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MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR THIRD-ORDER THREE-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. This paper concerns the nonlinear third-order three-point boundary-value problem

$$u'''(t) + h(t)f(u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta),$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. First, we establish the existence of at least three positive solutions by using the well-known Leggett-Williams fixed point theorem. And then, we prove the existence of at least $2m-1$ positive solutions for arbitrary positive integer m.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves or gravity driven flows and so on [5]. Recently, third-order boundary value problems (BVPs for short) have received much attention. For example, [3, 4, 8, 11, 15] discussed some third-order two-point BVPs, while [1, 2, 12, 13, 14] studied some third-order three-point BVPs. In particular, Anderson [1] obtained some existence results of positive solutions for the BVP

$$x'''(t) = f(t, x(t)), \quad t_1 \le t \le t_3, \tag{1.1} \quad \boxed{\texttt{0.1}}$$

$$x(t_1) = x'(t_2) = 0, \quad \gamma x(t_3) + \delta x''(t_3) = 0 \tag{1.2} \quad \boxed{\texttt{0.2}}$$

by using the well-known Guo-Krasnoselskii fixed point theorem [6, 9] and Leggett-Williams fixed point theorem [10]. In 2005, the author in [13] established various results on the existence of single and multiple positive solutions to some third-order differential equations satisfying the following three-point boundary conditions

$$x(0) = x'(\eta) = x''(1) = 0, \tag{1.3} \quad \boxed{\texttt{0.3}}$$

where $\eta \in [\frac{1}{2}, 1)$. The main tool in [13] was the Guo-Krasnoselskii fixed point theorem.

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Recently, motivated by the above-mentioned excellent works, we [7] considered the third-order three-point BVP

$$u'''(t) + h(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1.4}$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \tag{1.5}$$

where $0 < \eta < 1$. By using the Guo-Krasnoselskii fixed point theorem, we obtained the existence of at least one positive solution for the BVP (1.4)–(1.5) under the assumption that $1 < \alpha < \frac{1}{\eta}$ and f is either superlinear or sublinear.

In this paper, we will continue to study the BVP (1.4)–(1.5). First, some existence criteria for at least three positive solutions to the BVP (1.4)–(1.5) are established by using the well-known Leggett-Williams fixed point theorem. And then, for arbitrary positive integer m, existence results for at least $2m-1$ positive solutions are obtained.

In the remainder of this section, we state some fundamental concepts and the Leggett-Williams fixed point theorem.

Let E be a real Banach space with cone P. A map $\sigma : P \to [0,+\infty)$ is said to be a nonnegative continuous concave functional on P if σ is continuous and

$$\sigma(tx + (1-t)y) \geq t\sigma(x) + (1-t)\sigma(y)$$

for all x, $y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and σ be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_a = \{x \in P : \|x\| < a\},$$
$$P(\sigma, a, b) = \{x \in P : a \leq \sigma(x),\ \|x\| \leq b\}.$$

Theorem 1.1 (Leggett-Williams fixed point theorem)**.** *Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and σ be a nonnegative continuous concave functional on P such that $\sigma(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < d < a < b \leq c$ such that*

- (i) *$\{x \in P(\sigma,a,b) : \sigma(x) > a\} \neq \emptyset$ and $\sigma(Ax) > a$ for $x \in P(\sigma,a,b)$;*
- (ii) *$\|Ax\| < d$ for $\|x\| \leq d$;*
- (iii) *$\sigma(Ax) > a$ for $x \in P(\sigma,a,c)$ with $\|Ax\| > b$.*

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P_c}$ satisfying

$$\|x_1\| < d, \quad a < \sigma(x_2), \quad \|x_3\| > d, \quad \sigma(x_3) < a.$$

2. Preliminary Lemmas

In this section, we present several important lemmas whose proof can be found in [7].

Lemma 2.1. *Let $\alpha\eta \neq 1$. Then for $y \in C[0,1]$, the BVP*

$$u'''(t) + y(t) = 0, \quad t \in (0,1), \tag{2.1}$$
$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta) \tag{2.2}$$

has a unique solution $u(t) = \int_0^1 G(t,s)y(s)ds$, *where*

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts-s^2)(1-\alpha\eta)+t^2s(\alpha-1), & s \le \min\{\eta,t\}, \\ t^2(1-\alpha\eta)+t^2s(\alpha-1), & t \le s \le \eta, \\ (2ts-s^2)(1-\alpha\eta)+t^2(\alpha\eta-s), & \eta \le s \le t, \\ t^2(1-s), & \max\{\eta,t\} \le s \end{cases} \tag{2.3}$$ 2.30

is called the Green's function.

For convenience, we denote

$$g(s) = \frac{1+\alpha}{1-\alpha\eta} s(1-s), \quad s \in [0,1]. \tag{2.4}$$ 2.05

For the Green's function $G(t,s)$, we have the following two lemmas.

lem2.2 **Lemma 2.2.** *Let* $1 < \alpha < \frac{1}{\eta}$. *Then for any* $(t,s) \in [0,1] \times [0,1]$,

$$0 \le G(t,s) \le g(s).$$

lem2.3 **Lemma 2.3.** *Let* $1 < \alpha < \frac{1}{\eta}$. *Then for any* $(t,s) \in [\frac{\eta}{\alpha}, \eta] \times [0,1]$,

$$\gamma g(s) \le G(t,s),$$

where $0 < \gamma = \frac{\eta^2}{2\alpha^2(1+\alpha)} \min\{\alpha-1, 1\} < 1$.

3. Main results

In the remainder of this paper, we assume that the following conditions are satisfied:

(A1) $1 < \alpha < \frac{1}{\eta}$;
(A2) $f \in C([0,\infty), [0,\infty))$;
(A3) $h \in C([0,1], [0,\infty))$ and is not identical zero on $[\frac{\eta}{\alpha}, \eta]$.

For convenience, we let

$$D = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)ds,$$
$$C = \min_{t\in[\frac{\eta}{\alpha},\eta]} \int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)h(s)ds.$$

thm3.1 **Theorem 3.1.** *Assume that there exist numbers* d_0, d_1 *and* c *with* $0 < d_0 < d_1 < \frac{d_1}{\gamma} < c$ *such that*

$$f(u) < \frac{d_0}{D}, \quad u \in [0, d_0], \tag{3.1}$$ 1

$$f(u) > \frac{d_1}{C}, \quad u \in [d_1, \frac{d_1}{\gamma}], \tag{3.2}$$ 2

$$f(u) < \frac{c}{D}, \quad u \in [0, c]. \tag{3.3}$$ 2.1

Then the BVP (1.4)–(1.5) *has at least three positive solutions.*

Proof. Let the Banach space $E = C[0,1]$ be equipped with the norm

$$\|u\| = \max_{0\le t\le 1} |u(t)|.$$

We denote

$$P = \{u \in E : u(t) \geq 0,\ t \in [0,1]\}.$$

Then, it is obvious that P is a cone in E. For $u \in P$, we define

$$\sigma(u) = \min_{t \in [\frac{\eta}{\alpha}, \eta]} u(t)$$

and

$$Au(t) = \int_0^1 G(t,s)h(s)f(u(s))ds, \quad t \in [0,1]. \tag{3.4}$$ 3.1

It is easy to check that σ is a nonnegative continuous concave functional on P with $\sigma(u) \leq \|u\|$ for $u \in P$ and that $A : P \to P$ is completely continuous and fixed points of A are solutions of the BVP (1.4)–(1.5).

We first assert that if there exists a positive number r such that $f(u) < \frac{r}{D}$ for $u \in [0,r]$, then $A : \overline{P_r} \to P_r$. Indeed, if $u \in \overline{P_r}$, then for $t \in [0,1]$,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)h(s)f(u(s))ds \\
&< \frac{r}{D}\int_0^1 G(t,s)h(s)ds \\
&\leq \frac{r}{D}\max_{t\in[0,1]}\int_0^1 G(t,s)h(s)ds = r.
\end{aligned}$$

Thus, $\|Au\| < r$, that is, $Au \in P_r$. Hence, we have shown that if (3.1) and (3.3) hold, then A maps $\overline{P_{d_0}}$ into P_{d_0} and $\overline{P_c}$ into P_c.

Next, we assert that $\{u \in P(\sigma, d_1, d_1/\gamma) : \sigma(u) > d_1\} \neq \emptyset$ and $\sigma(Au) > d_1$ for all $u \in P(\sigma, d_1, d_1/\gamma)$. In fact, the constant function

$$\frac{d_1 + d_1/\gamma}{2} \in \{u \in P(\sigma, d_1, d_1/\gamma) : \sigma(u) > d_1\}.$$

Moreover, for $u \in P(\sigma, d_1, d_1/\gamma)$, we have

$$d_1/\gamma \geq \|u\| \geq u(t) \geq \min_{t\in[\frac{\eta}{\alpha},\eta]} u(t) = \sigma(u) \geq d_1$$

for all $t \in [\frac{\eta}{\alpha}, \eta]$. Thus, in view of (3.2), we see that

$$\begin{aligned}
\sigma(Au) &= \min_{t\in[\frac{\eta}{\alpha},\eta]}\int_0^1 G(t,s)h(s)f(u(s))ds \\
&\geq \min_{t\in[\frac{\eta}{\alpha},\eta]}\int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)h(s)f(u(s))ds \\
&> \frac{d_1}{C}\min_{t\in[\frac{\eta}{\alpha},\eta]}\int_{\frac{\eta}{\alpha}}^{\eta} G(t,s)h(s)ds = d_1
\end{aligned}$$

as required.

Finally, we assert that if $u \in P(\sigma, d_1, c)$ and $\|Au\| > d_1/\gamma$, then $\sigma(Au) > d_1$. To see this, we suppose that $u \in P(\sigma, d_1, c)$ and $\|Au\| > d_1/\gamma$, then, by Lemma 2.2

and Lemma 2.3, we have

$$\begin{aligned}\sigma(Au) &= \min_{t\in[\frac{\eta}{\alpha},\eta]}\int_0^1 G(t,s)h(s)f(u(s))ds\\ &\geq \gamma\int_0^1 g(s)h(s)f(u(s))ds \geq \gamma\int_0^1 G(t,s)h(s)f(u(s))ds\end{aligned}$$

for all $t\in[0,1]$. Thus

$$\sigma(Au)\geq \gamma\max_{t\in[0,1]}\int_0^1 G(t,s)h(s)f(u(s))ds = \gamma\|Au\| > \gamma\frac{d_1}{\gamma} = d_1.$$

To sum up, all the hypotheses of the Leggett-Williams theorem are satisfied. Hence A has at least three fixed points, that is, the BVP (1.4)–(1.5) has at least three positive solutions u, v, and w such that

$$\|u\| < d_0, \quad d_1 < \min_{t\in[\frac{\eta}{\alpha},\eta]} v(t), \quad \|w\| > d_0, \quad \min_{t\in[\frac{\eta}{\alpha},\eta]} w(t) < d_1.$$

□

thm3.2 **Theorem 3.2.** *Let m be an arbitrary positive integer. Assume that there exist numbers d_i $(1\leq i\leq m)$ and a_j $(1\leq j\leq m-1)$ with $0<d_1<a_1<\frac{a_1}{\gamma}<d_2<a_2<\frac{a_2}{\gamma}<\dots<d_{m-1}<a_{m-1}<\frac{a_{m-1}}{\gamma}<d_m$ such that*

$$f(u) < \frac{d_i}{D}, \quad u\in[0,d_i], \quad 1\leq i\leq m, \tag{3.5}$$

4.1

$$f(u) > \frac{a_j}{C}, \quad u\in[a_j,\frac{a_j}{\gamma}], \quad 1\leq j\leq m-1. \tag{3.6}$$

4.2

Then, the BVP (1.4)–(1.5) *has at least $2m-1$ positive solutions in $\overline{P_{d_m}}$.*

Proof. We use induction on m. First, for $m=1$, we know from (3.5) that $A:\overline{P_{d_1}}\to P_{d_1}$, then, it follows from Schauder fixed point theorem that the BVP (1.4)–(1.5) has at least one positive solution in $\overline{P_{d_1}}$.

Next, we assume that this conclusion holds for $m=k$. In order to prove that this conclusion also holds for $m=k+1$, we suppose that there exist numbers d_i $(1\leq i\leq k+1)$ and a_j $(1\leq j\leq k)$ with $0<d_1<a_1<\frac{a_1}{\gamma}<d_2<a_2<\frac{a_2}{\gamma}<\dots<d_k<a_k<\frac{a_k}{\gamma}<d_{k+1}$ such that

$$f(u) < \frac{d_i}{D}, \quad u\in[0,d_i],\ 1\leq i\leq k+1, \tag{3.7}$$

5

$$f(u) > \frac{a_j}{C}, \quad u\in[a_j,\frac{a_j}{\gamma}],\ 1\leq j\leq k. \tag{3.8}$$

6

By assumption, the BVP (1.4)–(1.5) has at least $2k-1$ positive solutions u_i $(i=1,2,\dots,2k-1)$ in $\overline{P_{d_k}}$. At the same time, it follows from Theorem 3.1, (3.7) and (3.8) that the BVP (1.4)–(1.5) has at least three positive solutions u, v, and w in $\overline{P_{d_{k+1}}}$ such that

$$\|u\| < d_k, \quad a_k < \min_{t\in[\frac{\eta}{\alpha},\eta]} v(t), \quad \|w\| > d_k, \quad \min_{t\in[\frac{\eta}{\alpha},\eta]} w(t) < a_k.$$

Obviously, v and w are different from u_i $(i=1,2,\dots,2k-1)$. Therefore, the BVP (1.4)–(1.5) has at least $2k+1$ positive solutions in $\overline{P_{d_{k+1}}}$, which shows that this conclusion also holds for $m=k+1$. □

Example 3.3. We consider the BVP

$$u'''(t) + 24f(u(t)) = 0, \quad t \in (0,1), \tag{3.9}$$ 10

$$u(0) = u'(0) = 0, \quad u'(1) = \frac{3}{2}u'(\frac{1}{2}), \tag{3.10}$$ 11

where

$$f(u) = \begin{cases} \frac{u^2+1}{28}, & u \in [0, \frac{1}{2}], \\ \frac{275}{56}u - \frac{135}{56}, & u \in [\frac{1}{2}, 1], \\ 2u^{\frac{1}{4}} + \frac{1}{2}, & u \in [1, 90], \\ \frac{u-90}{20}(160 \cdot 110^{\frac{1}{8}} - 2 \cdot 90^{\frac{1}{4}} - \frac{1}{2}) + 2 \cdot 90^{\frac{1}{4}} + \frac{1}{2}, & u \in [90, 110], \\ 160u^{\frac{1}{8}}, & u \in [110, \infty). \end{cases}$$

A simple calculation shows that

$$D = 11, \quad C = \frac{11}{27}, \quad \gamma = \frac{1}{90}.$$

Let $m = 3$. If we choose

$$d_1 = \frac{1}{2}, \quad d_2 = 90.1, \quad d_3 = 11000, \quad a_1 = 1, \quad a_2 = 110,$$

then the conditions (3.5) and (3.6) are satisfied. Therefore, it follows from Theorem 3.2 that the BVP (3.9)–(3.10) has at least five positive solutions.

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