{i=1}^d \int_{\Omega} M_i(\partial_i u / \lambda) \, dx < \infty \end{aligned}$$

This shows that  $\partial_i u / \lambda_0 \in \mathcal{L}_{M_i}(\Omega)$  for  $i = 1, \dots, d$ . To obtain (4.8), by using Young's inequality and the estimate (2.8), we deduce

$$\begin{aligned} \alpha \sum_{i=1}^d \int_{\Omega} M_i(\partial_i u / \lambda_0) \, dx &\leq \sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i u \, dx \\ &\leq \sum_{i=1}^d \left( \frac{\lambda_0^2}{\alpha} \rho_2^2 \|\varphi_0\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla \varphi|^2 + \frac{\alpha}{4} \int_{\Omega} |\partial_i u / \lambda_0|^2 \right) \\ &\leq C_*(\varphi_0, \alpha, \lambda_0, \rho_2) + \frac{\alpha}{2} \sum_{i=1}^d \int_{\Omega} M_i(\partial_i u / \lambda_0) \end{aligned}$$

and thus (4.8) holds for  $C_0 = 2C_*(\varphi_0, \alpha, \lambda_0, \rho_2)/\alpha$ . To obtain (4.9), first notice that from the previous two inequalities we obtain

$$\sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i u \, dx \leq \alpha C_0. \quad (4.10)$$

Then, because of (3.2), for any  $\phi \in W_0^1 E_{\mathcal{M}}(\Omega)$  such that  $\sum_{i=1}^d \|\partial_i \phi\|_{(M_i)} = 1/(k+1)$  we have

$$0 \leq \sum_{i=1}^d \int_{\Omega} (a_i(x, \omega, \partial_i u) - a_i(x, \omega, \partial_i \phi)) (\partial_i u - \partial_i \phi) \, dx.$$

Owing to (4.10) and Young's inequality, we deduce

$$\begin{aligned} &\sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i \phi \, dx \\ &\leq \sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i u \, dx - \sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i \phi) (\partial_i u - \partial_i \phi) \, dx \\ &\leq \alpha C_0 + \sum_{i=1}^d \int_{\Omega} |a_i(x, \omega, \partial_i \phi) \partial_i u| \, dx + \sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i \phi) \partial_i \phi \, dx \\ &\leq \alpha C_0 + 2\gamma \lambda_0 \sum_{i=1}^d \int_{\Omega} \left[ \bar{M}_i \left( \frac{a_i(x, \omega, \partial_i \phi)}{2\gamma} \right) + M_i(\partial_i u / \lambda_0) \right] \, dx \\ &\quad + 2\gamma \sum_{i=1}^d \int_{\Omega} \left[ \bar{M}_i \left( \frac{a_i(x, \omega, \partial_i \phi)}{2\gamma} \right) + M_i(|\partial_i \phi|) \right] \, dx, \end{aligned}$$

where  $\gamma$  is the constant appearing in (3.1). Since

$$\sum_{i=1}^d \bar{M}_i \left( \frac{a_i(x, \omega, \partial_i \phi)}{2\gamma} \right) \leq \frac{1}{2} \sum_{i=1}^d (\bar{M}_i(a_0(x)) + M_i(k|\partial_i \phi|)) \quad \text{a.e. in } \Omega,$$

using (2.3) we obtain

$$\sum_{i=1}^d \int_{\Omega} \bar{M}_i \left( \frac{a_i(x, \omega, \partial_i \phi)}{2\gamma} \right) dx \leq \frac{1}{2} \sum_{i=1}^d \int_{\Omega} \bar{M}_i(a_0(x)) dx + \frac{1}{2} = C_2.$$

Notice that  $C_2$  does not depend on  $\omega$ . Therefore, gathering all these estimates, we deduce that for all  $\phi \in W_0^1 E_{\mathcal{M}}(\Omega)$  such that  $\sum_{i=1}^d \|\partial_i \phi\|_{(M_i)} = 1/(k+1)$  we have

$$\sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i \phi dx \leq C_1,$$

from which, by considering the dual norm on  $L_{\bar{M}_i}(\Omega)$ , for each  $i = 1, \dots, d$ , we obtain the estimate (4.9).

Now we introduce the operator  $G: \omega \in E_{M_0}(\Omega) \rightarrow G(\omega) = u \in W_0^1 L_{\mathcal{M}}(\Omega)$ , with  $u$  being the unique solution to (4.7). Our strategy is to show that  $G$  satisfies the conditions of Schauder's fixed point theorem. From Corollary 2.8,  $W_0^1 L_{\mathcal{M}}(\Omega) \hookrightarrow E_{M_0}(\Omega)$  with compact embedding. Consequently,  $G$  maps  $E_{M_0}(\Omega)$  into itself and, due to the estimates (4.8),  $G$  is a compact operator. Moreover, from Corollary 2.10 and (4.8) we have, for  $R > 0$  large enough  $G(B_R) \subset B_R$  where  $B_R = \{v \in E_{M_0}(\Omega) : \|v\|_{M_i} \leq R, \text{ for } i = 1, \dots, d\}$ .

To complete the proof, it remains to show that  $G$  is a continuous operator. Indeed, let  $(\omega_n) \subset B_R$  such that  $\omega_n \rightarrow \omega$  strongly in  $E_{M_0}(\Omega)$  and consider the corresponding functions to  $\omega_n$ , that is,  $u_n = G(\omega_n)$ ,  $\varphi_n$ ,  $u = G(\omega)$ , and  $\varphi$ . Since the injection  $E_{M_0}(\Omega) \subset L^2(\Omega)$  is continuous, we can assume that for a subsequence, still denoted in the same way, it is  $\omega_n \rightarrow \omega$  a.e. in  $\Omega$ . Then, putting  $F_n = \rho(\omega_n) \varphi_n \nabla \varphi_n$  and  $F = \rho(\omega) \varphi \nabla \varphi$ , it is easy to check that  $F_n \rightarrow F$  strongly in  $L^2(\Omega)^d$ . On the other hand, since  $(u_n) \subset W_0^1 L_{\mathcal{M}}(\Omega)$  is bounded in this space, from Corollary 2.8, there exist a subsequence, still denoted in the same way, and a function  $U \in E_{M_0}(\Omega)$  such that  $u_n \rightarrow U$  strongly in  $E_{M_0}(\Omega)$  and a.e. in  $\Omega$ . We need to prove that  $u = U$ . To do so, we first need the following result.

**Lemma 4.3.** *There exists a subsequence, still denoted in the same way, such that  $\nabla u_n \rightarrow \nabla U$  a.e. in  $\Omega$ .*

This result is a special case of Proposition 4.4 below. For the proof of Lemma 4.3, one may repeat the arguments given in the proof of Proposition 4.4.

As a consequence of Lemma 4.3, there exists a subsequence, still denoted in the same way such that

$$a_i(x, \omega_n, \partial_i u_n) \rightarrow a_i(x, \omega, \partial_i U) \quad \text{a.e. in } \Omega, \text{ for } i = 1, \dots, d.$$

On the other hand, from the estimate (4.9),  $(a_i(x, \omega_n, \partial_i u_n)) \subset L_{M_0}(\Omega)$  is bounded in this space and thus, there exist a subsequence, still denoted in the same way, and functions  $\Phi_i \in L_{\bar{M}_i}(\Omega)$ ,  $i = 1, \dots, d$ , such that

$$a_i(x, \omega_n, \partial_i u_n) \rightarrow \Phi_i \quad \text{weak-* in } L_{\bar{M}_i}(\Omega),$$

and therefore  $\Phi_i = a_i(x, \omega, \partial_i U)$  for  $i = 1, \dots, d$ .

By using all these convergences and passing to the limit in the equation of  $u_n$ , that is,

$$\sum_{i=1}^d \int_{\Omega} a_i(x, \omega_n, \partial_i u_n) \partial_i \phi dx = - \int_{\Omega} \rho(\omega_n) \varphi_n \nabla \varphi_n \nabla \phi dx,$$

we obtain

$$\sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i U) \partial_i \phi \, dx = - \int_{\Omega} \rho(\omega) \varphi \nabla \varphi \nabla \phi \, dx,$$

Since  $u = G(\omega)$ , we also have

$$\sum_{i=1}^d \int_{\Omega} a_i(x, \omega, \partial_i u) \partial_i \phi \, dx = - \int_{\Omega} \rho(\omega) \varphi \nabla \varphi \nabla \phi \, dx,$$

Thus,

$$\sum_{i=1}^d \int_{\Omega} [a_i(x, \omega, \partial_i u) - a_i(x, \omega, \partial_i U)] \partial_i \phi \, dx = 0 \quad \text{for all } \phi \in W_0^1 L_{\mathcal{M}}(\Omega).$$

By taking  $\phi = u - U$  and using (3.2) we deduce that  $u = U$ . In particular, it is the whole sequence  $(u_n)$  that converges to  $u$  in  $E_{M_0}(\Omega)$  and this shows the continuity of the operator  $G$ . This completes the proof of Theorem 4.2.  $\square$

**Proof of Theorem 4.1.** The proof is divided into several steps. We start by introducing a sequence of approximate problems and derive some a priori estimates for the respective solutions to these approximate problems, then we show two intermediate results, namely, the strong convergence in  $L^1(\Omega)$ , up to a subsequence, of both  $\partial_i u_n$  for  $i = 1, \dots, d$  and  $\varphi_n$ , where  $(u_n, \varphi_n)$  is a weak solution to the  $n$ -th approximate problem of (1.1). The passing to the limit in the approximate problems will yield the main result.

**Step 1.** Approximate problems and a priori estimates. We define the truncation function at height  $K > 0$ ,  $T_K$ , by

$$T_K(s) = s \text{ if } |s| \leq K, \quad T_K(s) = K \frac{s}{|s|} \text{ if } |s| > K,$$

and we introduce the following regularization of the data, for every  $n \in \mathbb{N}$ ,

$$\rho_n(s) = \rho(s) + \frac{1}{n}, \tag{4.11}$$

The  $n$ -th approximate problem is

$$- \sum_{i=1}^d \partial_i (a_i(x, u_n, \partial_i u_n)) = \rho_n(u_n) |\nabla \varphi_n|^2 \quad \text{in } \Omega, \tag{4.12}$$

$$\operatorname{div}(\rho_n(u_n) \nabla \varphi_n) = 0 \quad \text{in } \Omega, \tag{4.13}$$

$$u_n = 0, \quad \varphi_n = \varphi_0 \quad \text{on } \partial\Omega, \tag{4.14}$$

In view of (3.6), we have that

$$n^{-1} \leq \rho_n(s) \leq \bar{\rho} + 1 = \rho_3, \quad \text{for all } s \in \mathbb{R}. \tag{4.15}$$

Thus, we can apply Theorem 4.2 to deduce the existence of a weak solution  $(u_n, \varphi_n)$  to (4.12)-(4.14).

By the maximum principle we have

$$\|\varphi_n\|_{L^\infty(\Omega)} \leq \|\varphi_0\|_{L^\infty(\Omega)}, \tag{4.16}$$

therefore, there exists a function  $\varphi \in L^\infty(\Omega)$  and a subsequence, still denoted in the same way, such that

$$\varphi_n \rightarrow \varphi \quad \text{weakly-* in } L^\infty(\Omega). \tag{4.17}$$

Multiplying (4.13) by  $\varphi_n - \varphi_0 \in H_0^1(\Omega)$  and integrating in  $\Omega$  we obtain

$$\begin{aligned} \int_{\Omega} \rho_n(u_n) |\nabla \varphi_n|^2 &= \int_{\Omega} \rho_n(u_n) \nabla \varphi_n \nabla \varphi_0 \, dx \\ &\leq \left( \int_{\Omega} \rho_n(u_n) |\nabla \varphi_n|^2 \right)^{1/2} \left( \int_{\Omega} \rho_n(u_n) |\nabla \varphi_0|^2 \right)^{1/2} \\ &\leq \left( \int_{\Omega} \rho_n(u_n) |\nabla \varphi_n|^2 \right)^{1/2} \left( \rho_3 \int_{\Omega} |\nabla \varphi_0|^2 \right)^{1/2}, \end{aligned}$$

hence

$$\int_{\Omega} \rho_n(u_n) |\nabla \varphi_n|^2 \, dx \leq C_1, \quad \text{for all } n \geq 1, \tag{4.18}$$

where  $C_1 = C_1(\rho_3, \|\varphi_0\|_{H^1(\Omega)})$ . Consequently,  $(\rho_n(u_n) \nabla \varphi_n)$  is bounded in  $L^2(\Omega)$ . Thus, there exists a function  $\Phi \in L^2(\Omega)^d$  and a subsequence, still denoted in the same way, such that

$$\rho_n(u_n) \nabla \varphi_n \rightharpoonup \Phi \text{ weakly in } L^2(\Omega)^d. \tag{4.19}$$

This weak limit function  $\Phi \in L^2(\Omega)^d$  is in fact the third component of the triplet appearing in the Definition 3.1 of a capacity solution.

Taking  $u_n$  as a test function in (4.12), we obtain

$$\sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i u_n \, dx = - \sum_{i=1}^d \int_{\Omega} \rho_n(u_n) \varphi_n \partial_i \varphi_n \partial_i u_n \, dx. \tag{4.20}$$

Since  $u_n \in W_0^1 L_{\mathcal{M}}(\Omega)$  and  $\varphi \in H^1(\Omega) \subset W^1 L_{\overline{\mathcal{M}}}(\Omega)$ , there exist  $\lambda_n > 0$  such that for  $i = 1, \dots, d$  one has  $\partial_i u_n / \lambda_n \in \mathcal{L}_{M_i}(\Omega)$ . By (2.8), (3.3), (3.5), (4.15), (4.16), and Young's inequality, we obtain

$$\begin{aligned} &\alpha \sum_{i=1}^d \int_{\Omega} M_i(\partial_i u_n / \lambda_0) \, dx \\ &\leq \sum_{i=1}^d \int_{\Omega} \rho_n(u_n) |\varphi_n| |\nabla \varphi_n| |\partial_i u_n| \, dx \\ &\leq \sum_{i=1}^d \int_{\Omega} \sqrt{\rho_3} \|\varphi_0\|_{L^\infty(\Omega)} \sqrt{\rho_n(u_n)} |\nabla \varphi_n| |\partial_i u_n| \, dx \\ &\leq \sum_{i=1}^d \left( \frac{\rho_3 \|\varphi_0\|_{L^\infty(\Omega)}^2 \lambda_n^2}{\alpha} \int_{\Omega} \rho_n(u_n) |\nabla \varphi_n(u_n)|^2 + \frac{\alpha}{2} \int_{\Omega} M_i(\partial_i u_n / \lambda_n), \, dx \right) \end{aligned}$$

which implies that  $\partial_i u_n / \lambda_0 \in \mathcal{L}_{M_i}(\Omega)$ . In particular, we may take  $\lambda_n = \lambda_0$  for all  $n \geq 1$  and then

$$\sum_{i=1}^d \int_{\Omega} M_i(\partial_i u_n / \lambda_0) \, dx \leq C, \tag{4.21}$$

where  $C$  is a positive constant not depending on  $n$ . Thus, from Corollary 2.10 and (4.21), the sequence  $(u_n)$  is bounded in  $W_0^1 L_{\mathcal{M}}(\Omega)$ , and since the embedding  $W_0^1 L_{\mathcal{M}}(\Omega) \hookrightarrow E_{M_0}(\Omega)$  is compact, there exist a subsequence of  $(u_n)$ , still denoted in the same way, and a function  $u \in E_{M_0}(\Omega)$  such that

$$u_n \rightarrow u \text{ strongly in } E_{M_0}(\Omega) \text{ and a.e. in } \Omega. \tag{4.22}$$

On the other hand, since  $(\nabla u_n) \subset L_{\mathcal{M}}(\Omega)$  is bounded and using (4.22), we also have that, up to a subsequence,

$$\nabla u_n \rightarrow \nabla u \quad \text{weak-* in } L_{\mathcal{M}}(\Omega) \quad (4.23)$$

Now let  $\phi \in W_0^1 E_{\mathcal{M}}(\Omega)$  be such that  $\sum_{i=1}^d \|\partial_i \phi\|_{(M_i)} = 1/(k+1)$ . In view of the monotonicity of  $\mathbf{a} = (a_1, \dots, a_d)$ , we easily find that

$$\begin{aligned} & \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i \phi \\ & \leq \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i u_n - \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i \phi) (\partial_i u_n - \partial_i \phi) \\ & \leq C + \sum_{i=1}^d \int_{\Omega} |a_i(x, u_n, \partial_i \phi) \partial_i u_n| + \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i \phi) \partial_i \phi, \end{aligned} \quad (4.24)$$

The last integrals in (4.24) are bounded with respect to  $n$ . Indeed, for the first one, owing to Young's inequality

$$\sum_{i=1}^d \int_{\Omega} |a_i(x, u_n, \partial_i \phi) \partial_i u_n| \leq 2\gamma \lambda_0 \sum_{i=1}^d \int_{\Omega} \left[ \bar{M}_i \left( \frac{a_i(x, u_n, \partial_i \phi)}{2\gamma} \right) + M_i(\partial_i u_n / \lambda_0) \right],$$

by using (3.1) we have

$$2\gamma \sum_{i=1}^d \bar{M}_i \left( \frac{a_i(x, u_n, \partial_i \phi)}{2\gamma} \right) \leq \gamma \sum_{i=1}^d (\bar{M}_i(a_0(x)) + M_i(k\partial_i \phi)),$$

and thus  $\sum_{i=1}^d \int_{\Omega} |a_i(x, u_n, \partial_i \phi) \partial_i u_n| \leq C$ , for all  $n \geq 1$  and  $\phi \in W_0^1 E_{\mathcal{M}}(\Omega)$  such that  $\sum_{i=1}^d \|\partial_i \phi\|_{(M_i)} = 1/(k+1)$ . In the same way, we can show that the second integral in (4.24) is bounded. Gathering all these estimates, and using the dual norm, it is easily deduced that, for  $i = 1, \dots, d$ ,

$$(a_i(x, u_n, \partial_i u_n)) \quad \text{is bounded in } L_{\bar{M}_i}(\Omega). \quad (4.25)$$

Thus, we have that, for a subsequence still denoted in the same way, there exists  $\delta_i \in L_{\bar{M}_i}(\Omega)$ , for each  $i = 1, \dots, d$ , such that

$$a_i(x, u_n, \partial_i u_n) \rightarrow \delta_i \quad \text{weak-* in } L_{\bar{M}_i}(\Omega). \quad (4.26)$$

**Step 2.** Introduction of regularized sequences and the almost everywhere convergence of the gradients. By using Lemma 2.12, there exists a sequence  $(v_j) \subset \mathcal{D}(\Omega)$  such that

- (1)  $v_j \rightarrow u$  in  $W_0^1 L_{\mathcal{M}}(\Omega)$  for the modular convergence;
- (2)  $v_j \rightarrow u$  and  $\partial_i v_j \rightarrow \partial_i u$  a.e. in  $\Omega$  for  $i = 1, \dots, d$ .

We will establish the following result.

**Proposition 4.4.** *Let  $(u_n, \varphi_n)$  be a solution of the approximate problem (4.12)-(4.14). Then there exists a subsequence, still denoted in the same way, such that for  $i = 1, \dots, d$ ,*

$$\partial_i u_n \rightarrow \partial_i u \quad \text{a.e. in } \Omega, \quad (4.27)$$

as  $n$  tends to  $+\infty$ .

*Proof.* We denote by  $\chi_s^j$  and  $\chi_s$ , respectively, the characteristic functions of the sets

$$\Omega_s^j = \left\{ x \in \Omega : \sum_{i=1}^d |\partial_i T_K(v_j)| \leq s \right\} \quad \text{and} \quad \Omega_s = \left\{ x \in \Omega : \sum_{i=1}^d |\partial_i T_K(u)| \leq s \right\}.$$

We denote by  $\epsilon(n, j)$  and  $\epsilon(n, j, s)$  any quantities such that

$$\limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \epsilon(n, j) = 0, \quad \limsup_{s \rightarrow \infty} \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \epsilon(n, j, s) = 0.$$

For  $\nu > 0$  and  $i, j, n \geq 1$  using the test function  $\varphi_{n,j}^\nu = T_\nu(u_n - T_K(v_j))$  in (4.12), we obtain

$$\sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i T_\nu(u_n - T_K(v_j)) \, dx = \int_{\Omega} \rho_n(u_n) |\nabla \varphi_n|^2 \varphi_{n,j}^\nu \, dx. \quad (4.28)$$

From (4.18) it follows that

$$\sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i T_\nu(u_n - T_K(v_j)) \, dx \leq C_1 \nu. \quad (4.29)$$

On the other hand

$$\begin{aligned} & \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i T_\nu(u_n - T_K(v_j)) \, dx \\ &= \sum_{i=1}^d \int_{\{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i (u_n - T_K(v_j)) \, dx \\ &= \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i (u_n - T_K(v_j)) \, dx \\ & \quad + \sum_{i=1}^d \int_{\{|u_n| \leq K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i (u_n - T_K(v_j)) \, dx \\ &= \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) (\partial_i T_K(u_n) - \partial_i T_K(v_j)) \, dx \\ & \quad + \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i u_n \, dx \\ & \quad - \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i T_K(v_j) \, dx. \end{aligned}$$

Then, using (3.3), we have

$$\begin{aligned} & \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i T_\nu(u_n - T_K(v_j)) \, dx \\ & \geq \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) (\partial_i T_K(u_n) - \partial_i T_K(v_j)) \, dx \\ & \quad - \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i T_K(v_j) \, dx. \end{aligned} \quad (4.30)$$

Also, in the set  $\{|u_n - T_K(v_j)| \leq \nu\}$ , we have  $|u_n| \leq |u_n - T_K(v_j)| + |T_K(v_j)| \leq \nu + K$ , and thus, we can write

$$\begin{aligned} & \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, u_n, \partial_i u_n) \partial_i T_K(v_j) \, dx \\ &= \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, T_{\nu+K}(u_n), \partial_i T_{\nu+K}(u_n)) \partial_i T_K(v_j) \, dx. \end{aligned} \quad (4.31)$$

By (4.31), (4.30) becomes

$$\begin{aligned} & \sum_{i=1}^d \int_{\Omega} a_i(x, u_n, \partial_i u_n) \partial_i T_{\nu}(u_n - T_K(v_j)) \, dx \\ & \geq \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) (\partial_i T_K(u_n) - \partial_i T_K(v_j)) \, dx \\ & \quad - \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, T_{\nu+K}(u_n), \partial_i T_{\nu+K}(u_n)) \partial_i T_K(v_j). \end{aligned} \quad (4.32)$$

We put

$$J_1 = \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, T_{\nu+K}(u_n), \partial_i T_{\nu+K}(u_n)) \partial_i T_K(v_j) \, dx.$$

Since  $(a_i(x, T_{K+\nu}(u_n), \partial_i T_{K+\nu}(u_n)))_n$  is bounded in  $L_{\bar{M}_i}(\Omega)$ , for each  $i = 1, \dots, d$ , we obtain, for certain  $l_{K+\nu}^i \in L_{\bar{M}_i}(\Omega)$ , and up to a subsequence, that

$$a_i(x, T_{K+\nu}(u_n), \partial_i T_{K+\nu}(u_n)) \rightarrow l_{K+\nu}^i \quad \text{weakly-* in } L_{\bar{M}_i}(\Omega).$$

Consequently,

$$\begin{aligned} & \sum_{i=1}^d \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \nu\}} a_i(x, T_{\nu+K}(u_n), \partial_i T_{\nu+K}(u_n)) \partial_i T_K(v_j) \, dx \\ & \rightarrow \sum_{i=1}^d \int_{\{|u| \geq K\} \cap \{|u - T_K(v_j)| \leq \nu\}} l_{K+\nu}^i \partial_i T_K(v_j) \, dx \end{aligned}$$

as  $n$  approaches infinity. Using Lemma 2.3, we obtain, as  $j$  tends to infinity, that

$$\begin{aligned} & \sum_{i=1}^d \int_{\{|u| \geq K\} \cap \{|u - T_K(v_j)| \leq \nu\}} l_{K+\nu}^i \partial_i T_K(v_j) \\ & \rightarrow \sum_{i=1}^d \int_{\{|u| \geq K\} \cap \{|u - T_K(u)| \leq \nu\}} l_{K+\nu}^i \partial_i T_K(u) = 0. \end{aligned}$$

since  $\partial_i T_K(u) = 0$  in the set  $\{|u| \geq K\}$ . This implies

$$J_1 = \epsilon(n, j). \quad (4.33)$$

Using (4.29) and (4.33) in (4.32), we obtain

$$\begin{aligned} & \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) (\partial_i T_K(u_n) - \partial_i T_K(v_j)) \, dx \\ & \leq C_1 \nu + \epsilon(n, j). \end{aligned} \quad (4.34)$$

On the other hand,

$$\begin{aligned}
 & \sum_{i=1}^d \int_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n))(\partial_i T_K(u_n) - \partial_i T_K(v_j)) \, dx \\
 &= \sum_{i=1}^d \int_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n))(\partial_i T_K(u_n) - \partial_i T_K(v_j)\chi_j^s) \\
 & \quad + \sum_{i=1}^d \int_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n))(\partial_i T_K(v_j)\chi_j^s - \partial_i T_K(v_j)) \\
 &= J_2 + J_3. \tag{4.35}
 \end{aligned}$$

The second integral,  $J_3$  tends to 0 as, first  $n$ , then  $j$  approach infinity. Indeed, since

$$a_i(x, T_K(u_n), \partial_i T_K(u_n)) \rightharpoonup l_K^i \quad \text{weakly-* in } (L_{M_i}(\Omega))$$

and

$$\begin{aligned}
 & (\partial_i T_K(v_j)\chi_j^s - \partial_i T_K(v_j))\chi_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} \\
 & \rightarrow (\partial_i T_K(v_j)\chi_j^s - \partial_i T_K(v_j))\chi_{\{|T_K(u)-T_K(v_j)|\leq\nu\}}
 \end{aligned}$$

strongly in  $E_{M_i}(\Omega)$  as  $n \rightarrow \infty$ , for  $i = 1, \dots, d$ , it follows that

$$\lim_{n \rightarrow \infty} J_3 = \sum_{i=1}^d \int_{\{|T_K(u)-T_K(v_j)|\leq\nu\}} l_K^i \cdot (\partial_i T_K(v_j)\chi_j^s - \partial_i T_K(v_j)) \, dx.$$

Finally letting  $j$ , then  $s$ , approach infinity, we deduce that

$$J_3 = \epsilon(n, j, s). \tag{4.36}$$

Consequently, from (4.34), (4.35), and (4.36), we have

$$\begin{aligned}
 J_2 &= \sum_{i=1}^d \int_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n))(\partial_i T_K(u_n) - \partial_i T_K(v_j)\chi_j^s) \\
 &\leq C_1\nu + \epsilon(n, j, s). \tag{4.37}
 \end{aligned}$$

Let  $S_n$  be the non-negative expression

$$S_n = \sum_{i=1}^d (a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(u))) (\partial_i T_K(u_n) - \partial_i T_K(u)),$$

and for each  $0 < \theta < 1$ , we write  $I_{n,r} = \int_{\Omega_r} S_n^\theta \, dx$ . We have

$$I_{n,r} = \int_{\Omega_r} S_n^\theta \chi_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} + \int_{\Omega_r} S_n^\theta \chi_{\{|T_K(u_n)-T_K(v_j)|>\nu\}}. \tag{4.38}$$

By using Hölder's inequality, the second term of the right-side hand is bounded by

$$\left( \int_{\Omega_r} S_n \, dx \right)^\theta \cdot \left( \int_{\Omega_r} \chi_{\{|T_K(u_n)-T_K(v_j)|>\nu\}} \, dx \right)^{1-\theta}.$$

Note that

$$\begin{aligned}
 \int_{\Omega_r} S_n \, dx &= \sum_{i=1}^d \left[ \int_{\Omega_r} a_i(x, T_K(u_n), \partial_i T_K(u_n)) \partial_i T_K(u_n) \, dx \right. \\
 & \quad \left. - \int_{\Omega_r} a_i(x, T_K(u_n), \partial_i T_K(u_n)) \partial_i T_K(u) \, dx \right]
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega_r} a_i(x, T_K(u_n), \partial_i T_K(u)) \partial_i T_K(u) \, dx \\
& - \int_{\Omega_r} a_i(x, T_K(u_n), \partial_i T_K(u)) \partial_i T_K(u_n) \, dx \Big].
\end{aligned}$$

Since, for each  $i = 1, \dots, d$ , the quantity  $(a_i(x, T_K(u_n), \partial_i T_K(u_n)))_n$  is bounded in  $L_{M_i}(\Omega)$ ,  $(\partial_i T_K(u_n))_n$  is bounded in  $L_{M_i}(\Omega)$ , and  $(a_i(x, T_K(u_n), \partial_i T_K(u)))_n$  is bounded in  $E_{M_i}(\Omega_r)$ , it follows that  $(S_n)$  is bounded in  $L^1(\Omega_r)$ . Then there exists a constant  $C_3 > 0$  such that

$$\int_{\Omega_r} S_n^\theta \chi_{\{|T_K(u_n) - T_K(v_j)| > \nu\}} \, dx \leq C_3 \operatorname{meas}\{|T_K(u_n) - T_K(v_j)| > \nu\}^{1-\theta}. \quad (4.39)$$

Using again Hölder's inequality, we deduce that

$$\begin{aligned}
& \int_{\Omega_r} S_n^\theta \chi_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} \, dx \\
& \leq \left( \int_{\Omega_r} 1 \, dx \right)^{1-\theta} \left( \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_r} S_n \, dx \right)^\theta \\
& \leq C_4 \left( \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_r} S_n \, dx \right)^\theta.
\end{aligned} \quad (4.40)$$

From (4.39) and (4.40), we obtain

$$\begin{aligned}
I_{n,r} & \leq C_3 \operatorname{meas}\{|T_K(u_n) - T_K(v_j)| > \nu\}^{1-\theta} \\
& + C_4 \left( \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_r} S_n \, dx \right)^\theta.
\end{aligned} \quad (4.41)$$

Let  $s \geq r > 0$ . We have

$$\begin{aligned}
& \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_r} S_n \, dx \\
& \leq \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_s} S_n \, dx \\
& = \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\} \cap \Omega_s} [a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(u)) \chi_s] \\
& \quad \times [\partial_i T_K(u_n) - \partial_i T_K(u) \chi_s] \, dx \\
& \leq \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} [a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(u)) \chi_s] \\
& \quad \times [\partial_i T_K(u_n) - \partial_i T_K(u) \chi_s] \, dx \\
& = \sum_{i=1}^d \left[ \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} [a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(v_j)) \chi_j^s] \right. \\
& \quad \times [\partial_i T_K(u_n) - \partial_i T_K(v_j) \chi_j^s] \, dx \\
& \quad + \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) \cdot [\partial_i T_K(v_j) \chi_j^s - \partial_i T_K(u) \chi_s] \, dx \\
& \quad \left. + \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} [a_i(x, T_K(u_n), \partial_i T_K(v_j)) \chi_j^s - a_i(x, T_K(u_n), \partial_i T_K(u)) \chi_s^s] \right]
\end{aligned}$$

$$\begin{aligned} & \times \partial_i T_K(u_n) \, dx \\ & - \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(v_j) \chi_j^s) \cdot \partial_i T_K(v_j) \chi_j^s \, dx \\ & + \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u) \chi_s) \cdot \partial_i T_K(u) \chi_s \, dx \Big] \\ & = \sum_{i=1}^d (I_{i,1} + I_{i,2} + I_{i,3} + I_{i,4} + I_{i,5}) = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

where  $I_k = \sum_{i=1}^d I_{i,k}$  for  $1 \leq k \leq 5$ . We will take the limit first in  $n$ , then in  $j$  and  $s$ , as they tend to infinity in these last five integrals.

**Estimate for  $I_1$ .** We rewrite

$$\begin{aligned} I_1 &= \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} [a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(v_j) \chi_j^s)] \\ & \quad \times [\partial_i T_K(u_n) - \partial_i T_K(v_j) \chi_j^s] \\ &= \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(u_n)) [\partial_i T_K(u_n) - \partial_i T_K(v_j) \chi_j^s] \\ & \quad - \sum_{i=1}^d \int_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}} a_i(x, T_K(u_n), \partial_i T_K(v_j) \chi_j^s) [\partial_i T_K(u_n) - \partial_i T_K(v_j) \chi_j^s] \\ &= J_2 - J_4. \end{aligned}$$

The estimate for  $J_2$  is given by (4.37). As  $J_4$  is concerned, we may repeat the same arguments above as we did for  $J_3$  to obtain that  $J_4 = \epsilon(n, j, s)$ . Therefore,

$$I_1 \leq C\nu + \epsilon(n, j, s) \tag{4.42}$$

**Estimate for  $I_2$ .** Since  $(a_i(x, T_K(u_n), \partial_i T_K(u_n)))_n$  converges to  $l_K^i$  weakly-\* in  $L_{\bar{M}_i}(\Omega)$  and  $((\partial_i T_K(v_j) \chi_j^s - \partial_i T_K(u) \chi_s) \chi_{\{|T_K(u_n) - T_K(v_j)| \leq \nu\}})_n$  converges to  $(\partial_i T_K(v_j) \chi_j^s - \partial_i T_K(u) \chi_s) \chi_{\{|T_K(u) - T_K(v_j)| \leq \nu\}}$  strongly in  $E_{M_i}(\Omega)$ , we obtain

$$I_2 = \sum_{i=1}^d \int_{\{|T_K(u) - T_K(v_j)| \leq \nu\}} l_K^i (\partial_i T_K(v_j) \chi_j^s - \partial_i T_K(u) \chi_s) \, dx + \epsilon(n).$$

By letting now  $j \rightarrow \infty$ , and using Lemma 2.3, we obtain that

$$I_2 = \epsilon(n, j). \tag{4.43}$$

**Estimates for  $I_3$ – $I_5$ .** Using similar arguments as above yields

$$I_3 = \epsilon(n, j), \tag{4.44}$$

$$I_4 = - \sum_{i=1}^d \int_{\Omega} a_i(x, T_K(u), \partial_i T_K(u) \chi_s) \partial_i T_K(u) \chi_s + \epsilon(n, j, s), \tag{4.45}$$

$$I_5 = \sum_{i=1}^d \int_{\Omega} a_i(x, T_K(u), \partial_i T_K(u) \chi_s) \partial_i T_K(u) \chi_s + \epsilon(n, j, s). \tag{4.46}$$

Gathering estimates (4.41)-(4.46), we obtain

$$I_{n,r} \leq (C\nu + \epsilon(n, j, s))^\theta + C_3 \text{meas}\{|T_K(u_n) - T_K(v_j)| > \nu\}^{1-\theta}. \quad (4.47)$$

By taking, first the lim sup with respect to  $n \rightarrow \infty$ , then  $j \rightarrow \infty$ ,  $s \rightarrow \infty$  and  $\nu \rightarrow 0$  yields

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sum_{i=1}^d \int_{\Omega_r} \left[ (a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(u))) \right. \\ & \left. \times (\partial_i T_K(u_n) - \partial_i T_K(u)) \right]^\theta dx = 0. \end{aligned}$$

Consequently, there exist a subsequence, still denoted in the same way, and a negligible subset  $Z \subset \Omega$  such that for all  $x \in \Omega \setminus Z$  one has

$$\begin{aligned} & \sum_{i=1}^d (a_i(x, T_K(u_n), \partial_i T_K(u_n)) - a_i(x, T_K(u_n), \partial_i T_K(u))) \\ & \times (\partial_i T_K(u_n) - \partial_i T_K(u)) \rightarrow 0. \end{aligned} \quad (4.48)$$

Let  $x \in \Omega \setminus Z$  be fixed. Then, according to assumption (A5), the sequence  $(\partial_i T_K(u_n)(x))_n \subset \mathbb{R}$  is bounded for  $i = 1, \dots, d$ . By extracting a convergent subsequence, still denoted in the same way, for some  $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{R}^d$ , we have that  $\partial_i T_K(u_n)(x) \rightarrow \zeta_i$ , for  $i = 1, \dots, d$ . Passing to the limit in (4.48) yields

$$\sum_{i=1}^d (a_i(x, T_K(u)(x), \zeta_i) - a_i(x, T_K(u)(x), \partial_i T_K(u)(x))) \cdot (\zeta_i - \partial_i T_K(u)(x)) = 0,$$

which, according to (3.2), it is only possible if  $\zeta_i = \partial_i T_K(u)(x)$  for  $i = 1, \dots, d$ . Therefore, for any  $K > 0$ , we have deduced that, up to a subsequence,  $\nabla T_K(u_n) \rightarrow \nabla T_K(u)$  a.e. in  $\Omega$ . Since  $K$  is arbitrary, we finally obtained the desired result. This completes the proof of Proposition 4.4.  $\square$

**Remark 4.5.** A straightforward consequence of Proposition 4.4 is that, owing to (4.26),  $\delta_i = a_i(x, u, \partial_i u)$  that is,

$$a_i(x, u_n, \partial_i u_n) \rightarrow a_i(x, u, \partial_i u) \quad \text{weakly-}^* \text{ in } L_{\bar{M}_i}(\Omega). \quad (4.49)$$

**Step 4.**  $L^1$ -convergence of  $(\varphi_n)$ . In this step, we will show that  $\varphi_n \rightarrow \varphi$  strongly in  $L^1(\Omega)$  up to a subsequence. The strongly convergence of  $(\varphi_n)$  in  $L^1(\Omega)$  is based in the next results which generalize that of González Montesinos and Ortega Gallego [13, Lemma 4] which, in its turn, generalize the original results due to Xu in [17].

**Lemma 4.6** ([12]). *Let  $M_i$  be an  $N$ -function for each  $i = 1, \dots, d$  which admits the representation  $M_i(t) = \int_0^{|t|} m_i(s) ds$  and such that  $s \leq m_i(s)$  for all  $s > 0$  and all  $i = 1, \dots, d$ . Let  $M_0$  be the  $N$ -function defined in (2.6). Let  $(u_n)$  be a bounded sequence in  $W^1 L_{\mathcal{M}}(\Omega)$  such that  $u_n \rightarrow u$  strongly in  $E_{M_0}(\Omega)$ . Then there exists a subsequence  $(u_{n(j)}) \subset (u_n)$  such that, for every  $\epsilon > 0$ , there exists a constant value  $\mathbf{M} = \mathbf{M}(\epsilon)$  and a function  $\psi \in W^{1,1}(\Omega)$  satisfying the following properties:*

$$0 \leq \psi \leq 1, \quad (4.50)$$

$$\|\psi - 1\|_{L^1(\Omega)} + \|\nabla \psi\|_{L^1(\Omega)} \leq \epsilon, \quad (4.51)$$

$$|u|, |u_{n(j)}| \leq \mathbf{M} \quad \text{on } \{\psi > 0\} \text{ for all } j \geq 1. \quad (4.52)$$

**Lemma 4.7** ([13]). *Let  $(u_n, \varphi_n)$  be a weak solution to the system (4.12)-(4.14),  $u \in E_{M_0}(\Omega)$  and  $\varphi \in L^\infty(\Omega)$  the limit functions appearing, respectively, in (4.17) and (4.22). Then, for any function  $S \in \mathcal{C}_0^1(\mathbb{R})$ , there exists a subsequence, still denoted in the same way, such that*

$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \quad \text{weakly in } H^1(\Omega). \quad (4.53)$$

Moreover, if  $0 \leq S \leq 1$ , then there exists a constant  $C > 0$ , independent of  $S$ , such that

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \rho_n(u_n) |\nabla[S(u_n)\varphi_n - S(u)\varphi]|^2 \leq C \|S'\|_{\infty} (1 + \|S'\|_{\infty}). \quad (4.54)$$

**Lemma 4.8.** *There exists a subsequence  $(\varphi_{n(j)}) \subset (\varphi_n)$  such that*

$$\lim_{j \rightarrow \infty} \int_{\Omega} |\varphi_{n(j)} - \varphi| = 0. \quad (4.55)$$

The proof of this result is a straightforward adaptation to that of [12, Lemma 5.7].

**Step 5.** Passing to the limit. According to (4.17), (4.19), (4.23), and (4.25), it is straightforward that condition (A9) of Definition 1 is fulfilled. The convergences in Proposition 4.4 and Lemma 4.8 lead us to (A10), and to obtain (A11), using Proposition 4.4 and Lemma 4.8 again with (4.53), it is sufficient to let  $j$  approach infinity in the expression

$$S(u_{n(j)})\rho_{n(j)}(u_{n(j)})\nabla\varphi_{n(j)} = \rho_{n(j)}(u_{n(j)})[\nabla(S(u_{n(j)})\varphi_{n(j)}) - \varphi_{n(j)}\nabla S(u_{n(j)})].$$

This completes the proof of Theorem 4.1.

**Remark 4.9.** Condition (3.4) is not necessary when the constant  $k$  appearing in (3.1) is less than 1.

## 5. NUMERICAL APPROXIMATION

In practical situations, it is very interesting to know the behavior of the solution  $(u, \varphi)$  of (1.1) for different choices of the functions  $a_i(x, s, \zeta)$ ,  $i = 1, \dots, d$ , and  $\rho(s)$ , not only from the quantitative standpoint but from the qualitative one as well. This information can be then used in order to design a thermistor useful for a particular task, for instance, for not letting pass the electric current (by self-destruction) in the event of an unexpected voltage increase.

The numerical resolution of a problem like (1.1) faces certain challenges. One reason is that this system is strongly coupled through nonlinear terms. Also, we do not know whether or not the solution given by Theorem 4.1 is unique.

In this section, we describe a numerical algorithm based on the least squares method to obtain a numerical approximation of a solution to a system like (1.1)

with certain mixed Dirichlet-natural boundary conditions in  $d = 1, 2$  or  $3$ , namely,

$$\begin{aligned} -\sum_{i=1}^d \partial_i(a_i(x, u, \partial_i u)) &= \rho(u)|\nabla\varphi|^2 \quad \text{in } \Omega, \\ \operatorname{div}(\rho(u)\nabla\varphi) &= 0 \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \Gamma_D, \quad \sum_{i=1}^d a_i(x, u, \partial_i u)n_i = 0 \quad \text{on } \Gamma_N, \\ \varphi &= \varphi_0 \quad \text{on } \Gamma_D, \quad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{5.1}$$

where  $\Omega \subset \mathbb{R}^d$  is a Lipschitz bounded domain with boundary  $\partial\Omega = \Gamma_D \cup \Gamma_N$ ,  $\Gamma_D \cap \Gamma_N = \emptyset$ ,  $\Gamma_N$  being an open set with respect to the induced topology of  $\partial\Omega$ ,  $\int_{\Gamma_0} > 0$ ,  $u_0 \geq 0$  is a constant value,  $\varphi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ ,  $n = n(x)$  is the unit, outer normal vector at  $x \in \Gamma_N$  (a.e. in  $\Gamma_N$ ),  $n = (n_1, \dots, n_d)$ , and the functions  $a_1, \dots, a_d$  and  $\rho$  satisfy the assumptions (A1)–(A7) and (A9)–(A11). A similar analysis like the one developed in the previous sections yields to the existence of a capacity solution to the system (5.1).

**5.1. Least squares and conjugate gradient method.** In [14] the authors implemented a fixed-point iterative method to obtain the numerical approximation of a system like (5.1) for  $d = 2$  in which  $\Gamma_N = \emptyset$  and the elliptic operator of the first equation is the anisotropic  $\mathbf{p}$ -Laplacian, that is  $a_1 = a_1(\partial_1 u)$  and  $a_2 = a_2(\partial_2 u)$  are given by

$$a_1(\zeta) = |\zeta|^{p_1-2}\zeta, \quad a_2(\zeta) = |\zeta|^{p_2-2}\zeta, \quad \text{for all } \zeta \in \mathbb{R}, \tag{5.2}$$

for some  $p_1, p_2 \in \mathbb{R}$  with  $p_1, p_2 \geq 2$ . The numerical simulations described in [14] based in the referred iterative method have shown good convergence properties for values of  $p_1$  and  $p_2$  in the interval  $(2, 5)$ . However, when we try to apply this same technique when  $a_1$  or  $a_2$  have not a polynomial growth, the algorithm does not converge. Instead, we have developed a technique based on the least squares method [4, 5]. The application of this technique needs more regularity to both  $\mathbf{a} = (a_1, \dots, a_d)$  and  $\rho$ , namely,

(A12) for a.e.  $x \in \Omega$ , all  $\zeta \in \mathbb{R}$  and all  $i = 1, \dots, d$ , the function  $s \in \mathbb{R} \rightarrow a_i(x, s, \zeta)$  is of class  $\mathcal{C}^1$ .

(A13) for a.e.  $x \in \Omega$ , all  $s \in \mathbb{R}$  and all  $i = 1, \dots, d$  the function  $\zeta \in \mathbb{R} \rightarrow a_i(x, s, \zeta)$  is of class  $\mathcal{C}^1$ .

(A14)  $\rho \in \mathcal{C}^1(\mathbb{R})$ .

We apply the finite element method to approximate the functions in the spaces involved in the solution of (5.1). We first consider polygonal/polyhedral approximations of  $\Omega$ ,  $\Gamma_D$  and  $\Gamma_N$  (all of them still denoted in the same way). Let  $\mathcal{T}_h = \{T_j\}_{j=1}^{N_T}$  be a triangulation of the domain  $\Omega$ ,  $N_T = \operatorname{card} \mathcal{T}_h$ . We consider the discrete space

$$V_h = \{v_h \in \mathcal{C}(\bar{\Omega}) : v_h|_{T_j} \in \mathcal{P}_\ell(T_j), \text{ for } j = 1, \dots, N_T\}, \tag{5.3}$$

where  $\ell \geq 1$  is an integer and  $\mathcal{P}_\ell(T_j)$  is the space of polynomial functions in  $T_j$  of degree  $\ell$ . Notice that  $V_h \subset W^{1,\infty}(\Omega) \subset W^1L_{\mathcal{M}}(\Omega)$ . We consider the projection of  $\varphi_0 \in H^1(\Omega)$  onto  $V_h$  and we still denote this projection as  $\varphi_0$ . We also consider the subspace  $V_{0h} \subset V_h$  defined as  $V_{0h} = \{v_h \in V_h : v_h|_{\Gamma_D} = 0\}$ .

Now we introduce the functional  $J: u_0 + V_{0h} \rightarrow \mathbb{R}$  as

$$J(u_h) = \frac{1}{2} \int_{\Omega} |\nabla \xi_h|^2, \tag{5.4}$$

where  $\xi_h = \xi_h(u_h)$  is defined in the following way. First, we compute  $\varphi_h \in V_h$  as the unique solution to the problem

$$\begin{aligned} &\text{Find } \varphi_h \in V_h \text{ such that } \varphi = \varphi_0 \text{ on } \Gamma_D \text{ and} \\ &\int_{\Omega} \rho(u_h) \nabla \varphi_h \nabla \phi_h = 0, \text{ for all } \phi_h \in V_h \text{ such that } \phi_h|_{\Gamma_D} = 0. \end{aligned} \tag{5.5}$$

Then,  $\xi_h \in V_h$  is the unique solution to the problem

$$\begin{aligned} &\text{Find } \xi_h \in V_h \text{ such that } \xi_h = 0 \text{ on } \Gamma_D \text{ and} \\ &\int_{\Omega} \nabla \xi_h \nabla v_h = \sum_{i=1}^d \int_{\Omega} a_i(x, u_h, \partial_i u_h) \partial_i v_h - \int_{\Omega} \rho(u_h) |\nabla \varphi_h|^2 v_h, \end{aligned} \tag{5.6}$$

for all  $v_h \in V_h$  such that  $v_h|_{\Gamma_D} = 0$ .

Clearly,  $J(u_h) \geq 0$  for all  $u_h \in u_0 + V_{0h}$  and  $J(u_h) = 0$  if and only if  $(u_h, \varphi_h)$  is the solution to the discrete problem

$$\begin{aligned} &u_h \in V_h, \quad u_h = u_0 \text{ on } \Gamma_D, \quad \varphi_h \in V_h, \quad \varphi = \varphi_0 \text{ on } \Gamma_D \text{ and} \\ &\sum_{i=1}^d \int_{\Omega} a_i(x, u_h, \partial_i u_h) \partial_i v_h = \int_{\Omega} \rho(u_h) |\nabla \varphi_h|^2 v_h, \\ &\text{for all } v_h \in V_h \text{ such that } v_h|_{\Gamma_D} = 0. \end{aligned} \tag{5.7}$$

$$\int_{\Omega} \rho(u_h) \nabla \varphi_h \nabla \phi_h = 0, \quad \text{for all } \phi_h \in V_h \text{ such that } \phi_h|_{\Gamma_D} = 0.$$

Problem (5.7) is a nonlinear discrete version of the variational formulation of problem (5.1). Thus, if  $(u_h, \varphi_h)$  is a solution to (5.7) then  $u_h$  is a global minimizer of the functional  $J$  on  $u_0 + V_{0h}$ . To obtain an approximation of the solution to (5.7), we generate a minimizing sequence  $(u_h^m)_{m \geq 1} \subset u_0 + V_{0h}$  so that the non-negative sequence  $(J(u_h^m))_{m \geq 1} \subset \mathbb{R}$  is a decreasing sequence. We can do this by means of the Polack-Ribière version of the conjugate gradient method [4, 5]. Denoting by  $J'(u_h) \in V'_{0h}$  the derivative of  $J$  at  $u_h \in u_0 + V_{0h}$ ,  $V'_{0h}$  being the dual space of  $V_{0h}$ , this algorithm consists of the following four steps:

**Step 1.** Initialization. Given  $u_h^0 \in u_0 + V_{0h}$ , compute  $g_h^0 \in V_{0h}$  as the solution to the variational problem

$$\begin{aligned} &g_h^0 \in V_h, \quad g_h^0 = 0 \quad \text{on } \Gamma_D \quad \text{and} \\ &\int_{\Omega} \nabla g_h^0 \nabla v_h = \langle J'(u_h^0), v_h \rangle, \quad \text{for all } v_h \in V_h \text{ such that } v_h|_{\Gamma_D} = 0. \end{aligned} \tag{5.8}$$

and put  $z_h^0 = g_h^0$ .

Then, for  $m \geq 0$ , assuming that  $u_h^m, g_h^m$  and  $z_h^m$ , are already known, compute  $u_h^{m+1}, g_h^{m+1}$  and  $z_h^{m+1}$  by:

**Step 2.** Descent.

$$u_h^{m+1} = u_h^m - \lambda_m z_h^m, \tag{5.9}$$

$\lambda_m > 0$  being the value where the function  $\lambda \in [0, \infty) \rightarrow J(u_h^m - \lambda z_h^m)$  attains its minimum, that is

$$J(u_h^m - \lambda_m z_h^m) \leq J(u_h^m - \lambda z_h^m), \quad \text{for all } \lambda \in [0, \infty). \quad (5.10)$$

**Step 3.** Construction of the new descent direction. Let  $g_h^{m+1} \in V_{0h}$  be the solution to the variational problem

$$\begin{aligned} g_h^{m+1} &\in V_h, \quad g_h^{m+1} = 0 \text{ on } \Gamma_D \text{ and} \\ \int_{\Omega} \nabla g_h^{m+1} \nabla v_h &= \langle J'(u_h^{m+1}), v_h \rangle, \quad \text{for all } v_h \in V_h \text{ such that } v_h|_{\Gamma_D} = 0. \end{aligned} \quad (5.11)$$

Then, define the number  $\gamma_m \in \mathbb{R}$  and the new descent direction  $z_h^{m+1}$  as follows

$$\gamma_m = \frac{\int_{\Omega} \nabla g_h^{m+1} \nabla (g_h^{m+1} - g_h^m)}{\int_{\Omega} |\nabla g_h^m|^2}, \quad (5.12)$$

$$z_h^{m+1} = g_h^{m+1} + \gamma_m z_h^m, \quad (5.13)$$

**Step 4.** Stopping test. If a certain stopping test is not satisfied, then increase  $m$  by one, go to Step 2 and proceed.

For the generation of the sequence  $(g_h^m)$  we first need to compute the gradient of  $J$  at  $u_h^m$ . To do so, let  $u_h \in u_0 + V_{0h}$  and  $w_h \in V_{0h}$ , then from (5.4) we obtain

$$\langle J'(u_h), w_h \rangle = \int_{\Omega} \nabla \langle \xi'_h, w_h \rangle \nabla \xi_h, \quad (5.14)$$

where  $\xi'_h \in \mathcal{L}(V_{0h})$  is the derivative of  $\xi_h$  with respect to  $u_h$  (here,  $\mathcal{L}(V_{0h})$  stands for the space of linear and continuous mappings from  $V_{0h}$  into itself). Owing to (5.6), and taking into account that  $a_i = a_i(x, s, \zeta)$  a.e. in  $\Omega$ , for all  $s \in \mathbb{R}$  and  $\zeta \in \mathbb{R}$ , we deduce the problem for  $\langle \xi'_h, w_h \rangle \in V_{0h}$

$$\begin{aligned} &\langle \xi'_h, w_h \rangle \in V_h \text{ such that } \langle \xi'_h, w_h \rangle = 0 \text{ on } \Gamma_D \text{ and} \\ &\int_{\Omega} \nabla \langle \xi'_h, w_h \rangle \nabla v_h \\ &= \sum_{i=1}^d \left[ \int_{\Omega} \partial_s a_i(x, u_h, \partial_i u_h) w_h \partial_i v_h + \int_{\Omega} \partial_{\zeta} a_i(x, u_h, \partial_i u_h) \partial_i w_h \partial_i v_h \right] \\ &\quad - \int_{\Omega} \rho'(u_h) w_h |\nabla \varphi_h|^2 v_h - 2 \int_{\Omega} \rho(u_h) \nabla \langle \varphi'_h, w_h \rangle \nabla \varphi_h v_h, \\ &\text{for all } v_h \in V_h \text{ such that } v_h|_{\Gamma_D} = 0. \end{aligned} \quad (5.15)$$

In our computations, we neglected the last term in the right side of (5.15). The reason is twofold. First, this term has appeared to be of a smaller order in front of the other terms in this equation and, secondly, the computation of  $\langle J'(u_h), w_h \rangle$  becomes much simpler.

Thus, making  $v_h = \xi_h$  in (5.15) yields

$$\begin{aligned} & \langle J'(u_h), w_h \rangle \\ &= \sum_{i=1}^d \left[ \int_{\Omega} \partial_s a_i(x, u_h, \partial_i u_h) w_h \partial_i \xi_h + \int_{\Omega} \partial_{\zeta} a_i(x, u_h, \partial_i u_h) \partial_i w_h \partial_i \xi_h \right] \\ & \quad - \int_{\Omega} \rho'(u_h) w_h |\nabla \varphi_h|^2 \xi_h \end{aligned} \tag{5.16}$$

The expression (5.16) is then used to compute the sequence  $(g_h^m)_{m \geq 1}$  from the solution of problems (5.8) and (5.11).

**Remark 5.1.** Notice that the description of the full algorithm described above only involves functions lying in the discrete space  $V_h$  defined in (5.3). One may check that each term in the above expressions makes sense from the assumptions (A1)–(A11), (A12)–(A14), and from the fact that  $V_h \subset W^{1,\infty}(\Omega)$ . If we try to develop the same algorithm within the spaces  $W^1L_{\mathcal{M}}(\Omega)$  and  $H^1(\Omega)$  then, in general, these terms would not be well defined, and this algorithm would be meaningless.

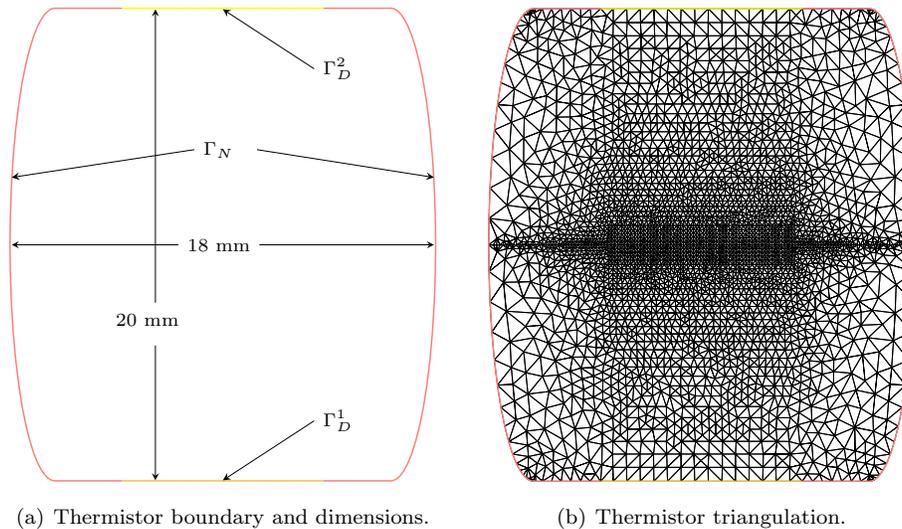


FIGURE 1. Description of the thermistor boundary  $\partial\Omega = \Gamma_D^1 \cup \Gamma_D^2 \cup \Gamma_N$  and the mesh used in the finite element simulations.

**5.2. Numerical simulations.** The algorithm described in the previous section has been implemented to compute the numerical approximation of a solution to problem (5.1). To do so, we have used the Freefem++ software package ([9]). These numerical simulations have been carried out in the 2D domain  $\Omega$  of Figure 1(a) and with the following data.

$$a_1 = a_1(\zeta) = |\zeta|^{p-2}\zeta, \quad a_2 = a_2(\zeta) = 2\beta\zeta e^{\beta|\zeta|^2},$$

for the particular choices  $p = 2.8, 3$  and  $3.2$ , and  $\beta = 10^{-7}$ . The electric conductivity  $\rho$  is given by

$$\rho(s) = 10e^{-|s-30|/20}, \quad s \in \mathbb{R}.$$

The domain  $\Omega$  is a barrel shape set in  $\mathbb{R}^2$ . In Figure 1(a) we may find the description of the boundary and the actual dimensions of  $\Omega$ . The Dirichlet boundary  $\Gamma_D$  has two connected components, namely,  $\Gamma_D^1$  and  $\Gamma_D^2$ , and the natural boundary  $\Gamma_N$  is the complement set of  $\Gamma_D$  in  $\partial\Omega$ .

As far as the boundary conditions are concerned, we have taken

$$u_0 = 30, \quad \varphi_0 = \begin{cases} 0 & \text{on } \Gamma_D^1, \\ V_0 & \text{on } \Gamma_D^2, \end{cases}$$

where  $V_0$  is a constant voltage, namely,  $V_0 = 10$  V or  $V_0 = 40$  V.

Figure 1(b) shows the finite element triangulation of  $\Omega$  from which the discrete space  $V_h$  is defined as in (5.3). It consists of 6,382 triangles, and 3,257 vertices.

**Remark 5.2.** One may wonder why the mesh density in Figure 1(b) is higher along the horizontal central line than elsewhere. The reason is that we first try the algorithm described in the previous section for the case of the  $\vec{\mathbf{p}}$ -Laplacian operator where the functions  $a_1$  and  $a_2$  are given by (5.2). We checked the convergence of this algorithm for this case. However, the error function  $\xi_h$  presented a singularity along this horizontal central line. This led us to consider a denser mesh along this central line, then obtaining better results.

In the definition of the space  $V_h$  we have taken  $\ell = 1$ . On the other hand, the initial guess  $u_h^0$  of the initialization stage (Step 1) have been  $u_h^0 = u_0$  in all cases.

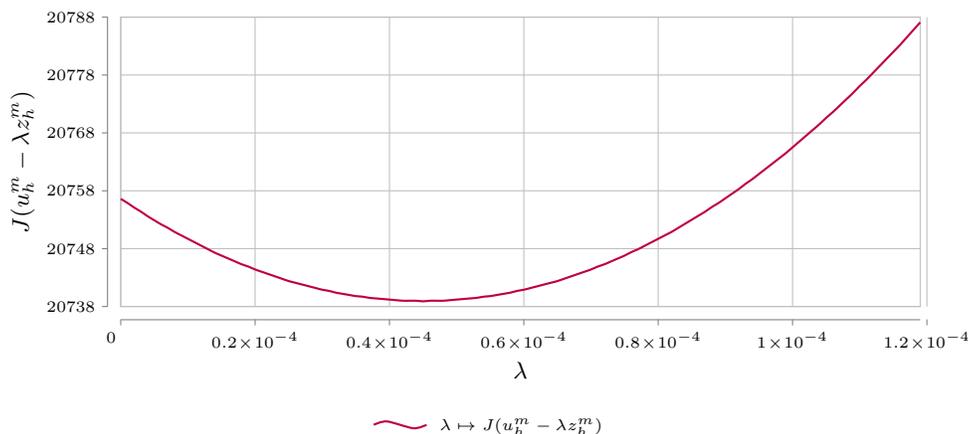


FIGURE 2. The function  $\lambda \in [0, +\infty) \mapsto J(u_h^m - \lambda z_h^m)$  is plotted around the optimal value  $\lambda_m$  for  $m = 40$ ,  $p = 3.0$  and  $V_0 = 10$ .

Figure 2 shows the behavior of the function  $J$  along a descent direction at iteration  $m = 40$  for  $p = 3.0$  and  $V_0 = 10$ . This plot puts in evidence one of the issues concerning the execution of this algorithm in this context: the range of values near  $\lambda_m$  is very small versus the actual values of the error function  $J(u_h^m - \lambda z_h^m)$ . This is not a particular case: it has happened in the six situations we have run, that is,  $p \in \{2.8, 3.0, 3.2\}$  and  $V_0 \in \{10, 40\}$ , and for each  $m \geq 0$ .

We have used two different stopping tests (see Step 4 above). The first one indicates a maximum number of iterations,  $N_{\max}$ , so that if  $m > N_{\max}$  then the algorithm stops and we keep  $u_h^{N_{\max}}$  and  $\varphi_h^{N_{\max}}$  as an approximation of the corresponding problem. The second stopping test compares the value of  $\gamma_m$ , given in (5.12), with a very small value, say  $\varepsilon_{\min} > 0$ . In this case, if  $\gamma_m < \varepsilon_{\min}$ , the algorithm stops as well, and we keep  $u_h^m$  and  $\varphi_h^m$  as an approximation of the corresponding problem. This second test avoids underflow situations: whenever  $\gamma_{m_0} < \varepsilon_{\min}$  occurs, it would yield  $u_h^{m_0+1} = u_h^{m_0}$ , and the sequence becomes stationary for  $m \geq m_0$ . We took  $N_{\max} = 10,000$  and  $\varepsilon_{\min} = 2 \times 10^{-19}$ .

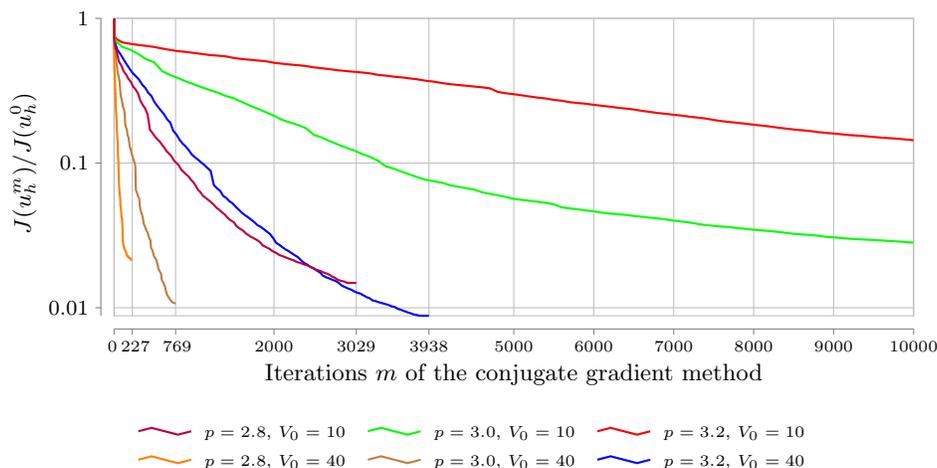


FIGURE 3. Descent of the normalized error functional  $J(u_h^m)/J(u_h^0)$  along the iterations generated by the conjugate gradient method  $u_h = u_h^m$ .

Figure 3 plots the descent of  $J/J(u_h^0)$  as a function of the iterations for the six different cases we have considered. Only when  $p = 3.0$  and  $p = 3.2$ , with  $V_0 = 10$  in both cases, the maximum number of iterations  $N_{\max}$  was reached and the convergence is very slow. In the other cases, though the convergence is much faster, the execution was stopped at iterations  $m = 227$  ( $p = 2.8, V_0 = 40$ ),  $m = 769$  ( $p = 3.0, V_0 = 40$ ),  $m = 3,029$  ( $p = 2.8, V_0 = 10$ ) and  $m = 3,938$  ( $p = 3.2, V_0 = 40$ ), respectively.

In Figure 4 six different iterations of the sequence  $(u_h^m)$  are shown for  $p = 3.0$  and  $V_0 = 10$ . Starting from  $u_h^0 = 30$ , these iterations seem to have an increasing character. Notice the different scale in each plot.

Figures 5 and 6 show the numerical approximation  $(u_h^m, \varphi_h^m)$  obtained from the conjugate gradient algorithm described in the previous section where  $m$  is the index of the last computed iteration. Figure 5 plots the distribution of the temperature  $u_h$  in the six considered cases, whereas Figure 6 plots the corresponding potential  $\varphi_h$ . Obviously, it is expected that the higher the voltage  $V_0$  the higher the maximum temperature. For this reason, we think that the algorithm has underestimated the computed temperature in the case  $p = 2.8, V_0 = 40$ . We also remark that for  $V_0 = 10$  the maximum temperature decreases with  $p$ , whereas for  $V_0 = 40$  is just the contrary. This behavior may be interesting for the design of a particular

thermistor that should switch off the current passing through itself in the event of an unexpected voltage increase. In this situation, the thermistor is used as a fuse protecting certain circuit components which are much more expensive than a single thermistor.

## 6. CONCLUSIONS

We have analyzed a nonlinear strongly coupled system of two partial differential equations of elliptic type, the second equation not being uniformly elliptic. This system is a generalization of the so-called thermistor problem in which the physical unknowns are the temperature,  $u$ , and the electric potential,  $\varphi$ , in a semiconductor device. The special anisotropic structure of the operator in the first equation leads us to consider this analysis in the framework of the anisotropic Orlicz-Sobolev spaces. On the other hand, since the second equation is not uniformly elliptic, we have introduced the concept of capacity solution adapted to this situation, and show an existence result of a capacity solution.

To obtain a numerical solution of this problem, we first consider a projection of the original problem from a straightforward application of the finite element method. This yields a discrete variational formulation in certain finite dimension vector spaces.

We have described a least squares method for the numerical approximation of this discrete variational formulation. The minimizing sequence is generated by means of the Polack-Ribière version of the conjugate gradient method. We have implemented this whole algorithm in a 2D domain by using the Freefem++ software package and run some numerical simulations for different choices of the functions  $a_1(x, s, \zeta)$  and  $a_2(x, s, \zeta)$ , and for a given conductivity  $\rho(s)$ . These numerical results may provide the necessary information in order to design a specific thermistor in an electric circuit.

Finally, the convergence of the minimizing sequence obtained from this algorithm may be very slow. This could be improved by the introduction of some preconditioning technique, which may be considered in future works.

## ACKNOWLEDGEMENTS

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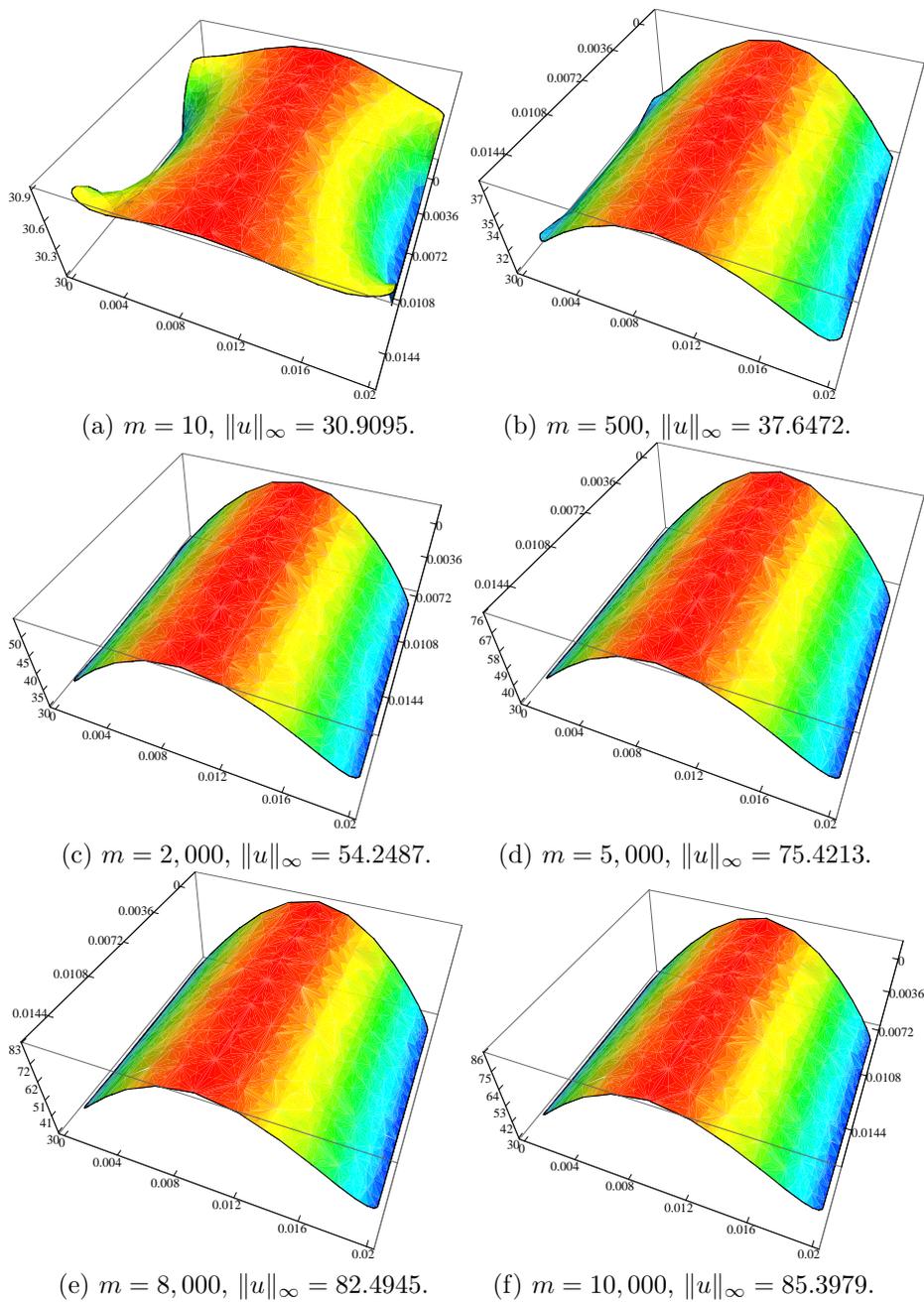


FIGURE 4. Iterations  $m = 10, 500, 2,000, 8,000$  and  $10,000$  of the conjugate gradient method: temperature  $u$  for  $p = 3$  and voltage  $V_0 = 10$ .

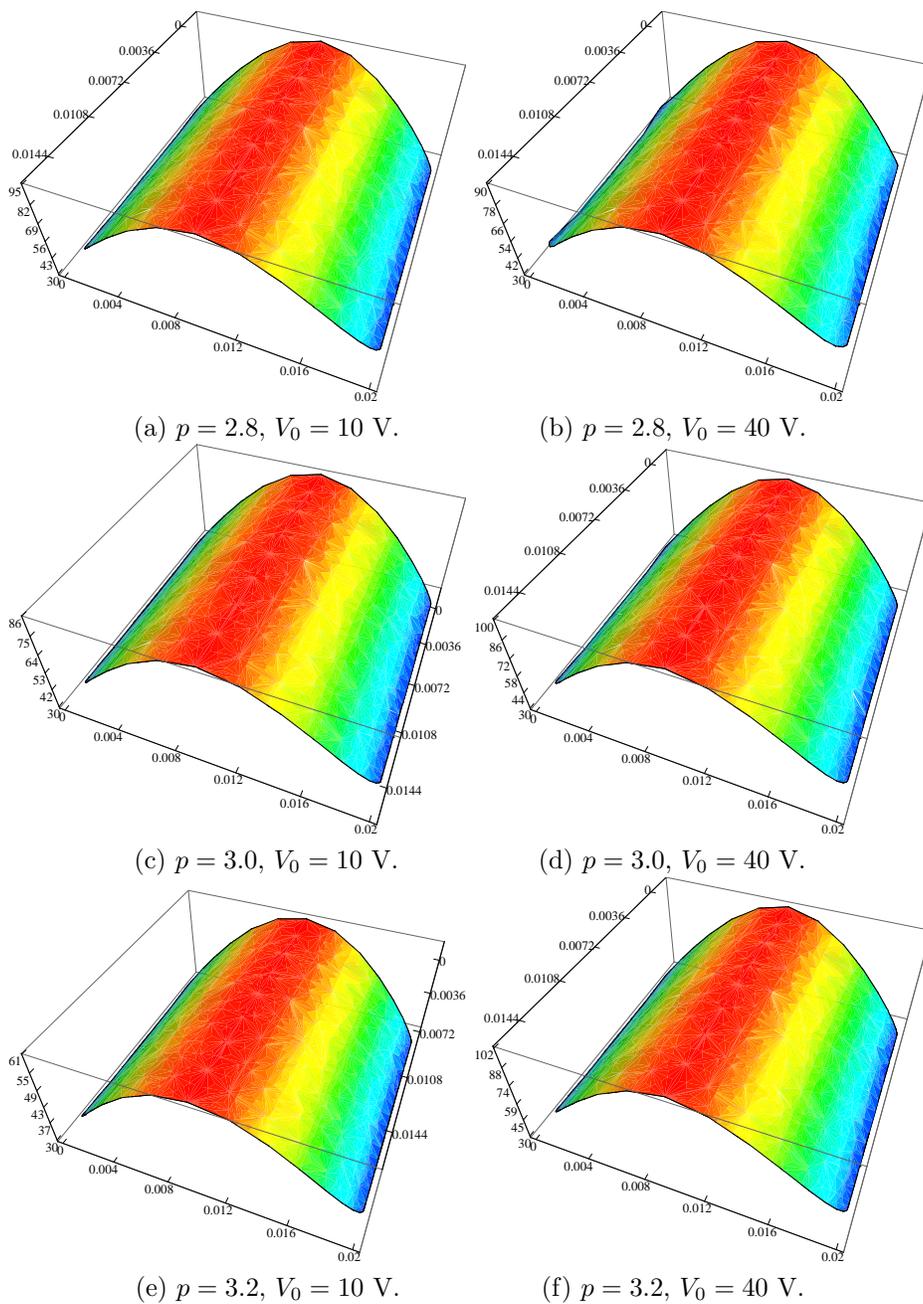


FIGURE 5. Distribution of the temperature  $u$  for  $p = 2.8, 3.0$  and  $3.2$  and voltages  $V_0 = 10$  (left) and  $V_0 = 40$  (right).

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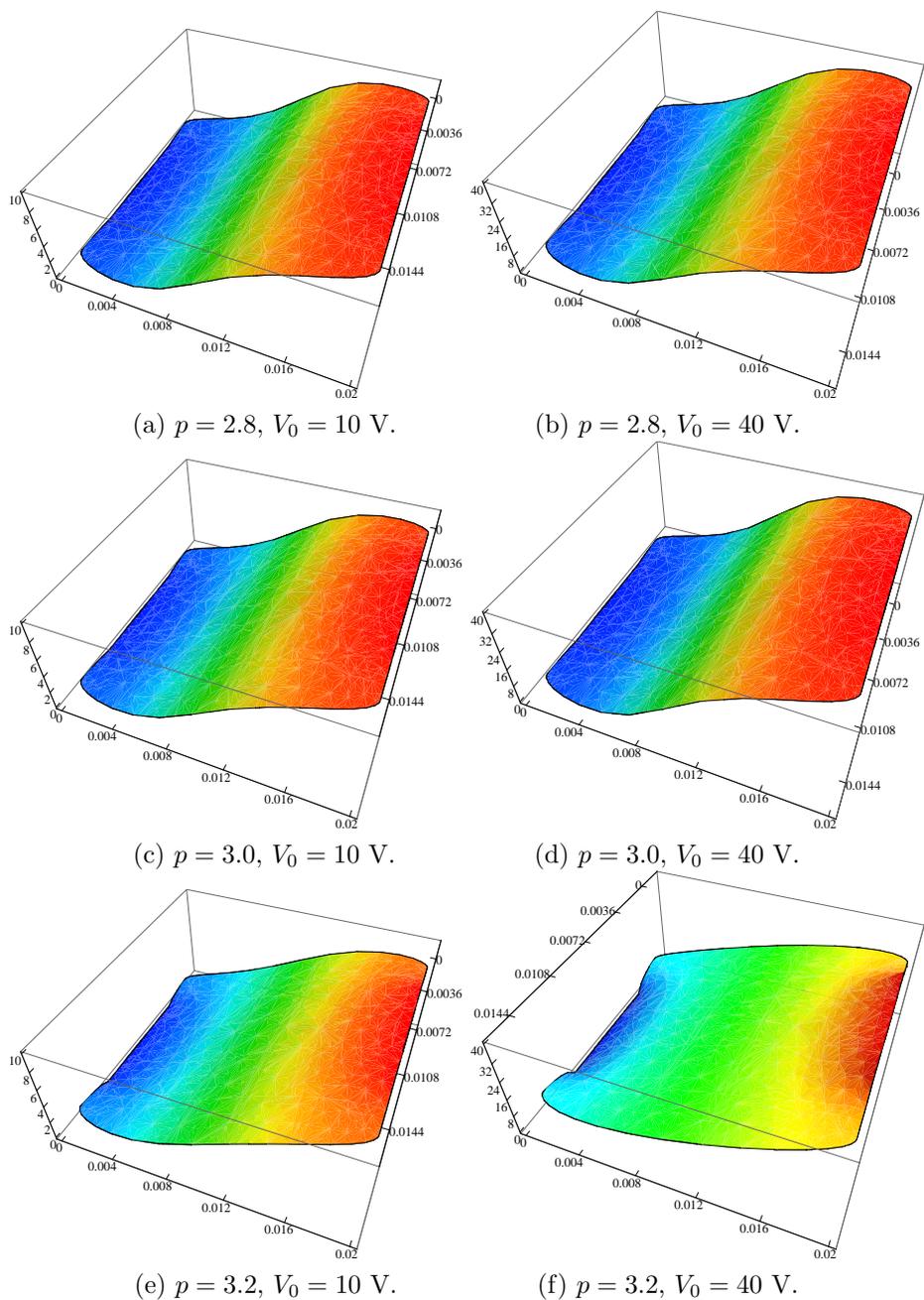


FIGURE 6. Potential  $\varphi$  for  $p = 2.8, 3.0$  and  $3.2$  and voltages  $V_0 = 10$  (left) and  $V_0 = 40$  (right).

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