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IDENTIFICATION OF TWO DEGENERATE TIME- AND SPACE-DEPENDENT KERNELS IN A PARABOLIC EQUATION

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Abstract. An inverse problem to determine two degenerate time- and space-dependent kernels in a parabolic integro-differential equation is considered. Observation data involves given values of the solution of the equation in a finite number of points over the time. Existence and uniqueness of a solution to the inverse problem is proved.

1. Introduction

Heat flow in materials with memory is governed by parabolic integro-differential equations containing time-dependent (and in the case of non-homogeneity also space-dependent) memory (or relaxation) kernels [1, 12, 13, 15]. These kernels are often unknown in the practice. To determine them, inverse problems are used.

Various problems to identify time-dependent memory kernels in parabolic equations have been studied in a number of papers (see [4, 5, 7, 8, 11] and references therein). When the kernels are both time- and space-dependent, inverse problems based on restricted Dirichlet-to-Neumann map in general case [6] and single trace measurements in stratified cases [2] are in the use.

In some context the kernels can be degenerate, i.e. representable as finite sums of products of known space-dependent functions times unknown time-dependent coefficients. This is so when either the medium is piecewise continuous or a problem for a general kernel is replaced by a related problem for an approximated kernel. The unknown coefficients are recovered by a finite number of measurements of certain time-dependent characteristics of the solution of the direct problem. In [9, 10] inverse problems of such a type were studied. These papers deal with the simplified case when the model contains only the relaxation kernel of heat flux. However, a more precise model of a material with thermal memory involves two relaxation kernels contained in basic constitutive relations: kernels of internal energy and heat flux [4, 5, 8, 13].

In the present paper we study an inverse problem to determine degenerate non-homogeneous relaxation kernels of internal energy and heat flux occurring in a parabolic equation governing heat flow in materials with memory. To recover the kernels, we make use of a finite number of measurements of temperature in fixed

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points over the time. As in [9, 10], we apply the fixed-point argument in weighted norms adjusted to the problem in the Laplace domain. Due to the structure of the problem, the kernels of internal energy and heat flux are recovered with different level of regularity. The corresponding problem with flux-type additional information is more complicated, and not covered by this paper.

In Section 2 we formulate the direct and inverse problems and in Section 3 apply the Laplace transform them. In Section 4 we rewrite the transformed problems in the fixed-point form. Sections 5 and 6 contain some auxiliary results for the direct problem. Main existence and uniqueness results for the inverse problem are included in Section 7 of the paper.

2. Physical background and formulation of problem.

We consider the heat flow in a rigid nonhomogeneous bar consisting of a material with thermal memory. For a sake of simplicity we assume the rod to be of the unit length. Then, in the linear approximation the system of constitutive relations and the heat balance equation read as

$$e(x,t) = \beta(x)u(x,t) + \int_0^t n(x,t-\tau)u(x,\tau)\,d\tau\,, \tag{2.1}$$

$$q(x,t) = -\lambda(x)u_x(x,t) + \int_0^t m(x,t-\tau)u_x(x,\tau)\,d\tau\,, \tag{2.2}$$

$$\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}q(x,t) = r(x,t), \tag{2.3}$$

respectively, where $x \in (0,1)$ is the space coordinate and $t \in \mathbb{R}$ is the time [4, 5, 8, 13] Here u is the temperature of the bar, which is assumed to be zero for $t < 0$, e is the internal energy, q the heat flux and r is the heat supply. Moreover, β and λ stand for the heat capacity and the heat conduction coefficient, respectively. The model contains two memory kernels n and m, being the relaxation kernels of the integral energy and the heat flux, respectively.

From (2.1)–(2.3), we obtain the parabolic integro-differential equation

$$\begin{aligned}
&\beta(x)\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial t}\int_0^t n(x,t-\tau)u(x,\tau)\,d\tau \\
&= \frac{\partial}{\partial x}(\lambda(x)u_x(x,t)) - \frac{\partial}{\partial x}\int_0^t m(x,t-\tau)u_x(x,\tau)\,d\tau + r(x,t), \quad x \in (0,1),\ t > 0.
\end{aligned} \tag{2.4}$$

The function $u(x,t)$ is assumed to satisfy the initial conditions

$$u(x,0) = \varphi(x), \quad x \in (0,1) \tag{2.5}$$

and the Dirichlet boundary conditions

$$u(0,t) = f_1(t),\ u(1,t) = f_2(t), \quad t > 0 \tag{2.6}$$

with given functions φ on $[0,1]$ and f_j, $j = 1,2$ on $[0,\infty)$. Equation (2.4) with the conditions (2.5) and (2.6) form the direct problem for the temperature u.

In an inverse problem we seek for the kernels n and m. We restrict ourselves to the case of the kernels in the following degenerate forms

$$n(x,t) = \sum_{j=1}^{N_1} \nu_j(x)n_j(t), \quad m(x,t) = \sum_{k=1}^{N_2} \mu_k(x)m_k(t)\,, \tag{2.7}$$

where $\nu_j,\ j=1,\dots,N_1,\ \mu_k,\ k=1,\dots,N_2$ are given x-dependent functions and $n_j,\ j=1,\dots,N_1,\ m_k,\ k=1,\dots,N_2$ are unknown time-dependent coefficients. Formulas (2.7) hold, for instance, when the medium is piecewise continuous, where n_j and m_k are characteristic functions or smooth approximations of characteristic functions of the subdomains of homogeneity. In general case (2.7) can be interpreted as finite-dimensional approximations of the actual kernels.

We are going to recover the unknowns n_j and m_k by the measurement of the temperature u in $N = N_1 + N_2$ different interior points $x_i \in (0,1)$, $i = 1,\dots,N$, i.e., by the additional conditions

$$u(x_i,t) = h_i(t), \quad i = 1,\dots,N,\ t > 0\,, \tag{2.8}$$

where h_i are given functions. Summing up, the relations (2.4)–(2.8) form the inverse problem for n and m.

3. Application of Laplace transform

Applying the Laplace transform to the equation (2.4) with initial condition (2.5) and taking in consideration (2.7) we obtain

$$\begin{aligned} &\beta(x)[pU(x,p)-\varphi(x)] + p\sum_{j=1}^{N_1} N_j(p)\nu_j(x)U(x,p) \\ &= \frac{\partial}{\partial x}(\lambda(x)U_x(x,p)) - \sum_{k=1}^{N_2} M_k(p)\frac{\partial}{\partial x}(\mu_k(x)U_x(x,p)) + R(x,p) \end{aligned} \tag{3.1}$$

where

$$U(x,p) = \mathcal{L}u(x,t) = \int_0^\infty e^{-pt}u(x,t)\,dt,\ \operatorname{Re}p > \sigma,\ N_j = \mathcal{L}n_j,\ M_k = \mathcal{L}m_k,\ R = \mathcal{L}r.$$

Here σ is taken so that the images of the Laplace transform exist in a half-plane $\operatorname{Re}p > \sigma$. In the sequel we will study the direct and inverse problems in the Laplace domain and show the existence and uniqueness for these problems in a half-plane $\operatorname{Re}p > \sigma$ with sufficiently large σ.

The boundary conditions (2.6) are transformed to

$$U(0,p) = F_1(p), \quad U(1,p) = F_2(p), \quad \operatorname{Re}p > \sigma \tag{3.2}$$

where $F_j = \mathcal{L}f_j$, $j = 1,2$.

The goal of this section is to rewrite the problem for U in a form of a system of integral equations. To this end we represent equation (3.1) in the form

$$\begin{aligned} (LU)(x,p) &= p\sum_{j=1}^{N_1} N_j(p)\nu_j(x)U(x,p) + \sum_{k=1}^{N_2} M_k(p)\frac{\partial}{\partial x}(\mu_k(x)U_x(x,p)) \\ &\quad - R(x,p) - \beta(x)\varphi(x) \end{aligned} \tag{3.3}$$

introducing the differential operator

$$(LU)(x,p) = \frac{\partial}{\partial x}(\lambda(x)U_x(x,p)) - \beta(x)pU(x,p), \quad x \in (0,1).$$

Let us denote by $G(x,y,p)$ the Green function of operator L with homogeneous boundary conditions

$$L_yG(x,y,p) = \delta(x,y), \quad x \in (0,1), y \in (0,1)\,, \tag{3.4}$$

$$G(x,0,p)=G(x,1,p)=0\,. \tag{3.5}$$

Then the solution of (3.3) is given by

$$\begin{aligned} U(x,p) = &\sum_{j=1}^{N_1} N_j(p)\int_0^1 G(x,y,p)\nu_j(y)pU(y,p)\,dy \\ &+\sum_{k=1}^{N_2} M_k(p)\int_0^1 G(x,y,p)\frac{\partial}{\partial y}\big(\mu_k(y)U_y(y,p)\big)\,dy - F(x,p) \end{aligned}\tag{3.6}$$

where

$$\begin{aligned} F(x,p) = &\int_0^1 G(x,y,p)[\beta(y)\varphi(y)+R(y,p)]\,dy \\ &+\lambda(0)G_y(x,0,p)F_1(p)-\lambda(1)G_y(x,1,p)F_2(p)\,. \end{aligned}\tag{3.7}$$

Integrating the integrals in the second sum of (3.6) by parts and observing (3.5) we have

$$\begin{aligned} U(x,p) = &\sum_{j=1}^{N_1} pN_j(p)\int_0^1 G(x,y,p)\nu_j(y)U(y,p)\,dy \\ &-\sum_{k=1}^{N_2} M_k(p)\int_0^1 G_y(x,y,p)\mu_k(y)U_y(y,p)\,dy - F(x,p). \end{aligned}\tag{3.8}$$

Further, differentiating (3.6) with respect to x we obtain the equation for $U_x(x,p)$

$$\begin{aligned} U_x(x,p) = &\sum_{j=1}^{N_1} pN_j(p)\int_0^1 G_x(x,y,p)\nu_j(y)U(y,p)\,dy \\ &+\sum_{k=1}^{N_2} M_k(p)\int_0^1 G_x(x,y,p)\frac{\partial}{\partial y}(\mu_k(y)U_y(y,p))\,dy - F_x(x,p). \end{aligned}\tag{3.9}$$

We split the second integral in (3.9) into two parts, from 0 to x and from x to 1, and integrate them by parts. Taking into consideration the equalities $G_x(x,0,p)=G_x(x,1,p)=0$, $0<x<1$, following from (3.5), and the jump relation

$$G_x(x,x-0,p)-G_x(x,x+0,p)=\frac{1}{\lambda(x)},\quad 0<x<1$$

(see [14, p. 169]) we get

$$\begin{aligned} &\int_0^1 G_x(x,y,p)\frac{\partial}{\partial y}(\mu_k(y)U_y(y,p))\,dy \\ &= G_x(x,y,p)\mu_k(y)U_y(y,p)\Big|_0^x + G_x(x,y,p)\mu_k(y)U_y(y,p)\Big|_x^1 \\ &\quad -\int_0^1 G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy \\ &= \frac{\mu_k(x)}{\lambda(x)}U_x(x,p) - \int_0^1 G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy\,. \end{aligned}\tag{3.10}$$

Thus, by (3.9) and (3.10) we have the following equation for $U_x(x,p)$

$$\begin{aligned} &U_x(x,p) \\ &= \frac{1}{\lambda(x)} \sum_{k=1}^{N_2} M_k(p)\mu_k(x)U_x(x,p) + \sum_{j=1}^{N_1} pN_j(p) \int_0^1 G_x(x,y,p)\nu_j(y)U(y,p)\,dy \\ &\quad - \sum_{k=1}^{N_2} M_k(p) \int_0^1 G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy - F_x(x,p) \end{aligned} \tag{3.11}$$

with F given by (3.7). Summing up, (3.8) and (3.11) form a system of integral equations for functions $U(x,p)$ and $U_x(x,p)$.

4. Fixed-point systems for inverse and direct problems

In this section we deduce a fixed-point system for the inverse problem in the Laplace domain and transform further the system for U and U_x. Applying Laplace transform to additional conditions (2.8) yields

$$U(x_i,p) = H_i(p), \quad i = 1,\dots,N = N_1 + N_2. \tag{4.1}$$

Thus, from equation (3.8) we obtain

$$\begin{aligned} &\sum_{k=1}^{N_1} pN_k(p)\, p \int_0^1 G(x_i,y,p)\nu_k(y)pU(y,p)\,dy \\ &- \sum_{l=1}^{N_2} \sqrt{p}M_l(p)\, \sqrt{p} \int_0^1 G_y(x_i,y,p)\mu_l(y)pU_y(y,p)\,dy \\ &= p^2[H_i(p) + F(x_i,p)] \end{aligned} \tag{4.2}$$

for $i = 1,\dots,N$, where F, given by (3.7), depends only on the data of the problem.

Firstly, let us study the behaviour of this equation in the process $\operatorname{Re} p \to \infty$. Suppose a priori that the inverse problem has a solution n_k, m_l with the following properties

1. n_k are differentiable, implying $pN_k(p) \to n_k(0)$ as $\operatorname{Re} p \to \infty$;
2. $\sqrt{p}M_l(p) \to 0$ as $\operatorname{Re} p \to \infty$;
3. the solution of the direct problem u corresponding to these n_k, m_l satisfies the relation $\frac{\partial}{\partial t}u(\cdot,t) \in C^1[0,1]$ for $t > 0$.

From 3 and the initial condition (2.5) we have $pU(x,p) \to \varphi(x)$ and $pU_x(x,p) \to \varphi'(x)$ as $\operatorname{Re} p \to \infty$. Using these asymptotic relations, the items 1, 2 above and the assertions (5.9) and (5.12) of Lemmas 5.1 and 5.2 below we obtain the equalities

$$-\sum_{k=1}^{N_1} n_k(0) \frac{1}{\beta(x_i)} \nu_k(x_i)\varphi(x_i) = \lim_{\operatorname{Re} p \to \infty} p^2[H_i(p) + F(x_i,p)], \tag{4.3}$$

for $i = 1,\dots,N = N_1 + N_2$, from (4.2) in the process $\operatorname{Re} p \to \infty$, which form a system for initial values $n_k(0)$ of the unknowns n_k.

Remark 4.1. Observing (3.8) we see that $-F(x,p)$ is the Laplace transform of a function $\tilde{u}(x,t)$, which is the solution of the direct problem (2.4) - (2.6) with lacking kernels ($n(x,t) = m(x,t) \equiv 0$). Therefore, according to the basic properties of the Laplace transform (see [3]), the limits in (4.3) exist and are finite under conditions

$$h_i(0) - \varphi(x_i) = 0, \quad i = 1, \dots, N$$

and $h_i - \tilde{u}(x_i, \cdot)$, $i = 1, \dots, N$, are twice continuously differentiable. In particular, the relations $h_i(0) - \varphi(x_i) = 0$ can be regarded as consistency conditions. Moreover, under these conditions

$$\lim_{\operatorname{Re} p \to \infty} p^2 [H_i(p) + F(x_i, p)] = \lim_{t \to 0^+} \big(h_i'(t) - \frac{\partial}{\partial t} \tilde{u}(x_i, t)\big), \quad i = 1, \dots, N .$$

System (4.3) has a unique solution provided

$$\begin{aligned} &\operatorname{rank} \Big(\big(\frac{1}{\beta(x_i)} \nu_k(x_i) \varphi(x_i) \big)_{k=1,\dots,N_1}, \lim_{\operatorname{Re} p \to \infty} p^2 [H_i(p) + F(x_i, p)] \Big)_{i=1,\dots,N} \\ &= \operatorname{rank} \big(\frac{1}{\beta(x_i)} \nu_k(x_i) \varphi(x_i) \big)_{k,\, i=1,\dots,N_1} = N_1 . \end{aligned} \tag{4.4}$$

Thus, (4.4) is a necessary condition for the inverse problem to have a unique solution with the properties 1–3.

The system (4.2) suggests that the kernels n_k and m_l can be determined simultaneously with higher smoothness in n_k than in m_l. Therefore we define

$$Q_k(p) = pN_k(p) - n_k(0) = \mathcal{L}(n_k') \tag{4.5}$$

and derive a fixed-point system for Q_k, M_l. Observing (4.3), (4.5) and having Lemma 5.2 in mind we obtain from (4.2) the system

$$\begin{aligned} &\sum_{k=1}^{N_1} Q_k(p) \frac{1}{\beta(x_i)} \nu_k(x_i) \varphi(x_i) + \sum_{l=1}^{N_2} M_l(p) \frac{1}{\beta(x_i)} (\mu_l(y) \varphi'(y))' \Big|_{y=x_i} \\ &= \sum_{k=1}^{N_1} [Q_k(p) + n_k(0)] \Big[\int_0^1 pG(x_i, y, p) \nu_k(y) [pU(y, p) - \varphi(y)] dy \\ &\quad + \int_0^1 pG(x_i, y, p) \nu_k(y) \varphi(y)\, dy + \frac{1}{\beta(x_i)} \nu_k(x_i) \varphi(x_i) \Big] \\ &\quad - \sum_{l=1}^{N_2} M_l(p) \Big[\int_0^1 pG_y(x_i, y, p) \mu_l(y) [pU_y(y, p) - \varphi'(y)]\, dy \\ &\quad - \int_0^1 pG(x_i, y, p) (\mu_l(y) \varphi'(y))'\, dy - \frac{1}{\beta(x_i)} (\mu_l(y) \varphi'(y))' \Big|_{y=x_i} \Big] \\ &\quad - p^2 [H_i(p) + F(x_i, p)] + \lim_{\operatorname{Re} q \to \infty} q^2 [H_i(q) + F(x_i, q)], \quad i = 1, \dots, N. \end{aligned} \tag{4.6}$$

In view of assertion (5.12) of Lemma 5.2 below and the relation $pU \to \varphi$ as $\operatorname{Re} p \to \infty$ the first row in (4.6) is the principal part of this system. Therefore we introduce the matrix $\Gamma = (\gamma_{ik})_{i,k=1,\dots,N}$ related to this principal part, where

$$\gamma_{ik} = \begin{cases} \frac{1}{\beta(x_i)} \nu_k(x_i) \varphi(x_i), & k = 1, \dots, N_1, \\ \frac{1}{\beta(x_i)} (\mu_{k-N_1}(y) \varphi'(y))' \big|_{y=x_i}, & k = N_1 + 1, \dots, N \end{cases} \tag{4.7}$$

and assume $\det\Gamma \neq 0$. Further, we introduce the unified notation for the unknowns

$$Z_k = \begin{cases} Q_k, & k = 1,\dots,N_1, \\ M_{k-N_1}, & k = N_1+1,\dots,N, \end{cases} \tag{4.8}$$

$Z = (Z_1,\dots,Z_N)$ and vanishing with $\operatorname{Re} p \to \infty$ functions

$$B^0[Z](x,p) = pU[Z](x,p) - \varphi(x)\,, \quad B^1[Z](x,p) = pU_x[Z](x,p) - \varphi'(x)\,, \tag{4.9}$$

where $U[Z](x,p)$ is the Laplace transform of the Z-dependent solution of the direct problem. Now system (4.6) can be written in the fixed-point form

$$Z = \Gamma^{-1}\mathcal{F}(Z)\,, \tag{4.10}$$

where $\mathcal{F}(Z) = (\mathcal{F}_1(Z),\dots,\mathcal{F}_N(Z))$,

$$\begin{aligned} &\mathcal{F}_i[Z](p) \\ &= \sum_{k=1}^{N_1} Z_k(p)\Big[\int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy \\ &\quad + \int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\Big] \\ &\quad + \sum_{k=N_1+1}^{N} Z_k(p)\Big[-\int_0^1 pG_y(x_i,y,p)\mu_{k-N_1}(y)B^1[Z](y,p)\,dy \\ &\quad + \int_0^1 pG(x_i,y,p)(\mu_{k-N_1}(y)\varphi'(y))'\,dy + \frac{1}{\beta(x_i)}(\mu_{k-N_1}(x)\varphi'(x))'\Big|_{x=x_i}\Big] \\ &\quad + \sum_{k=1}^{N_1} n_k(0)\int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy + \widehat{\Psi}_i(p), \quad i=1,\dots,N \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \widehat{\Psi}_i(p) = {}& \sum_{k=1}^{N_1} n_k(0)\Big[\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\Big] \\ & - p^2[H_i(p) + F(x_i,p)] + \lim_{\operatorname{Re} q\to\infty} q^2[H_i(q) + F(x_i,q)]. \end{aligned} \tag{4.12}$$

For future analysis we need a proper fixed-point system for the quantities $B^0[Z]$ and $B^1[Z]$, too. From (3.8) in view of the definitions of Z and $B^0[Z]$ we have

$$\begin{aligned} &B^0[Z](x,p) \\ &= \sum_{k=1}^{N_1} Z_k(p)\int_0^1 G(x,y,p)\nu_k(y)[B^0[Z](y,p) + \varphi(y)]\,dy \\ &\quad - \sum_{k=N_1+1}^{N} Z_k(p)\int_0^1 G_y(x,y,p)\mu_{k-N_1}(y)[B^1[Z](y,p) + \varphi'(y)]\,dy \\ &\quad + \sum_{k=1}^{N_1} n_k(0)\int_0^1 G(x,y,p)\nu_k(y)B^0[Z](y,p)\,dy + \Phi^0(x,p) \end{aligned} \tag{4.13}$$

with

$$\Phi^0(x,p)=\sum_{k=1}^{N_1} n_k(0)\int_0^1 G(x,y,p)\nu_k(y)\varphi(y)\,dy-pF(x,p)-\varphi(x). \tag{4.14}$$

From (3.11) we obtain

$$\begin{aligned}
B^1[Z](x,p)=&\sum_{k=1}^{N_1} Z_k(p)\int_0^1 G_x(x,y,p)\nu_k(y)[B^0[Z](y,p)+\varphi(y)]\,dy\\
&+\sum_{k=N_1+1}^{N} Z_k(p)\Big\{\frac{\mu_{k-N_1}(x)}{\lambda(x)}B^1[Z](x,p)\\
&-\int_0^1 G_xy(x,y,p)\mu_{k-N_1}(y)[B^1[Z](y,p)+\varphi'(y)]\,dy\Big\}\\
&+\sum_{k=1}^{N_1} n_k(0)\int_0^1 G_x(x,y,p)\nu_k(y)B^0[Z](y,p)\,dy\\
&+\sum_{k=N_1+1}^{N} Z_k(p)\frac{\mu_{k-N_1}(x)\varphi'(x)}{\lambda(x)}+\widetilde{\Phi}^1(x,p)
\end{aligned}$$

with

$$\widetilde{\Phi}^1(x,p)=\sum_{k=1}^{N_1} n_k(0)\int_0^1 G_x(x,y,p)\nu_k(y)\varphi(y)\,dy-pF_x(x,p)-\varphi'(x).$$

For the function $B^0[Z]$, which in contrast to $B^1[Z]$ does not contain a space derivative of U, we need a certain higher regularity in the time variable. To this end we will assume that the free term Φ^0 can be decomposed as follows

$$\Phi^0(x,p)=B^{0,0}(x,p)+\widetilde{\Phi}(x,p)\,, \tag{4.15}$$

where $|B^{0,0}(x,p)|\le\frac{\text{Const}}{|p|}$ and $|\widetilde{\Phi}(x,p)|\le\frac{\text{Const}}{|p|^\alpha}$ with some $\alpha>1$ for $\operatorname{Re}p>\sigma_0$, $x\in[0,1]$.

Remark 4.2. To clarify under what conditions the decomposition (4.15) is valid, let us consider separately two addends

$$\begin{gathered}
\Phi_1^0(x,p)=\sum_{k=1}^{N_1} n_k(0)\int_0^1 G(x,y,p)\nu_k(y)\varphi(y)\,dy\,,\\
\Phi_2^0(x,p)=-pF(x,p)-\varphi(x)
\end{gathered}$$

in the formula of the function $\Phi^0(x,p)$. Assuming $\nu(x),\varphi(x)\in C^1[0,1]$, the first addend $\Phi_1^0(x,p)$ satisfies (4.15) in view of the assertion (5.12) of Lemma 5.2. Indeed, due to (5.12) we have

$$\Phi_1^0(x,p)=-\frac{1}{p}\sum_{k=1}^{N_1}\frac{\nu_k(x)\varphi(x)}{\beta(x)}+\widetilde{\Phi}_1(x,p)\,,$$

where $|\widetilde{\Phi}_1(x,p)|\le\text{const}/|p|^{3/2}$. Next, as

$$\mathcal{L}^{-1}\Phi_2^0(x,p)=\mathcal{L}^{-1}(-pF(x,p)-\varphi(x))=\frac{\partial}{\partial t}\tilde{u}(x,t),$$

where $\tilde{u}(x,t)$ is the solution of the direct problem (2.4) - (2.6) without kernels m and n (see Remark 4.1), and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform, then by well-known properties of the Laplace transform (cf [3]) the addend $\Phi_2^0(x,t)$ satisfies the condition (4.15) with $\alpha=2$, if $\tilde{u}(x,t)$ is twice continuously differentiable with respect to t.

We split $B^0[Z]$ into the sum

$$B^0[Z](x,p) = B^{0,0}(x,p) + B^{0,1}[Z](x,p)\,, \tag{4.16}$$

where for $B^{0,1}$ we will require that $|B^{0,1}[Z](x,p)| \le \frac{\mathrm{Const}}{|p|^\alpha}$ for $\operatorname{Re} p > \sigma_0$, $x\in[0,1]$.

From (4.13) and (3.11) in view of (4.15), (4.16) and the definitions of Z and $B^1[Z]$ we deduce the following fixed-point equation for the vector $B[Z]=(B^{0,1}[Z],B^1[Z])$:

$$B[Z] = A[Z]B[Z] + b[Z]\,, \tag{4.17}$$

where $A[Z]=(A^0[Z],A^1[Z])$ is the Z-dependent linear operator of B with the components

$$\begin{aligned}(A^0[Z]B)(x,p) &= \sum_{k=1}^{N_1}(Z_k(p)+n_k(0))\int_0^1 G(x,y,p)\nu_k(y)B^{0,1}(y,p)\,dy\\ &\quad - \sum_{k=N_1+1}^{N} Z_k(p)\int_0^1 G_y(x,y,p)\mu_{k-N_1}(y)B^1(y,p)\,dy,\end{aligned} \tag{4.18}$$

$$\begin{aligned}&(A^1[Z]B)(x,p)\\ &= \sum_{k=1}^{N_1}(Z_k(p)+n_k(0))\int_0^1 G_x(x,y,p)\nu_k(y)B^{0,1}(y,p)\,dy\\ &\quad + \sum_{k=N_1+1}^{N} Z_k(p)\Big[\frac{\mu_{k-N_1}(x)}{\lambda(x)}B^1(x,p) - \int_0^1 G_{xy}(x,y,p)\mu_{k-N_1}(y)B^1(y,p)\,dy\Big]\end{aligned} \tag{4.19}$$

and $b[Z]=(b^0[Z],b^1[Z])$ is the Z-dependent B-free term with the components

$$\begin{aligned}b^0[Z](x,p) &= \sum_{k=1}^{N_1} Z_k(p)\int_0^1 G(x,y,p)\nu_k(y)[B^{0,0}(y,p)+\varphi(y)]\,dy\\ &\quad - \sum_{k=N_1+1}^{N} Z_k(p)\int_0^1 G_y(x,y,p)\mu_{k-N_1}(y)\varphi'(y)\,dy + \Phi^{0,1}(x,p),\end{aligned} \tag{4.20}$$

$$\begin{aligned}&b^1[Z](x,p)\\ &= \sum_{k=1}^{N_1} Z_k(p)\int_0^1 G_x(x,y,p)\nu_k(y)[B^{0,0}(y,p)+\varphi(y)]\,dy\\ &\quad + \sum_{k=N_1+1}^{N} Z_k(p)\Big[\frac{\mu_{k-N_1}(x)\varphi'(x)}{\lambda(x)} - \int_0^1 G_{xy}(x,y,p)\mu_{k-N_1}(y)\varphi'(y)\,dy\Big] + \Phi^1(x,p)\end{aligned} \tag{4.21}$$

and

$$\Phi^{0,1}(x,p) = \sum_{k=1}^{N_1} n_k(0) \int_0^1 G(x,y,p)\nu_k(y) B^{0,0}(y,p)\,dy + \widetilde{\Phi}(x,p), \tag{4.22}$$

$$\begin{aligned}\Phi^1(x,p) = \sum_{k=1}^{N_1} n_k(0) \int_0^1 &G_x(x,y,p)\nu_k(y)[B^{0,0}(y,p)+\varphi(y)]\,dy \\ &- pF_x(x,p) - \varphi'(x).\end{aligned} \tag{4.23}$$

5. Functional spaces and estimation of Green function

To analyse the direct and inverse problems we define the spaces

$$\mathcal{A}_{\gamma,\sigma} = \{V : V(p) \text{ is holomorphic on } \operatorname{Re} p > \sigma,\ \|V\|_{\gamma,\sigma} < \infty\}, \tag{5.1}$$

where

$$\|V\|_{\gamma,\sigma} = \sup_{\operatorname{Re} p > \sigma} |p|^\gamma |V(p)|$$

and

$$(\mathcal{A}_{\gamma,\sigma})^N = \{V = (V_1,\dots,V_N) : V_k(p) \in \mathcal{A}_{\gamma,\sigma}, k = 1,\dots,N\} \tag{5.2}$$

with the norm

$$\|V\|_{\gamma,\sigma} = \sum_{k=1}^N \|V_k\|_{\gamma,\sigma}\,, \quad V \in (\mathcal{A}_{\gamma,\sigma})^N.$$

We note that $\mathcal{A}_{\gamma,\sigma} \subset \mathcal{A}_{\gamma,\sigma'}$, $(\mathcal{A}_{\gamma,\sigma})^N \subset (\mathcal{A}_{\gamma,\sigma'})^N$ and $\|\cdot\|_{\gamma,\sigma'} \le \|\cdot\|_{\gamma,\sigma}$ if $\sigma' > \sigma$. Let α be a real number such that

$$1 < \alpha < \frac{3}{2}\,. \tag{5.3}$$

Moreover, let $c = (c_1,\dots,c_N)$ be a given vector. We will search the solution $Z = (Z_1,\dots,Z_N)$ of (4.10) on the space

$$\mathcal{M}_{c,\sigma} = \{Z : Z = \frac{c}{p} + V(p),\ V \in (\mathcal{A}_{\alpha,\sigma})^N\}\,. \tag{5.4}$$

Furthermore, we introduce the spaces of x- and p-dependent functions

$$\mathcal{B}_{\gamma,\sigma} = \{F(x,p) : F(x,\cdot) \in \mathcal{A}_{\gamma,\sigma} \text{ for } x \in [0,1]\,,\ F(\cdot,p) \in C[0,1] \text{ for } \operatorname{Re} p > \sigma\} \tag{5.5}$$

with the norms

$$\|F\|_{\gamma,\sigma} = \max_{0\le x\le 1} \sup_{\operatorname{Re} p>\sigma} |p|^\gamma |F(x,p)|.$$

Let α' be a given number such that

$$\alpha < \alpha' < \frac{3}{2}\,. \tag{5.6}$$

We are going to solve the equation (4.17) for the pair $B = (B^{0,1}, B^1)$ in the space $\mathcal{B}_\sigma = \mathcal{B}_{\alpha',\sigma} \times \mathcal{B}_{1,\sigma}$ with the norm

$$\|B\|_\sigma = \|B^{0,1}\|_{\alpha',\sigma} + \|B^1\|_{1,\sigma}\,. \tag{5.7}$$

For estimating the Green function we use the following lemmas proved in [10].

Lemma 5.1. *Let* $\lambda, \beta \in C^2[0,1]$ *and* $\lambda, \beta > 0$ *in* $[0,1]$. *Then*

$$K_1 = \sup_{0\le x\le 1,\ \mathrm{Re}\, p>0} |p| \int_0^1 |G(x,y,p)|\, dy < \infty\,, \tag{5.8}$$

$$K_2 = \sup_{0\le x\le 1\ \ \mathrm{Re}\, p>0} \sqrt{|p|} \int_0^1 |G_x(x,y,p)|\, dy < \infty\,, \tag{5.9}$$

$$K_3 = \sup_{0\le x\le 1,\ \mathrm{Re}\, p>0} \sqrt{|p|} \int_0^1 |G_y(x,y,p)|\, dy < \infty\,, \tag{5.10}$$

$$K_4 = \sup_{0\le x\le 1,\ \mathrm{Re}\, p>0} \int_0^1 |G_{xy}(x,y,p)|\, dy < \infty\,. \tag{5.11}$$

Lemma 5.2. *Let* $\lambda, \beta \in C^2[0,1]$, $\lambda, \beta > 0$ *in* $[0,1]$ *and* $V \in C^1[0,1]$. *Then*

$$\sup_{\mathrm{Re}\, p>0} \Big|\sqrt{|p|}\Big(\int_0^1 pG(x,y,p)V(y)\, dy + \frac{V(x)}{\beta(x)}\Big)\Big| \le K_5 \|V\|_{C^1[0,1]} \tag{5.12}$$

for all $x \in (0,1)$, *where* K_5 *is independent of* x *in every closed subinterval of* $(0,1)$, *and*

$$\sup_{\mathrm{Re}\, p>0} \Big|\sqrt{|p|}\Big(\int_0^1 G_{xy}(x,y,p)V(y)\, dy - \frac{V(x)}{\lambda(x)}\Big)\Big| \le K_6 \|V'\|_{C^1[0,1]} \tag{5.13}$$

for all $x \in [0,1]$, *where* K_6 *is independent of* x.

6. Analysis of direct problem

Let us assume the following hypotheses:

$\lambda, \beta \in C^2[0,1]$, $\lambda, \beta > 0$; Φ^0 given by (4.14) admits the decomposition (4.15), where $B^{0,0} \in \mathcal{B}_{1,\sigma_0}$ and $\widetilde{\Phi} \in \mathcal{B}_{\alpha',\sigma_0}$ with some $\sigma_0 \ge 1$ and α, α' that satisfy (5.3) (5.6); Φ^1 given by (4.23) belongs to $\mathcal{B}_{1,\sigma_0}$; $\nu_k \in C[0,1]$, $k = 1,\dots,N_1$, $\mu_l \in C^1[0,1]$, $l = 1,\dots,N_2$; $\varphi \in C^2[0,1]$. (6.1)

Lemma 6.1. *Let the assumptions* (6.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ *then the vector function* $b[Z] = (b^0[Z], b^1[Z])$, *given by* (4.20), (4.21), *belongs to* $\mathcal{B}_{\sigma_0}$ *and satisfies the estimate*

$$\|b[Z]\|_\sigma \le C_1\big[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha'}}\big(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big)\big] \tag{6.2}$$

with $\sigma \ge \sigma_0$, *where* C_1 *is a constant and* $|c| = \sum_{k=1}^N |c_k|$. *Moreover, for every* $\sigma \ge \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$ *the difference* $b[Z^1] - b[Z^2]$ *fulfils the estimate*

$$\|b[Z^1] - b[Z^2]\|_\sigma \le C_2\, \frac{1}{\sigma^{\alpha-\alpha'+\frac{1}{2}}} \|V^1 - V^2\|_{\alpha,\sigma} \tag{6.3}$$

with a constant C_2.

Lemma 6.2. *Let the assumptions* (6.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ *then the linear operator* $A[Z] = (A^0[Z], A^1[Z])$, *defined by* (4.18), (4.19), *is bounded in* $\mathcal{B}_\sigma$ *and satisfies the estimate*

$$\|A[Z]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \le C_3\big[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac{1}{2}}}\big] \tag{6.4}$$

for any $\sigma \ge \sigma_0$ with a constant C_3. Moreover, taking $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$, the estimate for difference

$$\|(A[Z^1] - A[Z^2])\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \le C_4 \frac{1}{\sigma^\alpha} \|V^1 - V^2\|_{\alpha,\sigma} \tag{6.5}$$

holds for any $\sigma \ge \sigma_0$ with a constant C_4.

Proofs of Lemmas 6.1 and 6.2 are presented in the appendix of this paper.

Due to Lemmas 6.1, 6.2 and the contraction principle equation (4.17) has a unique solution $B = B[Z] \in \mathcal{B}_\sigma$ provided $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ and $\sigma \ge \sigma_0$ satisfy the relation

$$\eta(Z,\sigma) := \frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac12}} < \frac{1}{C_3}\,. \tag{6.6}$$

Furthermore, from (4.17) we have $\|B[Z]\|_\sigma \le (1 - \|A[Z]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma})^{-1} \|b[Z]\|_\sigma$. This in view of (6.2), (6.4) and (6.6) yields the estimate

$$\|B[Z]\|_\sigma \le C_1 \big\{ 1 - C_3 \big[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac12}} \big] \big\}^{-1} \big[1 + \frac{1}{\sigma^{\frac32-\alpha'}} \big(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}} \big) \big] \tag{6.7}$$

for the solution of (4.17).

Next let us find an estimate for $B[Z^1] - B[Z^2]$. Let $\sigma \ge \sigma_0$ and $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2$ be such that (6.6) is valid for V replaced by V^1 and V^2 i.e. $\eta(Z^j,\sigma) < \frac{1}{C_3}$, $j = 1,2$. Subtracting equation (4.17) for $Z = Z^2$ from the corresponding equation for $Z = Z^1$ we have

$$B[Z^1] - B[Z^2] = A[Z^2](B[Z^1] - B[Z^2]) + (A[Z^1] - A[Z^2])B[Z^1] + b[Z^1] - b[Z^2].$$

This implies

$$\begin{aligned} &\|B[Z^1] - B[Z^2]\|_\sigma \\ &\le (1 - \|A[Z^2]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma})^{-1} [\|A[Z^1] - A[Z^2]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \|B[Z^1]\|_\sigma + \|b[Z^1] - b[Z^2]\|_\sigma]. \end{aligned}$$

Using in this relation the estimates (6.3) - (6.7) we obtain

$$\begin{aligned} &\|B[Z^1] - B[Z^2]\|_\sigma \\ &\le C_5 \big[1 - C_3 \big(\frac{|c|}{\sigma} + \frac{\|V^2\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac12}} \big) \big]^{-1} \\ &\quad \times \big\{ \frac{1}{\sigma^{\alpha-\alpha'+\frac12}} + \frac{1}{\sigma^\alpha} \big[1 - C_3 \big(\frac{|c|}{\sigma} + \frac{\|V^1\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac12}} \big) \big]^{-1} \\ &\quad \times \big[1 + \frac{1}{\sigma^{\frac32-\alpha'}} \big(|c| + \frac{\|V^1\|_{\alpha,\sigma}}{\sigma^{\alpha-1}} \big) \big] \big\} \|V^1 - V^2\|_{\alpha,\sigma} \end{aligned} \tag{6.8}$$

with a constant C_5. Summing up, we have proved the following theorem.

Theorem 6.3. *Let the assumptions* (6.1) *hold. Then there exists a constant $C_3 > 0$ depending on the data of equation* (4.17) *such that for any $\sigma \ge \sigma_0$ and $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$, that satisfies the inequality* (6.6)*, equation* (4.17) *has a unique solution $B[Z] = (B^{0,1}[Z], B^1[Z])$ in $\mathcal{B}_\sigma$. This solution satisfies estimate* (6.7)*. Moreover, for every $\sigma \ge \sigma_0$ and $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$ such that $\eta(Z^j,\sigma) < \frac{1}{C_3}$, $j = 1,2$, the difference $B[Z^1] - B[Z^2]$ fulfils estimate* (6.8)*.*

7. Existence and uniqueness for inverse problem

In this section we study the inverse problem in the fixed-point form (4.10) in the Laplace domain and thereupon infer a result for the inverse problem (2.4) - (2.8) in the time domain.

Due to the decomposition (4.16) the full Z-free term of the operator $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_N)$ given by (4.11) is $\Psi = (\Psi_1, \dots, \Psi_N)$, where

$$\Psi_i(p) = \widehat{\Psi}_i(p) + \sum_{k=1}^{N_1} n_k(0) \int_0^1 pG(x_i, y, p)\nu_k(y) B^{0,0}(y,p)\, dy. \tag{7.1}$$

and $\widehat{\Psi}_i$ is defined in (4.12)

Theorem 7.1. *Assume that* (6.1) *holds and*

$$\nu_k \in C^1[0,1],\ k = 1, \dots, N_1, \quad \mu_l \in C^2[0,1],\ l = 1, \dots, N_2 \quad \varphi \in C^3[0,1]. \tag{7.2}$$

Moreover, let $\det \Gamma \neq 0$ *for* Γ*, given by* (4.7)*, and*

$$\Psi = \frac{d}{p} + Y \in \mathcal{M}_{d,\sigma_0} \tag{7.3}$$

with some $d \in \mathrm{R}^N$*. Then there exists* $\sigma_1 \geq \sigma_0$ *such that equation* (4.10) *has a unique solution* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma_1}$*. Here* $c = \Gamma^{-1} d$*.*

Proof. Setting $c = \Gamma^{-1} d$ and observing (4.16), problem (4.10) in $\mathcal{M}_{c,\sigma}$ is equivalent to the following equation for V in $(\mathcal{A}_{\alpha,\sigma})^N$,

$$V = F(V)\,, \tag{7.4}$$

where $F = \Gamma^{-1} F_1$ and

$$\begin{aligned}
&(F_1(V))_i(p)\\
&= \sum_{k=1}^{N_1} \big(\frac{c_k}{p} + V_k(p)\big)\Big\{ \int_0^1 pG(x_i, y, p)\nu_k(y)[B^{0,0}(y,p)\\
&\quad + B^{0,1}[Z](y,p)]dy + \int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\, dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\Big\}\\
&\quad + \sum_{k=N_1+1}^{N} \big(\frac{c_k}{p} + V_k(p)\big)\Big\{ - \int_0^1 pG_y(x_i,y,p)\mu_{k-N_1}(y) B^1[Z](y,p)\, dy\\
&\quad + \int_0^1 pG_y(x_i,y,p)(\mu_{k-N_1}(y)\varphi'(y))'\, dy + \frac{1}{\beta(x_i)}(\mu_{k-N_1}(x)\varphi'(x))'\Big|_{x=x_i}\Big\}\\
&\quad + \sum_{k=1}^{N_1} n_k(0) \int_0^1 pG(x_i,y,p)\nu_k(y) B^{0,1}[Z](y,p)\, dy + Y_i(p)\,, \quad i = 1, \dots, N.
\end{aligned} \tag{7.5}$$

We will prove the assertion of theorem using the fixed-point argument in the balls

$$D_{\alpha,\sigma}(\rho) = \{V \in (\mathcal{A}_{\alpha,\sigma})^N : \|V\|_{\alpha,\sigma} \leq \rho\}.$$

Multiplying by $|p|^\alpha$ in (7.5) and estimating we have

$$\begin{aligned}
&|p|^\alpha |(F_1(V))_i(p)|\\
&\leq \sum_{k=1}^{N_1} \Big(\frac{|c_k|}{|p|^{\frac{3}{2}-\alpha}} + \frac{|p|^\alpha |V_k(p)|}{\sqrt{|p|}}\Big)\Big\{|p| \int_0^1 |G(x_i,y,p)| dy
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\|\nu_k(y)\|_{C[0,1]}}{\sqrt{|p|}}\Big[|p| \max_{0\le y\le 1}|B^{0,0}(y,p)| + \frac{1}{|p|^{\alpha'-1}}|p|^{\alpha'}\max_{0\le y\le 1}|B^{0,1}[Z](y,p)|\Big]\\
&+ \sqrt{|p|}\Big|\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\Big|\Big\}\\
&+ \sum_{k=N_1+1}^{N}\Big(\frac{|c_k|}{|p|^{\frac32-\alpha}} + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\frac12}}\Big)\Big[\sqrt{|p|}\int_0^1 |G_y(x_i,y,p)|dy\\
&\times \|\mu_{k-N_1}\|_{C[0,1]}|p|\max_{0\le y\le 1}|B^1[Z](y,p)|\\
&+ \sqrt{|p|}\Big|\int_0^1 pG(x_i,y,p)(\mu_{k-N_1}(y)\varphi'(y))'dy + \frac{1}{\beta(x_i)}(\mu_{k-N_1}(x)\varphi'(x))'\Big|_{x=x_i}\Big|\Big]\\
&+ \sum_{k=1}^{N_1}|n_k(0)|\,|p|\int_0^1|G(x,y,p)|dy\|\nu_k\|_{C[0,1]}\frac{1}{|p|^{\alpha'-\alpha}}|p|^{\alpha'}\max_{0\le x\le 1}|B^{0,1}[Z](x,p)|\\
&+ |p|^{\alpha}|Y_i(p)|, \quad i=1,\dots,N\,.
\end{aligned}
$$

Using the assumptions (6.1), (7.2), (7.3), the assertions (5.8), (5.10) and (5.12) of Lemmas 5.1, 5.2 and the definition of the norm $\|\cdot\|_{\gamma,\sigma}$ we obtain

$$
\begin{aligned}
&|p|^{\alpha}|(F_1(V))_i(p)|\\
&\le \sum_{k=1}^{N_1}\Big(\frac{|c_k|}{|p|^{\frac32-\alpha}} + \frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}}\Big)\Big\{K_1\frac{1}{\sqrt{|p|}}\|\nu_k\|_{C[0,1]}\\
&\quad\times\big[\|B^{0,0}\|_{1,\sigma_0} + \frac{\|B^{0,1}[Z]\|_{\alpha',\sigma}}{|p|^{\alpha'-1}}\big] + K_5\|\nu_k\varphi\|_{C^1[0,1]}\Big\}\\
&\quad+ \sum_{k=N_1+1}^{N}(\frac{|c_k|}{|p|^{\frac32-\alpha}} + \frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}})\Big(K_3\|\mu_{k-N_1}\|_{C[0,1]}\|B^1[Z]\|_{1,\sigma}\\
&\quad+ K_5\|(\mu_{k-N_1}\varphi')'\|_{C^1[0,1]}\Big) + \sum_{k=1}^{N_1}|n_k(0)|\,K_1\|\nu_k\|_{C[0,1]}\frac{\|B^{0,1}[Z]\|_{\alpha',\sigma}}{|p|^{\alpha'-\alpha}}\\
&\quad+ \|Y_i(p)\|_{\alpha,\sigma_0}, \quad i=1,\dots N
\end{aligned}
$$

for $\mathrm{Re}\,p>\sigma$, $\sigma\ge\sigma_0$, $x\in[0,1]$. Taking here the supremum over $\mathrm{Re}\,p>\sigma$, $x\in[0,1]$, observing the relation $|p|^{\gamma}>\sigma^{\gamma}$ for $\mathrm{Re}\,p>\sigma$, which holds in the cases $\gamma=3/2-\alpha$, $\alpha'-1$, $\alpha'-\alpha$ due to the assumed inequalities (5.3) (5.6), and the inequality $1/\sigma^{\alpha'-1}\le 1$, which holds due to $\sigma\ge\sigma_0\ge 1$, we get

$$
\begin{aligned}
&\|F_1(V)\|_{\alpha,\sigma}\\
&\le C_6\big\{\big(\frac{|c|}{\sigma^{\frac32-\alpha}} + \frac{\|V\|_{\alpha,\sigma}}{\sqrt{\sigma}}\big)(\|B[Z]\|_{\sigma}+1) + \frac{1}{\sigma^{\alpha'-\alpha}}\|B[Z]\|_{\sigma}\big\} + \|Y\|_{\alpha,\sigma_0}, \quad \sigma\ge\sigma_0
\end{aligned}
\tag{7.6}
$$

with a constant C_6 depending on the data of the problem.

Further, let us suppose that $V\in D_{\alpha,\sigma}(\rho)$, where σ and ρ satisfy

$$
\eta_0(\rho,\sigma) := \frac{|c|}{\sigma} + \frac{\rho}{\sigma^{\alpha}} + \frac{1}{\sigma^{\alpha'-\frac12}} < \frac{1}{C_3} \tag{7.7}
$$

and $\sigma \geq \sigma_0$. Then (6.6) holds, hence we can apply estimate (6.7) of Theorem 6.3 for $\|B[Z]\|_\sigma$. Plugging (6.7) into (7.6) and estimating $\|V\|_{\alpha,\sigma}$ by ρ we have

$$\begin{aligned}\|F_1(V)\|_{\alpha,\sigma} \leq C_6\Big\{&\big(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{1}{\sigma^{\alpha'-\alpha}} + \frac{\rho}{\sqrt{\sigma}}\big)C_1[1-\eta_0(\rho,\sigma)C_3]^{-1}\\ &\times \big[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha'}}(|c| + \frac{\rho}{\sigma^{\alpha-1}})\big] + \frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{\rho}{\sqrt{\sigma}}\Big\} + \|Y\|_{\alpha,\sigma_0}.\end{aligned}$$

From this inequality, due to the equality $F = \Gamma^{-1}F_1$, we see that for every $\rho > \rho_0 := |\Gamma^{-1}|\|Y\|_{\alpha,\sigma_0}$ there exists $\sigma_2 = \sigma_2(\rho) \geq \sigma_0$ such that the inequalities $\eta_0(\rho,\sigma) < 1/C_3$ and $\|FV\|_{\alpha,\sigma} \leq \rho$ hold for any $\sigma \geq \sigma_2(\rho)$. Consequently,

$$F : D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \quad \text{for } \rho > \rho_0 \text{ and } \sigma \geq \sigma_2(\rho). \tag{7.8}$$

Next, we prove that F is a contraction. From (7.5) with $Z = \frac{c}{p} + V$ and $\widetilde{Z} = \frac{c}{p} + \widetilde{V}$ we have

$$\begin{aligned}&\big(F_1(V) - F_1(\widetilde{V})\big)_i(p)\\ &= \sum_{k=1}^{N_1}(V_k(p) - \widetilde{V_k}(p))\Big\{\int_0^1 pG(x_i,y,p)\nu_k(y)\big[B^{0,0}(y,p) + B^{0,1}[Z](y,p)\big]\,dy\\ &\quad + \int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\Big\}\\ &\quad + \sum_{k=1}^{N_1}(\frac{c_k}{p} + \widetilde{V_k}(p))\int_0^1 pG(x_i,y,p)\nu_k(y)[B^{0,1}[Z] - B^{0,1}[\widetilde{Z}]](y,p)\,dy\\ &\quad + \sum_{k=N_1+1}^{N}(V_k(p) - \widetilde{V_k}(p))\Big\{-\int_0^1 pG_y(x_i,y,p)\mu_{k-N_1}(y)B^1[Z](y,p)\,dy\\ &\quad + \int_0^1 pG_y(x_i,y,p)(\mu_{k-N_1}(y)\varphi'(y))'\,dy + \frac{1}{\beta(x_i)}(\mu_{k-N_1}(x)\varphi'(x))'\Big|_{x=x_i}\Big\}\\ &\quad - \sum_{k=N_1+1}^{N}(\frac{c_k}{p} + \widetilde{V_k}(p))\int_0^1 pG_y(x_i,y,p)\mu_{k-N_1}(y)[B^1[Z] - B^1[\widetilde{Z}]](y,p)\,dy\\ &\quad + \sum_{k=1}^{N_1} n_k(0)\int_0^1 pG(x_i,y,p)\nu_k(y)[B^{0,1}[Z] - B^{0,1}[\widetilde{Z}]](y,p)\,dy\,,\end{aligned} \tag{7.9}$$

$i = 1,\dots,N$. Performing similar operations as above in deriving (7.6) we obtain from (7.9) the estimate

$$\begin{aligned}&\|F_1(V) - F_1(\widetilde{V})\|_{\alpha,\sigma}\\ &\leq C_7\Big\{\frac{\|V-\widetilde{V}\|_{\alpha,\sigma}}{\sqrt{\sigma}}(\|B[Z]\|_\sigma + 1)\\ &\quad + \big(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{\|\widetilde{V}\|_{\alpha,\sigma}}{\sqrt{\sigma}}\big)\|B[Z] - B[\widetilde{Z}]\|_\sigma + \frac{1}{\sigma^{\alpha'-\alpha}}\|B[Z] - B[\widetilde{Z}]\|_\sigma\Big\}\end{aligned}$$

for $\sigma \geq \sigma_0$ with a constant C_7. Supposing that $V, \widetilde{V} \in D_{\alpha,\sigma}(\rho)$ with $\sigma \geq \sigma_0$ and ρ such that (7.7) hold by the estimates (6.7) and (6.8) of Theorem 6.3 we have

$$\|F_1(V) - F_1(\widetilde{V})\|_{\alpha,\sigma}$$

$$
\begin{aligned}
&\leq C_7\Big\{\frac{1}{\sqrt{\sigma}}\Big(1+\frac{C_1}{1-\eta_0(\rho,\sigma)C_3}\big[1+\frac{1}{\sigma^{\frac{3}{2}-\alpha'}}(|c|+\frac{\rho}{\sigma^{\alpha-1}})\big]\Big)\\
&\quad+\big(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}}+\frac{\rho}{\sqrt{\sigma}}+\frac{1}{\sigma^{\alpha'-\alpha}}\quad\big)\frac{C_5}{1-\eta_0(\rho,\sigma)C_3}\\
&\quad\times\Big[\frac{1}{\sigma^{\alpha-\alpha'+\frac{1}{2}}}+\frac{1}{\sigma^{\alpha}(1-\eta_0(\rho,\sigma)C_3)}\big(1+\frac{1}{\sigma^{\frac{3}{2}-\alpha'}}(|c|+\frac{\rho}{\sigma^{\alpha-1}})\big)\Big]\Big\}\|V-\widetilde{V}\|_{\alpha,\sigma}\,.
\end{aligned}
$$

The coefficient of $\|V-\widetilde{V}\|_{\alpha,\sigma}$ on the right-hand side of this estimate approaches zero as $\sigma\to\infty$ for a fixed $\rho>0$. Hence, for every $\rho>0$ there exists $\sigma_3=\sigma_3(\rho)\geq\sigma_0$, such that the inequality $\eta_0(\rho,\sigma)<1/C_3$ holds and $F=\Gamma^{-1}F_1$ is a contraction in the ball $D_{\alpha,\sigma}(\rho)$ for $\rho>0$ and $\sigma\geq\sigma_3(\rho)$. This together with (7.8) shows that equation (7.4) has a unique solution V in every ball $D_{\alpha,\sigma}(\rho)$, where $\rho>\rho_0$ and $\sigma\geq\sigma_4(\rho)=\max(\sigma_2(\rho);\sigma_3(\rho))$. This proves the existence assertion of theorem with $\sigma_1=\sigma_4(2\rho_0)$.

It remains to prove that the solution of (7.4) is unique in the whole space $(\mathcal{A}_{\alpha,\sigma_1})^N$. Suppose that (7.4) has two solutions V^1 and V^2 in $(\mathcal{A}_{\alpha,\sigma_1})^N$. Let us define $\bar\rho:=\max(2\rho_0;\|V^1\|_{\alpha,\sigma_1};\|V^2\|_{\alpha,\sigma_1})$ and $\bar\sigma:=\max(\sigma_1;\sigma_4(\bar\rho))$. Then we have $\|V^j\|_{\alpha,\sigma_1}\leq\bar\rho$, $j=1,2$. Since the norm $\|.\|_{\alpha,\sigma}$ is nonincreasing with respect to σ and $\bar\sigma\geq\sigma_1$, from this relation we derive

$$
\|V_j\|_{\alpha,\bar\sigma}\leq\bar\rho\implies V^j\in D_{\alpha,\bar\sigma(\bar\rho)}\,,\quad j=1,2.
$$

But due to $\bar\rho>\rho_0$ and $\bar\sigma\geq\sigma_4(\bar\rho)$, the uniqueness in the ball $D_{\alpha,\bar\sigma(\bar\rho)}$ has already been shown. Thus, $V^1=V^2$. Theorem 7.1 is proved. □

Finally, applying the well-known results about the invertibility of the Laplace transform [3] we deduce the following corollary from Theorem 7.1.

Corollary 7.2. *Let conditions* (4.4) *hold yielding the unique initial values $n_j(0)$ for the unknowns n_j, $k=1,\dots,N_1$ from system* (4.3). *Moreover, let the assumptions of Theorem 7.1 be satisfied for the functions λ_k, μ_l, φ and the quantities Φ^0, Φ^1, Ψ given by formulas* (4.14), (4.23), (7.1) *with* (4.14), (3.7) *in terms of the Laplace transforms R, F_1, F_2, H_i of the data of inverse problem* (2.4) - (2.8).

Then inverse problem (2.4)–(2.8) *has the unique solution (n,m) with coefficients n_j and m_k of the form*

$$
\begin{gathered}
n_j(t)=n_j(0)+c_jt+\frac{1}{2\pi i}\int_0^t\int_{\xi-i\infty}^{\xi+i\infty}e^{\tau p}V_j(p)\,dp\,d\tau\,,\quad k=1,\dots N_1\,,\\
m_k(t)=c_{k+N_1}+\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}e^{tp}V_{k+N_1}(p)\,dp,,\quad k=1,\dots N_2\,,
\end{gathered}
$$

where $\xi>\sigma_1>1$, $c=(c_1,\dots,c_N)\in\mathrm{R}^N$, $c=\Gamma^{-1}d$ with d from (7.3), *$V=(V_1,\dots,V_N)\in(\mathcal{A}_{\alpha,\sigma_1})^N$, $N=N_1+N_2$. The functions n_j are continuously differentiable and m_k are continuous for $t\geq0$. Moreover, $n_j'(0)=c_j$, $j=1,\dots,N_1$ and $m_k(0)=c_{k+N_1}$, $k=1,\dots,N_2$.*

8. Appendix

Proof of Lemma 6.1. . Let us start with the estimation of $b^0[Z]$. Substituting $\frac{c}{p}+V$ for Z in (4.20), multiplying by $|p|^{\alpha'}$ and estimating we have

$$
\begin{aligned}
&|p|^{\alpha'}|b^0[Z](x,p)| \\
&\le \sum_{k=1}^{N_1}\big(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\big)|p|\int_0^1 \big|G(x,y,p)\big|dy \\
&\quad \times \|\nu_k\|_{C[0,1]}\Big[\frac{1}{|p|^{3-\alpha'}}\,|p|\max_{0\le y\le 1}|B^{0,0}(y,p)| + \frac{\|\varphi\|_{C[0,1]}}{|p|^{2-\alpha'}}\Big] \\
&\quad + \sum_{k=N_1+1}^{N}\big(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\big)\sqrt{|p|}\int_0^1 |G_y(x,y,p)|dy\|\mu_{k-N_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{|p|^{\frac{3}{2}-\alpha'}} \\
&\quad + |p|^{\alpha'}|\Phi^{0,1}(x,p)|.
\end{aligned}
\tag{8.1}
$$

Note that (6.1) implies $\Phi^{0,1}\in\mathcal{B}_{\alpha',\sigma_0}$ for the function $\Phi^{0,1}$ defined in (4.22). Using this relation, the assertions (5.8), (5.10) of Lemma 5.1 and the definitions of the norms $\|\cdot\|_{\gamma,\sigma}$, $\|\cdot\|_{\sigma}$ we obtain from (8.1)

$$
\begin{aligned}
&|p|^{\alpha'}|b^0[Z](x,p)| \\
&\le \sum_{k=1}^{N_1}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{|p|^{\alpha-1}}\big)K_1\|\nu_k\|_{C[0,1]}\big[\frac{\|B^{0,0}\|_{1,\sigma}}{|p|^{3-\alpha'}} + \frac{\|\varphi\|_{C[0,1]}}{|p|^{2-\alpha'}}\big] \\
&\quad + \sum_{k=N_1+1}^{N}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{|p|^{\alpha-1}}\big)K_3\|\mu_{k-N_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{|p|^{\frac{3}{2}-\alpha'}} + \|\Phi^{0,1}\|_{\alpha',\sigma_0}
\end{aligned}
$$

for $\operatorname{Re}p>\sigma$, $\sigma\ge\sigma_0$, $x\in[0,1]$. Taking here the supremum over $\operatorname{Re}p>\sigma$, $x\in[0,1]$ and observing the relation $|p|^{\gamma}>\sigma^{\gamma}$ for $\operatorname{Re}p>\sigma$, which holds in the cases $\gamma=\alpha-1$, $3-\alpha'$, $2-\alpha'$, $3/2-\alpha'$ due to (5.3) (5.6), we have

$$
\begin{aligned}
\|b^0[Z]\|_{\alpha',\sigma} &\le \sum_{k=1}^{N_1}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big)K_1\|\nu_k\|_{C[0,1]}\big[\frac{\|B^{0,0}\|_{1,\sigma}}{\sigma^{3-\alpha'}} + \frac{\|\varphi\|_{C[0,1]}}{\sigma^{2-\alpha'}}\big] \\
&\quad + \sum_{k=N_1+1}^{N}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big)K_3\|\mu_{k-N_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{\sigma^{\frac{3}{2}-\alpha'}} + \|\Phi^{0,1}\|_{\alpha',\sigma_0}
\end{aligned}
$$

for $\sigma\ge\sigma_0$. Finally, observing that $\sigma^{\gamma'}\ge\sigma^{\gamma}$ for $\gamma'>\gamma$, because $\sigma\ge\sigma_0\ge 1$ we arrive at the relation

$$
\|b^0[Z]\|_{\alpha',\sigma}\le\frac{C_8}{\sigma^{\frac{3}{2}-\alpha'}}\big(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big) + \|\Phi^{0,1}\|_{\alpha',\sigma_0}\,,\quad \sigma\ge\sigma_0 \tag{8.2}
$$

with a constant C_8 depending on $K_1, K_3, \nu, \mu, B^{0,0}, \varphi$.

Next we perform similar transformations with $b^1[Z]$ in (4.21) multiplying by $|p|$ instead of $|p|^{\alpha'}$. We have

$$
|p||b^1[Z](x,p)| \le \sum_{k=1}^{N_1}\big(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\big)\sqrt{|p|}\int_0^1 |G_x(x,y,p)|dy
$$

$$\times \|\nu_k\|_{C[0,1]}\big[\frac{1}{|p|^{\frac{3}{2}}}|p|\max_{0\le y\le 1}|B^{0,0}(y,p)| + \frac{\|\varphi\|_{C[0,1]}}{\sqrt{p}}\big]$$
$$+\sum_{k=N_1+1}^{N}\big(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\big)\frac{1}{\sqrt{|p|}}\sqrt{|p|}\Big|\frac{\mu_{k-N_1}(x)\varphi'(x)}{\lambda(x)}$$
$$-\int_0^1 G_{xy}(x,y,p)\mu_{k-N_1}(y)\varphi'(y)\,dy\Big| + |p||\Phi^1(x,p)|.$$

Using here the assumption (6.1), the assertions (5.9) and (5.13) of Lemmas 5.1, 5.2 and taking the supremum over $\operatorname{Re} p>\sigma$, $x\in[0,1]$ we obtain

$$\begin{aligned}\|b^1[Z]\|_{1,\sigma} \le \sum_{k=1}^{N_1}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big)K_2\|\nu_k\|_{C[0,1]}\times\big[\frac{\|B^{0,0}\|_{1,\sigma}}{\sigma^{\frac{3}{2}}} + \frac{\|\varphi\|_{C[0,1]}}{\sigma^{\frac{1}{2}}}\big]\\
+\sum_{k=N_1+1}^{N}\big(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big)K_6\|(\mu_{k-N_1}\varphi')'\|_{C[0,1]}\frac{1}{\sqrt{\sigma}} + \|\Phi^1\|_{1,\sigma_0}\end{aligned}$$

for $\sigma\ge\sigma_0$. This yields

$$\|b^1[Z]\|_{1,\sigma} \le \frac{C_9}{\sqrt{\sigma}}\big(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\big) + \|\Phi^1\|_{1,\sigma_0}, \quad \sigma\ge\sigma_0 \tag{8.3}$$

with a constant C_9.

In particular, (8.2) and (8.3) imply $b[Z]=(b^0[Z],b^1[Z])\in\mathcal{B}_\sigma$ for $\sigma\ge\sigma_0$ and estimate (6.2). To prove (6.3) we denote $Z=Z^1-Z^2$. Then the components $b^0[Z]$ and $b^1[Z]$ of the vector $b[Z]=b[Z^1]-b[Z^2]$ are expressed by the formulas (4.20) with $\Phi^{0,1}=0$ and (4.21) with $\Phi^1=0$, respectively. Using the estimates (8.2) and (8.3) for the components of $b[Z]$ and observing that $Z=\frac{c}{p}+V$ with $c=0$ and $V=V^1-V^2$ we deduce (6.3). The proof is complete. □

Proof of Lemma 6.2. First we show that the linear operator $A[Z]=(A^0[Z],A^1[Z])$, given by (4.18), (4.19), is bounded in $\mathcal{B}_\sigma$ and satisfies estimate (6.4). From (4.18) by $Z=\frac{c}{p}+V$ we get

$$\begin{aligned}|p|^{\alpha'}|(A^0[Z]B)(x,p)| \le \sum_{k=1}^{N_1}\big(\frac{|c_k|}{|p|^2} + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha+1}} + \frac{|n_k(0)|}{|p|}\big)\\
\times|p|\int_0^1|G(x,y,p)|dy\cdot\|\nu_k\|_{C[0,1]}\,|p|^{\alpha'}\max_{0\le y\le 1}|B^{0,1}(y,p)|\\
+\sum_{k=N_1+1}^{N}\big(\frac{|c_k|}{|p|^{\frac{5}{2}-\alpha'}} + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\frac{3}{2}-\alpha'+\alpha}}\big)\sqrt{|p|}\int_0^1|G_y(x,y,p)|dy\\
\times\|\mu_{k-N_1}\|_{C[0,1]}\,|p|\max_{0\le y\le 1}|B^1(y,p)|\,.\end{aligned}$$

Using Lemma 5.1 and taking the supremum over $\operatorname{Re} p>\sigma$, $x\in[0,1]$ we deduce

$$\begin{aligned}\|A^0[Z]B\|_{\alpha',\sigma} \le \sum_{k=1}^{N_1}\big(\frac{|c_k|}{\sigma^2} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha+1}} + \frac{|n_k(0)|}{\sigma}\big)K_1\|\nu_k\|_{C[0,1]}\|B^{0,1}\|_{\alpha',\sigma}\\
+\sum_{k=N_1+1}^{N}\big(\frac{|c_k|}{\sigma^{\frac{5}{2}-\alpha'}} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\frac{3}{2}-\alpha'+\alpha}}\big)K_3\|\mu_{k-N_1}\|_{C[0,1]}\|B^1\|_{1,\sigma}\end{aligned}$$

for $\sigma \geq \sigma_0$. This due to $\alpha' > 1/2$ and $\sigma_0 \geq 1$ implies

$$\|A^0[Z]B\|_{\alpha',\sigma} \leq C_{10}\Big[\frac{|c|}{\sigma^{\frac{5}{2}-\alpha'}} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\frac{3}{2}-\alpha'+\alpha}} + \frac{|n(0)|}{\sigma}\Big]\|B\|_\sigma, \quad \sigma \geq \sigma_0\,, \tag{8.4}$$

where C_{10} is a constant. Further, from (4.19) we derive

$$\begin{aligned}
&|p||(A^1[Z]B)(x,p)| \\
&\leq \sum_{k=1}^{N_1}\Big(\frac{|c_k|}{|p|^{\frac{1}{2}+\alpha'}} + \frac{|p|^\alpha|V_k(p)|}{|p|^{\alpha+\alpha'-\frac{1}{2}}} + \frac{|n_k(0)|}{|p|^{\alpha'-\frac{1}{2}}}\Big) \\
&\quad \times \sqrt{|p|}\int_0^1 |G_x(x,y,p)|dy \cdot \|\nu_k\|_{C[0,1]}|p|^{\alpha'} \max_{0\leq y\leq 1}|B^{0,1}(y,p)| \\
&\quad + \sum_{k=N_1+1}^{N}\Big(\frac{|c_k|}{|p|} + \frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\Big)\Big[\frac{\|\mu_{k-N_1}\|_{C[0,1]}}{\lambda_0}|p|\max_{0\leq y\leq 1}|B^1(y,p)| \\
&\quad + \int_0^1 |G_{xy}(x,y,p)|dy \cdot \|\mu_{k-N_1}\|_{C[0,1]}|p|\max_{0\leq x\leq 1}|B^1(x,p)|\Big]\,.
\end{aligned}$$

Here $\lambda_0 := \min_{0\leq x\leq 1}\lambda(x) > 0$ because $\lambda \in C[0,1]$, $\lambda(x) > 0$, by assumption. Using Lemma 5.1, taking the supremum over $\operatorname{Re} p > \sigma$, $x \in [0,1]$ and observing the inequalities $\alpha' > 1/2$ and $\sigma_0 \geq 1$ we get

$$\|A^1[Z]B\|_{1,\sigma} \leq C_{11}\Big[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{|n(0)|}{\sigma^{\alpha'-\frac{1}{2}}}\Big]\|B\|_\sigma, \quad \sigma \geq \sigma_0 \tag{8.5}$$

with a constant C_{11}.

Putting estimates (8.4) and (8.5) together and taking the inequalities $\alpha' < 3/2$, $\sigma \geq \sigma_0$ into account we have

$$\|A[Z]B\|_\sigma \leq C_{12}\Big[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{|n(0)|}{\sigma^{\alpha'-\frac{1}{2}}}\Big]\|B\|_\sigma, \quad \sigma \geq \sigma_0 \tag{8.6}$$

with a constant C_{12}. Due to this relation $A[Z]$ is bounded in $\mathcal{B}_\sigma$ and satisfies estimate (6.4).

It remains to prove (6.5). Denoting $Z = Z^1 - Z^2$ the components $A^0[Z]$ and $A^1[Z]$ of the vector $A[Z] = A[Z^1] - A[Z^2]$ are expressed by the formulas (4.18) and (4.19), respectively, containing $n_k(0) = 0$. Using the estimate (8.6) for $A[Z]$ and observing that $Z = \frac{c}{p} + V$ with $c = 0$ and $V = V^1 - V^2$ we deduce (6.5). The lemma is proved. $\square$

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