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NONEXISTENCE RESULTS FOR WEIGHTED p-LAPLACE EQUATIONS WITH SINGULAR NONLINEARITIES

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Abstract. In this article we present some nonexistence results concerning stable solutions to the equation

$$\operatorname{div}\left(w(x)|\nabla u|^{p-2}\nabla u\right) = g(x)f(u) \quad \text{in } \mathbb{R}^N,\ p \geq 2$$

when $f(u)$ is either $u^{-\delta} + u^{-\gamma}$ with $\delta, \gamma > 0$ or $e^{1/u}$ where w, g are suitable weight functions.

1. Introduction

Elliptic problems with singular nonlinearity have been a subject of extensive research which is evident by the bast literature available in the field. Following the paper by Crandall et al [2], questions of existence, uniqueness, regularity, multiplicity and asymptotic behavior have been widely explored. Interested readers may find the papers by Boccardo-Orsina [1], Lazer-Mckenna [11], Mohammed [10] and the reference therein very helpful. In this article we are interested in the study of nonexistence of stable solutions to two singular equations. To put it in perspective let us look at related literature available concerning this type of equations. In Ma-Wei [9], the problem

$$\Delta u = u^{-\delta} \quad \text{in } \mathbb{R}^n$$

was considered for any $\delta > 0$ and among many other properties it was proved that the problem does not admit a positive stable solution provided

$$2 \leq n < 2 + \frac{4}{1+\delta}(\delta + \sqrt{\delta^2 + \delta}).$$

A generalized problem related to the Liouville theorem for stable solution of the equation $-\Delta u = f(u)$ was studied by Dupaigne-Farina [5] and showed to admit no bounded stable solution provided $f \geq 0$ and $1 \leq n \leq 4$. Similar results about nonexistence were also provided in higher dimension which requires convexity requirement on f was done in [6]. Guo and Mei [8] studied the problem

$$\Delta_p u = u^{-\delta} \text{ in } \mathbb{R}^n$$

and showed the nonexistence of stable solution for some range of $\delta > 0$ and $2 \leq p < n$ among other results. Chen et al [3] generalized the results of Guo and Mei [8] for the problem

$$\Delta_p u = f(x)u^{-\delta} \text{ in } \mathbb{R}^n$$

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for some $f \in L^1_{\text{loc}}$ such that f behaves like a radial function for large enough x. This was recently generalized by Le et al [12] for the weighted p-laplace equation. Readers may also find the paper by Du and Guo [7] a good read related to singular problems. In this work we consider the problem

$$-\operatorname{div}\big(w(x)|\nabla u|^{p-2}\nabla u\big) = g(x)f(u) \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N \tag{1.1}$$

where $f(u)$ is either $-u^{-\delta} - u^{-\gamma}$ to be denoted by $(1.1)_s$, or $-e^{1/u}$ to be denoted by $(1.1)_e$, and $\delta, \gamma > 0$. We also assume $p \geq 2$, $N \geq 1$ and the weight functions $w, g \in L^1_{\text{loc}}(\mathbb{R}^N)$ both positive a.e. in $\mathbb{R}^N$ such that $g^{-1} \in L^\infty(\mathbb{R}^N)$ unless otherwise mentioned.

Definition 1.1. We say that $u \in C^1(\mathbb{R}^N)$ is a weak solution to (1.1) if $u > 0$ in $\mathbb{R}^N$ and for all $\varphi \in C^1_c(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}\nabla u \nabla\varphi\,dx = \int_{\mathbb{R}^N} g(x)\varphi(x)f(u)\,dx. \tag{1.2}$$

Definition 1.2. A weak solution u of (1.1) is said to be stable if for all $\varphi \in C^1_c(\mathbb{R}^N)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}|\nabla\varphi|^2\,dx + (p-2)\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-4}|\langle\nabla u, \nabla\varphi\rangle|^2\,dx \\ &- \int_{\mathbb{R}^N} g(x)f'(u)\varphi^2(x)\,dx \geq 0. \end{aligned} \tag{1.3}$$

Therefore, if u is a stable solution of equation (1.1) then

$$\int_{\mathbb{R}^N} g(x)f'(u)\varphi^2\,dx \leq (p-1)\int_{\mathbb{R}^N} w(x)|\nabla u|^{p-2}|\nabla\varphi|^2\,dx. \tag{1.4}$$

In this article we provide suitable conditions on the weight functions w, g to guarantee the nonexistence of stable solutions for (1.1) with a power nonlinearity and an exponential nonlinearity both of which are singular as the solutions approaches 0. We achieve this by using special test functions which are adapted from Dupaigne [4] to arrive at a contradiction.

Notation.

Equation (1.1) will be denoted by $(n)_s$ for $f(u) = -u^{-\delta} - u^{-\gamma}$ with $\delta, \gamma > 0$; while (1.1) will be denoted by $(n)_e$ for $f(u) = -e^{1/u}$, with $n = 1, 2, 3, 4$. Without loss of generality, we assume $0 < \delta \leq \gamma$.

We denote by $B_r(0)$ to be the ball centered at 0 with radius $r > 0$. c as a generic constant whose values may vary depending on the situation. If c depends on ϵ we denote it by c_ϵ. Moreover, if c depends on $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n \geq 2)$ then we denote it by $c(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

For p_1, $p_2 > 0$ such that $p_1 \geq 1$, by $\|w\|^{p_1}_{L^{p_1}(B_{2r}(0))} = o(r^{p_2})$, we mean there exist a fixed constant $c > 0$ independent of r such that $\|w\|^{p_1}_{L^{p_1}(B_{2r}(0))} \leq cr^{p_2}$.

We say that (δ, p) belongs to the class

(1) P_a if either $\delta \geq 1$ for $2 \leq p < 3$ or $\delta > 1$ for $p = 3$ or $\delta > \frac{(p-1)^2}{4}$ for $p > 3$ respectively.
(2) P_b if either $\delta > 0$ for $p = 2$ or $\delta > \frac{(p-1)^2}{4}$ for $p > 2$ respectively.

We denote

$$s_p = \begin{cases} \delta + \sqrt{\delta^2 + \delta} & \text{for } p = 2, \\ \frac{2\delta}{(p-1)} - \frac{p-1}{2} & \text{for } p > 2. \end{cases}$$

Note that if (δ, p) belongs to either P_a or P_b, then we have $s_p > 0$.

We denote $S_p^a = 1 + \frac{2s_p}{p-1+\delta}$ and $S_p^b = 1 + \frac{2s_p}{p-1+\gamma}$ for $p \geq 2$. We denote $m = \|g^{-1}\|_\infty$.

Let us fix a constant M such that $M > 0$ for $p = 2$ and $0 < M < \frac{4}{(p-1)^2}$ for $p > 2$ respectively. Denote

$$t_p = \begin{cases} \frac{1}{M} + \sqrt{\frac{1}{M} + \frac{1}{M^2}} & \text{for } p = 2, \\ \frac{2}{M(p-1)} - \frac{p-1}{2} & \text{for } p > 2. \end{cases} \tag{1.5}$$

We define the number $T_p = 1 + \frac{2t_p}{p}$ for $p \geq 2$. Observe that $t_p > 0$ for any $p \geq 2$ and therefore $T_p > 1$.

Throughout this article $\psi_R \in C_c^1(\mathbb{R}^N)$ is a test function such that

$$0 \leq \psi_R \leq 1 \text{ in } \mathbb{R}^N, \quad \psi_R = 1 \text{ in } B_R(0),$$
$$\psi_R = 0 \text{ in } \mathbb{R}^N \setminus B_{2R}(0)$$

with

$$|\nabla \psi_R| \leq \frac{c}{R}$$

for some constant $c > 0$ (independent of R).

2. Main results

Theorem 2.1 (Caccioppoli type Estimate)**.** *Let $u \in C^1(\mathbb{R}^N)$ be a stable solution to $(1.1)_s$. Then the following holds:*

1. *Let (δ, p) belong to the class P_a such that $\delta < \gamma$. Assume that $0 < u \leq 1$ in $\mathbb{R}^N$. Then for any $\beta \in (0, s_p)$, there exists a constant $c = c(\beta, p, m) > 0$ such that for every $\psi \in C_c^1(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, we have*
$$\int_{\mathbb{R}^N} g(x) (\frac{\psi}{u})^{2\beta+p-1+\delta}\, dx \leq c \int_{\mathbb{R}^N} w^{\theta_a'}(x) |\nabla \psi|^{p\theta_a'}\, dx \tag{2.1}$$
where
$$\theta_a = \frac{2\beta + p - 1 + \delta}{2\beta}, \quad \theta_a' = \frac{2\beta + p - 1 + \delta}{p - 1 + \delta}.$$
2. *Let (δ, p) belong to the class P_b such that $\delta < \gamma$ where $\gamma \geq 1$. Assume that $u \geq 1$ in $\mathbb{R}^N$. Then for any $\beta \in (0, s_p)$, there exists a constant $c = c(\beta, p, m) > 0$ such that for every $\psi \in C_c^1(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, we have*
$$\int_{\mathbb{R}^N} g(x) (\frac{\psi}{u})^{2\beta+p-1+\gamma}\, dx \leq c \int_{\mathbb{R}^N} w^{\theta_b'}(x) |\nabla \psi|^{p\theta_b'}\, dx \tag{2.2}$$
where
$$\theta_b = \frac{2\beta + p - 1 + \gamma}{2\beta}, \quad \theta_b' = \frac{2\beta + p - 1 + \gamma}{p - 1 + \gamma}.$$

(3) *Let (δ, p) belong to the class P_a such that $\delta = \gamma \geq 1$. Assume that $u > 0$ in $\mathbb{R}^N$. Then for any $\beta \in (0, s_p)$, there exists a constant $c = c(\beta, p, m) > 0$ such that for every $\psi \in C_c^1(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, we have*

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \leq c \int_{\mathbb{R}^N} w^{\theta'_a}(x)|\nabla\psi|^{p\theta'_a}\,dx \tag{2.3}$$

where

$$\theta_a = \frac{2\beta+p-1+\delta}{2\beta}, \quad \theta'_a = \frac{2\beta+p-1+\delta}{p-1+\delta}.$$

Theorem 2.2. *Let (δ, p) belong to the class P_a such that $\delta < \gamma$ and $u \in C^1(\mathbb{R}^N)$ such that $0 < u \leq 1$ in $\mathbb{R}^N$. Assume that there exist a fixed constant $\lambda_p^a > 0$ such that*

$$\|w\|_{L^{S_p^a}(B_{2R}(0))}^{S_p^a} = o(R^{\lambda_p^a}), \quad \lambda_p^a < pS_p^a.$$

Then u is not a stable solution of $(1.1)_s$.

Theorem 2.3. *Let (δ, p) belong to the class P_b such that $\delta < \gamma$ with $\gamma \geq 1$ and $u \in C^1(\mathbb{R}^N)$ such that $u \geq 1$ in $\mathbb{R}^N$. Assume that there exist a fixed constant $\lambda_p^b > 0$ such that*

$$\|w\|_{L^{S_p^b}(B_{2R}(0))}^{S_p^b} = o(R^{\lambda_p^b}), \quad \lambda_p^b < pS_p^b.$$

Then u is not a stable solution of $(1.1)_s$.

Theorem 2.4. *Let (δ, p) belong to the class P_a such that $\delta = \gamma \geq 1$ and $u \in C^1(\mathbb{R}^N)$ such that $u > 0$ in $\mathbb{R}^N$. Assume that there exist a fixed constant $\lambda_p^c > 0$ such that*

$$\|w\|_{L^{S_p^a}(B_{2R}(0))}^{S_p^a} = o(R^{\lambda_p^c}) \ \text{and}\ \lambda_p^c < pS_p^a.$$

Then u is not a stable solution of $(1.1)_s$.

Remark 2.5. For $w \in L^\infty(\mathbb{R}^N)$, one can choose $\lambda_p^a, \lambda_p^b, \lambda_p^c$ equal to the dimension N of $\mathbb{R}^N$ in Theorems 2.2, 2.3 and 2.4.

Theorem 2.6 (Caccioppoli type estimate)**.** *Let $u \in C^1(\mathbb{R}^N)$ be a bounded stable solution to $(1.1)_e$ such that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M$ for some positive constant M. Then for any $\beta \in (0, t_p)$, there exists a constant $c = c(\beta, p, m) > 0$ such that for every $\psi \in C_c^1(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$ in $\mathbb{R}^N$, we have*

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx \leq c \int_{\mathbb{R}^N} w^{\theta'}|\nabla\psi|^{p\theta'}\,dx \tag{2.4}$$

where $\theta = \frac{2\beta+p}{2\beta}$ and $\theta' = \frac{2\beta+p}{p}$.

Theorem 2.7. *Let $u \in C^1(\mathbb{R}^N)$ be positive such that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M$ for some positive constant M. Assume that there exists a fixed constant $\mu_p > 0$ such that*

$$\|w\|_{L^{T_p}(B_{2R}(0))}^{T_p} = o(R^{\mu_p}), \quad \mu_p < pT_p.$$

Then u is not a stable solution of $(1.1)_e$.

Remark 2.8. For $w \in L^\infty(\mathbb{R}^N)$, one can choose $\mu_p = N$ in Theorem 2.7.

3. Proof of main results

Before proving the main results, we prove an important lemma.

Lemma 3.1. *Assume $p \geq 2$ and let $u \in C^1(\mathbb{R}^N)$ be a stable solution to (1.1). Fix a constant $\beta > 0$. Then for every given $\epsilon \in (0, 2\beta + p - 1)$, there exists a constant $c_\epsilon = c_\epsilon(\beta, p) > 0$ such that for any nonnegative $\psi \in C_c^1(\mathbb{R}^N)$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^N} g(x) f'(u) u^{-2\beta-p+2} \psi^p \, dx \\ &\leq c_\epsilon \int_{\mathbb{R}^N} w(x) u^{-2\beta} |\nabla \psi|^p \, dx \\ &\quad - \frac{(p-1)(\beta + \frac{p}{2} - 1)^2 + \epsilon}{2\beta + p - 1 - \epsilon} \int_{\mathbb{R}^N} g(x) f(u) u^{-2\beta-p+1} \psi^p \, dx. \end{aligned} \tag{3.1}$$

Proof. Suppose $u \in C^1(\mathbb{R}^N)$ be a stable solution of equation (1.1) and $\psi \in C_c^1(\mathbb{R}^N)$ be nonnegative. We prove the lemma in two steps.

Step 1: Suppose $u \in C^1(\mathbb{R}^N)$ is a weak solution of (1.1) and let $\alpha > 0$. Choosing $\varphi = u^{-\alpha}\psi^p$ as a test function in the weak form (1.2), since

$$\nabla \varphi = -\alpha u^{-\alpha-1} \psi^p \nabla u + p \psi^{p-1} u^{-\alpha} \nabla \psi$$

we obtain

$$\begin{aligned} &\alpha \int_{\mathbb{R}^N} w(x) u^{-\alpha-1} \psi^p |\nabla u|^p \, dx \\ &\leq p \int_{\mathbb{R}^N} w(x) u^{-\alpha} \psi^{p-1} |\nabla u|^{p-1} |\nabla \psi| \, dx - \int_{\mathbb{R}^N} g(x) u^{-\alpha} f(u) \psi^p \, dx. \end{aligned} \tag{3.2}$$

Now using Young's inequality for $\epsilon \in (0, \alpha)$ we obtain

$$\begin{aligned} &p \int_{\mathbb{R}^N} w(x) u^{-\alpha} \psi^{p-1} |\nabla u|^{p-1} |\nabla \psi| \, dx \\ &= p \int_{\mathbb{R}^N} (w^{\frac{1}{p'}} u^{\frac{-\alpha-1}{p'}} |\nabla u|^{p-1} \psi^{p-1})(w^{\frac{1}{p}} u^{\frac{p-\alpha-1}{p}} |\nabla \psi|) dx \\ &\leq \epsilon \int_{\mathbb{R}^N} w(x) u^{-\alpha-1} \psi^p |\nabla u|^p \, dx + c_\epsilon \int_{\mathbb{R}^N} w(x) u^{p-\alpha-1} |\nabla \psi|^p \, dx. \end{aligned}$$

Plugging this estimate in (3.2) and defining

$$A = \int_{\mathbb{R}^N} w(x) u^{-\alpha-1} \psi^p |\nabla u|^p \, dx, \quad B = \int_{\mathbb{R}^N} w(x) u^{p-\alpha-1} |\nabla \psi|^p \, dx,$$

we obtain

$$(\alpha - \epsilon) A \leq c_\epsilon B - \int_{\mathbb{R}^N} g(x) u^{-\alpha} f(u) \psi^p \, dx. \tag{3.3}$$

Step 2: Suppose $u \in C^1(\mathbb{R}^N)$ is a stable solution of (1.1) and let $\beta > 0$. Choosing $\varphi = u^{-\beta-\frac{p}{2}+1} \psi^{\frac{p}{2}}$ as a test function in the stability equation (1.4), since

$$\nabla \varphi = -(\beta + \frac{p}{2} - 1) u^{-\beta-\frac{p}{2}} \psi^{\frac{p}{2}} \nabla u + \frac{p}{2} \psi^{\frac{p-2}{2}} u^{-\beta-\frac{p}{2}+1} \nabla \psi$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)f'(u)u^{-2\beta-p+2}\psi^p\,dx\\
&\le (p-1)(\beta+\frac{p}{2}-1)^2\int_{\mathbb{R}^N} w(x)u^{-2\beta-p}\psi^p|\nabla u|^p\,dx\\
&\quad+(p-1)\frac{p^2}{4}\int_{\mathbb{R}^N} w(x)u^{-2\beta-p+2}\psi^{p-2}|\nabla u|^{p-2}|\nabla\psi|^2\,dx\\
&\quad+p(p-1)(\beta+\frac{p}{2}-1)\int_{\mathbb{R}^N} w(x)u^{-2\beta-p+1}\psi^{p-1}|\nabla u|^{p-1}|\nabla\psi|\,dx\\
&=: X+Y+Z.
\end{aligned}\tag{3.4}$$

Let $\alpha=2\beta+p-1$. Then we have $X=(p-1)(\beta+\frac{p}{2}-1)^2A$. Now we prove the required estimate for $p=2$ and $p>2$ separately in the cases a and b below.

Case a. Let $p=2$. Therefore

$$Y=\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^2dx=B.$$

Using the exponents $p'=\frac{p}{p-1}$ and p in Young's inequality, we obtain

$$\begin{aligned}
Z&=2\beta\int_{\mathbb{R}^N} w(x)u^{-2\beta-1}\psi|\nabla u||\nabla\psi|\,dx\\
&=2\beta\int_{\mathbb{R}^N}(w^{\frac12}\psi u^{-\beta-1}|\nabla u|)(w^{\frac12}u^{-\beta}|\nabla\psi|)\,dx\\
&\le\epsilon\int_{\mathbb{R}^N} w(x)u^{-2\beta-2}\psi^2|\nabla u|^2\,dx+c_\epsilon\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^2\,dx\\
&=\epsilon A+c_\epsilon B.
\end{aligned}$$

Now putting $\alpha=2\beta+1>0$ in (3.3) we obtain

$$A\le\frac{1}{2\beta+1-\epsilon}\Big\{c_\epsilon B-\int_{\mathbb{R}^N} g(x)f(u)u^{-2\beta-1}\psi^2\,dx\Big\}.\tag{3.5}$$

Using (3.5) together with the above estimates on Y and Z in (3.4), we obtain

$$\int_{\mathbb{R}^N} g(x)f'(u)u^{-2\beta}\psi^2\,dx\le c_\epsilon B-\frac{\beta^2+\epsilon}{2\beta+1-\epsilon}\int_{\mathbb{R}^N} g(x)f(u)u^{-2\beta-1}\psi^2\,dx.$$

Case b. Let $p>2$. Therefore, using the exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ in Young's inequality, we have

$$\begin{aligned}
Y&=(p-1)\frac{p^2}{4}\int_{\mathbb{R}^N} w(x)u^{-2\beta-p+2}\psi^{p-2}|\nabla u|^{p-2}|\nabla\psi|^2dx\\
&=(p-1)\frac{p^2}{4}\int_{\mathbb{R}^N}(w^{\frac{p-2}{p}}|\nabla u|^{p-2}\psi^{p-2}u^{\frac{(-2\beta-p)(p-2)}{p}})(w^{\frac{2}{p}}u^{\frac{-4\beta}{p}}|\nabla\psi|^2)\,dx\\
&\le\frac{\epsilon}{2}\int_{\mathbb{R}^N} w(x)u^{-2\beta-p}\psi^p|\nabla u|^p\,dx+\frac{c_\epsilon}{2}\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^p\,dx\\
&=\frac{\epsilon}{2}A+\frac{c_\epsilon}{2}B.
\end{aligned}$$

Also using the exponents $p' = \frac{p}{p-1}$ and p in Young's inequality, we obtain

$$\begin{aligned}
Z &= p(p-1)(\beta+\frac{p}{2}-1)\int_{\mathbb{R}^N} w(x)u^{-2\beta-p+1}\psi^{p-1}|\nabla u|^{p-1}|\nabla\psi|\,dx\\
&= p(p-1)(\beta+\frac{p}{2}-1)\int_{\mathbb{R}^N}(w^{\frac{1}{p'}}\psi^{p-1}u^{y}|\nabla u|^{p-1})(w^{\frac{1}{p}}u^{y'}|\nabla\psi|)\,dx\\
&\le \frac{\epsilon}{2}\int_{\mathbb{R}^N} w(x)u^{-2\beta-p}\psi^{p}|\nabla u|^{p}\,dx + \frac{c_\epsilon}{2}\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^{p}\,dx\\
&= \frac{\epsilon}{2}A + \frac{c_\epsilon}{2}B,
\end{aligned}$$

where $y = -\frac{(2\beta+p)(p-1)}{p}$ and $y' = -2\beta-p+1+\frac{(2\beta+p)(p-1)}{p}$.

Now putting $\alpha = 2\beta+p-1>0$ in (3.3) we obtain

$$A \le \frac{1}{2\beta+p-1-\epsilon}\Big\{c_\epsilon B - \int_{\mathbb{R}^N} g(x)f(u)u^{-2\beta-p+1}\psi^{p}\,dx\Big\}. \tag{3.6}$$

Using (3.6) together with the above estimates on Y and Z in (3.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)f'(u)u^{-2\beta-p+2}\psi^{p}\,dx\\
&\le c_\epsilon B - \frac{(p-1)(\beta+\frac{p}{2}-1)^2+\epsilon}{2\beta+p-1-\epsilon}\int_{\mathbb{R}^N} g(x)f(u)u^{-2\beta-p+1}\psi^{p}\,dx.
\end{aligned}$$

□

Proof of Theorem 2.1. Let $u\in C^1(\mathbb{R}^N)$ be a positive stable solution to $(1.1)_s$. Then by Lemma 3.1, using that $0<\delta\le\gamma$ and $f(u) = -u^{-\delta}-u^{-\gamma}$ in the inequality (3.1), we obtain

$$\beta_\epsilon\int_{\mathbb{R}^N} g(x)\big(\frac{1}{u^\delta}+\frac{1}{u^\gamma}\big)u^{-2\beta-p+1}\psi^{p}\,dx \le c_\epsilon\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^{p}dx$$

where

$$\beta_\epsilon = \Big(\delta - \frac{(p-1)(\beta+\frac{p}{2}-1)^2+\epsilon}{2\beta+p-1-\epsilon}\Big).$$

Observe that for every $\beta\in(0,s_p)$,

$$\lim_{\epsilon\to0}\beta_\epsilon = \Big(\delta-\frac{(p-1)(\beta+\frac{p}{2}-1)^2}{2\beta+p-1}\Big)>0.$$

Therefore fixing $\beta\in(0,s_p)$, we can choose an $\epsilon\in(0,1)$ such that $\beta_\epsilon>0$. Hence we have

$$\int_{\mathbb{R}^N} g(x)\big(\frac{1}{u^\delta}+\frac{1}{u^\gamma}\big)u^{-2\beta-p+1}\psi^{p}\,dx \le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^{p}dx \tag{3.7}$$

for some positive constant c.

Case 1. Since $\delta<\gamma$ and $0<u\le1$ in $\mathbb{R}^{\mathbb{N}}$, for any $\beta\in(0,s_p)$, inequality (3.7) becomes

$$\int_{\mathbb{R}^N} g(x)u^{-2\beta-p+1-\delta}\psi^{p}\,dx \le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^{p}\,dx.$$

Replacing ψ by $\psi^{\frac{2\beta+p-1+\delta}{p}}$, we obtain

$$\int_{\mathbb{R}^N} g(x)\big(\frac{\psi}{u}\big)^{2\beta+p-1+\delta} \le c\Big(\frac{2\beta+p-1+\delta}{p}\Big)^{p}\int_{\mathbb{R}^N} w(x)u^{-2\beta}\psi^{2\beta+\delta-1}|\nabla\psi|^{p}dx$$

$$= c\int_{\mathbb{R}^N} (\frac{\psi}{u})^{2\beta} \left(w(x)|\nabla\psi|^p\psi^{\delta-1}\right) dx.$$

Choosing the exponents $\theta_a = \frac{2\beta+p-1+\delta}{2\beta}$, $\theta_a' = \frac{2\beta+p-1+\delta}{p-1+\delta}$ in Young's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \\
&\le \Big\{\epsilon\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}dx + c_\epsilon\int_{\mathbb{R}^N} w^{\theta_a'}g^{-\frac{\theta_a'}{\theta_a}}\psi^{(\delta-1)\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx\Big\}.
\end{aligned}$$

Now using that $g^{-1}\in L^\infty(\mathbb{R}^N)$, $0\le\psi\le 1$ and $\delta\ge 1$, we obtain the required inequality

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx.$$

Case 2. Since $\delta<\gamma$ and $u\ge 1$ in $\mathbb{R}^{\mathbb{N}}$, for any $\beta\in(0,s_p)$, inequality (3.7) becomes

$$\int_{\mathbb{R}^N} g(x)u^{-2\beta-p+1-\gamma}\psi^p\,dx \le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^p\,dx.$$

Replacing ψ by $\psi^{\frac{2\beta+p-1+\gamma}{p}}$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\gamma} &\le c\Big(\frac{2\beta+p-1+\gamma}{p}\Big)^p\int_{\mathbb{R}^N} w(x)u^{-2\beta}\psi^{2\beta+\gamma-1}|\nabla\psi|^p\,dx \\
&= c\int_{\mathbb{R}^N} (\frac{\psi}{u})^{2\beta}\left(w(x)|\nabla\psi|^p\psi^{\gamma-1}\right)\,dx.
\end{aligned}$$

Choosing the exponents $\theta_b = \frac{2\beta+p-1+\gamma}{2\beta}$, $\theta_b' = \frac{2\beta+p-1+\gamma}{p-1+\gamma}$ in Young's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\gamma}\,dx \\
&\le \Big\{\epsilon\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\gamma}\,dx + c_\epsilon\int_{\mathbb{R}^N} w^{\theta_b'}g^{-\frac{\theta_b'}{\theta_b}}\psi^{(\gamma-1)\theta_b'}|\nabla\psi|^{p\theta_b'}\,dx\Big\}.
\end{aligned}$$

Now using that $g^{-1}\in L^\infty(\mathbb{R}^N)$, $0\le\psi\le 1$ and $\gamma\ge 1$, we obtain the required inequality

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\gamma}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_b'}|\nabla\psi|^{p\theta_b'}\,dx. \tag{3.8}$$

Case 3. Since $\delta=\gamma$ and $u>0$ in $\mathbb{R}^{\mathbb{N}}$, for any $\beta\in(0,s_p)$, inequality (3.7) becomes

$$\int_{\mathbb{R}^N} g(x)u^{-2\beta-p+1-\delta}\psi^p\,dx \le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^p\,dx.$$

Replacing ψ by $\psi^{\frac{2\beta+p-1+\delta}{p}}$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx &\le c\left(\frac{2\beta+p-1+\delta}{p}\right)^p\int_{\mathbb{R}^N} w(x)u^{-2\beta}\psi^{2\beta+\delta-1}|\nabla\psi|^p dx \\
&= c\int_{\mathbb{R}^N} (\frac{\psi}{u})^{2\beta}\left(w(x)|\nabla\psi|^p\psi^{\delta-1}\right)\,dx.
\end{aligned}$$

Choosing the exponents $\theta_a = \frac{2\beta+p-1+\delta}{2\beta}$, $\theta_a' = \frac{2\beta+p-1+\delta}{p-1+\delta}$ in Young's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \\
&\le \Big\{\epsilon\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx + c_\epsilon\int_{\mathbb{R}^N} w^{\theta_a'} g^{-\frac{\theta_a'}{\theta_a}}\psi^{(\delta-1)\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx\Big\}.
\end{aligned}$$

Now using that $g^{-1}\in L^\infty(\mathbb{R}^N)$, $0\le\psi\le 1$ and $\delta\ge 1$, we obtain the required inequality

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx.$$

□

Proof of Theorem 2.2. By contradiction, let us suppose that u is a stable solution to $(1.1)_s$ such that $0<u\le 1$ in $\mathbb{R}^N$. Then by Theorem 2.1, we have

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx.$$

Choosing $\psi=\psi_R$, we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p-1+\delta}\,dx \le cR^{-p\theta_a'}\int_{B_{2R}(0)} w^{\theta_a'}\,dx. \tag{3.9}$$

Since $\theta_a' = \frac{2\beta+p-1+\delta}{p-1+\delta} < S_p^a$ and $\|w\|^{S_p^a}_{L^{S_p^a}(B_{2R}(0))} = o(R^{\lambda_p^a})$, we have

$$\int_{B_{2R}(0)} w^{\theta_a'}\,dx \le cR^{\frac{\theta_a'\lambda_p^a}{S_p^a}+\frac{N}{S_p^a}(S_p^a-\theta_a')}.$$

Hence from (3.9), we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p-1+\delta}\,dx \le cR^{\theta_a'(\frac{\lambda_p^a}{S_p^a}-p)+\frac{N}{S_p^a}(S_p^a-\theta_a')}$$

for some positive constant c independent of R. Now,

$$\lim_{\beta\to s_p}\Big\{\theta_a'(\frac{\lambda_p^a}{S_p^a}-p)+\frac{N}{S_p^a}(S_p^a-\theta_a')\Big\} = \lambda_p^a - pS_p^a < 0.$$

Hence, we can choose $\beta\in(0,s_p)$ such that

$$\theta_a'(\frac{\lambda_p^a}{S_p^a}-p)+\frac{N}{S_p^a}(S_p^a-\theta_a') < 0.$$

Therefore, letting $R\to\infty$ in the above integral inequality, we obtain

$$\int_{\mathbb{R}^N} g(x)(\frac{1}{u})^{2\beta+p-1+\delta}\,dx = 0,$$

which is a contradiction. □

Proof of Theorem 2.3. By contradiction, let us suppose that u is a stable solution to $(1.1)_s$ such that $u\ge 1$ in $\mathbb{R}^N$. Then by Theorem 2.1, we have

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\gamma}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_b'}|\nabla\psi|^{p\theta_b'}\,dx.$$

Choosing $\psi = \psi_R$, we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p-1+\gamma}\,dx \le cR^{-p\theta_b'} \int_{B_{2R}(0)} w^{\theta_b'}\,dx. \tag{3.10}$$

Since $\theta_b' = \frac{2\beta+p-1+\gamma}{p-1+\gamma} < S_p^b$ and $\|w\|_{L^{S_p^b}(B_{2R}(0))}^{S_p^b} = o(R^{\lambda_p^b})$, we have

$$\int_{B_{2R}(0)} w^{\theta_b'}\,dx \le cR^{\frac{\theta_b'\lambda_p^b}{S_p^b} + \frac{N}{S_p^b}(S_p^b - \theta_b')}.$$

Hence from (3.10), we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p-1+\gamma}\,dx \le cR^{\theta_b'(\frac{\lambda_p^b}{S_p^b} - p) + \frac{N}{S_p^b}(S_p^b - \theta_b')}$$

for some positive constant c independent of R. Now,

$$\lim_{\beta\to s_p} \Big\{\theta_b'(\frac{\lambda_p^b}{S_p^b} - p) + \frac{N}{S_p^b}(S_p^b - \theta_b')\Big\} = \lambda_p^b - pS_p^b < 0.$$

Hence, we can choose $\beta \in (0, s_p)$ such that

$$\theta_b'(\frac{\lambda_p^b}{S_p^b} - p) + \frac{N}{S_p^b}(S_p^b - \theta_b') < 0.$$

Therefore, letting $R \to \infty$ in the above integral inequality, we obtain

$$\int_{\mathbb{R}^N} g(x)(\frac{1}{u})^{2\beta+p-1+\gamma}\,dx = 0,$$

which is a contradiction. □

Proof of Theorem 2.4. By contradiction, let us suppose that u is a stable solution to $(1.1)_s$. Then by Theorem 2.1, we have

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p-1+\delta}\,dx \le c\int_{\mathbb{R}^N} w^{\theta_a'}|\nabla\psi|^{p\theta_a'}\,dx.$$

Now arguing exactly as in Theorem 2.2, we have the required result. □

Proof of Theorem 2.6. Let $u \in C^1(\mathbb{R}^N)$ be a bounded stable solution to $(1.1)_e$ such that $\|u\|_{L^\infty(\mathbb{R}^N)} \le M$. By Lemma 3.1 using the condition $0 < u \le M$ in $\mathbb{R}^N$ and $f(u) = -e^{1/u}$ in (3.1), we obtain

$$\beta_\epsilon \int_{\mathbb{R}^N} g(x)e^{1/u}u^{-2\beta-p+1}\psi^p\,dx \le c_\epsilon \int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^p\,dx,$$

where

$$\beta_\epsilon = \Big(\frac{1}{M} - \frac{(p-1)(\beta+\frac{p}{2}-1)^2+\epsilon}{2\beta+p-1-\epsilon}\Big).$$

Observe that

$$\lim_{\epsilon\to 0}\beta_\epsilon = \Big(\frac{1}{M} - \frac{(p-1)(\beta+\frac{p}{2}-1)^2}{2\beta+p-1}\Big) > 0 \quad \forall \beta \in (0, t_p).$$

Therefore fixing $\beta \in (0, t_p)$ we can choose an $\epsilon \in (0,1)$ such that $\beta_\epsilon > 0$. Now using the fact $e^{1/u} > \frac{1}{u}$ in the above integral inequality we obtain

$$\int_{\mathbb{R}^N} g(x)u^{-2\beta-p}\psi^p\,dx \le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}|\nabla\psi|^p\,dx.$$

Replacing ψ by $\psi^{\frac{2\beta+p}{p}}$ and using Young's inequality with exponents $\theta = \frac{2\beta+p}{2\beta}$ and $\theta' = \frac{2\beta+p}{p}$ for $\epsilon \in (0,1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx &\le c\int_{\mathbb{R}^N} w(x)u^{-2\beta}\psi^{2\beta}|\nabla\psi|^p\,dx\\ &= c\int_{\mathbb{R}^N} g^{1/\theta}(\frac{\psi}{u})^{2\beta}\big(g^{-1/\theta}w(x)|\nabla\psi|^p\big)\,dx\\ &\le \epsilon\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx + c_\epsilon\int_{\mathbb{R}^N} g^{-\theta'/\theta}w^{\theta'}|\nabla\psi|^{p\theta'}\,dx.\end{aligned}$$

Therefore we obtain the inequality

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx \le c\int_{\mathbb{R}^N} g^{-\theta'/\theta}w^{\theta'}|\nabla\psi|^{p\theta'}\,dx.$$

Since $g^{-1} \in L^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx \le c\int_{\mathbb{R}^N} w^{\theta'}|\nabla\psi|^{p\theta'}\,dx.$$

□

Proof of Theorem 2.7. By contradiction, let us suppose that u is a bounded stable solution to $(1.1)_e$ such that $0 < u \le M$ in $\mathbb{R}^N$. Then by Theorem 2.6, we have

$$\int_{\mathbb{R}^N} g(x)(\frac{\psi}{u})^{2\beta+p}\,dx \le c\int_{\mathbb{R}^N} w^{\theta'}|\nabla\psi|^{p\theta'}\,dx$$

where $\theta = \frac{2\beta+p}{2\beta}$ and $\theta' = \frac{2\beta+p}{p}$. Choosing $\psi = \psi_R$, we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p}\,dx \le cR^{-p\theta'}\int_{B_{2R}(0)} w^{\theta'}\,dx. \tag{3.11}$$

Since $\theta' = \frac{2\beta+p}{p} < T_p$ and $\|w\|^{T_p}_{L^{T_p}(B_{2R}(0))} = o(R^{\mu_p})$, we have

$$\int_{B_{2R}(0)} w^{\theta'}\,dx \le cR^{\frac{\theta'\mu_p}{T_p}+\frac{N}{T_p}(T_p-\theta')}.$$

Hence from (3.11), we obtain

$$\int_{B_R(0)} g(x)(\frac{1}{u})^{2\beta+p}\,dx \le cR^{\theta'(\frac{\mu_p}{T_p}-p)+\frac{N}{T_p}(T_p-\theta')}$$

for some positive constant c independent of R. Now,

$$\lim_{\beta\to t_p}\Big\{\theta'(\frac{\mu_p}{T_p}-p)+\frac{N}{T_p}(T_p-\theta')\Big\} = \mu_p - pT_p < 0.$$

Hence, we can choose $\beta \in (0, t_p)$ such that

$$\theta'(\frac{\mu_p}{T_p}-p)+\frac{N}{T_p}(T_p-\theta') < 0.$$

Therefore, letting $R \to \infty$ in the above integral inequality, we obtain

$$\int_{\mathbb{R}^N} g(x)(\frac{1}{u})^{2\beta+p}\,dx = 0,$$

which is a contradiction. □

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