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HIGHER ORDER CRITERION FOR THE NONEXISTENCE OF FORMAL FIRST INTEGRAL FOR NONLINEAR SYSTEMS

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Abstract. The main purpose of this article is to find a criterion for the nonexistence of formal first integrals for nonlinear systems under general resonance. An algorithm illustrates an application to a class of generalized Lokta-Volterra systems. Our result generalize the classical Poincaré's nonintegrability theorem and the existing results in the literature.

1. Introduction

Let us consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{C}^n, \tag{1.1}$$

where $f(\mathbf{x})$ is an n-dimensional vector-valued analytic function with $f(\mathbf{0}) = \mathbf{0}$.

In 1891, Poincaré [12] provided a criterion on the nonexistence of analytic or formal first integrals for system (1.1).

Theorem 1.1 (Poincaré's nonintegrability theorem)**.** *If the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Jacobian matrix $A = Df(\mathbf{0})$ are $\mathbb{N}$-non-resonant, i.e., they do not satisfy any resonant relations of the form*

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in \mathbb{Z}_+, \quad \sum_{j=1}^{n} k_j \geq 1,$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, then system (1.1) *has neither analytic nor formal first integrals in any neighborhood of the origin $\mathbf{x} = \mathbf{0}$.*

In 1996, Furta [6] gave another proof to Theorem 1.1 with the additional assumption that the matrix A can be diagonalized, and furthermore, he obtained a result about the nonexistence of formal integral for semi-quasi-homogeneous systems. In 2001, Shi and Li [13] presented a different proof to Theorem 1.1 without the assumption of diagonalization of A. In 2007, Shi [14] extended Theorem 1.1 and provided a necessary condition for system (1.1) to have a rational first integral. Cong et al [5] in 2011 generalized the Shi's result to a more general case. For related information, see [7] and references therein.

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We note that the above results are all obtained under the condition that the eigenvalues of A are non-resonant. How about the case when the eigenvalues of A are resonant? In 2003, Li et al [10] studied the case that A has a zero eigenvalue and others are non-resonant. In 2010, Liu et al [11] studied the case that several eigenvalues of A are zero and the others are non-resonant, and Li et al [9] studied the case that the eigenvalues of A are pairwise resonant. In this paper, we will extend the above results to the more general case, i.e., the eigenvalues of A are in general resonance. With the help of the theory of normal form [1] and power transformations [4], we give a criterion on the nonexistence of formal first integral for system (1.1) under the case that the eigenvalues of A are in general resonance.

This paper is organized as follows. In section 2, we will build a criterion of nonexistence of formal first integral of nonlinear systems under general resonance. In section 3, an algorithm to determine the nonexistence of formal first integrals of nonlinear systems will be illustrated by studying a class of generalized Lotka-Volterra systems.

2. Main results

System (1.1) can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x} + F(\mathbf{x}) \tag{2.1}$$

near some neighborhood of $\mathbf{x} = \mathbf{0}$, where $A = Df(\mathbf{0})$, $F(\mathbf{x}) = o(\|\mathbf{x}\|)$. Let $\Lambda = (\lambda_1, \dots, \lambda_n)^T$, and

$$\Theta = \big\{\mathbf{k} = (k_1, \dots, k_n)^T \in \mathbb{Z}^n : \langle \mathbf{k}, \Lambda\rangle = \sum_{i=1}^{n} k_i\lambda_i = 0\big\},$$

then Θ is a subgroup of $\mathbb{Z}^n$. Let rank Θ denote the number of the least generating elements of Θ.

We assume that A is a diagonalizable matrix, and without loss of generality, $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. By the Poincaré-Dulac normal form theory [1], there exists a formal transformation $\mathbf{x} = \mathbf{y} + h(\mathbf{y})$, such that system (2.1) can be changed to

$$\dot{y}_i = \lambda_i y_i + \tilde{F}^i(\mathbf{y}), \quad i = 1, \dots, n, \tag{2.2}$$

where $\mathbf{y} = (y_1, \dots, y_n)^T$,

$$\tilde{F}^i(\mathbf{y}) = \sum_{r=2}^{\infty} \sum_{|\mathbf{k}|=r, \langle \mathbf{k},\Lambda\rangle=\lambda_i} \tilde{F}^i_{\mathbf{k}} \mathbf{y}^{\mathbf{k}}, \quad i = 1, \dots, n, \tag{2.3}$$

here $|\mathbf{k}| = \sum_{i=1}^{m} k_i$, $\mathbf{y}^{\mathbf{k}} = y_1^{k_1} \dots y_n^{k_n}$, $\tilde{F}^i_{\mathbf{k}}$ are constant coefficients. For convenience, we give the following definition.

Definition 2.1. A function $\Phi(\mathbf{x})$ is resonant with respect to Λ, if it can be written as

$$\Phi(\mathbf{x}) = \sum_{i=1}^{\infty} \sum_{|\mathbf{k}|=i, \langle \mathbf{k},\Lambda\rangle=0} \Phi_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where $\Phi_{\mathbf{k}}$ are constants.

Lemma 2.2 ([9, 15])**.** *If system* (2.1) *admits a nontrivial formal first integral* $\Phi(\mathbf{x})$*, then system* (2.2) *has a nontrivial formal first integral* $\tilde{\Phi}(\mathbf{y})$ *which is resonant with respect to* Λ.

Lemma 2.3. *If* $\operatorname{rank}\Theta = m$ $(0 < m < n)$*, then there exist* $n-m$ *eigenvalues which are* $\mathbb{N}$*-non-resonant.*

Proof. Assume that any $n-m$ eigenvalues of matrix A are $\mathbb{N}$-resonant, then there exist $n-m$ constants $a_1^1, \dots, a_{n-m}^1 \in \mathbb{Z}_+$ with $a_1^1 + \dots + a_{n-m}^1 \neq 0$ such that

$$a_1^1\lambda_1 + \dots + a_{n-m}^1\lambda_{n-m} = 0.$$

Thus $\tau_{\mathbf{1}} = (a_1^1, \dots, a_{n-m}^1, 0, \dots, 0)^T \in \Theta$. Without loss of generality, we assume that $a_1^1 \neq 0$.

Analogously, there exist $n-m$ constants $a_1^2, \dots, a_{n-m}^2 \in \mathbb{Z}_+$ with $a_1^2 + \dots + a_{n-m}^2 \neq 0$ such that

$$a_2^2\lambda_2 + \dots + a_{n-m+1}^2\lambda_{n-m+1} = 0.$$

Thus $\tau_{\mathbf{2}} = (0, a_2^2, \dots, a_{n-m+1}^2, 0, \dots, 0)^T \in \Theta$. Similarly, we assume that $a_2^2 \neq 0$.

Repeating the above process, we obtain

$$\tau_{\mathbf{i}} = (0, \dots, 0, a_i^i, \dots, a_{n-m+i-1}^i, 0 \dots, 0)^T \in \Theta,\ a_i^i \neq 0,\ i = 1, \dots, m+1.$$

Obviously, $\tau_{\mathbf{1}}, \dots, \tau_{\mathbf{m+1}}$ are linear independent. This is contradict to the fact that rank $\Theta = m$, the proof is complete. ☐

For simplicity of presentation, we make the following assumption.

(H1) $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $\operatorname{rank}\Theta = m$ $(0 < m < n)$, and $\lambda_{m+1}, \dots, \lambda_n$ are $\mathbb{N}$-non-resonant. Let

$$\Theta' = \Big\{\mathbf{k} \in \Theta : \sum_{i=1}^n k_i \geq 1, \text{ and there exists } j \in \{1, \dots, n\} \text{ such that } \mathbf{k} + \mathbf{e}_j \geq 0\Big\},$$

here $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the natural bases of $\mathbb{R}^n$, and a vector $\mathbf{a} \geq 0$ means every component $a_i \geq 0$, for $i = 1, \dots, n$.

Lemma 2.4. *Assume* (H1) *holds, then there exist* m *linear independent vectors* $\tau_{\mathbf{1}}, \dots, \tau_{\mathbf{m}} \in \mathbb{Q}^n$*, such that*

(1) *for any* $\mathbf{k} \in \Theta$*, there exists* $\mathbf{a} = (a_1, \dots, a_m)^T \in \mathbb{Q}^m$*, such that* $\mathbf{k} = a_1\tau_{\mathbf{1}} + \dots + a_m\tau_{\mathbf{m}}$*;*
(2) *for any* $\mathbf{k} \in \Theta'$*, there exists* $\mathbf{a} = (a_1, \dots, a_m)^T \in \mathbb{Z}_+^m$*,* $|\mathbf{a}| \geq 1$*, such that* $\mathbf{k} = a_1\tau_{\mathbf{1}} + \dots + a_m\tau_{\mathbf{m}}$*.*

Proof. Let $B = (\alpha_{\mathbf{1}}, \dots, \alpha_{\mathbf{m}})^T$, $P_1 = (\alpha_{\mathbf{m+1}}, \dots, \alpha_{\mathbf{n}})^T$, here $\alpha_{\mathbf{1}}, \dots, \alpha_{\mathbf{m}}$ are the least generating elements of Θ, and $\alpha_{\mathbf{m+1}}, \dots, \alpha_{\mathbf{n}} \in \mathbb{Z}^n$ are fundamental solutions of the linear equations $B\mathbf{x} = \mathbf{0}$, then $P_1B^T = \mathbf{0}$. Let

$$R = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix}, \quad P = \begin{pmatrix} P_2 \\ P_1 \end{pmatrix},$$

where P_2 is a $m \times n$ sub-matrix of R such that P is invertible. Then $P_1 = (\mathbf{0}, E_{n-m})P$, i.e., $P_1P^{-1} = (\mathbf{0}, E_{n-m})$, where E_{n-m} denotes the $n-m$ order unit matrix. Let

$$\tau = (\tau_{\mathbf{1}}, \dots, \tau_{\mathbf{m}}) = P^{-1}\begin{pmatrix} E_m \\ \mathbf{0} \end{pmatrix}, \tag{2.4}$$

then $\tau_{\mathbf{1}},\dots,\tau_{\mathbf{m}}\in\mathbb{Q}^n$, and they are linear independent. Furthermore,

$$P_1\tau=P_1P^{-1}\begin{pmatrix}E_m\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & E_{n-m}\end{pmatrix}\begin{pmatrix}E_m\\ \mathbf{0}\end{pmatrix}=\mathbf{0},$$

therefore $\tau_{\mathbf{1}},\dots,\tau_{\mathbf{m}}$ are fundamental solutions of the linear equations

$$P_1\mathbf{x}=\mathbf{0}. \tag{2.5}$$

(1) For any $\mathbf{k}\in\Theta$, there exist $b_1,\dots,b_m\in\mathbb{R}$, such that $\mathbf{k}=b_1\alpha_{\mathbf{1}}+\dots+b_m\alpha_{\mathbf{m}}$. Therefore

$$P_1\mathbf{k}=b_1P_1\alpha_{\mathbf{1}}+\dots+b_mP_1\alpha_{\mathbf{m}}=P_1B^T(b_1,\dots,b_m)^T=\mathbf{0}.$$

This means that $\mathbf{k}$ is a solution of (2.5), thus there exists $\mathbf{a}=(a_1,\dots,a_m)^T\in\mathbb{Q}^m$, such that

$$\mathbf{k}=a_1\tau_{\mathbf{1}}+\dots+a_m\tau_{\mathbf{m}}.$$

(2) For any $\mathbf{k}\in\Theta'$, by (1), there exists $\mathbf{a}=(a_1,\dots,a_m)^T\in\mathbb{Q}^m$, such that $\mathbf{k}=a_1\tau_{\mathbf{1}}+\dots+a_m\tau_{\mathbf{m}}$. By (2.4), we have

$$\mathbf{k}=\tau\mathbf{a}=P^{-1}\begin{pmatrix}E_m\\ \mathbf{0}\end{pmatrix}\mathbf{a},$$

thus

$$\begin{pmatrix}\mathbf{a}\\ \mathbf{0}\end{pmatrix}=P\mathbf{k}=\begin{pmatrix}P_2\\ P_1\end{pmatrix}\mathbf{k}=\begin{pmatrix}P_2\mathbf{k}\\ \mathbf{0}\end{pmatrix}.$$

According to the choice of P_2, we obtain that $(a_1,\dots,a_m)^T\in\mathbb{Z}_+^m$, and $|\mathbf{a}|\geq 1$. $\square$

Lemma 2.5. *Assume* (H1) *holds, and $\tau_{\mathbf{1}},\dots,\tau_{\mathbf{m}}$ be given in Lemma 2.4. Then under the change of variables*

$$\begin{gathered}z_i=y_1^{\tau_{i1}}\dots y_n^{\tau_{in}},\quad i=1,\dots,m,\\ w_j=y_{m+j},\quad j=1,\dots,n-m,\end{gathered}\tag{2.6}$$

system (2.2) *becomes*

$$\begin{gathered}\dot z_i=z_i\bar F^i(\mathbf{z}),\quad i=1,\dots,m,\\ \dot w_j=\lambda_{m+j}w_j+w_j\bar F^{m+j}(\mathbf{z}),\quad j=1,\dots,n-m,\end{gathered}\tag{2.7}$$

where $\mathbf{z}=(z_1,\dots,z_m)$, $\bar F^i(\mathbf{z})$ $(i=1,\dots,n)$ are formal power series with respect to $\mathbf{z}$, $\bar F^i(\mathbf{0})=0$.

Proof. By Lemma 2.4 and (2.3), we have

$$\begin{gathered}\dot z_i=z_i\Big(\tau_{i1}\frac{\tilde F^1(\mathbf{y})}{y_1}+\dots+\tau_{in}\frac{\tilde F^n(\mathbf{y})}{y_n}\Big),\quad i=1,\dots,m,\\ \dot w_j=\lambda_{m+j}w_j+w_j\frac{\tilde F^{m+j}(\mathbf{y})}{y_{m+j}},\quad j=1,\dots,n-m,\end{gathered}\tag{2.8}$$

where

$$\frac{\tilde F^i(\mathbf{y})}{y_i}=\sum_{r=2}^{\infty}\sum_{|\mathbf{k}|=r,\langle\mathbf{k},\Lambda\rangle=\lambda_i}\tilde F^i_{\mathbf{k}}y_1^{k_1}\dots y_i^{k_i-1}\dots y_n^{k_n},\quad i=1,\dots,n. \tag{2.9}$$

Obviously, for every monomial $y_1^{k_1}\dots y_i^{k_i-1}\dots y_n^{k_n}$ in the above expression, the exponents $(k_1,\dots,k_i-1,\dots,k_n)\in\Theta'$. By Lemma 2.4, there exist $\mathbf{a}=(a_1,\dots,a_m)^T$ in $\mathbb{Z}_+^m$ with $|\mathbf{a}|\geq 1$, such that

$$(k_1,\dots,k_i-1,\dots,k_n)^T=a_1\tau_{\mathbf{1}}+\dots+a_m\tau_{\mathbf{m}}.$$

Therefore

$$\begin{aligned}\frac{\tilde{F}^i(\mathbf{y})}{y_i} &= \sum_{r=2}^{\infty} \sum_{|\mathbf{k}|=r, \langle \mathbf{k},\Lambda\rangle=\lambda_i} \tilde{F}^i_{\mathbf{k}} y_1^{a_1\tau_{11}+\cdots+a_m\tau_{m1}} \dots y_n^{a_1\tau_{1n}+\cdots+a_m\tau_{mn}} \\ &= \sum_{r=2}^{\infty} \sum_{|\mathbf{k}|=r, \langle \mathbf{k},\Lambda\rangle=\lambda_i} \tilde{F}^i_{\mathbf{k}} (y_1^{\tau_{11}} \dots y_n^{\tau_{1n}})^{a_1} \dots (y_1^{\tau_{m1}} \dots y_n^{\tau_{mn}})^{a_m} \\ &= \sum_{r=1}^{\infty} \sum_{|\mathbf{a}|=r} \bar{F}^i_{\mathbf{a}} z_1^{a_1} \dots z_m^{a_m}, \quad i=1,\dots,n.\end{aligned} \tag{2.10}$$

By (2.8) and (2.10), we can get (2.7), and the lemma is proved. □

Remark 2.6. It should be pointed out that there are similar arguments in [3, 4] as used in Lemma 2.4 and Lemma 2.5. Here, to ensure that $\bar{F}^i(\mathbf{z})$ $(i=1,\dots,n)$ in (2.7) are formal power series with respect to $\mathbf{z}$, we give a different way to calculate $\tau_{\mathbf{1}},\dots,\tau_{\mathbf{m}}$.

Lemma 2.7. *Assume that* (H1) *holds. If system* (2.1) *has a nontrivial first integral, then system* (2.7) *has a nontrivial formal first integral which is independent with* $w_1,\dots,w_{n-m}$*, and the system*

$$\dot{\mathbf{z}} = \mathbf{z}\bar{F}(\mathbf{z}) \tag{2.11}$$

has a nontrivial formal first integral, where $\mathbf{z}\bar{F}(\mathbf{z}) := (z_1\bar{F}^1(\mathbf{z}),\dots,z_m\bar{F}^m(\mathbf{z}))$.

Proof. Assume that (2.1) has a nontrivial formal first integral $\Phi(\mathbf{x})$, then by Lemma 2.2, $\tilde{\Phi}(\mathbf{y}) = \Phi(\mathbf{y}+h(\mathbf{y}))$ is a formal first integral of system (2.2), and $\tilde{\Phi}(\mathbf{y})$ is resonant with respect to Λ, therefore $\tilde{\Phi}(\mathbf{y})$ can be written as

$$\tilde{\Phi}(\mathbf{y}) = \sum_{i=1}^{\infty} \sum_{|\mathbf{k}|=i, \langle \mathbf{k},\Lambda\rangle=0} \tilde{\Phi}_{\mathbf{k}} y_1^{k_1} \dots y_n^{k_n}, \tag{2.12}$$

where $\tilde{\Phi}_{\mathbf{k}}$ are nonzero constants.

Note that for every monomial $y_1^{k_1}\dots y_n^{k_n}$ in (2.12), $(k_1,\dots,k_n)^T \in \Theta'$. By Lemma 2.4, there exists $\mathbf{a} = (a_1,\dots,a_m)^T \in \mathbb{Z}_+^m$, $|\mathbf{a}| \ge 1$, such that

$$(k_1,\dots,k_n)^T = a_1\tau_{\mathbf{1}} + \dots + a_m\tau_{\mathbf{m}}.$$

Thus

$$\begin{aligned}\tilde{\Phi}(\mathbf{y}) &= \sum_{i=1}^{\infty} \sum_{|\mathbf{k}|=i,\ \langle \mathbf{k},\Lambda\rangle=0} \tilde{\Phi}_{\mathbf{k}} y_1^{a_1\tau_{11}+\cdots+a_m\tau_{m1}} \dots y_n^{a_1\tau_{1n}+\cdots+a_m\tau_{mn}} \\ &= \sum_{i=1}^{\infty} \sum_{|\mathbf{k}|=i,\ \langle \mathbf{k},\Lambda\rangle=0} \tilde{\Phi}_{\mathbf{k}} (y_1^{\tau_{11}} \dots y_n^{\tau_{1n}})^{a_1} \dots (y_1^{\tau_{m1}} \dots y_n^{\tau_{mn}})^{a_m} \\ &= \sum_{i=1}^{\infty} \sum_{|\mathbf{k}|=i,\ \langle \mathbf{k},\Lambda\rangle=0} \tilde{\Phi}_{\mathbf{k}} z_1^{a_1} \dots z_m^{a_m} =: \Psi(\mathbf{z}).\end{aligned}$$

Since (2.2) is changed to (2.7) under the transformation (2.6), $\Psi(\mathbf{z})$ is a first integral of (2.7). It is clearly that $\Psi(\mathbf{z})$ is independent with $w_1,\dots,w_{n-m}$, so it is also a first integral of (2.11). □

Expand $\bar{F}(\mathbf{z})$ as

$$\bar{F}(\mathbf{z}) = \bar{F}_p(\mathbf{z}) + \bar{F}_{p+1}(\mathbf{z}) + \dots,$$

where $p \geq 1, \bar{F}_k(\mathbf{z})(k = p, p+1, \dots)$ are the k-th order homogeneous polynomial. We can get the following lemma easily.

Lemma 2.8. *If system* (2.11) *has a nontrivial formal first integral* $\Psi(\mathbf{z})$*, then the lowest order homogeneous terms* $\Psi_q(\mathbf{z})$ *is a first integral of system*

$$\dot{\mathbf{z}} = \mathbf{z}\bar{F}_p(\mathbf{z}). \tag{2.13}$$

System (2.13) can be treated as a quasi-homogeneous system of degree $p+1$ with the exponents $s_1 = \dots = s_m = 1$ (for more detail, see [6]). Let ξ be a balance of vector field $\mathbf{z}\bar{F}_p(\mathbf{z})$, i.e., ξ is a nonzero solution of the algebraic equations $\frac{1}{p}\xi + \xi\bar{F}_p(\xi) = 0$, then system (2.13) has a particular solution $\mathbf{z}_0(t) = t^{-\frac{1}{p}E_m}\xi$.

We make the change of variables

$$\mathbf{z}(t) = t^{-\frac{1}{p}E_m}(\xi + \mathbf{u}), \quad s = \ln t,$$

then system (2.13) reads

$$\frac{d\mathbf{u}}{ds} = K\mathbf{u} + \hat{F}(\mathbf{u}),$$

where

$$K = \begin{pmatrix} \frac{1}{p} + \bar{F}_p^1(\xi) & & \\ & \ddots & \\ & & \frac{1}{p} + \bar{F}_p^m(\xi) \end{pmatrix} + \begin{pmatrix} \xi_1 \frac{\partial \bar{F}_p^1(\xi)}{\partial z_1} & \dots & \xi_1 \frac{\partial \bar{F}_p^1(\xi)}{\partial z_m} \\ \vdots & \dots & \vdots \\ \xi_m \frac{\partial \bar{F}_p^m(\xi)}{\partial z_1} & \dots & \xi_m \frac{\partial \bar{F}_p^m(\xi)}{\partial z_m} \end{pmatrix}$$

is the so-called Kovalevskaya matrix associated to the balance ξ,

$$\hat{F}(\mathbf{u}) = \frac{1}{p}E_m\xi + (\xi + \mathbf{u})\bar{F}_p(\xi + \mathbf{u}) - \frac{\partial \mathbf{z}\bar{F}_p(\mathbf{z})}{\partial \mathbf{z}}\mathbf{u}\Big|_{\mathbf{z}=\xi} = O(\|\mathbf{u}\|^2).$$

Now we can state our main result.

Theorem 2.9. *Assume that* (H1) *holds,* ξ *is a balance of vector field* $\mathbf{z}\bar{F}_p(\mathbf{z})$*,* $\mu = (\mu_1, \dots, \mu_m)$ *is the eigenvalues of Kovalevskaya matrix associated to the balance* ξ*. Let*

$$\Omega = \big\{\mathbf{k} = (k_1, \dots, k_m)^T \in \mathbb{Z}_+^m : \langle \mathbf{k}, \mu \rangle = 0\big\}.$$

(1) *If rank* $\Omega = 0$*, then system* (2.1) *does not have any nontrivial formal first integrals in the neighborhood of* $\mathbf{x} = \mathbf{0}$*;*
(2) *If rank* $\Omega = l > 0$*, then system* (2.1) *has at most* l *functionally independent formal first integrals in the neighborhood of* $\mathbf{x} = \mathbf{0}$*.*

Proof. (1) Assume that system (2.1) admits a formal first integral in a neighborhood of $\mathbf{x} = \mathbf{0}$, then by Lemma 2.7, system (2.11) admits a formal first integral, and by Lemma 2.8, system (2.13) has a homogeneous first integral. According to [6, Theorem 1], we know that $\mu_1, \dots, \mu_m$ are $\mathbb{N}$-resonant, which contradicts with rank $\Omega = 0$.

(2) Assume that system (2.1) admits $l+1$ functional independent first integrals $\Phi^1(\mathbf{x}), \dots, \Phi^{l+1}(\mathbf{x})$ in a neighborhood of $\mathbf{x} = \mathbf{0}$. By Lemma 2.2, Lemma 2.5 and Lemma 2.7, after a sequence of transformations, we can see that system (2.11) have $l+1$ formal first integrals $\Psi^1(\mathbf{z}), \dots, \Psi^{l+1}(\mathbf{z})$. Since the sequence of transformations which transform (2.1) to (2.11) are local invertible, hence $\Psi^1(\mathbf{z}), \dots, \Psi^{l+1}(\mathbf{z})$ are functional independent in the neighborhood of $\mathbf{z} = \mathbf{0}$. According to Ziglin lemma

in [17], system (2.11) has $l+1$ first integrals $\tilde{\Psi}^1(\mathbf{z}),\dots,\tilde{\Psi}^{l+1}(\mathbf{z})$ whose lowest order homogeneous terms $\tilde{\Psi}^1_{q_1}(\mathbf{z}),\dots,\tilde{\Psi}^{l+1}_{q_{l+1}}(\mathbf{z})$ are functionally independent, and by Lemma 2.8, $\tilde{\Psi}^1_{q_1}(\mathbf{z}),\dots,\tilde{\Psi}^{l+1}_{q_{l+1}}(\mathbf{z})$ are $l+1$ first integrals of quasi-homogeneous system (2.13). By [8, Theorem B], we get rank $\Omega \ge l+1$, which contradicts with the assumption of theorem. The proof is complete. $\square$

3. Example

Based on the arguments in Section 2, one can give an algorithm to test whether a given system admits formal first integral or not, see the next example.

Example 3.1. Consider the generalized Lotka-Volterra system

$$
\begin{aligned}
\dot{x}_1 &= x_1 + a_1x_1x_1x_2 + a_2x_1x_3x_4 + a_3x_1x_5x_6,\\
\dot{x}_2 &= -x_2 + b_1x_2x_1x_2 + b_2x_2x_3x_4 + b_3x_2x_5x_6,\\
\dot{x}_3 &= \sqrt{2}x_3 + c_1x_3x_1x_2 + c_2x_3x_3x_4 + c_3x_3x_5x_6,\\
\dot{x}_4 &= -\sqrt{2}x_4 + d_1x_4x_1x_2 + d_2x_4x_3x_4 + d_3x_4x_5x_6,\\
\dot{x}_5 &= \sqrt{3}x_5 + e_1x_5x_1x_2 + e_2x_5x_3x_4 + e_3x_5x_5x_6,\\
\dot{x}_6 &= -\sqrt{3}x_6 + f_1x_6x_1x_2 + f_2x_6x_3x_4 + f_3x_6x_5x_6,
\end{aligned} \tag{3.1}
$$

where $a_i,\ b_i,\ c_i,\ d_i,\ e_i,\ f_i \in \mathbb{R},\ i=1,2,3$. Lotka-Volterra equations can be used to describe cooperation and competition between biological species in ecology. In 1988, Brenig [2] introduced generalized Lotka-Volterra equations. By transforming the original equations into a canonical form, he discussed the integrability of some type generalized Lotka-Volterra equations. As an application, we consider the nonexistence of formal first integral for above Lotka-Volterra system.

Obviously, the eigenvalues of Jocabi matrix at $\mathbf{x}=\mathbf{0}$ of system (3.1) are

$$
\lambda_1 = 1,\quad \lambda_2 = -1,\quad \lambda_3 = \sqrt{2},\quad \lambda_4 = -\sqrt{2},\quad \lambda_5 = \sqrt{3},\quad \lambda_6 = -\sqrt{3},
$$

and system (3.1) is a normal form. Let

$$
\Omega = \big\{\mathbf{k} = (k_1,\dots,k_6)^T \in \mathbb{Z}^6_+ : \sum_{i=1}^{6} k_i\lambda_i = 0\big\},
$$

$$
\Theta = \big\{\mathbf{k} = (k_1,\dots,k_6)^T \in \mathbb{Z}^6 : \sum_{i=1}^{6} k_i\lambda_i = 0\big\},
$$

then $\operatorname{rank}\Omega = \operatorname{rank}\Theta = 3$, and $(1,1,0,0,0,0)^T,\ (0,0,1,1,0,0)^T,\ (0,0,0,0,1,1)^T$ are the least generating elements of Θ. Making the change of variables

$$
z_1 = x_1x_2,\quad z_2 = x_3x_4,\quad z_3 = x_5x_6,\quad w_1 = x_2,\quad w_2 = x_4,\quad w_3 = x_6,
$$

system (3.1) becomes

$$
\begin{aligned}
\dot{z}_1 &= z_1(\alpha_1z_1 + \alpha_2z_2 + \alpha_3z_3),\\
\dot{z}_2 &= z_2(\beta_1z_1 + \beta_2z_2 + \beta_3z_3),\\
\dot{z}_3 &= z_3(\gamma_1z_1 + \gamma_2z_2 + \gamma_3z_3),\\
\dot{w}_1 &= -w_1 + b_1w_1z_1 + b_2w_1z_2 + b_3w_1z_3,\\
\dot{w}_2 &= -\sqrt{2}w_2 + e_1w_2z_1 + e_2w_2z_2 + e_3w_2z_3,\\
\dot{w}_3 &= -\sqrt{3}w_3 + f_1w_3z_1 + f_2w_3z_2 + f_3w_3z_3,
\end{aligned} \tag{3.2}
$$

where $\alpha_i = a_i + b_i$, $\beta_i = c_i + d_i$, $\gamma_i = e_i + f_i$, $i = 1, 2, 3$.

By Lemma 2.8, we need only to investigate the formal first integral for the system

$$\begin{aligned} \dot{z}_1 &= z_1(\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3), \\ \dot{z}_2 &= z_2(\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3), \\ \dot{z}_3 &= z_3(\gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3). \end{aligned} \tag{3.3}$$

System (3.3) is a quasi-homogeneous system of degree 2 with exponents $s_1 = s_2 = s_3 = 1$. Assume $\alpha_1 \neq 0$, then $\xi = (-\frac{1}{\alpha_1}, 0, 0)^T$ is a balance of system (3.3). Making the change of variables

$$z_1 = t^{-1}(-\frac{1}{\alpha_1} + u_1), \quad z_2 = t^{-1}u_2, \quad z_3 = t^{-1}u_3, \quad t = e^s, \tag{3.4}$$

system (3.3) reads

$$\begin{aligned} u_1' &= -u_1 - \frac{\alpha_2}{\alpha_1}u_2 - \frac{\alpha_3}{\alpha_1}u_3 + u_1(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3), \\ u_2' &= (1 - \frac{\beta_1}{\alpha_1})u_2 + u_2(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3), \\ u_3' &= (1 - \frac{\gamma_1}{\alpha_1})u_3 + u_3(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3), \end{aligned} \tag{3.5}$$

where $'$ means the derivative with respect to s. Here the corresponding Kovalevskaya matrix is

$$K = \begin{pmatrix} -1 & -\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_3}{\alpha_1} \\ 0 & 1 - \frac{\beta_1}{\alpha_1} & 0 \\ 0 & 0 & 1 - \frac{\gamma_1}{\alpha_1} \end{pmatrix}.$$

Obviously,

$$\mu_1 = -1, \quad \mu_2 = 1 - \frac{\beta_1}{\alpha_1}, \quad \mu_3 = 1 - \frac{\gamma_1}{\alpha_1}$$

are eigenvalues of K. Let

$$\Omega_1 = \big\{\mathbf{k} = (k_1, k_2, k_3)^T \in \mathbb{Z}_+^3 : \sum_{i=1}^{3} k_i \mu_i = 0\big\}.$$

By Theorem 2.9, we have the following results.

Theorem 3.2. *Assume $\alpha_1 \neq 0$.*

1. *If rank $\Omega_1 = 0$, then system* (3.1) *does not have any nontrivial formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$;*
2. *If rank $\Omega_1 = l_1 > 0$, then system* (3.1) *has at most l_1 functionally independent formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$.*

It is not difficult to see that if $\alpha_1, \beta_1, \gamma_1$ are $\mathbb{Z}$-non-resonant, and rank $\Omega_1 = 0$, by Theorem 3.2, we get nonexistence of formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$. Furthermore, we can obtain following conclusions.

Corollary 3.3.
1. *If α_1, β_1, γ_1 are $\mathbb{Z}$-non-resonant, then system* (3.1) *does not have any nontrivial formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$;*
2. *If $\alpha_1 = \beta_1 = \gamma_1 \neq 0$, $\alpha_2 = \alpha_3 = 0$ and $\frac{\gamma_3}{\beta_2} \notin \mathbb{Q}$, then system* (3.1) *does not have any nontrivial formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$.*
3. *If $\alpha_1 = \beta_1 = \gamma_1 \neq 0$, $\alpha_2 = \alpha_3 = 0$ and $\frac{\gamma_3}{\beta_2} \in \mathbb{Q}$, then system* (3.1) *has at most one nontrivial formal first integral in the neighborhood of $\mathbf{x} = \mathbf{0}$.*

Proof. The first case is obvious, we omit it. Let us consider the cases when $\alpha_1 = \beta_1 = \gamma_1 \neq 0$, and $\alpha_2 = \alpha_3 = 0$. In this situation, we need to consider

$$\begin{gathered} u_1' = -u_1 + \alpha_1 u_1^2, \\ u_2' = u_2(\alpha_1 u_1 + \beta_2 u_2 + \beta_3 u_3), \\ u_3' = u_3(\alpha_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3), \\ u_0' = -u_0, \end{gathered} \tag{3.6}$$

where $u_0 = e^{-s}$. It is clearly that $\mu_1 = -1$, $\mu_2 = 0$, $\mu_3 = 0$, and $\operatorname{rank}\Omega_1 = 2 > 0$. Let

$$\Theta_1 = \big\{\mathbf{k} = (k_1, k_2, k_3, k_4)^T \in \mathbb{Z}^4 : \sum_{i=1}^{4} k_i \mu_i = 0\big\},$$

where $\mu_4 = -1$.

By the Poincaré-Dulac normal form theory, system (3.6) can be reduced to the canonical form

$$\begin{gathered} u_1' = -u_1 + u_1\varphi^1(u_2, u_3), \\ u_2' = u_2(\beta_2 u_2 + \gamma_3 u_3) + \varphi^2(u_2, u_3), \\ u_3' = u_3(\gamma_2 u_2 + \gamma_3 u_3) + \varphi^3(u_2, u_3), \\ u_0' = -u_0, \end{gathered} \tag{3.7}$$

where $\varphi^1(u_2, u_3) = O(|(u_2, u_3)|)$, $\varphi^2(u_2, u_3) = O(|(u_2, u_3)|^2)$ and $\varphi^3(u_2, u_3) = O(|(u_2, u_3)|^3)$. The subsystem of system (3.7),

$$\begin{gathered} u_2' = u_2(\beta_2 u_2 + \beta_3 u_3) + \varphi^2(u_2, u_3), \\ u_3' = u_3(\gamma_2 u_2 + \gamma_3 u_3) + \varphi^3(u_2, u_3) \end{gathered}$$

is a semi-quasihomogeneous system with exponents $s_1 = s_2 = 1$, and its quasi-homogeneous cut is

$$\begin{gathered} u_2' = u_2(\beta_2 u_2 + \beta_3 u_3), \\ u_3' = u_3(\gamma_2 u_2 + \gamma_3 u_3). \end{gathered} \tag{3.8}$$

Note that if $\beta_2 \neq 0$, then $(-\frac{1}{\beta_2}, 0)$ is a balance of (3.8), and the corresponding Kovalevskaya matrix is

$$K' = \begin{pmatrix} -1 & -\frac{\beta_3}{\beta_2} \\ 0 & 1 - \frac{\gamma_3}{\beta_2} \end{pmatrix}.$$

Obviously, $\bar\mu_1 = -1$, $\bar\mu_2 = 1 - \frac{\gamma_3}{\beta_2}$ are eigenvalues of K'. Let

$$\Omega_2 = \big\{\mathbf{k} = (k_1, k_2)^T \in \mathbb{Z}_+^2 : k_1\bar\mu_1 + k_2\bar\mu_2 = 0\big\}.$$

By Theorem 2.9, we know that if $\operatorname{rank}\Omega_2 = 0$, then (3.1) does not have any nontrivial formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$; if rank $\Omega_2 = l_2 > 0$, then (3.1) has at most l_2 functionally independent formal first integrals in the neighborhood of $\mathbf{x} = \mathbf{0}$. Therefore we get the proofs of last two cases. □

Remark 3.4. If $\Omega_2 = l_2 > 0$, one can use the same idea as the change of variables (3.4) to get a new system like (3.5) to do more investigations. While, one particular case should be noted is that, if there exist $i_0 \in \mathbb{N}$, such that for every $j > i_0$, $\operatorname{rank}\Theta_j = \operatorname{rank}\Theta_{i_0} = l_{i_0} \geq 2$, we can not get the nonexistence of formal first integral for system (3.1). And this case always implies the partial existence of formal first integral for (3.1), i.e., (3.1) may have $l_{i_0} - 1$ formal first integrals in a neighborhood of $\mathbf{x} = \mathbf{0}$.

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