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ON THE VARIATIONAL STRUCTURE OF BREATHER SOLUTIONS II: PERIODIC MKDV EQUATION

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Abstract. We study the periodic modified KdV equation, where a periodic in space and time breather solution is known from the work of Kevrekidis et al. [19]. We show that these breathers satisfy a suitable elliptic equation, and we also discuss via numerics its spectral stability. We also identify a source of nonlinear instability for the case described in [19], and we conjecture that, even if spectral stability is satisfied, nonlinear stability/instability depends only on the sign of a suitable discriminant function, a condition that is trivially satisfied in the case of non-periodic (in space) mKdV breathers. Finally, we present a new class of breather solution for mKdV, believed to exist from geometric considerations, and which is periodic in time and space, but has nonzero mean, unlike standard breathers.

1. Introduction

1.1. **Setting of the problem.** In this article, we consider the breather solution of the periodic modified Korteweg-de Vries (mKdV) equation

$$u_t + (u_{xx} + u^3)_x = 0, \quad u(t,x) \in \mathbb{R},\ t \in \mathbb{R},\ x \in \mathbb{T}_x := \mathbb{R}/L\mathbb{Z}; \tag{1.1}$$

Here $L > 0$ is a fixed length to be determined later on. The above equation is a well-known *completely integrable* model [1, 17, 22], with infinitely many conserved quantities, and a suitable Lax-pair formulation.

Solutions of (1.1) are an invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $u(t-t_0, x-x_0)$ is also a solution. Additionally, if $c > 0$ is any number, then $\sqrt{c}\, u(c^{3/2}t, \sqrt{c}x)$ is also solution.

In addition to standard solitons, mKdV (1.1) do have *periodic in space breather solutions* [19].

Definition 1.1 (Periodic breather)**.** A KKSH periodic breather is a solution of (1.1) of the form

$$B = B(t,x;\alpha,\beta,k,m) := 2\sqrt{2}\partial_x\Big[\arctan\Big(\frac{\beta}{\alpha}\frac{\mathrm{sn}(\alpha(x+\delta t),k)}{\mathrm{nd}(\beta(x+\gamma t),m)}\Big)\Big],$$
$$\delta := \alpha^2(1+k) + 3\beta^2(m-2), \quad \gamma := 3\alpha^2(1+k) + \beta^2(m-2),$$

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where $\mathrm{sn}(\cdot)$ and $\mathrm{nd}(\cdot)$ are the standard Jacobi elliptic functions.

(See (2.1) for a more detailed description and general definition of periodic breather.) These breathers can be written using only two parametric variables, say β and k, and have a characteristic period $L = L(\beta, k)$, with $L \to +\infty$ as $k \to 0$.

1.2. **Main results.** In [7, 6, 8], we showed that aperiodic (in space) mKdV breathers are stable in their energy space. In [9], we also showed evidence that sine-Gordon breathers are stable. In this paper, we will make use of a combined theoretical-numerical approach to conclude why periodic breathers may and may not be stable in some particular regimes. We will need to introduce the following assumptions:

(A1) The kernel of a linearized operator around a breather is nondegenerate and it satisfies the gap condition; and

(A2) There is a unique simple negative eigenvalue associated to this linear operator. (See p. 11 for a precise description.)

(A3) The following sign condition is satisfied: if $M_{\#}[B]$ is the mass of the breather solution in terms of β and k, and a_1, a_2 are "variational" parameters given in (2.8)-(2.9), then

$$\frac{\partial_k a_1 \partial_\beta M_{\#}[B] - \partial_\beta a_1 \partial_k M_{\#}[B]}{\partial_k a_1 \partial_\beta a_2 - \partial_k a_2 \partial_\beta a_1} > 0. \tag{1.2}$$

(See (3.12) for more details.)

Theorem 1.2. *Under spectral assumptions* (A1)–(A3)*, KKSH breathers are stable for L-periodic H^2-perturbations.*

We also present numerical evidence (see Subsection 3.3) that assumptions (A1) and (A2) are valid in generality. Assumption (A3) is only verified in a certain regime of parameters, therefore it is expected that KKHS breathers may and may not be stable, according to some particular condition. In Figure 9 we describe numerically the meaning of assumption (A3). In particular, it is inferred that for k small enough (depending only on β), assumption (A3) is satisfied. Additionally, note that the condition $k \to 0^+$ is equivalent to take the spatial period $L \to +\infty$, and formally recovering the standard mKdV aperiodic breather. (See (2.6) for this fact.)

Moreover, we believe that the lack or invalidity of assumption (A3) is precisely the source of instability in KKHS breathers, in the sense that if this assumption is not satisfied, then they should be unstable, as it happens in the soliton case in the regime of supercritical nonlinearities (see [30, Lemma 1.6] for example), and just in agreement with the numerical computations performed by Kevrekidis et al. [19], for which (1.2) is not satisfied.

1.3. **Discussion.** Assumption (A3) above is a sort of generalization of the Weinstein's sign condition for soliton solutions, but it is different from the former because it also considers a certain variation of energy (and not only mass), which has been written above in terms of the mass only. Additionally, this sign condition can be evaluated for non-periodic breathers such as mKdV and SG, and it is trivially satisfied, see e.g. [9, eq. (3.23)] for the corresponding computation in the SG case.

Additionally, assumption (A3) is needed for ensuring that one can "replace" the first eigenfunction appearing from assumption (A2), which is an instability direction, by the breather itself. The advantage of this replacement comes from the fact that a perturbative dynamics lying in the L^2 orthogonal space to the breather allows to control variations of the scalings of the breather, and moreover, their dynamics has formally *quadratic variation* in time, meaning that if the error in our initial data is of order η, then any bad variation of the scalings will be of order at most η^2, and therefore negligible for the stability result. At the rigorous level, the conservation of mass allows to control the orthogonal direction to the breather itself in terms of quadratic terms only. The procedure to replace the first eigenfunction by the breather is standard, but needs to ensure that a certain denominator is never zero, see e.g. [9, eq. (3.30)]. Under assumption (A3), such a denominator is never zero. Recall that a similar result has been proved in [6, eq. (4.24)], see [16, 14, 30] for more examples in the case of soliton solutions. Adding this new assumption, our method of proof works as in the SG case.

1.4. **New breathers.** Finally, Section 4 deals with a new class of mKdV periodic breathers, which have the nonstandard property of being of nonzero mean.

Theorem 1.3. *Given any parameter $\mu > 0$, there exists a periodic in space breather solution B_μ, which is solution of* (1.1) *and satisfies the decomposition $B_\mu = \mu + \widetilde{B}_\mu$, where $\widetilde{B}_\mu$ has zero mean.*

For an explicit formula for B_μ, the reader can consult Definition 4.1. As it is explained in the last comments of this paper, it also satisfies a proper fourth order elliptic equation.

This new breather has been conjectured to exist by several works on curvature motion of closed curves on the plane, see e.g. [5, 21] and references therein. However, its description does not follow the ideas from [19], because KKHS breathers are zero mean solutions. Instead we use, in a slightly different way, the method of proof employed in our work [8], which links zero mean breathers with the corresponding zero solution of the equation. In order to find a breather with a nonzero mean, we use as starting point the nonzero constant solution μ, and then apply twice a suitable Bäcklund transformation, as is done in [8]. Since the mean of a modified KdV solution is a conserved quantity, this property will be also preserved by the Bäcklund transformation, leading to the desired solution with the property sought. Concerning this new breather, in this work we only study its simplest properties, and describe its main differences with KKSH breathers, leaving its deep understanding, by length reasons, to a forthcoming publication. For the moment, we only advance that these breathers satisfy a suitable elliptic equation, as any other breather in this article. Additionally, we conjecture that this breather should be as stable as the constant solution $u = \mu$ is for the mKdV periodic dynamics.

1.5. **Previous results.** If one studies perturbations of solitons in (1.1) and more general equations, the concepts of *orbital*, and *asymptotic stability* emerge naturally. In particular, since energy and mass are conserved quantities, it is natural to expect that solitons are stable in a suitable energy space. Indeed, H^1-stability of mKdV and more general solitons and multi-solitons has been considered e.g. in Benjamin [11], Bona-Souganidis-Strauss [12], Weinstein [32], Maddocks-Sachs [23], Martel-Merle-Tsai [24], Martel-Merle [25] and M. [29]. L^2-stability of KdV solitons has been proved by Merle-Vega [28]. Moreover, asymptotic stability properties for gKdV equations have been studied by Pego-Weinstein [31] and Martel-Merle [26, 27], among many other authors.

We also mention that, in addition to Theorems [9, Thm. 1.3], (1.2) and (1.3), the more involved problem of asymptotic stability for breathers could been also considered, as far as a good and rigorous understanding of the associated spectral problem is at hand. Usually, these spectral properties are harder to establish than the ones involved in the stability problem (because the convergence problem requires the use of weighted functions, which destroy most of breather's algebraic properties). In addition, breathers can have zero, positive or negative velocity, which means that they do not necessarily decouple from radiation. However, it is worth to mention that if the velocity of a periodic mKdV breather is positive, then there is local strong asymptotic stability in the energy space, see [6].

This article is organized as follows. Section 2, and in particular, Theorem 2.1 are devoted to the proof that periodic KKHS breathers satisfy suitable elliptic equations, and we find its variational structure. In Section 3, after some numerical tests, we sketch the proof of Theorem 1.2 (see Theorem 3.1) and we conjecture (see p. 15) that KKHS breathers have a dual stability/instability regime. Moreover, assuming the validity of some numerical computations, we show that KKHS are stable in a particular set of parameters. Finally, in Section 4 we present a new kind of periodic mKdV breather whose main property is the fact that it has nonzero mean.

2. Periodic mKdV breathers

2.1. **Definitions.** We consider now the case of the periodic (in space) mKdV equation. Periodic mKdV breathers (or KKSH breathers), in the sense of Definition 1.1, were found by Kevrekidis, Khare, Saxena and Herring [18, 19] by using elliptic functions and a matching of free parameters. More precisely, we consider the equation (1.1) where

$$u : \mathbb{R}_t \times \mathbb{T}_x \mapsto \mathbb{R}_x,$$

is periodic in space, and $\mathbb{T}_x = \mathbb{T} = \mathbb{R}/L\mathbb{Z} = (0, L)$ denotes a torus with period L, to be fixed later. Given $\alpha, \beta > 0$, $x_1, x_2 \in \mathbb{R}$ and $k, m \in [0, 1]$, KKSH breathers are given by the explicit formula [18]

$$B = B(t, x; \alpha, \beta, k, m, x_1, x_2) := \partial_x \tilde{B} := 2\sqrt{2}\partial_x \Big[\arctan\Big(\frac{\beta}{\alpha} \frac{\operatorname{sn}(\alpha y_1, k)}{\operatorname{nd}(\beta y_2, m)} \Big) \Big], \quad (2.1)$$

with $\operatorname{sn}(\cdot, k)$ and $\operatorname{nd}(\cdot, m)$ the standard Jacobi elliptic functions of elliptic modulus k and m, respectively, but now

$$y_1 := x + \delta t + x_1, \quad y_2 := x + \gamma t + x_2, \quad (2.2)$$

$$\delta := \alpha^2(1 + k) + 3\beta^2(m - 2), \quad \gamma := 3\alpha^2(1 + k) + \beta^2(m - 2). \quad (2.3)$$

See Figure 1 for a description of a KKHS breather solution and [2, 13] for a more detailed account on the Jacobi elliptic functions sn and nd presented in (2.1). Additionally, in order to be a periodic solution of mKdV, the parameters m, k, α and β must satisfy the commensurability conditions on the spatial periods

$$\frac{\beta^4}{\alpha^4} = \frac{k}{1 - m}, \quad K(k) = \frac{\alpha}{2\beta} K(m), \quad (2.4)$$

where K denotes the complete elliptic integral of the first kind, defined as [13]

$$K(r) := \int_0^{\pi/2} (1 - r \sin^2(s))^{-1/2} ds = \int_0^1 ((1 - t^2)(1 - rt^2))^{-1/2} dt, \quad (2.5)$$

and which satisfies
$$K(0)=\frac{\pi}{2},\quad \lim_{k\to 1^-}K(k)=\infty.$$
Under these assumptions, the spatial period is given by
$$L:=\frac{4}{\alpha}K(k)=\frac{2}{\beta}K(m). \tag{2.6}$$
Note that conditions (2.4) formally imply that B has only four independent parameters (e.g. β,k and translations x_1,x_2). Additionally, if we assume that the ratio β/α stays bounded, we have that k approaches 0 as m is close to 1. Using this information, the standard non periodic mKdV breather [6, eq. (1.8)] can be formally recovered as the limit of very large spatial period $L\to+\infty$, obtained e.g. if $k\to 0$. In that sense, we can think of (2.1) as a nontrivial periodic bifurcation at infinity of the aperiodic mKdV breather.

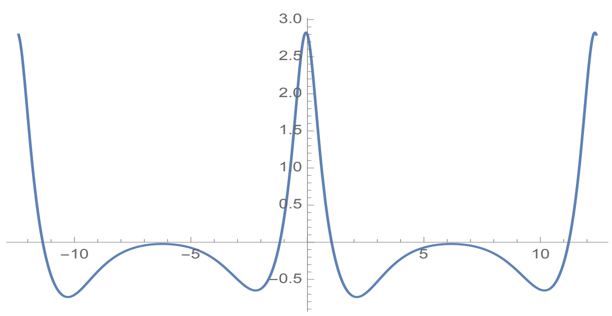

FIGURE 1. Graph of the KKSH breather at time $t=0$, for $\beta=1$, $k=0.001$, $m=0.999$ and $L=12.398$.

It is important to mention that conditions (2.4) impose a particular set of restrictions on k and m. In terms of k, one has $k\in[0,k^*)$, with $k^*\sim 0.058$, while m is decreasing with respect to k, with $m(k=0)=1$ and $m(k\sim k^*)\sim 0$. Below, Figure 2 describes the behavior of k and m more clearly.

In what follows, we will use the convention that
$$m=m(k),\quad \alpha=\alpha(\beta,k), \tag{2.7}$$
obtained by solving $m=m(k)$ in (2.4) numerically, and then solving for α algebraically.

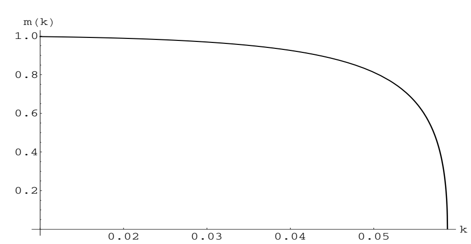

FIGURE 2. Graph of m as a function of k obtained solving conditions (2.4). Although m runs from 0 to 1, not all values of $k\in[0,1]$ are allowed, being the limiting value of $k\sim 0.05883626$.

2.2. **Variational characterization.** Our first result is the following theorem.

Theorem 2.1. *Assume* (2.4). *Let* B *be any KKSH breather. Define*

$$a_1 := 2(\beta^2(2-m) - \alpha^2(1+k)) = -\frac{1}{2}(\delta+\gamma), \tag{2.8}$$

$$a_2 := \alpha^4(1+k^2-26k) + 2\alpha^2\beta^2(2-m)(1+k) + \beta^4 m^2. \tag{2.9}$$

Then B *satisfies the generalized nonlinear elliptic equation*

$$B_{(4x)} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - a_1(B_{xx}+B^3) + a_2 B = 0. \tag{2.10}$$

See Section 5 for the proof of this result. We emphasize that the first condition in (2.4) is essential for the proof of (2.10). However, we do not need the second one. Note additionally that a_1 and a_2 converge to the corresponding constants for the aperiodic mKdV case when $k \to 0$ and $m \to 1$, see [6, eq. (1.8)].

Now we introduce the conserved quantities

$$M_\#[u](t) := \frac{1}{2}\int_{\mathbb{T}} u^2(t,x)dx, \tag{2.11}$$

$$E_\#[u](t) := \frac{1}{2}\int_{\mathbb{T}} u_x^2(t,x)dx - \frac{1}{4}\int_{\mathbb{T}} u^4(t,x)dx, \tag{2.12}$$

$$F_\#[u](t) := \frac{1}{2}\int_{\mathbb{T}} u_{xx}^2(t,x)dx - \frac{5}{2}\int_{\mathbb{T}} u^2u_x^2(t,x)dx + \frac{1}{4}\int_{\mathbb{T}} u^6(t,x)dx, \tag{2.13}$$

which are preserved in the space

$$H^2(\mathbb{T}) := \{v \in H^2(0,L) : v(0) = v(L),\ v_x(0) = v_x(L)\}, \tag{2.14}$$

if $u(t,\cdot) \in H^2(\mathbb{T})$ is a solution of (1.1). The proof of this result is straightforward if we work by density in a space of smooth functions and we note that $u(t,0) = u(t,L)$ and $u_x(t,0) = u_x(t,L)$ for all time imply

$$u_t(t,0) = u_t(t,0), \quad u_{xt}(t,0) = u_{xt}(t,L),$$

and therefore, using (1.1), $u_{xxx}(t,0) = u_{xxx}(t,L)$.

Corollary 2.2. *KKSH breathers are critical points of the functional*

$$\mathcal{H}_\#[u] := F_\#[u](t) + a_1E_\#[u](t) + a_2M_\#[u](t),$$

defined in the space $H^2(\mathbb{T})$ *and preserved along the mKdV periodic flow.*

As in the previous sections, the next step is the study of the nonlinear stability of this solution. However, we must emphasize that the existence of a suitable elliptic equation does not imply stability. Even worse, breathers with standard spectrum may be nonlinearly unstable. Indeed, we present below compelling evidence that periodic mKdV breathers are *unstable* under a suitable type of periodic perturbations, at least for L not so large. (If L is large it seems that the periodic breather is close in a certain topology to the aperiodic breather, which satisfies stability properties in a very general open neighborhood.) Our results do agree with the numerical ones obtained by Kevrekidis et al. [18, 19], and our numerical computations below. In this case, the periodic character of the solution leads to nontrivial interactions between adjacent breathers, which probably play an important role in the instability character of this solution.

3. Spectral analysis of the periodic mKdV breather

3.1. **Mathematical description.** In this section we give further evidence of the stable-unstable character of the KKSH breather solution depending on the parameters phase space. First of all, let us notice that, thanks to (2.10), the linearized operator for a KKSH breather is given by the expression

$$\begin{aligned}\mathcal{L}_\#[z] := z_{(4x)} + (5B^2 - a_1)z_{xx} + 10BB_xz_x \\ + \Big(a_2 + 5B_x^2 + 10BB_{xx} + \frac{15}{2}B^4 - 3a_1B^2\Big)z.\end{aligned} \tag{3.1}$$

This operator is defined acting on functions in $H^2(\mathbb{T})$, $\mathbb{T} = (0, L)$, see (2.14), and it reduces to the standard aperiodic mKdV breather operator (see [6, (4.1)]) as the length of the interval tends to infinity (or $k \to 0$). The constants a_1 and a_2 were introduced in (2.8) and (2.9), and we assume the convention (2.7).

In the following lines, we analyze the spectral stability of the KKSH breather, namely the understanding of the spectrum of $\mathcal{L}_\#$ in (3.1). First, we prove some useful expression for the mass of a breather. As a second step, we compute numerically the spectrum of $\mathcal{L}_\#$, and conclude that it has the desired spectral properties. Then, we analyze which property is the main responsible of the KKSH stability.

3.2. **Mass calculations.** The purpose of this paragraph is to compute the mass of the KKSH breather as a function of k and β. This explicit function will be essential for the study of the nonlinear stability of the solution.

First of all, recall the breather profile in (2.1). Since the mass $M_\#$ in (2.11) is conserved, we can simply assume $t = x_1 = x_2 = 0$ in (2.1)-(2.2). Consider the functions F and G defined in (5.9), and the length of the interval L defined in (2.6). We have from (5.13),

$$\frac{1}{2}\int_0^L B^2 = 4\alpha^2\beta^2\frac{F}{G}(x=L) - 4\alpha^2\beta^2\frac{F}{G}(x=0) - 2\alpha^2\frac{k}{\beta^2}\int_0^L(\beta^2\operatorname{sn}_1^2 - \alpha^2\operatorname{nd}_2^2). \tag{3.2}$$

Now, note that from (2.4) and the periodic character of the involved functions, we have

$$\begin{aligned}\frac{1}{2}\int_0^L B^2 = 4\beta m\frac{\operatorname{cd}(\frac{\beta}{2}L,m)\operatorname{sd}(\frac{\beta}{2}L,m)}{\operatorname{nd}(\frac{\beta}{2}L,m)} - 4\beta m\frac{\operatorname{cn}(\frac{\beta}{2}L,m)\operatorname{sn}(\frac{\beta}{2}L,m)}{\operatorname{dn}(\frac{\beta}{2}L,m)} \\ - 8\alpha K(k) + 4\alpha E(A(2K(k),k),k) + 4\beta E\big(A\big(\frac{\beta}{2}L,m\big),m\big),\end{aligned} \tag{3.3}$$

where E denotes the *complete elliptic integral of the second kind* [13], defined as

$$E(r) := \int_0^{\pi/2}(1 - r\sin^2(s))^{1/2}ds = \int_0^1(1-t^2)^{-1/2}(1-rt^2)^{1/2}dt, \tag{3.4}$$

and A is the *Jacobi amplitude*, that can be defined by

$$A(x,r) := \int_0^x dn(s,r)ds. \tag{3.5}$$

Some simple exact values for E are $E(0) = \pi/2$ and $E(1) = 1$, and for A are $A(0,r) = 0$, $A(K(r),r) = \frac{\pi}{2}$. Note also that $E(A(2K(k),k),k) = E(A(0,k) + \pi, k) = 2E(k)$. Using (2.4), (3.3) simplifies as follows:

$$\frac{1}{2}\int_0^L B_0^2 = 4\beta E(m) + 8\alpha E(k) - 8\alpha K(k). \tag{3.6}$$

We are able to go one step further in the simplification of (3.6), using (2.4) again. We have

$$\frac{k}{1-m} = \frac{1}{16}\frac{K(m)^4}{K(k)^4}. \tag{3.7}$$

Hence, with these relations, (3.6) simplifies to the compact expression:

$$M_{\#}[B] = \frac{1}{2}\int_0^L B_0^2 = 4\beta\Big(E(m) + 4\frac{K(k)}{K(m)}(E(k)-K(k))\Big). \tag{3.8}$$

When $k \to 0$, we have $m \to 1$, and $M_{\#}[B] \to 4\beta$, which is the value of the mass of the aperiodic mKdV breather solution in the real line (see [6, p.6, Lemma 2.1]).

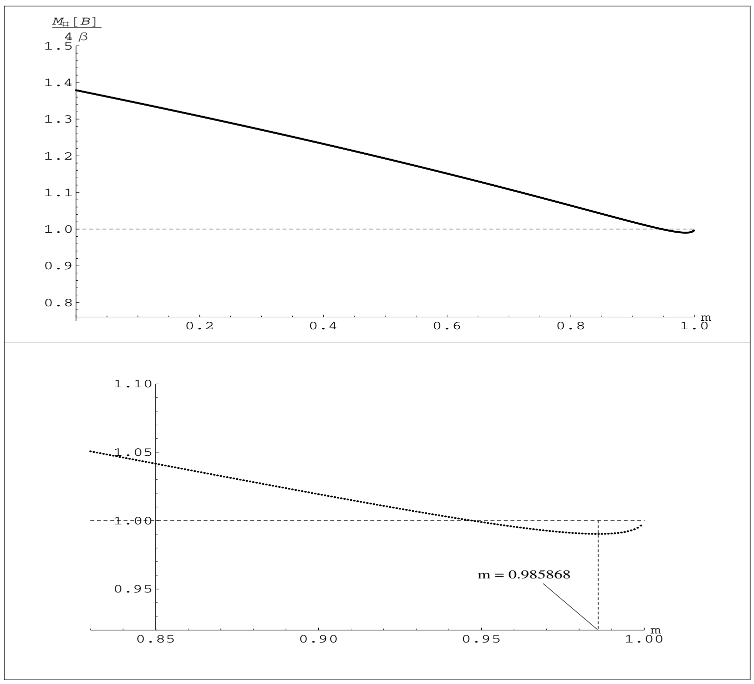

FIGURE 3. Mass of the periodic breather $M_{\#}[B]/4\beta$ as a function of m. Note that the resulting function is decreasing except for $m \gtrsim 0.98$ (see zoom figure below), corresponding to $k \lesssim 0.025$, a parameter region very close to the stable, aperiodic mKdV breather.

For $k \neq 0, m \neq 1$, the dependence of the periodic mass (3.8) with respect to the parameter m is computed in the following way: for each value of m, we solve numerically the implicit equation (3.7) in k. We then substitute these two pairs of values (m,k) verifying (3.7) inside the expression (3.8). The resulting plot of (3.8) versus m is given in Figure 3.

3.3. **Numerical analysis.** Given the complicated functions that define the KKSH breather and its related linearized operator, a rigorous description of the spectra of $\mathcal{L}_{\#}$ (3.1) has escaped to us. We perform then some numerical simulations to get a good understanding of these spectral properties. We could expect, given the instability results in [19], that the KKSH breather should give rise to a very different spectrum, with respect to the previous aperiodic breathers. However, as we will show later, this is not the case.

In the following lines we explain the main ideas of the numerical method used to compute the eigenvalues of $\mathcal{L}_{\#}$. The core of the algorithm is the same as in the previous paper [9], the main difference being the required test functions, which now must be periodic on $[0, L]$. We have used the classical orthonormal basis of $L^2(0, L)$:

$$\Big\{\frac{1}{\sqrt{L}},\ \sqrt{\frac{2}{L}}\cos\Big(\frac{2\pi n x}{L}\Big),\ \sqrt{\frac{2}{L}}\sin\Big(\frac{2\pi n x}{L}\Big)\Big\},$$

where $n \in \{1, \dots, N\}$. For the standard numerical computations, it is sufficient to take $N = 40$, although when approaching the critical values $k \to k^*$ $(m \to 0)$ or $k \to 0$ $(m \to 1)$, more and more test functions are naturally required.

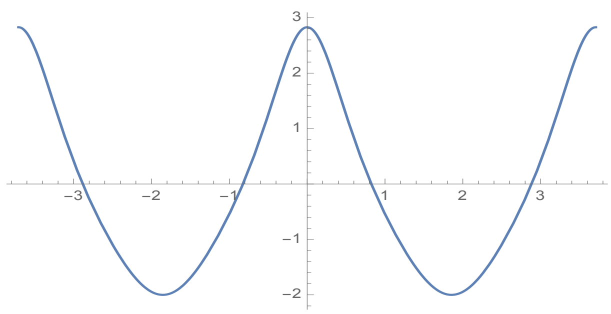

FIGURE 4. Periodic KKSH breather [19] at time $t = 0$ for $\beta = 1$, $m = 0.5$, $k = 0.057$, here $L \sim 3.71$. Although numerically unstable [19], this breather leads to a linearized operator $\mathcal{L}_{\#}$ that possesses a "standard" one negative eigenvalue and a two dimensional kernel (3.9).

First of all, we run a specific computation for the case described in [19], which the authors present as numerically unstable. In this case, $\beta = 1$, $m = 0.5$, $k = 0.057$, and $L \sim 3.71$ (see Figure 4). Numerically, we have found that this breather possesses the same spectral structure of all our previous breather solutions. To be more precise, running our numerical algorithm with $N = 40$ test functions, we have found the following approximations of the first four eigenvalues of $\mathcal{L}_{\#}$:

$$\{-4.86,\ -1.23 \times 10^{-8},\ 3.22 \times 10^{-10},\ 35.35\}. \tag{3.9}$$

Clearly the two components of the kernel are recovered with high precision (recall that a second negative eigenvalue but very close to zero is unlikely just by continuity arguments on the coefficients of the original breather), and a distinctive negative eigenvalue appears, far from the kernel itself. This behavior repeats for all cases we have studied.

As a second test, we perform several eigenvalue computations for the same parameter $\beta = 1$, and k moving. For the most difficult case, the one where k approaches the critical value ~ 0.0588, we obtain the results described in Figure 5. Also, in Figure 1, we exactly describe those eigenvalues, that we obtain for different

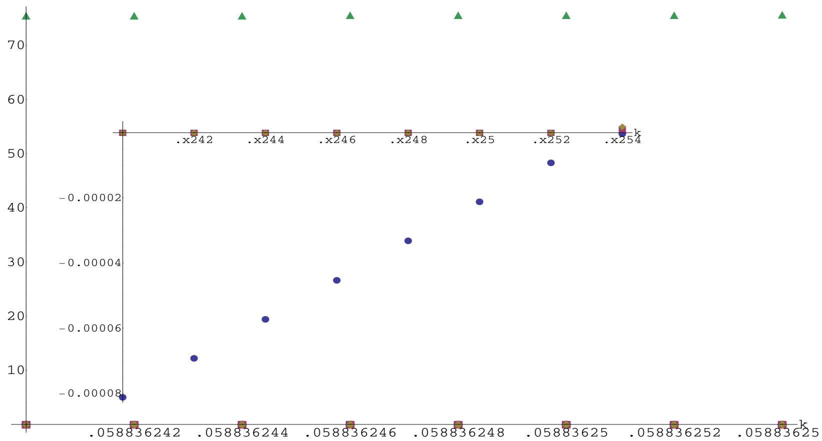

FIGURE 5. The double zero kernel (purple box and brown diamond) and the first positive eigenvalue (green triangle) of the linearized operator $\mathcal{L}_\#$ around a periodic KKSH breather for $N = 40$, $\beta = 1$ and k increasing. Note that the numerical method returns only two eigenvalues very close to zero, as expected from the conjectured linear spectral stability. Inside, the representation of the negative (blue circle) and double zero kernel. We used the notation $.x242 \equiv 0.058836242$, $x = 058836$.

values of k. It is important to mention that we always get one negative eigenvalue and a two dimensional kernel, as well as a clearly defined spectral gap.

k	1st. eig	2nd	3rd	4th
$.x240$	-0.00008	$-7.750 \cdot 10^{-10}$	$-1.565 \cdot 10^{-9}$	75.490
$.x242$	-0.00006	$-6.449 \cdot 10^{-10}$	$-1.561 \cdot 10^{-9}$	75.502
$.x244$	-0.00005	$-5.952 \cdot 10^{-10}$	$-1.558 \cdot 10^{-9}$	75.514
$.x246$	-0.00004	$-6.333 \cdot 10^{-10}$	$-1.554 \cdot 10^{-9}$	75.528
$.x248$	-0.00003	$-6.830 \cdot 10^{-10}$	$-1.552 \cdot 10^{-9}$	75.544
$.x250$	-0.00002	$-6.160 \cdot 10^{-10}$	$-1.546 \cdot 10^{-9}$	75.563
$.x252$	$-9.189 \cdot 10^{-6}$	$-6.774 \cdot 10^{-10}$	$-1.541 \cdot 10^{-9}$	75.589
$.x254$	$-3.463 \cdot 10^{-7}$	$-1.135 \cdot 10^{-6}$	$-1.869 \cdot 10^{-6}$	75.640

TABLE 1. The first four eigenvalues of $\mathcal{L}_\#$ for $\beta = 1$, $x_1 = 0.1$, $x_2 = 0$, and k varying from $.x240 \equiv 0.058836240$, $x = 058836$ to $.x254$, as in Figure 5. All computations were made with $N = 40$ test functions. The third and fourth columns represent approximate kernel of $\mathcal{L}_\#$.

In the intermediate case, for pairs (k, m), with $k = 0.01, \dots, 0.04$, we obtain the results described in Figure 6. Also, in the Table 2, we describe the eigenvalues of $\mathcal{L}_\#$ that we obtain for different values of k. Once again, we obtain one negative eigenvalue and a two dimensional kernel.

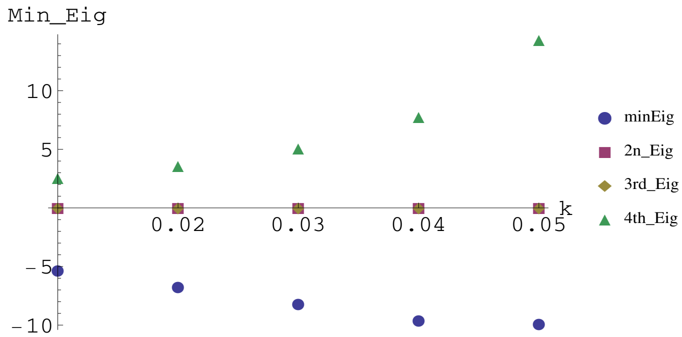

FIGURE 6. For intermediate values of k: we plot the negative eigenvalue, the double zero kernel and the fourth eigenvalue of the linearized operator $\mathcal{L}_{\#}$ around a periodic KKSH breather, for $\beta = 1$ and k increasing from 0.01 to 0.05, as expressed before in Table 2. Note that the numerical method returns only two eigenvalues very close to zero, as expected from the conjectured linear spectral stability. Computations were made with $N = 50$ test functions.

k	1st eig.	2nd	3rd	4th
0.01	-5.343	$1.023 \cdot 10^{-6}$	0.0001	2.540
0.02	-6.751	$5.905 \cdot 10^{-10}$	$2.683 \cdot 10^{-6}$	3.561
0.03	-8.216	$-2.651 \cdot 10^{-9}$	$1.163 \cdot 10^{-6}$	5.067
0.04	-9.623	$3.997 \cdot 10^{-9}$	$4.896 \cdot 10^{-8}$	7.756
0.05	-9.922	$-2.173 \cdot 10^{-7}$	$-1.143 \cdot 10^{-8}$	14.329

TABLE 2. The first four eigenvalues of $\mathcal{L}_{\#}$ for $\beta = 1$, $x_1 = 0.1$, $x_2 = 0$, and k varying, as corresponding to Figure 6. All computations were made with $N = 50$ test functions. The third and fourth columns represent the approximate kernel of $\mathcal{L}_{\#}$.

Finally, for the case, with k small, i.e. $k \approx 10^{-3}$, we obtain in Figure 7 the description of the discrete spectra for the first four eigenvalues for the linearized operator $\mathcal{L}_{\#}$. Also, in the Table 3, we show the explicit numerical eigenvalues which correspond to these small values of k. Once again, we obtain one negative eigenvalue and a two dimensional kernel.

Therefore, we can certainly claim that there is strong evidence that, according to numerical simulations, KKSH breathers have standard spectrum, in the sense that they have only one negative eigenvalue, and a nondegenerate, two-dimensional kernel. For clarity reasons, we present a rigorous statement of both properties for the case of the periodic KKHS breather.

- (A4) (Nondegeneracy of the kernel) For each $k \in (0, k^*)$, $x_1, x_2 \in \mathbb{R}$ and $\beta > 0$, $\ker \mathcal{L}_{\#}$ is spanned by the two elements $\partial_{x_1} B$ and $\partial_{x_2} B$; and there is a uniform gap between the kernel and the bottom of the positive spectrum;
- (A5) (Unique, simple negative eigenvalue) For each $k \in (0, k^*)$, $x_1, x_2 \in \mathbb{R}$ and $\beta > 0$, the operator $\mathcal{L}_{\#}$ has a unique simple, negative eigenvalue

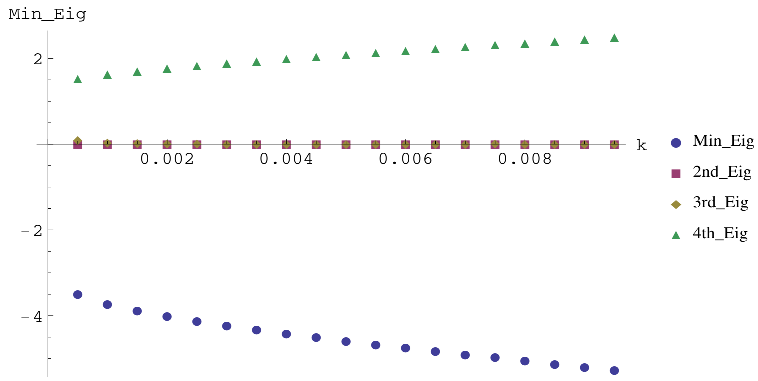

FIGURE 7. For k (*small*): the negative eigenvalue, the double zero kernel and the fourth eigenvalue for the linearized operator $\mathcal{L}_{\#}$ around a periodic KKSH breather, for $\beta = 1$ and k increasing from 0.0005 to 0.0095, as expressed before in Table 3. Note that the numerical method returns only two eigenvalues very close to zero, as expected from the conjectured linear spectral stability. Computations were made with $N = 50$ test functions.

k	1st eig.	2nd$\cdot 10^{-6}$	3rd	4th
0.0095	-5.271	1.265	0.0001	2.495
0.0085	-5.127	1.962	0.0002	2.405
0.0075	-4.971	1.248	0.0142	2.316
0.0065	-4.828	5.092	0.0005	2.225
0.0055	-4.671	8.609	0.0009	2.134
k	1st eig.	2nd$\cdot 10^{-5}$	3rd	4th
0.0045	-4.505	1.531	0.0017	2.040
0.0035	-4.327	2.919	0.0034	1.941
0.0025	-4.127	6.234	0.0073	1.833
0.0015	-3.883	16.47	0.0198	1.708
0.0005	-3.497	79.70	0.0983	1.530

TABLE 3. The first four eigenvalues for the linearized operator $\mathcal{L}_{\#}$ around a periodic KKSH breather for $\beta = 1$, $x_1 = 0.1$, $x_2 = 0$, and k varying in a sample of points of Figure 7. All computations were made with $N = 50$ test functions. The third and fourth columns represent respectively the approximate kernel of $\mathcal{L}_{\#}$.

$\lambda_1 = \lambda_1(\beta, k, x_1, x_2) < 0$ associated to the unit L^2-norm eigenfunction B_{-1}. Moreover, there is $\lambda_1^0 < 0$ depending on β and k only, such that $\lambda_1 \leq \lambda_1^0$ for all x_1, x_2.

3.4. **Duality stability/instability.** Now the main problem is to figure out where our nonlinear stability proof does/does not work. For this purpose, the *discriminant*

$$D = D(\beta, k) := \partial_k a_1 \partial_\beta a_2 - \partial_k a_2 \partial_\beta a_1 \quad (a_1, a_2 \text{ from } (2.8) - (2.9)), \tag{3.10}$$

is the key element to check.

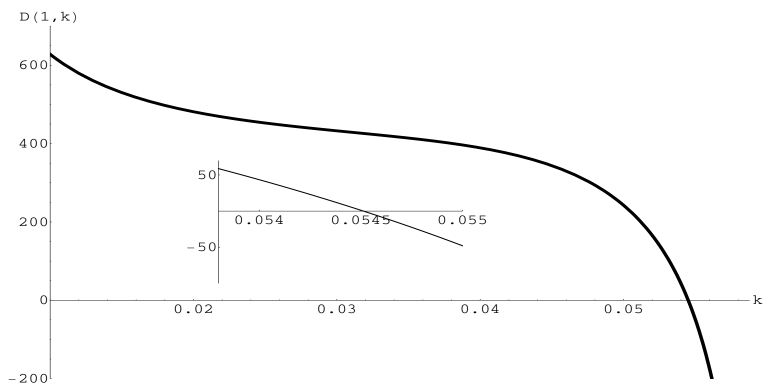

FIGURE 8. Discriminant function $D(1,k)$ in terms of k, as described in (3.10). Note that $D(1,k)$ changes sign (see zoom figure inside) for $k \geq k_* \sim 0.0545$.

To explain why this element is important, let us notice that from (2.10) and (3.1), we readily have (compare with [9, Corollary (3.7)])

$$\mathcal{L}_\#(\partial_k B) = \partial_k a_1(B_{xx} + B^3) - \partial_k a_2 B,$$
$$\mathcal{L}_\#(\partial_\beta B) = \partial_\beta a_1(B_{xx} + B^3) - \partial_\beta a_2 B.$$

Therefore, as soon as $D \neq 0$,

$$B_{0,\#} := \frac{1}{D}(\partial_k a_1 \partial_\beta B - \partial_\beta a_1 \partial_k B), \tag{3.11}$$

satisfies the equation

$$\mathcal{L}_\#(B_{0,\#}) = -B,$$

see also [6, Corollary 4.5]. Using this fact, we can easily prove, as in [9, Proposition (3.11)], that the eigenfunction associated to the negative eigenvalue of $\mathcal{L}_\#$ can be replaced by the breather itself, which has better behavior in terms of error controlling, unlike the first eigenfunction. This simple fact allows us to prove the nonlinear stability result as in the standard approach, without using scaling modulations. Recall that using the first eigenfunction as orthogonality condition does not guaranty a suitable control on the scaling modulation parameter, because the control given by this direction might be not good enough to close the stability estimates. However, the breather can be used as an alternative direction, and all these previous arguments remain valid, exactly as in [6], provided the Weinstein's sign condition

$$\int_0^L B_{0,\#} B > 0 \quad \big(\text{or equivalently } \int_0^L B_{0,\#\mathcal{L}_\#[B_{0,\#}]<0}\big), \tag{3.12}$$

do hold. Using (3.11), we are lead to the understanding of the quantity

$$\begin{aligned}\int_0^L B_{0,\#} B &= \frac{1}{D}\int_0^L (\partial_k a_1 \partial_\beta B - \partial_\beta a_1 \partial_k B) B \\ &= \frac{1}{D}(\partial_k a_1 \partial_\beta M_\#[B] - \partial_\beta a_1 \partial_k M_\#[B]) =: HG(\beta, k),\end{aligned} \tag{3.13}$$

where $M_\#[B]$ was computed in (3.8). Recall that a_1 and a_2 are almost explicit from (2.8)-(2.9). An exact expression for $HG(\beta,k)$ has escaped to us, however, we can graph this new function in some interesting cases. In particular, for the case considered in [19], we assume $\beta = 1$ and we graph $D = D(1,k)$ and $HG(1,k)$, to obtain the results in Figure 8 and Figure 9. We note that condition (3.12) holds provided k is small enough. However, the values for which $HG(\beta,k) > 0$ do not coincide with the values for which the standard Weinstein's condition (positive derivative with respect to the scaling), deduced from Figure 3, holds true, and this is totally natural for the case of breathers, as it was explained in [6, Corollary 2.2].

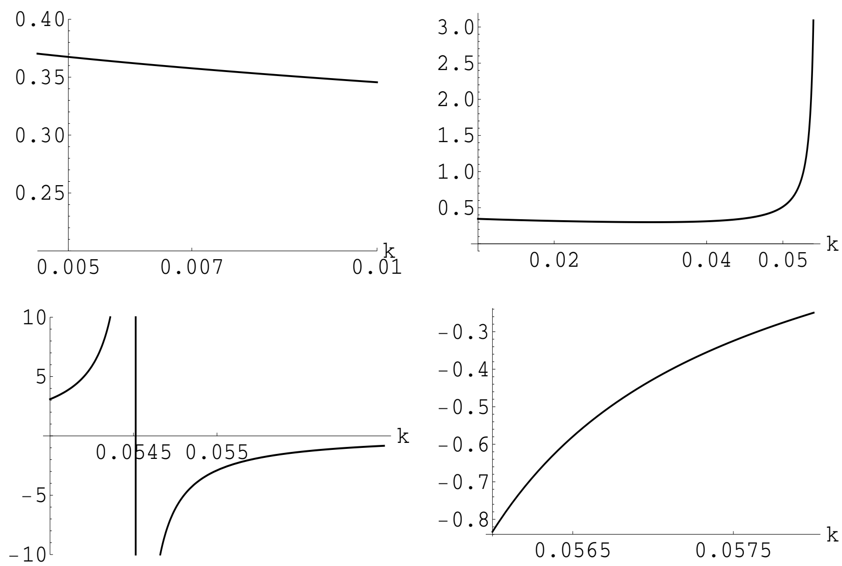

FIGURE 9. Weinstein's condition $HG(\beta,k)$ (3.13), for the periodic KKSH breather, in the case $\beta = 1$, for $k \in [0.0045, 0.01]$ (above left), for $k \in [0.01, 0.054]$ (above right), $k \in [0.054, 0.056]$ (below left), and $k \in [0.056, 0.058]$ (below right). In order to run our argument for a stability proof [6], we require $HG(1,k) > 0$, which is only satisfied for $k < k_*$, where $k_* \sim 0.0545$ is the approximate point where $D(1,k)$ vanishes (see Figure 8). Note also that the case $k \sim 0.057$ assumed in [19], that leads to instability, is not included in the stability region described in Theorem 3.1, but in the region where $HG(1,k) < 0$.

Now we are ready to fully state assumption (A3) described in the introduction of this paper.

(A6) (Positive generalized Weinstein's condition) The following generalized Weinstein's type sign condition is satisfied: if $M_\#[B]$ is the mass of the breather solution (3.8) in terms of β and k, and a_1, a_2 are the variational parameters given in (2.8)-(2.9), then

$$HG(\beta,k) = \frac{1}{D}(\partial_k a_1 \partial_\beta M_\#[B] - \partial_\beta a_1 \partial_k M_\#[B]) > 0. \tag{3.14}$$

Now we can present a rigorous statement for Theorem 1.2.

Theorem 3.1. *Under assumptions (A4)–(A6), KKHS breathers are orbitally stable under small, L-periodic H^2 perturbations. More precisely, there are $\eta_0 > 0$ and $K_0 > 0$, only depending on β and k, such that if $0 < \eta < \eta_0$ and if u_0 is L-periodic with*

$$\|u_0 - B(t = 0, \cdot; 0, 0)\|_{H^2(\mathbb{T})} < \eta,$$

then there are real-valued parameters $x_1(t)$ and $x_2(t)$ for which the global H^2, L-periodic solution $u(t)$ of (1.1) with initial data u_0 satisfies

$$\sup_{t\in\mathbb{R}} \|u(t) - B(t, \cdot; x_1(t), x_2(t))\|_{H^2(\mathbb{T})} < K_0\eta,$$

with similar estimates for the derivatives of the shift parameters x_1, x_2.

Proof. The proof of this result is completely similar to the proof of [9, Theorem (2.5)], after following the same steps (see also [6] for a proof in the scalar mKdV case). Assumption (A3) is used to ensure that an expression like [9, eq. (3.30)] has a nonzero denominator. □

Additionally, we conjecture the following alternative stability theory for KKSH breathers:

Conjecture 3.2. *Assume that $HG(\beta, k) < 0$. Then B is unstable under small $H^2(\mathbb{T})$ perturbations.*

Examples of unstable structures that have a nondegenerate kernel and only one negative eigenvalue are solitons for the nonlinear Klein-Gordon equation in $\mathbb{R}_t \times \mathbb{R}^d_x$:

$$u_{tt} - \Delta u + u + u^p = 0, \quad p > 1.$$

Note that this result cannot be deduced from the Grillakis-Shatah-Strauss method, since breathers are not simple solitary waves. Here, unlike our case, the lack of stability is related to the absence of a scaling symmetry controlling the negative direction (appearing because of the negative eigenvalue). Another type of instability result suggested very recently is motivated by the existence of an stable "internal mode" which triggers a nonlinear unstable dynamics in NLS and discrete models. See the works [15, 20] for further reading and more references.

4. Periodic mKdV breathers with nonzero mean

4.1. **Introduction.** Although KKSH breathers are periodic in space, the are still zero-mean solutions. We would like to see if nonzero mean periodic breather solutions may exist. (Non periodic and nonzero mean breather solutions of mKdV were already known, see [3, 5].) By periodic breather we refer to the object in Definition 1.1, that is, any solution that is periodic in time and space, having two independent and different space variables (i.e. not being a one profile solution being translated over time, as solitons are), and finally, having oscillatory behavior, unlike standard 2-soliton solutions. Numerical evidence of the existence of these solutions was given by the first author in [5], since these solutions are connected with the (nonzero) curvature of the planar curve evolving according to a particular law. For more details about this connection, see e.g. [5].

For the case of the mKdV equation, we have been able to obtain a new set of periodic breather solutions of (1.1) *with nonzero mean.* More explicitly,

Definition 4.1. Given $c_1, c_2, \mu > 0$, p, q nonzero integers, with p, q coprime, such that the following *commensurability* condition is satisfied[1]

$$\frac{2\mu^2 - c_1}{2\mu^2 - c_2} = \frac{p^2}{q^2}, \tag{4.1}$$

we define the breather $B = B(t, x; c_1, c_2, \mu, p, q)$ by the formula

$$B := \mu + 2\sqrt{2}\partial_x \arctan\Big(\frac{f(t,x)}{g(t,x)}\Big), \tag{4.2}$$

where

$$\rho := \frac{\sqrt{c_1} + \sqrt{c_2}}{\sqrt{c_1} - \sqrt{c_2}},$$

$$f(t,x) := -\sqrt{2}\mu\rho\Big(\sqrt{c_1} - \sqrt{c_2} + \sqrt{2\mu^2 - c_2}\tan y_2 - \sqrt{2\mu^2 - c_1}\tan y_1\Big),$$

$$g(t,x) := 2\mu^2 + \Big(\sqrt{2\mu^2 - c_1}\tan y_1 - \sqrt{c_1}\Big)\Big(\sqrt{2\mu^2 - c_2}\tan y_2 - \sqrt{c_2}\Big).$$

Here

$$y_1 = \frac{1}{2}\sqrt{2\mu^2 - c_1}(x - \delta t), \quad y_2 = \frac{1}{2}\sqrt{2\mu^2 - c_2}(x - \gamma t), \tag{4.3}$$

and the speeds are

$$\delta := \mu^2 + c_1, \quad \gamma := \mu^2 + c_2.$$

Note that condition (4.1) is imposed in order to obtain a truly periodic solution, see the formulae for f and g. The spatial period of this breather is given by $L = \frac{2\pi q}{\sqrt{2\mu^2 - c_1}}$. See also Figure 10-11 for some drawings of different breather solutions, depending on the parameters c_1, c_2, μ, p and q.

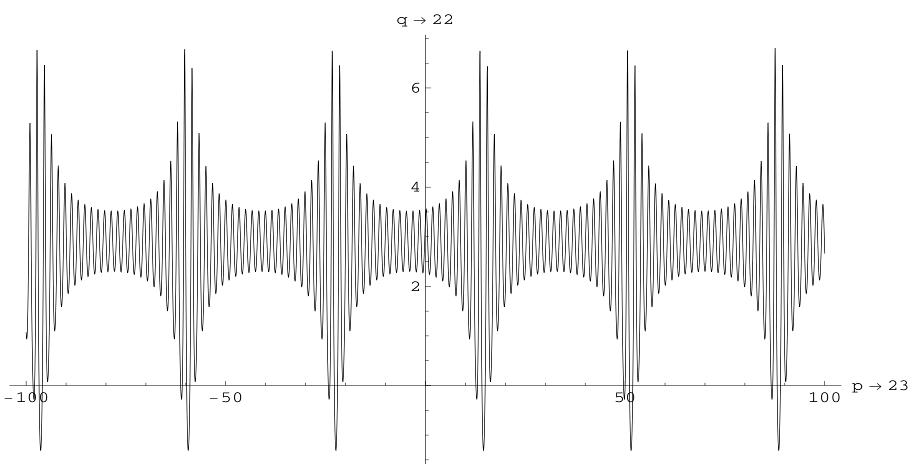

FIGURE 10. Periodic breather of (1.1) with non zero mean. The parameters here are $c_1 = 1.65$, $c_2 = 2.95$, $p = 23$, $q = 22$, and the period L is ~ 35.7

Recall that KKSH breathers cannot have any admissible set of parameters k and m; they are constrained by conditions in (2.4). Here, the only condition that we

[1] Note that each c_i is less than $2\mu^2$.

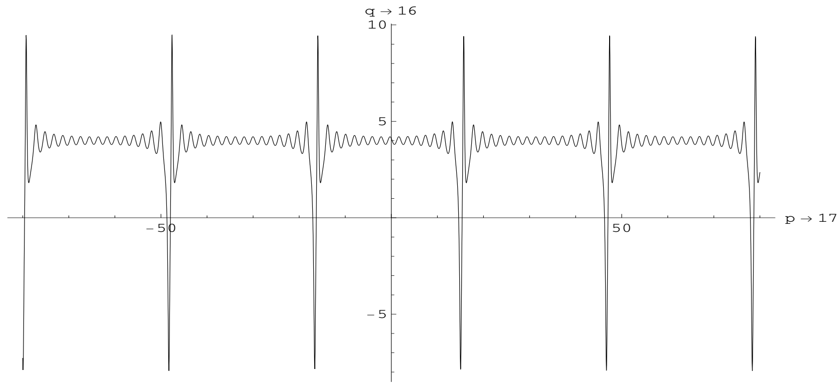

FIGURE 11. Periodic breather of (1.1) with non zero mean, for parameters $c_1 = 20.65$, $c_2 = 21.95$, $p = 17$, $q = 16$.

need to satisfy is (4.1). Therefore, given nonnegative integers p and q, with p and q coprime, and given $c_1 < 2\mu^2$, then there is a unique c_2 solution to (4.1). In that sense, μ being fixed, the breather in (4.2) has three different degrees of freedom (in addition to shifts), two of them being discrete. Finally, in terms of p and q, the larger these integers are, the more oscillatory the breather solution is. Note additionally that this breather has not been constructed by using Jacobi functions, but only standard periodic functions, a fact that simplifies many computations.

It is also relevant to mention that the construction of a breather solution with different patterns as the usually required can be very involved. For example, a different class of periodic breather was discovered by Blank et al. in [10]. This particular solution is constructed for the so called ϕ^4 model, by assuming that the equation is no longer autonomous, but it has suitable, well-chosen periodic coefficients.

4.2. **Sketch of proof of Theorem 1.3.** We will use the Bäcklund Transformation for mKdV (see [8] for a rigorous setting of the computations below) to construct this solution. Given a solution u_0 of the form $u_0 = \partial_x \widetilde{u}_0$ of mKdV and fixed constants $a_1, a_2 \in \mathbb{R}$, $a_1 \neq a_2$, we can construct a second $u_1 = \partial_x \widetilde{u}_1$ and third solution $u_2 = \partial_x \widetilde{u}_2$ by setting

$$u_1 - u_0 = a_1 \sin\Big(\frac{\widetilde{u}_1 + \widetilde{u}_0}{\sqrt{2}}\Big), \quad \text{and} \quad u_2 - u_0 = a_2 \sin\Big(\frac{\widetilde{u}_2 + \widetilde{u}_0}{\sqrt{2}}\Big).$$

These two solutions can be combined to construct a fourth solution u through the *permutability* condition [8]:

$$\tan\Big(\frac{\widetilde{u} - \widetilde{u}_0}{2\sqrt{2}}\Big) = -\Big(\frac{a_1 + a_2}{a_1 - a_2}\Big) \tan\Big(\frac{\widetilde{u}_2 - \widetilde{u}_1}{2\sqrt{2}}\Big). \tag{4.4}$$

Fix $\mu > 0$. Starting with the constant solution $u_0 = \mu$, $\widetilde{u}_0 = \mu x$ and two constants $a_1 = \sqrt{2c_1}$, $a_2 = \sqrt{2c_2}$, $c_1, c_2 > 0$, we have

$$u_1 - \mu = \sqrt{2c_1} \sin\Big(\frac{\tilde{u}_1 + \mu x}{\sqrt{2}}\Big), \quad u_2 - \mu = \sqrt{2c_2} \sin\Big(\frac{\tilde{u}_2 + \mu x}{\sqrt{2}}\Big). \tag{4.5}$$

Looking for a solution of the form $u_i = \partial_x \tilde{u}_i$, $i = 1, 2$, we obtain

$$\begin{aligned} \widetilde{u}_i(t,x) = -\mu x + 2\sqrt{2}\arctan\Big(\frac{1}{2\mu}\Big(- \sqrt{2c_i} \\ + \sqrt{4\mu^2 - 2c_i}\tan\Big(\frac{\sqrt{4\mu^2-2c_i}}{2\sqrt{2}}(x-(\mu^2+c_i)t)\Big)\Big)\Big). \end{aligned} \tag{4.6}$$

The factor $(\mu^2 + c_i)t$ in the above solution appears as a constant of integration, and it is chosen in such a form that u_i is actually solution to mKdV. Calling $y_1 = x + \delta t$, $y_2 = x + \gamma t$ as in (4.3), and $\rho = \frac{\sqrt{c_1}+\sqrt{c_2}}{\sqrt{c_1}-\sqrt{c_2}}$, the desired breather (4.2) is obtained by using (4.4) with $u_0 = \mu$, u_1, u_2 from (4.6):

$$B = \mu + 2\sqrt{2}\partial_x \arctan\Big(\frac{f(t,x)}{g(t,x)}\Big),$$

where,

$$\begin{aligned} f(t,x) &:= -\sqrt{2}\mu\rho\Big(\sqrt{c_1} - \sqrt{c_2} + \sqrt{2\mu^2 - c_2}\tan y_2 - \sqrt{2\mu^2-c_1}\tan y_1\Big), \\ g(t,x) &:= 2\mu^2 + \Big(\sqrt{2\mu^2-c_1}\tan y_1 - \sqrt{c_1}\Big)\Big(\sqrt{2\mu^2-c_2}\tan y_2 - \sqrt{c_2}\Big). \end{aligned}$$

The fact that this is a solution of mKdV is a tedious, lengthy but straightforward computation.

4.3. **Final comments.** A detailed study of this new breather solution will be done elsewhere. For the moment, we advance that B (4.2) satisfies the elliptic ODE

$$\begin{aligned} &B_{4x} - (c_1 + c_2 - 4\mu^2)(B_{xx} + 3\mu(B-\mu)^2 + (B-\mu)^3) \\ &+ (c_1 - 2\mu^2)(c_2 - 2\mu^2)(B-\mu) + 5(B-\mu)B_x^2 + 5(B-\mu)^2 B_{xx} \\ &+ \frac{3}{2}(B-\mu)^5 + 5\mu B_x^2 + \frac{15}{2}\mu(B-\mu)^4 + 10\mu(B-\mu)B_{xx} + 10\mu^2(B-\mu)^3 = 0. \end{aligned}$$

In particular, B is a critical point of the functional

$$\mathcal{H}_{\mu p}[w](t) := F_{\mu p}[w](t) + (c_1 + c_2 - 4\mu^2)E_{\mu p}[w](t) + (c_1 - 2\mu^2)(c_2 - 2\mu^2)M_{\mu p}[w](t),$$

where $M_{\mu p}$ and $E_{\mu p}$ are defined as follows:

$$M_{\mu p}[w](t) := \frac{1}{2}\int_{\mathbb{R}} (w-\mu)^2(t,x)dx = M_{\mu p}[w](0), \tag{4.7}$$

$$E_{\mu p}[w](t) := \frac{1}{2}\int_{\mathbb{R}} w_x^2 - \mu\int_{\mathbb{R}}(w-\mu)^3 - \frac{1}{4}\int_{\mathbb{R}}(w-\mu)^4 = E_{\mu p}[w](0), \tag{4.8}$$

$$\begin{aligned} F_{\mu p}[w](t) := \frac{1}{2}\int_{\mathbb{R}} w_{xx}^2 dx - 5\mu\int_{\mathbb{R}}(w-\mu)w_x^2\,dx + \frac{5}{2}\mu^2\int_{\mathbb{R}}(w-\mu)^4 dx \\ - \frac{5}{2}\int_{\mathbb{R}}(w-\mu)^2 w_x^2 dx + \frac{3\mu}{2}\int_{\mathbb{R}}(w-\mu)^5 + \frac{1}{4}\int_{\mathbb{R}}(w-\mu)^6 dx \end{aligned} \tag{4.9}$$

is a third conserved quantity for the mKdV equation. Finally, it is interesting to note that we can recover the aperiodic breather with nonzero mean (see [3, 4]) choosing in (4.2) complex conjugate scalings $\sqrt{c_1} = \beta + i\alpha$, $\sqrt{c_2} = \beta - i\alpha$. For the sake of simplicity, we only show a picture of it in Figure 12.

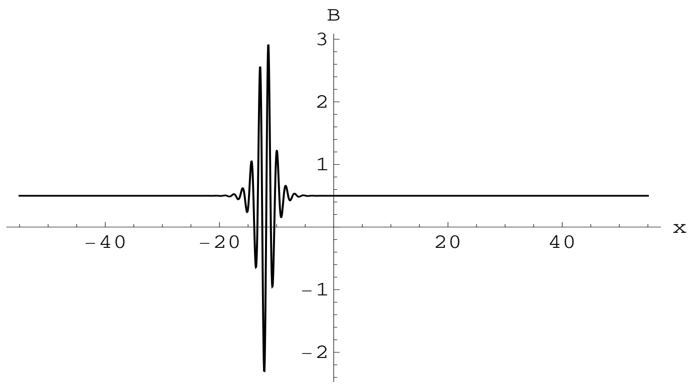

FIGURE 12. Aperiodic mKdV breather with non zero mean at $t = 0.2$ with $\beta = 1$, $\alpha = 4$, $\mu = 0.5$, obtained from the periodic breather (4.2).

5. Appendix: Proof of Theorem 2.1

Let B be a periodic KKSH breather. Without loss of generality, we can assume $x_1 = x_2 = 0$, and after taking time derivative we assume $t = 0$, since (1.1) is invariant under space and time translations, as well as (2.1). Recall that from (2.1)

$$\tilde{B}_t := \delta\tilde{B}_1 + \gamma\tilde{B}_2,$$

where $\tilde{B}_j := \partial_{x_j}\tilde{B}$. We also have [2, 13]:

$$\operatorname{sn}'(s,k) = \operatorname{cn}(s,k)\operatorname{dn}(s,k), \quad \Big(\operatorname{dn}(s,k) := \frac{1}{\operatorname{nd}(s,k)}\Big),$$
$$\operatorname{cn}'(s,k) = -\operatorname{sn}(s,k)\operatorname{dn}(s,k), \quad \operatorname{dn}'(s,k) = -k\operatorname{sn}(s,k)\operatorname{cn}(s,k),$$

and $\operatorname{cn}(0,k) = \operatorname{dn}(0,k) = 1$, $\operatorname{sn}(0,k) = 0$.

We start with some notation. Let

$$\operatorname{sn}_1 := \operatorname{sn}(\alpha y_1,k), \quad \operatorname{cn}_1 := \operatorname{cn}(\alpha y_1,k), \quad \operatorname{dn}_1 := \operatorname{dn}(\alpha y_1,k);$$
$$\operatorname{sn}_2 := \operatorname{sn}(\beta y_2,m), \quad \operatorname{cn}_2 := \operatorname{cn}(\beta y_2,m), \quad \operatorname{dn}_2 := \operatorname{dn}(\beta y_2,m).$$

We have

$$\tilde{B} = 2\sqrt{2}\arctan\Big(\frac{\beta}{\alpha}\operatorname{sn}_1\operatorname{dn}_2\Big).$$

Define

$$h := \alpha\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2 - \beta m\operatorname{sn}_1\operatorname{sn}_2\operatorname{cn}_2, \quad \tilde{h} := \alpha\delta\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2 - \beta m\gamma\operatorname{sn}_1\operatorname{sn}_2\operatorname{cn}_2, \tag{5.1}$$

so that

$$\begin{aligned} h_x = -\Big[&\alpha^2\operatorname{sn}_1\operatorname{dn}_2(k\operatorname{cn}_1^2 + \operatorname{dn}_1^2) + 2\alpha\beta m\operatorname{cn}_1\operatorname{dn}_1\operatorname{sn}_2\operatorname{cn}_2 \\ &+ \beta^2 m\operatorname{sn}_1\operatorname{dn}_2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)\Big], \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} h_t = -\Big[&\alpha^2\delta\operatorname{sn}_1\operatorname{dn}_2(k\operatorname{cn}_1^2 + \operatorname{dn}_1^2) + \alpha\beta m(\delta+\gamma)\operatorname{cn}_1\operatorname{dn}_1\operatorname{sn}_2\operatorname{cn}_2 \\ &+ \beta^2 m\gamma\operatorname{sn}_1\operatorname{dn}_2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)\Big]. \end{aligned} \tag{5.3}$$

Similarly,

$$g := \alpha^2 + \beta^2 \operatorname{sn}_1^2 \operatorname{dn}_2^2, \quad g_x = 2\beta^2 \operatorname{sn}_1 \operatorname{dn}_2(\alpha \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2), \tag{5.4}$$

$$g_t = 2\beta^2 \operatorname{sn}_1 \operatorname{dn}_2(\alpha\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m\gamma \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2) = 2\beta^2 \operatorname{sn}_1 \operatorname{dn}_2 \tilde{h}. \tag{5.5}$$

Then

$$B = \frac{2\sqrt{2}\alpha\beta h}{g}, \quad B(x=0) = 2\sqrt{2}\beta,$$

$$\tilde{B}_t = \delta\tilde{B}_1 + \gamma\tilde{B}_2 = \frac{2\sqrt{2}\alpha\beta\tilde{h}}{g}, \quad \tilde{B}_t(x=0) = 2\sqrt{2}\beta\delta.$$

Similarly,

$$B_x = \frac{2\sqrt{2}\alpha\beta\hat{h}}{g^2}, \quad \hat{h} := h_x g - g_x h. \tag{5.6}$$

It is not difficult to check that $h_x(x=0) = g_x(x=0) = 0$. In particular $B_x(x = 0) = 0$. From this identity we see that

$$B_{xx}(x=0) = \frac{2\sqrt{2}\alpha\beta\hat{h}_x}{g^2}(x=0) = -2\sqrt{2}\beta[(2+3m)\beta^2 + (1+k)\alpha^2].$$

First of all, from (1.1) and (2.3) we have

$$\tilde{B}_t + B_{xx} + B^3 = (\tilde{B}_t + B_{xx} + B^3)(x=0) = 0.$$

This identity can be proved to hold *for any* $t, x_1, x_2 \in \mathbb{R}$. On the other hand, if

$$\mathcal{M} := \frac{1}{2}\int_0^x B^2, \quad \mathcal{M}_t = \int_0^x BB_t,$$

we have

$$B\tilde{B}_t - \mathcal{M}_t + \frac{1}{2}B_x^2 + \frac{1}{4}B^4 = (B\tilde{B}_t + \frac{1}{2}B_x^2 + \frac{1}{4}B^4)(x=0) =: \frac{1}{2}c_0,$$

where for $t = x_1 = x_2 = 0$ we have that c_0 is explicitly given by

$$c_0 := 16\beta^2[\alpha^2(1+k) + \beta^2(3m-4)]. \tag{5.7}$$

However, in the general case, c_0 may depend on time.

Replacing in (2.10) we obtain

$$\begin{aligned}
&B_{(4x)} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - a_1(B_{xx} + B^3) + a_2 B \\
&= -(B_t + 3B^2B_x)_x + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - a_1(B_{xx} + B^3) + a_2 B \\
&= -B_{tx} - BB_x^2 + 2B^2B_{xx} + \frac{3}{2}B^5 - a_1(B_{xx} + B^3) + a_2 B \\
&= -B_{tx} - B(c_0 - 2B\tilde{B}_t + 2\mathcal{M}_t - \frac{1}{2}B^4) - 2B^2(B^3 + \tilde{B}_t) \\
&\quad + \frac{3}{2}B^5 + a_1\tilde{B}_t + a_2 B \\
&= -B_{tx} - 2B\mathcal{M}_t + a_1\tilde{B}_t + (a_2 - c_0)B.
\end{aligned} \tag{5.8}$$

Now we prove that this last quantity is identically zero. We compute $\mathcal{M}_t$. Denote

$$F := \frac{1}{2\alpha}\operatorname{sn}_1 \operatorname{sn}_1' + \frac{1}{2\beta}\operatorname{nd}_2 \operatorname{nd}_2', \quad G := \beta^2 \operatorname{sn}_1^2 + \alpha^2 \operatorname{nd}_2^2, \tag{5.9}$$

where, with a slight abuse of notation we denote

$$\mathrm{sn}_1' := \mathrm{sn}'(\alpha y_1, k), \quad \mathrm{cn}_1' := \mathrm{cn}'(\alpha y_1, k), \quad \mathrm{dn}_1' := \mathrm{dn}'(\alpha y_1, k);$$

and so on. We claim that

$$B^2 = 8\alpha^2\beta^2 (\frac{F}{G})_x - 4\alpha^2 \frac{k}{\beta^2}(\beta^2 \mathrm{sn}_1^2 - \alpha^2 \mathrm{nd}_2^2). \tag{5.10}$$

Indeed, from (2.1) we have

$$B^2 = 8\alpha^2\beta^2 \frac{\alpha^2 \mathrm{nd}_2^2 \mathrm{sn}_1'^2 + \beta^2 \mathrm{sn}_1^2 \mathrm{nd}_2'^2 - 2\alpha\beta \mathrm{sn}_1 \mathrm{nd}_2 \mathrm{sn}_1' \mathrm{nd}_2'}{(\beta^2 \mathrm{sn}_1^2 + \alpha^2 \mathrm{nd}_2^2)^2}. \tag{5.11}$$

Note that

$$\begin{aligned} F_xG - FG_x &= \frac{1}{2}(\mathrm{sn}_1'^2 + \mathrm{sn}_1 \mathrm{sn}_1'' + \mathrm{nd}_2'^2 + \mathrm{nd}_2 \mathrm{nd}_2'')(\beta^2 \mathrm{sn}_1^2 + \alpha^2 \mathrm{nd}_2^2) \\ &\quad - (\beta \mathrm{sn}_1 \mathrm{sn}_1' + \alpha \mathrm{nd}_2 \mathrm{nd}_2')(\beta \mathrm{sn}_1 \mathrm{sn}_1' + \alpha \mathrm{nd}_2 \mathrm{nd}_2'). \end{aligned}$$

Now using the well-known JEF identities (see [2, 13]) and replacing above we obtain

$$\begin{aligned} F_xG - FG_x &= \alpha^2 \mathrm{nd}_2^2 \mathrm{sn}_1'^2 + \beta^2 \mathrm{sn}_1^2 \mathrm{nd}_2'^2 - 2\alpha\beta \mathrm{sn}_1 \mathrm{sn}_1' \mathrm{nd}_2 \mathrm{nd}_2' \\ &\quad + \frac{1}{2}(\beta^2 k \mathrm{sn}_1^6 + \alpha^2 k \mathrm{nd}_2^2 \mathrm{sn}_1^4 + \beta^2(m-1) \mathrm{sn}_1^2 \mathrm{dn}_2^4 + \alpha^2(m-1) \mathrm{dn}_2^6) \\ &= \alpha^2 \mathrm{nd}_2^2 \mathrm{sn}_1'^2 + \beta^2 \mathrm{sn}_1^2 \mathrm{nd}_2'^2 - 2\alpha\beta \mathrm{sn}_1 \mathrm{nd}_2 \mathrm{sn}_1' \mathrm{nd}_2' \\ &\quad + \frac{k}{2} \mathrm{sn}_1^4(\beta^2 \mathrm{sn}_1^2 + \alpha^2 \mathrm{nd}_2^2) + \frac{(m-1)}{2} \mathrm{nd}_2^4(\beta^2 \mathrm{sn}_1^2 + \alpha^2 \mathrm{nd}_2^2). \end{aligned}$$

Therefore, from (5.11),

$$B^2 = 8\alpha^2\beta^2 \frac{(F_xG - FG_x)}{G^2} - 4\alpha^2\beta^2 \frac{k \mathrm{sn}_1^4 + (m-1) \mathrm{nd}_2^4}{(\beta^2 \mathrm{sn}_1^2 + \alpha^2 \mathrm{nd}_2^2)}. \tag{5.12}$$

Since $\frac{\beta^4}{\alpha^4} = \frac{k}{1-m}$, (5.12) simplifies as follows:

$$B^2 = 8\alpha^2\beta^2 (\frac{F}{G})_x - 4\alpha^2 \frac{k}{\beta^2}(\beta^2 \mathrm{sn}_1^2 - \alpha^2 \mathrm{nd}_2^2),$$

as desired. From (5.10) we have

$$\frac{1}{2}\int_0^x B^2 = 4\alpha^2\beta^2 \frac{F}{G} - 4\alpha^2\beta^2 \frac{F}{G}(0) - 2\alpha^2 \frac{k}{\beta^2}\int_0^x (\beta^2 \mathrm{sn}_1^2 - \alpha^2 \mathrm{nd}_2^2). \tag{5.13}$$

Since

$$\frac{F}{G} = \frac{\beta \mathrm{sn}_1 \mathrm{cn}_1 \mathrm{dn}_1 \mathrm{dn}_2^2 + \alpha m \mathrm{sn}_2 \mathrm{cn}_2 \mathrm{nd}_2}{2\alpha\beta g} =: \frac{\bar{h}}{2\alpha\beta g},$$

we obtain

$$\begin{aligned} \mathcal{M}_t &= 2\alpha\beta \frac{(g\bar{h}_t - g_t\bar{h})}{g^2} - 2\alpha\beta\Big(\frac{(g\bar{h}_t - g_t\bar{h})}{g^2}(x=0)\Big)\Big|_{t=0} \\ &\quad - 2\alpha^2 \frac{k}{\beta^2}(\delta\beta^2 \mathrm{sn}_1^2 - \gamma\alpha^2 \mathrm{nd}_2^2 + \gamma\alpha^2). \end{aligned}$$

The second term above can be computed explicitly. We have

$$\mathcal{M}_t = 2\alpha\beta \frac{(g\bar{h}_t - g_t\bar{h})}{g^2} - 2\beta^2(\delta + m\gamma) - 2\alpha^2 \frac{k}{\beta^2}(\delta\beta^2 \mathrm{sn}_1^2 - \gamma\alpha^2 \mathrm{nd}_2^2 + \gamma\alpha^2).$$

Let us calculate B_{tx}. Using (5.6), we have

$$B_{tx} = \frac{2\sqrt{2}\alpha\beta}{g^3}(g\hat{h}_t - 2g_t\hat{h}).$$

Now we compute the term $-B_{tx} - 2B\mathcal{M}_t$. We have

$$\begin{aligned} -B_{tx} - 2B\mathcal{M}_t = -\frac{2\sqrt{2}\alpha\beta}{g^3}\Big[&\hat{h}_t g - 2g_t\hat{h} + 4\alpha\beta(g\bar{h}_t - g_t\bar{h})h \\ &- 4\alpha^2\frac{k}{\beta^2}g^2(\delta\beta^2\operatorname{sn}_1^2 - \gamma\alpha^2\operatorname{nd}_2^2 + \gamma\alpha^2)h\Big] + 4\beta^2(\delta+m\gamma)B. \end{aligned} \tag{5.14}$$

We consider the term $\hat{h} + 2\alpha\beta h\bar{h}$. We have

$$\hat{h} + 2\alpha\beta h\bar{h} = h_x g + h(2\alpha\beta\bar{h} - g_x). \tag{5.15}$$

The term $2\alpha\beta\bar{h} - g_x$ reads now

$$\begin{aligned} 2\alpha\beta\bar{h} - g_x &= 2\alpha\beta(\beta\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 + \alpha m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2) \\ &\quad - 2\beta^2(\alpha\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 - m\beta\operatorname{sn}_1^2\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2) \\ &= 2\beta m\operatorname{sn}_2\operatorname{cn}_2(\alpha^2\operatorname{nd}_2 + \beta^2\operatorname{sn}_1^2\operatorname{dn}_2) \\ &= 2\beta m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2 g. \end{aligned}$$

Consequently, (5.15) $= g(h_x + 2\beta m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2 h)$, and replacing in (5.14),

$$\begin{aligned} &-B_{tx} - 2B\mathcal{M}_t \\ &= -\frac{2\sqrt{2}\alpha\beta}{g^2}\Big[\hat{h}_t - 2g_t(h_x + 2\beta m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2 h) + 4\alpha\beta\bar{h}_t h \\ &\quad - 4\alpha^2\frac{k}{\beta^2}g(\delta\beta^2\operatorname{sn}_1^2 - \gamma\alpha^2\operatorname{nd}_2^2 + \gamma\alpha^2)h\Big] + 4\beta^2(\delta+m\gamma)B. \end{aligned} \tag{5.16}$$

Since $\hat{h}_t = h_{tx}g + h_x g_t - g_{xt}h - g_x h_t$, we are left to compute the term

$$4\beta h(\alpha\bar{h}_t - \frac{1}{4\beta}g_{tx} - m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2 g_t) - h_x g_t - g_x h_t. \tag{5.17}$$

Note that

$$\alpha\bar{h}_t - \frac{1}{4\beta}g_{tx} = (\alpha\bar{h} - \frac{1}{4\beta}g_x)_t.$$

From (5.4) we have

$$\alpha\bar{h} - \frac{1}{4\beta}g_x = m\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2 g + \frac{1}{2}\beta\Big[\alpha\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 - \beta m\operatorname{sn}_1^2\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2\Big],$$

and

$$\begin{aligned} (5.17) &= 4\beta m(\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2)_t gh + 2\beta^2 h\Big[\alpha\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 - \beta m\operatorname{sn}_1^2\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2\Big]_t \\ &\quad - (h_x g_t + g_x h_t) \\ &= 4\beta^2 m\gamma(\operatorname{sn}_2\operatorname{cn}_2\operatorname{nd}_2)' gh + 2\beta^2 h\Big[\alpha\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 - \beta m\operatorname{sn}_1^2\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2\Big]_t \\ &\quad - (h_x g_t + g_x h_t). \end{aligned}$$

Now we have

$$\begin{aligned} &(\alpha\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{dn}_2^2 - \beta m\operatorname{sn}_1^2\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2)_t \\ &= \alpha^2\delta\operatorname{dn}_2^2[\operatorname{dn}_1^2(\operatorname{cn}_1^2 - \operatorname{sn}_1^2) - k\operatorname{sn}_1^2\operatorname{cn}_1^2] - 2\alpha\beta m(\delta+\gamma)\operatorname{sn}_1\operatorname{cn}_1\operatorname{dn}_1\operatorname{sn}_2\operatorname{cn}_2\operatorname{dn}_2 \end{aligned}$$

$$- \beta^2 m\gamma \operatorname{sn}_1^2[\operatorname{dn}_2^2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - m \operatorname{sn}_2^2 \operatorname{cn}_2^2],$$

and since $h = \alpha \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2$,

$$\begin{aligned}
&2\beta^2 h(\alpha \operatorname{sn}_1 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^2 - \beta m \operatorname{sn}_1^2 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2)_t \\
&= 2\beta^2 \Big[\alpha^3\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^3[\operatorname{dn}_1^2(\operatorname{cn}_1^2 - \operatorname{sn}_1^2) - k \operatorname{sn}_1^2 \operatorname{cn}_1^2] \\
&\quad - 2\alpha^2\beta m(\delta+\gamma) \operatorname{sn}_1 \operatorname{cn}_1^2 \operatorname{dn}_1^2 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2 \\
&\quad - \alpha\beta^2 m\gamma \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2[\operatorname{dn}_2^2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - m \operatorname{sn}_2^2 \operatorname{cn}_2^2] \\
&\quad - \alpha^2\beta m\delta \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2[\operatorname{dn}_1^2(\operatorname{cn}_1^2 - \operatorname{sn}_1^2) - k \operatorname{sn}_1^2 \operatorname{cn}_1^2] \\
&\quad + 2\alpha\beta^2 m^2(\delta+\gamma) \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{sn}_2^2 \operatorname{cn}_2^2 \operatorname{dn}_2 \\
&\quad + \beta^3 m^2\gamma \operatorname{sn}_1^3 \operatorname{sn}_2 \operatorname{cn}_2[\operatorname{dn}_2^2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - m \operatorname{sn}_2^2 \operatorname{cn}_2^2]\Big].
\end{aligned}$$

On the other hand, using (5.3), (5.2), (5.4) and (5.5),

$$\begin{aligned}
&- (g_t h_x + g_x h_t) \\
&= 2\beta^2 \operatorname{sn}_1 \operatorname{dn}_2 \Big[(\alpha\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m\gamma \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2)(\alpha^2 \operatorname{sn}_1 \operatorname{dn}_2(k \operatorname{cn}_1^2 + \operatorname{dn}_1^2) \\
&\quad + 2\alpha\beta m \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{sn}_2 \operatorname{cn}_2 + \beta^2 m \operatorname{sn}_1 \operatorname{dn}_2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)) \\
&\quad + (\alpha \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2)(\alpha^2\delta \operatorname{sn}_1 \operatorname{dn}_2(k \operatorname{cn}_1^2 + \operatorname{dn}_1^2) \\
&\quad + \alpha\beta m(\delta+\gamma) \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{sn}_2 \operatorname{cn}_2 + \beta^2 m\gamma \operatorname{sn}_1 \operatorname{dn}_2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)) \\
&= 2\beta^2 \Big[2\alpha^3\delta \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^3(k \operatorname{cn}_1^2 + \operatorname{dn}_1^2) \\
&\quad + \alpha^2\beta m(3\delta+\gamma) \operatorname{sn}_1 \operatorname{cn}_1^2 \operatorname{dn}_1^2 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2 \\
&\quad + \alpha\beta^2 m(\delta+\gamma) \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^3(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) \\
&\quad - \alpha^2\beta m(\delta+\gamma) \operatorname{sn}_1^3 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2(k \operatorname{cn}_1^2 + \operatorname{dn}_1^2) \\
&\quad - \alpha\beta^2 m^2(\delta+3\gamma) \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{sn}_2^2 \operatorname{cn}_2^2 \operatorname{dn}_2 - 2\beta^3 m^2\gamma \operatorname{sn}_1^3 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)\Big].
\end{aligned}$$

Rearranging similar terms, and using the identities [13]

$$\operatorname{sn}_1^2 + \operatorname{cn}_1^2 = 1, \quad k \operatorname{sn}_1^2 + \operatorname{dn}_1^2 = 1, \quad m \operatorname{cn}_2^2 + 1 - m = \operatorname{dn}_2^2, \dots,$$

and the fact that $k\alpha^4 = (1-m)\beta^4$, we obtain

$$\begin{aligned}
&2\beta^2 h(\alpha \operatorname{sn}_1 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^2 - \beta m \operatorname{sn}_1^2 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2)_t - (h_x g_t + g_x h_t) \\
&= 2\beta^2 \Big[\alpha^3\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2^3(\operatorname{dn}_1^2 + k \operatorname{sn}_1^2 \operatorname{cn}_1^2) + \beta^3 m^2\gamma \operatorname{sn}_1^3 \operatorname{sn}_2 \operatorname{cn}_2(\operatorname{sn}_2^2 \operatorname{dn}_2^2 - \operatorname{cn}_2^2) \\
&\quad - \alpha^2\beta m\gamma \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2 \operatorname{dn}_2^2(\operatorname{dn}_1^2 + k \operatorname{sn}_1^2 \operatorname{cn}_1^2) \\
&\quad - \alpha\beta^2 m\delta \operatorname{sn}_1^2 \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2(\operatorname{sn}_2^2 \operatorname{dn}_2^2 - \operatorname{cn}_2^2)\Big] \\
&= 2\beta^2(\alpha\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m\gamma \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2)(\alpha^2 \operatorname{dn}_2^2(\operatorname{dn}_1^2 + k \operatorname{sn}_1^2 \operatorname{cn}_1^2) \\
&\quad + \beta^2 m \operatorname{sn}_1^2(\operatorname{cn}_2^2 - \operatorname{sn}_2^2 \operatorname{dn}_2^2)) \\
&= 2\beta^2\tilde{h}[\alpha^2 \operatorname{dn}_2^2(1 - k \operatorname{sn}_1^4) + \beta^2 \operatorname{sn}_1^2 \operatorname{dn}_2^2 - (1-m)\beta^2 \operatorname{sn}_1^2 - \beta^2 m \operatorname{sn}_1^2 \operatorname{sn}_2^2 \operatorname{dn}_2^2] \\
&= 2\beta^2\tilde{h}[\operatorname{dn}_2^2\{\alpha^2(1 - k \operatorname{sn}_1^4) + \beta^2 \operatorname{sn}_1^2 - \beta^2 \operatorname{sn}_1^2(1 - \operatorname{dn}_2^2)\} - \alpha^4\beta^{-2} k \operatorname{sn}_1^2] \\
&= 2\beta^2\tilde{h}g(\operatorname{dn}_2^2 - \alpha^2\beta^{-2} k \operatorname{sn}_1^2).
\end{aligned}$$

From (5.16) we conclude that

$$
\begin{aligned}
&- B_{tx} - 2B\mathcal{M}_t \\
&= -\frac{2\sqrt{2}\alpha\beta}{g}\Big[h_{tx} + 2\tilde{h}(\beta^2 \operatorname{dn}_2^2 - \alpha^2 k \operatorname{sn}_1^2) + 4\beta^2 m\gamma(\operatorname{sn}_2 \operatorname{cn}_2 \operatorname{nd}_2)' h \\
&\quad - 4\alpha^2 \frac{k}{\beta^2}(\delta\beta^2 \operatorname{sn}_1^2 - \gamma\alpha^2 \operatorname{nd}_2^2 + \gamma\alpha^2)h\Big] + 4\beta^2(\delta + m\gamma)B.
\end{aligned} \tag{5.18}
$$

Note that [13] $(\operatorname{sn}_2 \operatorname{cn}_2 \operatorname{nd}_2)' = \operatorname{cn}_2^2 - \operatorname{sn}_2^2 + m \operatorname{sn}_2^2 \operatorname{cn}_2^2 \operatorname{nd}_2^2$. Consequently,

$$
\begin{aligned}
&4\beta^2 m\gamma(\operatorname{sn}_2 \operatorname{cn}_2 \operatorname{nd}_2)' - 4\alpha^2 \frac{k}{\beta^2}(\delta\beta^2 \operatorname{sn}_1^2 - \gamma\alpha^2 \operatorname{nd}_2^2 + \gamma\alpha^2) \\
&= 4\beta^2 m\gamma(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - 4\alpha^2 k\delta \operatorname{sn}_1^2 + 4\beta^2\gamma \operatorname{nd}_2^2(m^2 \operatorname{sn}_2^2 \operatorname{cn}_2^2 + k\frac{\alpha^4}{\beta^4}(1 - \operatorname{dn}_2^2)) \\
&= 4\beta^2 m\gamma(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - 4\alpha^2 k\delta \operatorname{sn}_1^2 + 4\beta^2 m\gamma \operatorname{sn}_2^2 \operatorname{nd}_2^2(m \operatorname{cn}_2^2 + k\frac{\alpha^4}{\beta^4}) \\
&= 4\beta^2 m\gamma(\operatorname{cn}_2^2 - \operatorname{sn}_2^2) - 4\alpha^2 k\delta \operatorname{sn}_1^2 + 4\beta^2 m\gamma \operatorname{sn}_2^2 = 4\beta^2 m\gamma \operatorname{cn}_2^2 - 4\alpha^2 k\delta \operatorname{sn}_1^2 .
\end{aligned}
$$

Finally we compute h_{tx}. From (5.3) we have

$$
\begin{aligned}
h_{tx} = -\Big[&\alpha^2(k \operatorname{cn}_1^2 + \operatorname{dn}_1^2)(\alpha\delta \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m(2\delta + \gamma) \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2) \\
&+ \beta^2 m(\operatorname{cn}_2^2 - \operatorname{sn}_2^2)(\alpha(\delta + 2\gamma) \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2 - \beta m\gamma \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2) \\
&- 4\alpha^2 k\delta \operatorname{sn}_1^2(\alpha \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2) - 4\beta^2\gamma(\beta m \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2) \operatorname{dn}_2^2\Big].
\end{aligned}
$$

Using the last two identities and some standard simplifications, (5.18) becomes

$$
\begin{aligned}
&- B_{tx} - 2B\mathcal{M}_t \\
&= -\frac{2\sqrt{2}\alpha\beta}{g}\Big[[-\alpha^2(1 + k)\delta + \beta^2((2 - m)\delta + 2m\gamma)](\alpha \operatorname{cn}_1 \operatorname{dn}_1 \operatorname{dn}_2) \\
&\quad + [\alpha^2(1 + k)(2\delta + \gamma) + \beta^2(2 - 3m)\gamma](\beta m \operatorname{sn}_1 \operatorname{sn}_2 \operatorname{cn}_2)\Big] + 4\beta^2(\delta + m\gamma)B \\
&= -a_1\tilde{B}_t - \tilde{a}_2 B,
\end{aligned}
$$

where a_1 is defined in (2.8), and

$$
\tilde{a}_2 = \alpha^4(1 + k)^2 - 2\alpha^2\beta^2(1 + k)(m + 6) + \beta^4(2 - m)(18 - m).
$$

Comparing this with (5.8) we have (c_0 is given by (5.7))

$$
a_2 = \tilde{a}_2 + c_0 = \alpha^4(1 + k)^2 + 2\alpha^2\beta^2(1 + k)(2 - m) + \beta^4(m^2 + 28m - 28).
$$

Finally we use that $k\alpha^4 = \beta^4(1 - m)$ to obtain

$$
a_2 = \alpha^4(1 + k^2 - 26k) + 2\alpha^2\beta^2(1 + k)(2 - m) + \beta^4 m^2,
$$

as in (2.9). The proof is complete.

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References

[1] M. Ablowitz, P. Clarkson; *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.

[2] M. Abramowitz, I. A. Stegun; *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York. Dover, 1972.

[3] M. A. Alejo; *Focusing mKdV breather solutions with nonvanishing boundary conditions by the Inverse Scattering Method*, Jour. NonLin. Math. Phys. **19**, no. 1 (2012), 1250009.

[4] M. A. Alejo; *On the ill-posedness of the Gardner equation*, J. Math. Anal. Appl., **396** no. 1 (2012), 256-260.

[5] M. A. Alejo; *Geometric breathers of the mKdV*, Acta App. Math., **121** no. 1 (2012), p.137–155.

[6] M. A. Alejo, C. Muñoz; *Nonlinear stability of mKdV breathers*, Comm. Math. Phys., **37** (2013), 2050–2080.

[7] M. A. Alejo, C. Muñoz; *On the nonlinear stability of mKdV breathers*, J. Phys. A: Math. Theor., **45** (2012), 432001.

[8] M. A. Alejo, C. Muñoz; *Dynamics of complex-valued modified KdV solitons with applications to the stability of breathers*, Analysis and PDE 8-3 (2015), 629–674.

[9] M. A. Alejo, C. Muñoz, José M. Palacios; *On the variational structure of breather solutions I: the Sine-Gordon case*, submitted, arxiv.org/pdf/1309.0625v2.pdf.

[10] C. Blank, M. Chirilus-Bruckner, V. Lescarret, G. Schneider; *Breather solutions in periodic media*, Comm. Math. Phys., **302** (3) p. 815–841 (2011).

[11] T. B. Benjamin; *The stability of solitary waves*, Proc. Roy. Soc. London A **328**, 153–183 (1972).

[12] J. L. Bona, P. Souganidis, W. Strauss; *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London, **411** (1987), 395–412.

[13] P. F. Byrd, M. D. Friedman; *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd ed., Springer-Verlag, New York and Heidelberg, 1971.

[14] R. Côte, C. Muñoz; *Multi-solitons for nonlinear Klein-Gordon equations*, Forum of Mathematics, Sigma, Vol. 2 (2014), e15 (38 pages).

[15] J. Cuevas-Maraver, P. G. Kevrekidis, D. E. Pelinovsky; *Nonlinear instabilities of multi-site breathers in Klein-Gordon lattices*, Stud. Appl. Math., online (2016)

[16] T. Duyckaerts, F. Merle; *Dynamic of threshold solutions for energy-critical NLS*, GAFA **18** (2008), 1787–1840.

[17] C. S. Gardner, M. D. Kruskal, R. Miura; *Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Math. Phys. **9**, no. 8 (1968), 1204–1209.

[18] P. G. Kevrekidis, A. Khare, A. Saxena, G. Herring; *On some classes of mKdV periodic solutions*, Journal of Physics A: Mathematical and General, **37** (2004), 10959-10965.

[19] P. G. Kevrekidis, A. Khare, A. Saxena; *Breather lattice and its stabilization for the modified Korteweg-de Vries equation*, Phys.Rev. E, **68** (2003), 0477011-0477014.

[20] P. G. Kevrekidis, D. E. Pelinovsky, A. Saxena; *When linear stability does not exclude nonlinear instability*, Phys. Rev. Lett. **114** (2015), 214101.

[21] A. Khare; *personal communication*.

[22] G. L. Lamb; *Elements of Soliton Theory*, Pure Appl. Math., Wiley, New York, 1980.

[23] J. H. Maddocks, R. L. Sachs; *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math., **46** (1993), 867–901.

[24] Y. Martel, F. Merle, T.P. Tsai; *Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations*, Comm. Math. Phys. **231** (2002), 347–373.

[25] Y. Martel, F. Merle; *Stability of two soliton collision for nonintegrable gKdV equations*, Comm. Math. Phys. **286** (2009), 39–79.

[26] Y. Martel, F. Merle; *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157**, no. 3 (2001), 219–254.

[27] Y. Martel, F. Merle; *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity **18** (2005), 55–80.

[28] F. Merle, L. Vega; *L^2 stability of solitons for KdV equation*, Int. Math. Res. Not., no. 13 (2003), 735–753.

[29] C. Muñoz; *The Gardner equation and the stability of the multi-kink solutions of the mKdV equation*, Discrete Contin. Dyn. Syst., **36** (2016), no. 7, 3811–3843.

[30] C. Muñoz; *Stability of integrable and nonintegrable structures*, Adv. Differential Equations, Volume 19, Number 9/10 (2014), 947–996.

[31] R. L. Pego, M. I. Weinstein; *Asymptotic stability of solitary waves*, Commun. Math. Phys. **164**, 305–349 (1994).

[32] M. I. Weinstein; *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68 .

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