Electronic Journal of Differential Equations, Vol. 2004(2004), No. 143, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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TWIN POSITIVE SOLUTIONS FOR FOURTH-ORDER TWO-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES

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Abstract. A new fixed point theorem on double cones is applied to obtain the existence of at least two positive solutions to

$$(\Phi_p(y''(t))'' - a(t)f(t, y(t), y''(t)) = 0, \quad 0 < t < 1,$$
$$y(0) = y(1) = 0 = y''(0) = y''(1),$$

where $f : [0, 1] \times [0, \infty) \times (-\infty, 0] \to R, a \in L^1([0, 1], (0, \infty))$. We also give some examples to illustrate our results.

1. Introduction

We study the existence of multiple positive solutions for the fourth-order two-point boundary-value problem

$$(\Phi_p(y''(t))'' - a(t)f(t, y(t), y''(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$
$$y(0) = y(1) = 0 = y''(0) = y''(1), \tag{1.2}$$

where the nonlinear term f is allowed to change sign, $a \in L^1([0, 1], (0, \infty)), \Phi_p x = |x|^{p-2}x, 1/p + 1/q = 1, p > 1$. When $p = 2$ Problem (1.1)-(1.2) describes the deformations of an elastic beam. The boundary conditions are given according to the control at the ends of the beam.

A great deal of research has been devoted to the existence of solutions for the fourth-order two point boundary value problem. We refer the reader to [1, 3, 4, 5, 6, 2] and their references. Aftabizadeh [1], [2], del Pino and Manasevich [3], Gupta [4, 5], Ma and Wang [6], Liu [9] have studied the existence problem of positive solutions of the following fourth-order two-point boundary-value problem

$$y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 < t < 1,$$
$$y(0) = y(1) = 0 = y''(0) = y''(1).$$

All the above works were done under assumption that the nonlinear term f is nonnegative.

2000 *Mathematics Subject Classification.* 34B10, 34B15.
Key words and phrases. Fourth-order two-point boundary-value problem; fixed point theorem; double cones; positive solution.

Submitted August 28, 2004. Published December 3, 2004.
Supported by grant 10371006 from the National Natural Sciences Foundation of China.

In this paper, we will impose growth conditions on f which ensure the existence of at least two positive solutions for (1.1)-(1.2). The key tool in our approach is the following fixed point theorem on double cones.

For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let

$$K_r = \{x \in K : \|x\| < r\}, \quad \partial K_r = \{x \in K : \|x\| = r\}.$$

Suppose $\alpha : K \to R^+$ is a continuously increasing functional, i.e., α is continuous and $\alpha(\lambda x) \le \alpha(x)$ for $\lambda \in (0, 1)$. Let

$$K(b) = \{x \in K : \alpha(x) < b\}, \quad \partial K(b) = \{x \in K : \alpha(x) = b\}$$

and $K_a(b) = \{x \in K : a < \|x\|, \alpha(x) < b\}$. The origin in X is denoted by θ.

Theorem 1.1 ([10]). *Let X be a real Banach space with norm $\|\cdot\|$ and $K, K' \subset X$ two solid cones with $K' \subset K$. Suppose $T : K \to K$ and $T' : K' \to K'$ are two completely continuous operators and $\alpha : K' \to R^+$ a continuously increasing functional satisfying $\alpha(x) \le \|x\| \le M\alpha(x)$ for all x in K', where $M \ge 1$ is a constants $b > a > 0$ such that*

(C1) $\|Tx\| < a$, *for* $x \in \partial K_a$
(C2) $\|T'x\| < a$, *for* $x \in \partial K'_a$ *and* $\alpha(T'x) > b$ *for* $x \in \partial K'(b)$
(C3) $Tx = T'x$, *for* $x \in K'_a(b) \cap \{u : T'u = u\}$

then T has at least two fixed points y_1 and y_2 in K such that

$$0 \le \|y_1\| < a < \|y_2\|, \quad \alpha(y_2) < b.$$

2. Existence of positive solutions

Lemma 2.1. *Suppose $g(\cdot) \in C[0, 1]$, then*

$$(\Phi_p(y''(t)))'' - g(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$y(0) = y(1) = 0 = y''(0) = y''(1), \tag{2.2}$$

has a unique solution

$$y(t) = \int_0^1 G(t, s)\Phi_q\big(\int_0^1 G(s, \tau)g(\tau)d\tau\big)ds, \tag{2.3}$$

where

$$G(t, s) = \begin{cases} t(1 - s), & 0 \le t \le s \le 1; \\ s(1 - t), & 0 \le s \le t \le 1. \end{cases}$$

Proof. Let $\Phi_p y''(t) = u(t)$, then (2.1)-(2.2) becomes

$$u'' - g(t) = 0, \quad 0 < t < 1;$$
$$u(0) = u(1) = 0.$$

It is clear that the above boundary-value problem has a unique solution,

$$u(t) = \int_0^1 G(t, s)g(s)ds,$$

where $G(t, s) = \begin{cases} t(1 - s), & 0 \le t \le s \le 1; \\ s(1 - t), & 0 \le s \le t \le 1. \end{cases}$. Then $\Phi_p y''(t) = u(t)$, i.e., $y''(t) = (\Phi_q u)(t)$. By the boundary condition we know that

$$y(t) = \int_0^1 G(t, s)(\Phi_q u)(s)ds = \int_0^1 G(t, s)\Phi_q\big(\int_0^1 G(s, r)g(r)dr\big)ds.$$

The proof is completed. $\square$

In this paper, we assume the following conditions:

(H1) $f : [0,1] \times [0,\infty) \times (-\infty, 0] \to R$ is continuous, $a \in L^1([0,1],(0,\infty))$

(H2) $a(t)f(t,0,0) \not\equiv 0$, $f(t,0,0) \geq 0$ for $t \in [0,1]$.

Let $X = \{x \in C^2[0,1] : x(0) = x(1) = 0 = x''(0) = x''(1)\}$. Then X is a Banach space with the norm $\|x\| = \sup_{t\in[0,1]} |x''(t)|$. Define

$$K = \{x \in X : x \text{ is nonnegative and concave on } [0,1]\},$$

and $K' = \{x \in X : x$ is nonnegative and concave on $[0,1]$, $\alpha(x) \geq \delta^{q-1}\|x\|$, $\delta \in (0,1/2)\}$, where $\alpha(x) = \min_{t\in[\delta,1-\delta]}\{-x''(t)\}$. Obviously $K, K' \subset X$ are two cones with $K' \subset K$. From the definition K', we know that $\alpha(x) \leq \|x\| \leq \frac{1}{\delta^{q-1}}\alpha(x)$, $\delta \in (0,1/2)$. Denote

$$(Tx)(t) = \Big[\int_0^1 G(t,s)\Phi_q\Big(\int_0^1 G(s,\tau)a(\tau)f(\tau,x(\tau),x''(\tau))d\tau\Big)ds\Big]^+,$$

where $B^+ = \max\{B, 0\}$.

$$(Ax)(t) = \int_0^1 G(t,s)\Phi_q\Big(\int_0^1 G(s,\tau)a(\tau)f(\tau,x(\tau),x''(\tau))d\tau\Big)ds.$$

For $x \in X$, define $\theta : X \to K$ by $(\theta x)(t) = \max\{x(t), 0\}$, then $T = \theta \circ A$. For $x \in K'$, let

$$(T'x)(t) = \int_0^1 G(t,s)\Phi_q\Big(\int_0^1 G(s,\tau)a(\tau)f^+(\tau,x(\tau),x''(\tau))d\tau\Big)ds,$$

where $f^+(t,x(t),x''(t)) = \max\{f(t,x(t),x''(t)), 0\}$.

Lemma 2.2. *For $x \in X$, we have $\|x\|_\infty \leq \|x''\|_\infty$ and $\|x'\|_\infty \leq \|x''\|_\infty$ where $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.*

Proof. From

$$x(t) = \int_0^1 G(t,s)\Phi_q\Big(\int_0^1 G(s,r)g(r)dr\Big)ds$$

and

$$x''(t) = -\Phi_q\Big(\int_0^1 G(t,r)g(r)dr\Big),$$

we have

$$\begin{aligned}
|x(t)| &= \Big|\int_0^1 G(t,s)\Phi_q\Big(\int_0^1 G(s,r)g(r)dr\Big)ds\Big| \\
&= \Big|\int_0^1 G(t,s)|x''(s)|ds\Big| \\
&\leq \|x''\|_\infty \int_0^1 G(t,s)ds \\
&= \|x''\|_\infty\Big[\int_0^t s(1-t)ds + \int_t^1 t(1-s)ds\Big] \\
&= \|x''\|_\infty\Big(\frac{t^2(1-t)}{2} + \frac{t(1-t)^2}{2}\Big).
\end{aligned}$$

So $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)| \le \frac{1}{8}\|x''\|_\infty < \|x''\|_\infty$. At the same time, from $x'(t) = \int_0^t x''(s)ds$, we have $\|x'\|_\infty = \sup_{t\in[0,1]} |x'(t)| \le \|x''\|_\infty$. The proof is complete. $\square$

Note that X is a Banach space with the norm $\|x\| = \sup_{t\in[0,1]} |x''(t)|$.

Lemma 2.3. *$T'(K') \subset K'$.*

Proof. For any $x \in K'$, it is clear that $(T'x)(t)$ is nonnegative from the definition of T'. From $(T'x)''(t) = -\Phi_q\big(\int_0^1 G(t,s)f^+(s,x(s),x''(s))ds\big)$, we know $(T'x)''(t) \le 0$. So $T'x$ is concave on $[0,1]$. Then

$$\begin{aligned}
-(T'x)''(t) &= \Phi_q\Big(\int_0^1 G(t,s)f^+(s,x(s),x''(s))ds\Big)\\
&\le \Phi_q\Big(\int_0^1 G(s,s)f^+(s,x(s),x''(s))ds\Big),
\end{aligned}$$

which implies

$$\|-(T'x)''\|_\infty \le \Phi_q\Big(\int_0^1 G(s,s)f^+(s,x(s),x''(s))ds\Big),$$

and

$$\begin{aligned}
\alpha(T'x) &= \min_{t\in[\delta,1-\delta]}[-(T'x)''(t)]\\
&= \min_{t\in[\delta,1-\delta]} \Phi_q\Big(\int_0^1 G(t,s)f^+(s,x(s),x''(s))ds\Big)\\
&= \min_{t\in[\delta,1-\delta]} \Phi_q\Big(\int_t^1 t(1-s)f^+(s,x(s),x''(s))ds\\
&\quad + \int_0^t s(1-t)f^+(s,x(s),x''(s))ds\Big)\\
&\ge \min_{t\in[\delta,1-\delta]} \Phi_q\Big(\int_t^1 \delta s(1-s)f^+(s,x(s),x''(s))ds\\
&\quad + \int_0^t \delta s(1-s)f^+(s,x(s),x''(s))ds\Big)\\
&= \Phi_q\Big(\delta\int_0^1 G(s,s)f^+(s,x(s),x''(s))ds\Big)\\
&\ge \delta^{q-1}\|T'x\|.
\end{aligned}$$

The proof is complete. $\square$

For convenience, we denote

$$\begin{gathered}
Q = \max_{t\in[0,1]}\Big\{\Phi_q\Big(\int_0^1 G(t,s)a(s)ds\Big)\Big\},\\
m = \min_{t\in[\delta,1-\delta]}\Big\{\Phi_q\Big(\int_\delta^{1-\delta} G(t,s)a(s)ds\Big)\Big\},\\
m_i = \min_{t\in[\delta_i,1-\delta_i]}\Big\{\Phi_q\Big(\int_{\delta_i}^{1-\delta_i} G(t,s)a(s)ds\Big)\Big\}.
\end{gathered}$$

It is clear that $0 < m < Q < \infty$.

From the continuity of $f, a \in L^1([0,1],(0,\infty))$. It is easy to see $A : K \to X$ and $T' : K' \to K'$ are completely continuous. So $T : K \to K$ is completely continuous.

Theorem 2.4. *Suppose (H1) and (H2) are satisfied and there exist a, b, d such that $0 < d < \delta^{q-1}a < a < \delta^{q-1}b < b$. Assume that f satisfies the following conditions:*

- (H3) *For $(t,u,v) \in [0,1] \times [0,b] \times [-b,-d]$, $f(t,u,v) \geq 0$*
- (H4) *For $(t,u,v) \in [0,1] \times [0,a] \times [-a,0]$, $f(t,u,v) < \Phi_p(\frac{a}{Q})$.*
- (H5) *For $(t,u,v) \in [0,1] \times [0,b] \times [-b,-\delta^{q-1}b]$, $f(t,u,v) \geq \Phi_p(\frac{b}{m})$.*

Then (1.1)-(1.2) *has at least two positive solutions y_1, y_2 such that*

$$0 < \|y_1\| < a < \|y_2\|, \quad \alpha(y_1) < \delta^{q-1}b, \quad \|y_1\|_\infty < a, \quad \|y_2\|_\infty < b \tag{2.4}$$

Proof. First we show that T has a fixed point $y_1 \in K$ with $\|y_1\| \leq a$. In fact, for any $y \in \partial K_a$, we have $\|y\| = a$. So $0 \leq y(t) \leq a, -a \leq y''(t) < 0, t \in [0,1]$. Let $I = \{t \in [0,1] : f(t, y(t), y''(t)) \geq 0\}$.

$$\begin{aligned}
\|Ty\| &= \max_{t\in[0,1]} |(Ty)''(t)| \\
&= \max_{t\in[0,1]} \max\big\{\Phi_q\big(\int_0^1 G(t,s)a(s)f(s,y(s),y''(s))ds\big), 0\big\} \\
&\leq \max_{t\in[0,1]} \Phi_q\big(\int_I G(t,s)a(s)f(s,y(s),y''(s))ds\big) \\
&\leq \Phi_q\Big(\max_{t\in[0,1], 0\leq u\leq a, -a\leq v\leq 0} f(t,u,v) \max_{t\in[0,1]}\big\{\int_I G(t,s)a(s)ds\big\}\Big) \\
&< \frac{a}{Q}\max_{t\in[0,1]}\big\{\Phi_q\big(\int_0^1 G(t,s)a(s)ds\big)\big\} \\
&= a.
\end{aligned}$$

The existence of y_1 is proved by using condition (C1) of Theorem 1.1 and $0 \leq y_1 \leq a$, $-a \leq y_1'' \leq 0$. Obviously, y_1 is a solution of (1.1)-(1.2). Suppose this is not true, then there is $t_0 \in (0,1)$ such that $y_1(t_0) \neq (Ay_1)(t_0)$. It must be $(Ay_1)(t_0) < 0 = y_1(t_0)$. Let (t_1, t_2) be the maximum interval such that $(Ay_1)(t_0) < 0$ for $t \in (t_1, t_2)$. We claim $[t_1, t_2] \neq [0,1]$ because of $a(t)f(t,0,0) \not\equiv 0$ for $t \in [0,1]$.

If $t_2 < 1$. $y_1(t) = 0, t \in [t_1, t_2]$. $(Ay_1)(t_2) = 0, (Ay_1)(t) < 0$, for $t \in (t_1, t_2)$. Then $(Ay_1)'(t_2) \geq 0$. For $t \in (t_1, t_2)$, we have

$$\begin{aligned}
(Ay_1)''(t) &= -\Phi_q(\int_0^1 G(t,s)a(s)f(s,y(s),y''(s))ds) \\
&= -\Phi_q(\int_0^1 G(t,s)a(s)f(s,0,0)ds) < 0.
\end{aligned}$$

So $(Ay_1)'(t)$ is decreasing for $t \in (t_1, t_2)$. Noticing $(Ay_1)'(t_2) \geq 0$, so $t_1 = 0$ and $(Ay_1)'(t) > 0, t \in [0, t_2), (Ay_1)(0) < 0$, which contradicts (1.2). If $t_1 > 0$. So $y_1(t) = 0, (Ay_1)(t_1) = 0, (Ay_1)(t) < 0$ for $t \in (t_1, t_2)$.$(Ay_1)'(t_1) \leq 0$.

$$\begin{aligned}
(Ay_1)''(t) &= -\Phi_q(\int_0^1 G(t,s)a(s)f(s,y(s),y''(s))ds) \\
&= -\Phi_q(\int_0^1 G(t,s)a(s)f(s,0,0)ds) \leq 0.
\end{aligned}$$

So $(Ay_1)'(t)$ is decreasing for $t \in (t_1, t_2)$. So $t_2 = 1, (Ay_1)(1) < 0$, which contradicts boundary condition (1.2). So y_1 is a solution of (1.1)-(1.2). We now show that (C2) of Theorem 1.1 is satisfied. For $x \in \partial K_a^{'}$, i.e., $\|x\| = a$, then $0 < x(t) < a$, $-a < x''(t) < 0$ for $t \in [0, 1]$.

$$\begin{aligned}
\|T'y\| &= \max_{t\in[0,1]} |(T'y)''(t)| \\
&= \max_{t\in[0,1]} \big\{\Phi_q\big(\int_0^1 G(t,s)a(s)f^+(s,y(s),y''(s))ds\big)\big\} \\
&\le \Phi_q\Big[\max\big\{f^+(t,u,v) : t \in [0,1], 0 \le u \le a, -a \le v \le 0\big\} \\
&\quad \times \max_{t\in[0,1]} \big\{\Phi_q\big(\int_0^1 G(t,s)a(s)ds\big)\big\}\Big] \\
&< \frac{a}{Q}\int_0^1 G(t,s)a(s)ds \\
&= a.
\end{aligned}$$

For $y \in \partial K'(\delta^{q-1}b)$, we have $\alpha(y) = \delta^{q-1}b$. So $\delta^{q-1}b \le \|y\| \le b$, i.e., $-b \le y''(t) \le -\delta^{q-1}b$ for $t \in [\delta, 1-\delta]$, at the same time $\|y\|_\infty \le b$. Then

$$\begin{aligned}
\alpha(T'y) &= -\min_{t\in[\delta,1-\delta]} (T'y)''(t) \\
&= \min_{t\in[\delta,1-\delta]} \Phi_q\big(\int_0^1 G(t,s)a(s)f^+(s,y(s),y''(s))ds\big) \\
&\ge \Phi_q\big(\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} G(t,s)a(s)f^+(s,y(s),y''(s))ds\big) \\
&> \Phi_q\Big(\min\big\{f(t,u,v) : t \in [0,1], u \in [0,b], v \in [-b, -\delta^{q-1}b]\big\} \\
&\quad \times \min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta} G(t,s)a(s)ds\Big) \\
&= b > \delta^{q-1}b.
\end{aligned}$$

Finally we show that (C_3) of Theorem 1.1 is also satisfied. Let $x \in K_a^{'}(\delta^{q-1}b) \cap \{u : T'u = u\}$, then $\|x\| < \frac{1}{\delta^{q-1}}\alpha(x)$. From $\alpha(x) \le \|x\| \le \frac{1}{\delta^{q-1}}\alpha(x)$, we have

$$\min_{t\in[\delta,1-\delta]}\{-x''(t)\} = \alpha(x) \ge \delta^{q-1}\|x\| > \delta^{q-1}a > d\,.$$

So $-x'' \in [d, b]$. From (H3), we have $f(t, u, v) = f^+(t, u, v)$, which implies $Ty = T'y$. Therefore, there exist two positive solutions y_1, y_2 satisfying (2.2). □

Remark. When $p = 2, a(t) \equiv 1, f(t, u, v) > 0, \delta = 1/4$, Theorem 2.4 reduces to [9, Theorem 3.1]. But our result shows at least two positive solutions, whereas there is at least one positive solution in B. Liu [9, Theorem 3.1].

Theorem 2.5. *Suppose (H1),(H2) hold. Also assume that*

(H6) $\delta_i \in (0, 1/2)$, $i = 1, 2, \dots, n$, $0 < \int_{\delta_i}^{1-\delta_i} a(s)ds < \infty$

(H7) *There exists constants* $a_i, b_i, d_i > 0$, $i = 1, 2, \dots, n$, *where* $0 < d_i < \delta^{q-1} a_i < a_i < \delta_i^{q-1} b_i < b_i < d_{i+1}$ *such that for* $i = 1, 2, \dots, n$, *we have*

$$f(t,u,v) \geq 0 \quad \text{for } (t,u,v) \in [0,1] \times [0,b_i] \times [-b_i, -d_i],$$
$$f(t,u,v) < \Phi_q(\frac{a_i}{Q}) \quad \text{for } (t,u,v) \in [0,1] \times [0,a_i] \times [-a_i, 0],$$
$$f(t,u,v) \geq \Phi_q(\frac{b_i}{m_i}) \quad \text{for } (t,u,v) \in [0,1] \times [0,b_i] \times [-b_i, -\delta_i^{q-1} b_i].$$

Then (1.1)-(1.2) *has at least* $n+1$ *positive solutions* $y_1, y_2, \dots, y_{n+1}$ *satisfying*

$$0 \leq \|y_1\| < a_1 < \|y_2\| \leq b_1, \quad \alpha(y_2) < \delta_1^{q-1} b_1, \quad 0 < \|y_1\|_\infty < a_1,$$
$$0 < \|y_2\|_\infty < b_1, \quad a_2 < \|y_3\|, \quad \alpha(y_3) < \delta_2^{q-1} b_2, \quad 0 < \|y_3\|_\infty < b_2,$$
$$\dots$$
$$a_n < \|y_{n+1}\|, \quad \alpha(y_{n+1}) < \delta_n^{q-1} b_n, \quad 0 < \|y_{n+1}\|_\infty < b_n.$$

Theorem 2.6. *Suppose (H1), (H2), (H6) hold. Also assume*

(H8) *There exists constants* a_i, $b_i > 0$, d, $i = 1, 2, \dots, n$, *where* $0 < d < \delta^{q-1} a_i < a_i < \delta_i^{q-1} b_i < b_i$, *such that for* $i = 1, 2, \dots, n$, *we have:*

$$f(t,u,v) \geq 0 \quad \text{for } (t,u,v) \in [0,1] \times [0,b_n] \times [-b_n, -d],$$
$$f(t,u,v) < \frac{a_i}{Q} \quad \text{for } (t,u,v) \in [0,1] \times [0,a_i] \times [-a_i, 0],$$
$$f(t,u,v) \geq \frac{b_i}{m_i} \quad \text{for } (t,u,v) \in [0,1] \times [0,b_i] \times [-b_i, -\delta_i^{q-1} b_i],$$

Then (1.1)-(1.2) *has at least* $2n$ *positive solutions* $y_1, y_2, \dots, y_{2n}$ *satisfying*

$$0 \leq \|y_1\| < a_1 < \|y_2\|, \quad \alpha(y_2) < \delta_1^{q-1} b_1 < \alpha(y_3),$$
$$0 < \|y_1\|_\infty < a_1, \quad 0 < \|y_2\|_\infty < b_1,$$
$$\|y_3\| < a_2 < \|y_4\|, \quad \alpha(y_4) < \delta_2^{q-1} b_2 < \alpha(y_5), \quad \|y_3\|_\infty < a_2, \quad \|y_4\|_\infty < b_2, \dots$$
$$\|y_{2n-1}\| < a_n < \|y_{2n}\|, \quad \alpha(y_{2n}) < \delta_n^{q-1} b_n, \quad \|y_{2n-1}\|_\infty < a_n, \quad \|y_{2n}\|_\infty < b_n.$$

Example. Consider the boundary-value problem

$$\begin{gathered} (\Phi_3 y''(t))'' - \Big[\frac{y + \pi/6}{6}\big(\sqrt{3}\cos(y'' + \frac{5}{12}\pi)\big)^{19} + \frac{t}{10}\Big] = 0, \quad 0 < t < 1, \\ y(0) = y(1) = 0 = y''(0) = y''(1), \end{gathered} \tag{2.5}$$

where $a(t) = t$, $f(t,u,v) = \frac{u+\pi/6}{6}(\sqrt{3}\cos(v + \frac{5}{12}\pi))^{19} + \frac{t}{10}$, $p = 3$, $q = 3/2$. Clearly f is allowed to change sign on $[0,1] \times [0,\infty) \times (-\infty, 0)$.

$$Q = \max_{t \in [0,1]} \Phi_{3/2}\Big(\int_0^1 G(t,s)a(s)ds\Big) = \max_{t \in [0,1]} \Big(\frac{t}{6}(-t^2 + 1)\Big)^{1/2} = (\frac{\sqrt{3}}{27})^{1/2} = 3^{-\frac{5}{4}}.$$

Note that

$$\begin{aligned} \int_\delta^{1-\delta} G(t,s)a(s)ds &= (1-t)\int_\delta^t s^2 ds + t\int_t^{1-\delta}(1-s)s ds \\ &= \frac{1}{6}[-t^3 + t(4\delta^3 - 3\delta^2 + 1) - 2\delta^3]. \end{aligned}$$

Let $\delta = 1/4$, $d = \pi/36$, $a = \pi/12$, $b = \pi/2$. It is clear that $d < \delta^{1/2} a < a < \delta^{1/2} b$. Then

$$m = \min_{t\in[\delta,1-\delta]} \Phi_{3/2}\Big(\int_{\delta}^{1-\delta} G(t,s)a(s)ds\Big) > \sqrt{\frac{1}{24}}\,.$$

For $(t,u,v) \in [0,1]\times[0,\pi/2]\times[-\pi/2,-\pi/36]$, we have $f(t,u,v) = \frac{u+\pi/6}{6}(\sqrt{3}\cos(v+\frac{5}{12}\pi))^{19} + \frac{t}{10} > 0$. So (H3) holds. For $(t,u,v) \in [0,1]\times[0,\pi/12]\times[-\pi/12,0]$, $f(t,u,v) = \frac{u+\pi/6}{6}\big(\sqrt{3}\cos(v+\frac{5}{12}\pi)\big)^{19} + \frac{t}{10} < \frac{\pi}{24}\times(\frac{\sqrt{3}}{2})^{19} + \frac{1}{10} < 0.6 < (\frac{\pi}{12}\times 3^{5/4})^2 = \Phi_3(a/Q)$. So (H4) holds. For $(t,u,v) \in [0,1]\times[0,\pi/2]\times[-\pi/2,-\pi/4]$, $f(t,u,v) = \frac{u+\pi/6}{6}\big(\sqrt{3}\cos(v+\frac{5}{12}\pi)\big)^{19} + \frac{t}{10} > (\pi\sqrt{6})^2 > \Phi_3(b/m)$. So (H5) holds. Thus by Theorem 2.4, this boundary-value problem has at least two positive solutions y_1, y_2 such that

$$0 < \|y_1\| < \frac{\pi}{12} < \|y_2\|, \quad \alpha(y_1) < \frac{\pi}{16}, \quad \|y_1\|_\infty < \frac{\pi}{12}, \quad \|y_2\|_\infty < \frac{\pi}{2}.$$

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