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EXISTENCE OF TRIVIAL AND NONTRIVIAL SOLUTIONS OF A FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATION

TIHOMIR GYULOV & STEPAN TERSIAN

Abstract. We study the multiplicity of nontrivial solutions for a semilinear fourth-order ordinary differential equation arising in spatial patterns for bistable systems. In the proof of our results, we use minimization theorems and Brezis–Nirenberg's linking theorem. We obtain also estimates on the minimizers of the corresponding functionals.

1. Introduction

In this paper, we study existence and multiplicity of solutions to the boundary-value problem for the fourth-order ordinary differential equation

$$\begin{gathered} u^{iv} + Au'' + Bu + f(x,u) = 0, \\ u(0) = u(L) = u''(0) = u''(L) = 0, \end{gathered} \tag{1.1}$$

where A and B are constants and $f(x,u)$ is a continuous function, defined in $\mathbb{R}^2$, whose potential $F(x,u) = \int_0^u f(x,t)dt$ satisfies suitable assumptions. The problem is motivated by the study of formation of spatial periodic patterns in bistable systems. In the study of spatial patterns an important role is played by a model equation, which is simpler than full equations describing the process. Recently, interest has turned to fourth-order parabolic differential equation, involving bistable dynamics, such as the extended Fisher-Kolmogorov (EFK) equation proposed by Coullet, Elphick & Repaux in 1987 and Dee & VanSaarlos in 1988. Another well known equation of this type is the Swift-Hohenberg (SH) equation proposed in 1977. With appropriate changes of variables, stationary solutions of these equations lead to the equation

$$u^{iv} - pu'' - u + u^3 = 0, \tag{1.2}$$

in which $p > 0$ corresponds to EFK equation and $p < 0$ to the SH equation. Solutions of Eq. (1.2) which are bounded on the real line have been recently studied by a variety of methods such as topological shooting method and variational methods [1, 3, 4, 7, 8, 9, 10].

When f is an even $2L$ periodic function with respect to x, and odd with respect to u, the $2L$ periodic extension $\bar{u}$ of the odd extension of the solution u of the

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problem (1.1) to the interval $[-L, L]$ yields a $2L$ periodic solution of (1.1). The solvability of (1.1) for some extension of (1.2) was studied in [3, 4, 7, 8, 10] by variational methods.

We suppose that $f(x,0) = 0$, $\forall x \in \mathbb{R}$ and the potential

$$F(x,u) = \int_0^u f(x,s)\,ds$$

satisfies following assumptions:

(H1) There is a number $p > 2$ and for each bounded interval I there is a constant $c > 0$ such that

$$F(x,u) \ge c|u|^p, \ \forall x \in I, \forall u \in \mathbf{R}$$

(H2) $F(x,u) = o(u^2)$ as $u \to 0$, uniformly with respect to x in bounded intervals.

A typical function that satisfies (H1) and (H2) is

$$f(x,u) = b(x)u|u|^{p-2}, \quad p > 2,$$

where $b(x)$ is a continuous, positive function.

Problem (1.1) has a variational structure and its solutions can be found as critical points of the functional

$$I(u;L) := \frac{1}{2}\int_0^L (u''^2 - Au'^2 + Bu^2)dx + \int_0^L F(x,u)dx \tag{1.3}$$

in the Sobolev space

$$X(L) := H^2(0,L) \cap H_0^1(0,L).$$

In this work we obtain nontrivial critical points of the functional I using Brezis-Nirenberg's linking theorem [2, 5]. Recall its statement. Let E be a Banach space with a direct sum decomposition $E = X \oplus Y$. The functional $J \in C^1(E,\mathbb{R})$ has a local linking at 0 if, for some $r > 0$

$$\begin{aligned} J(x) \le 0, \quad x \in X, \quad \|x\| \le r\,, \\ J(y) \ge 0, \quad y \in Y, \quad \|y\| \le r\,. \end{aligned}$$

Theorem 1.1 (Brezis and Nirenberg [2]). *Suppose that $J \in C^1(E,\mathbb{R})$ satisfies the (PS) condition and has a local linking at 0. Assume that J is bounded below and $\inf_E J < 0$. Then J has at least two nontrivial critical points.*

It is easy to see that if $4B \ge A^2$ and $f(x,u)u > 0$ for $x \ge 0$ and $u \ne 0$ the problem (1.1) has only the trivial solution. We shall assume $4B < A^2$ and study separately the cases $A \le 0$ (EFK equation) and $A > 0$ (SH equation). Our main results are as follows.

Theorem 1.2 (Nontrivial solutions). *Let the function $F(x,u)$ satisfy (H1) and (H2).*

(i) *Let $4B < A^2$, $A \le 0$, $B < 0$ and set $L_1 := \pi\sqrt{2}/\sqrt{A + \sqrt{A^2 - 4B}}$. If $L > L_1$, then problem* (1.1) *has at least two nontrivial solutions.*

(ii) *Let $4B < A^2$, $A > 0$, and set $L_1 := \pi\sqrt{2}/\sqrt{A + \sqrt{A^2 - 4B}}$. Then, problem* (1.1) *has at least two nontrivial solutions if either*

(a) *$B \le 0$ and $L > L_1$, or*

(b) *$B > 0$, and $L \in]nL_1, nM_1[$, where $M_1 := \pi\sqrt{2}/\sqrt{A - \sqrt{A^2 - 4B}}$.*

Theorem 1.3 (trivial solutions)**.** *Let the continuous function $f(x,u)$ satisfy the assumption $f(x,0)=0$ and $f(x,u)u>0, u\neq 0$ for $x\in[0,L]$.*

(i) *Let $4B<A^2$, $A\leq 0$,, set $L_1:=\pi\sqrt{2}/\sqrt{A+\sqrt{A^2-4B}}$ for $B<0$. Then problem* (1.1) *has only the trivial solution provided that one of the following holds: (a) $B\geq 0$, or (b) $B<0$ and $0<L\leq L_1$.*

(ii) *Let $4B<A^2$, $A>0$, set $L_1:=\pi\sqrt{2}/\sqrt{A+\sqrt{A^2-4B}}$ and*
$$h_n=\Big(\frac{(n^2+n)A}{2n^2+2n+1}\Big)^2,$$
$n\in\mathbb{N}\cup\{0\}$. Then problem (1.1) *has only the trivial solution provided that one of the following holds: (a) $B\leq 0$ and $0<L<L_1$, or (b) $h_n<B\leq h_{n+1}$ and $L\in T_{n+1}$, where T_{n+1} is a finite union of bounded intervals.*

Next, we consider the problem
$$\begin{gathered}u^{iv}+Au''+Bu+u^3=0,\quad 0<x<L,\\ u(0)=u(L)=u''(0)=u''(L)=0,\end{gathered}\tag{1.4}$$
which is related to stationary the EFK equation or to stationary the SH equation. The corresponding energy functional is
$$J(u;L)=\frac{1}{L}\Big\{\frac{1}{2}\int_0^L(u''^2-Au'^2+Bu^2)dx+\frac{1}{4}\int_0^L u^4dx\Big\}.$$
By Theorem 1.2 for $4B<A^2$, problem (1.4) has at least two nontrivial solutions if L belongs to infinite interval $]L_1,+\infty[$ if $A\leq 0$,or to a bounded interval $]nL_1,nM_1[$, if $A>0$, where L_1 and M_1 are depended on A and B. One of these nontrivial solutions is a nontrivial minimizer u_0 of the functional J. In this section we will estimate the average of L^2-norm of the minimizer u_0.

Let for $u\in X(L)$, let
$$|u|^2:=\frac{1}{L}\int_0^L u^2(x)dx.$$
Let $P(\xi)=\xi^4-A\xi^2+B$ be the symbol of the linear operator $\mathcal{L}u=u^{iv}+Au''+Bu$. By the proof of Theorem 1.2, if $L\in\Delta_n$, where Δ_n is an interval which is the set of solutions of the inequality $P_n(L)<0$, (1.4) has at least two nontrivial solutions. Moreover if $L\in\Delta_n$, there exist natural numbers $m,m+1,\dots,m+k$, $m\geq 1,k\geq 0$ depending on L such that $P_j(L)<0$ if and only if $j\in S=\{m,m+1,\dots,m+k\}$ and $P_j(L)\geq 0$ if and only if $j\notin S$. Let $E_{k+1}(L)$ be the finite dimensional subspace of $X(L)$
$$E_{k+1}(L)=\operatorname{span}\Big\{\sin(\frac{m\pi x}{L}),\dots,\sin(\frac{(m+k)\pi x}{L})\Big\},$$
and for $u\in X(L)$, $u=\overline{u}+\widetilde{u}$, $\overline{u}\in E_{k+1}$, $\widetilde{u}\in E_{k+1}^{\perp}$ be the orthogonal decomposition of u.

Theorem 1.4. *Let for a fixed $n\in\mathbb{N}$, let Δ_n be the set of solutions of the inequality $P_n(L)<0$. For $L\in\Delta_n$ let*
$$P_j(L)<0\quad \text{if } j\in S=\{m,m+1,\dots,m+k\},$$
and
$$p_n=P_{m_n}(L)=\min\{P_j(L):j\in S\}<0.$$

Then, if $L \in \Delta_n$, problem (1.4) has a nontrivial solution u_0, which is a minimizer of the functional J, and the following estimates hold:

(i) $-\frac{1}{4}p_n^2 \le J(u_0; L) \le -\frac{1}{6}p_n^2$
(ii) $\frac{2}{3}|p_n| \le |\overline{u}_0|^2 \le |u_0|^2 \le |p_n|$
(iii) $|\widetilde{u}_0|^2 \le (-\frac{2}{3} + \sqrt{\frac{2}{3}})|p_n|$
(iv) $J_1(\widetilde{u}_0; L) \le \frac{2}{9}p_n^2$.

This paper is organized as follows: In Section 2 we prove some auxiliary lemmas. In Section 3 we prove Theorem 1.2, and Theorem 1.3. In Section 3 we prove Theorem 1.4.

2. Preliminaries

We study the nonautonomous fourth-order ordinary differential equation

$$u^{iv} + Au'' + Bu + f(x,u) = 0, \quad 0 < x < L,$$

where $A \in \mathbb{R}$, $B \in \mathbb{R}$ are constants and $f(x,u)$ is a continuous function, whose potential $F(x,u) = \int_0^u f(x,t)dt$ is a nonnegative function which satisfies assumptions (H1) and (H2).

Let $X(L)$ be the Sobolev space

$$X(L) := \{H^2(0,L) : u(0) = u(L) = 0\}.$$

A weak solution of the problem (1.1) is a function $u \in X(L)$, such that

$$\int_0^L (u''v'' - Au'v' + Buv + f(x,u)v)dx = 0, \quad \forall v \in X(L).$$

One can prove that a weak solution of (1.1) is a classical solution of (1.1) (see [10, Proposition 1]. Weak Solutions of (1.1) are critical points of the functional $I : X(L) \to \mathbb{R}$,

$$I(u;L) := \frac{1}{2}\int_0^L (u''^2 - Au'^2 + Bu^2)dx + \int_0^L F(x,u)dx. \tag{2.1}$$

The following technical lemmata play an important role in further considerations.

Lemma 2.1. *We have the following: For $u \in X(L)$,*

$$\int_0^L u^2 dx \le \frac{L^{2k}}{\pi^{2k}} \int_0^L (u^{(k)})^2 dx, \quad k = 1, 2\,. \tag{2.2}$$

The scalar product

$$\langle u, v\rangle = \int_0^L u''v''dx, \quad u \in X(L),\ v \in X(L)$$

induces an equivalent norm in $X(L)$. The set of functions $\{\sin(\frac{n\pi x}{L}) : n \in \mathbb{N}\}$ is a complete orthogonal basis in $X(L)$.

Proof. The Poincáre type inequality (2.2) is proved in [6]. For $u \in X(L)$, we have

$$\int_0^L u'^2 dx = \int_0^L u' du = -\int_0^L uu''dx \le \frac{1}{2}\int_0^L (u^2 + u''^2)dx \le \frac{1}{2}\big(\frac{L^4}{\pi^4} + 1\big)\int_0^L u''^2 dx,$$

which shows that $[u] = \langle u, u\rangle^{1/2}$ is an equivalent norm in $X(L)$. The set of functions $\{\sin \frac{n\pi x}{L} : n \in \mathbb{N}\}$ is clearly orthogonal in $X(L)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. It is a complete orthogonal basis in $X(L)$. Indeed, let $v \in X(L)$ be such that

$$\langle v, \sin \frac{n\pi x}{L}\rangle = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$0 = \int_0^L v''(\sin \frac{n\pi x}{L})'' dx = \int_0^L v(\sin \frac{n\pi x}{L})^{(4)} dx = (\frac{n\pi}{L})^4 \int_0^L v \sin \frac{n\pi x}{L} dx,$$

and

$$\int_0^L v \sin \frac{n\pi x}{L} dx = 0, \quad \forall n \in \mathbb{N}.$$

Since $X(L) \subset L^2(0, L)$ and $\{\sin \frac{n\pi x}{L} : n \in \mathbb{N}\}$ is an orthogonal basis in $L^2(0, L)$ it follows that $v = 0$, which means that the set $\{\sin \frac{n\pi x}{L} : n \in \mathbb{N}\}$ is a complete orthogonal basis in $X(L)$. □

Lemma 2.2. *Let A, B be constants and $f(x, u)$ be a continuous function such that (H1) holds. Then the functional I is bounded from below and it satisfies the (PS) condition.*

Proof. Using Fourier series arguments and the previous lemma, we obtain that for $u \in X(L)$,

$$\begin{gathered} u = \sum_{k=1}^{\infty} c_k \sin \frac{k\pi x}{L}, \\ I(u; L) = \frac{L}{4} \sum_{k=1}^{\infty} c_k^2 P(\frac{k\pi}{L}) + \int_0^L F(x, u) dx, \end{gathered} \tag{2.3}$$

where $P(\xi) = \xi^4 - A\xi^2 + B$ is the symbol of the linear differential operator

$$\mathcal{L}(u) := u^{(4)} + Au'' + Bu.$$

Observe that $P(\xi)$ is bounded from below for any A and B

$$P(\xi) \geq B - \frac{A^2}{4}.$$

It follows from (2.3) and (H1) that $I(u; L) \geq 0$ if $4B \geq A^2$. If $4B < A^2$ we have

$$I(u; L) > \frac{1}{2}\big(B - \frac{A^2}{4}\big)\|u\|_{L^2}^2 + C_1(L)\|u\|_{L^2}^p. \tag{2.4}$$

From the elementary inequality

$$-ax^2 + bx^p \geq -a\frac{p-2}{p}(\frac{2a}{pb})^{\frac{2}{p-2}}$$

for $a > 0$, $b > 0$, $x > 0$ and $p > 2$, it follows that the right hand side of (2.4) is bounded from below by a negative constant.

Suppose now that $(u_n)_n$ is a (PS) sequence, i.e. there exists $c_1 > 0$ such that

$$c_1 > |I(u_n; L)| \quad \text{and} \quad I'(u_n; L) \to 0. \tag{2.5}$$

In what follows c_j will denote various positive constants. We have

$$I(u; L) = \frac{1}{4}\int_0^L u''^2 dx + \frac{1}{2}\bar{I}(u; L),$$

where

$$\bar{I}(u;L)=\frac{1}{2}\int_0^L(u''^2-2Au'^2+2Bu^2)dx+2\int_0^L F(x,u)dx.$$

As before the functional $\bar{I}$ is bounded from below and we have

$$c_1\geq\frac{1}{4}\int_0^L u_n''^2dx-c_2.$$

The sequence $(u_n)_n$ is a bounded sequence in $X(L)$ in view of Lemma 2.1. There exists a subsequence still denoted by $(u_n)_n$ and a function $u_0\in X(L)$ such that

$$u_n\rightharpoonup u_0\quad\text{in}\quad X(L),\tag{2.6}$$

and by Sobolev's embedding theorem

$$\begin{aligned}u_n&\to u_0\quad\text{in}\quad C^1[0,L],\\ u_n&\to u_0\quad\text{in}\quad L^2(0,L).\end{aligned}\tag{2.7}$$

Since $f(x,u)$ is continuous and $\{|u_n(x)|\}$ uniformly bounded in $[0,L]$, and letting $n\to\infty$ in

$$(I'(u_n;L),u_0)=\int_0^L(u_n''u_0''-Au_n'u_0'+Bu_nu_0+f(x,u_n)u_0)dx$$

we obtain

$$\int_0^L(u_0''^2-Au_0'^2+Bu_0^2+f(x,u_0)u_0)dx=0.\tag{2.8}$$

From the boundedness of $(u_n)_n$ in $X(L)$ and (2.8) it follows $(I'(u_n;L),u_n)\to 0$ and

$$\begin{aligned}\int_0^L u_n''^2dx&=(I'(u_n),u_n)+\int_0^L(Au_n'^2-Bu_n^2-f(x,u_n)u_n)dx\\ &\to\int_0^L(Au_0'^2-Bu_0^2-f(x,u_0)u_0)dx=\int_0^L u_0''^2dx,\end{aligned}$$

which implies that $\|u_n\|\to\|u_0\|$ and then $\|u_n-u_0\|\to 0$, which completes the proof of Lemma 2.2. $\square$

3. Existence results

The polynomial

$$p(\xi)=\xi^4-A\xi^2$$

and the real functions

$$p_n(L)=p(\frac{n\pi}{L})$$

play an important role in the sequel.

Let $A\leq 0$. The polynomial $p(\xi)$ is a positive increasing and convex function for $\xi>0$. The functions $p_n(L)$ are positive decreasing functions for every $n\in\mathbb{N}$ and

$$\begin{gathered}p_n(L)\to+\infty,\quad\text{as } L\to 0,\\ p_n(L)\to 0,\quad\text{as}\quad L\to+\infty.\end{gathered}$$

These functions are ordered as

$$0<p_1(L)<p_2(L)<\ldots<p_n(L)<\ldots$$

for every $L>0$, and some of their graphs are showm in Figure 1.

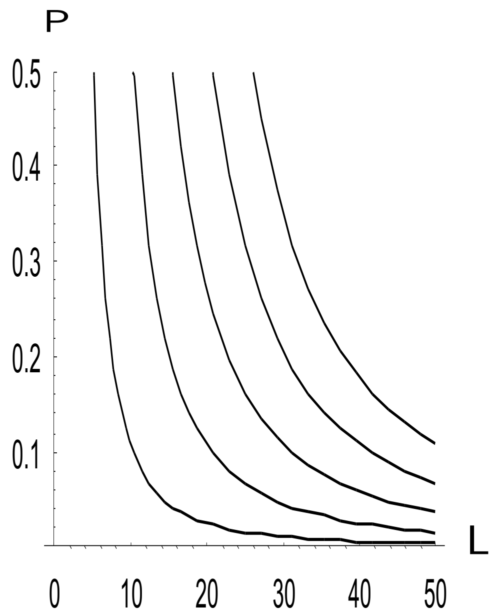

FIGURE 1. Graphs of functions $p_n(L) = (\frac{n\pi}{L})^4 + (\frac{n\pi}{L})^2$, $n = 1, 2, 3, 4$

Let $B < 0$. Then the equation $p_n(L) + B = 0$ has the unique solution

$$L_n = nL_1, \quad L_1 := \frac{\pi\sqrt{2}}{\sqrt{A + \sqrt{A^2 - 4B}}} \tag{3.1}$$

and

$$p_n(L) + B \geq 0 \quad \text{if } L \leq nL_1, \tag{3.2}$$
$$p_n(L) + B < 0 \quad \text{if } L > nL_1. \tag{3.3}$$

Let $A > 0$. Then the polynomial $p(\xi) = \xi^4 - A\xi^2$ is positive for $\xi > \sqrt{A}$ and it has a negative minimum $p_0 = -A^2/4$ at $\xi_0 = \sqrt{A/2}$. The functions $p_n(L)$ are decreasing if $0 < L < n\pi\sqrt{2/A}$ and increasing if $L > n\pi\sqrt{2/A}$, $p_n(L) > 0$ if $0 < L < n\pi/\sqrt{A}$ and $p_n(L) < 0$ if $L > n\pi/\sqrt{A}$. The graphs of functions $p_n(L)$ with $A = 1$ and $n = 1, 2, 3, 4$ are presented on Figure 2.

Lemma 3.1. *Let* $l_n := \frac{\pi}{\sqrt{A}}\sqrt{2n^2 + 2n + 1}$ *and* $L_1 := \pi\sqrt{2}/\sqrt{A + \sqrt{A^2 - 4B}}$. *Then we have the following results:*

(a)
$$\begin{gathered} p_n(L) = p_{n+1}(L) \Leftrightarrow L = l_n, \\ p_n(L) < p_{n+1}(L) \Leftrightarrow L < l_n, \\ p_n(L) > p_{n+1}(L) \Leftrightarrow L > l_n, \end{gathered} \tag{3.4}$$

and

$$q(L) = \inf\{p_n(L) : n \in \mathbb{N}\} = \begin{cases} p_1(L), & 0 < L \leq l_1, \\ p_{n+1}(L), & l_n < L \leq l_{n+1}. \end{cases} \tag{3.5}$$

(b) *Let* $B \leq 0$. *Then* $p_n(L) + B < 0$ *if and only if* $L > nL_1$.

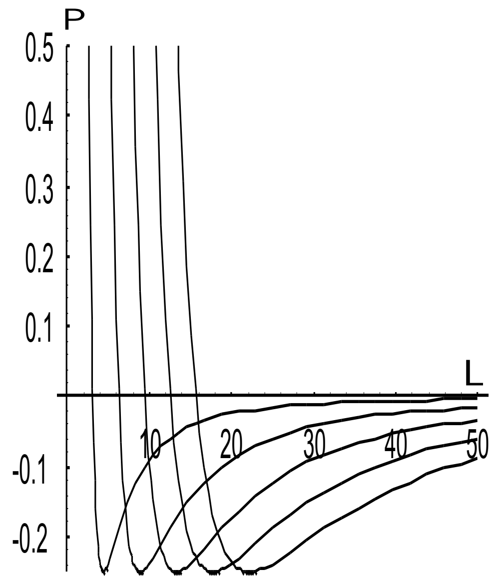

FIGURE 2. Graphs of functions $p_n(L) = (\frac{n\pi}{L})^4 - (\frac{n\pi}{L})^2$, $n = 1, 2, 3, 4$

(c) *Let $B > 0$ and $M_1 := \pi\sqrt{2}/\sqrt{A - \sqrt{A^2 - 4B}}$. Then $p_n(L) + B < 0$ if and only if $nL_1 < L < nM_1$.*

Proof. (a) The equation $p_n(L) = p_{n+1}(L)$ is equivalent to $(nx)^4 - A(nx)^2 = ((n+1)x)^4 - A((n+1)x)^2$, where $x = \pi/L$. A direct calculation shows that it is satisfied for $x^2 = A/(2n^2 + 2n + 1)$, i.e. for $L = l_n = \frac{\pi}{\sqrt{A}}\sqrt{2n^2 + 2n + 1}$. We have

$$p_n(l_n) = -\big(\frac{(n^2 + n)A}{2n^2 + 2n + 1}\big)^2 = -h_n$$

and (3.4) holds. From (3.4) it follows (3.5).
(b) To solve the inequality $p_n(L) + B < 0$ we set $x = \pi/L$, and assuming that $B \leq 0$ we are led to

$$(nx)^2 \in \big]0, \frac{A + \sqrt{A^2 - 4B}}{2}\big[,$$

equivalent to $L > nL_1$.
(c) To solve the same inequality in the case $B > 0$ we compute

$$(nx)^2 \in \big]\frac{A - \sqrt{A^2 - 4B}}{2}, \frac{A + \sqrt{A^2 - 4B}}{2}\big[$$

which is equivalent to $nL_1 < L < nM_1$. □

Proof of Theorem 1.2. Case (i). Let $L > L_1$. There exists a natural number n such that $nL_1 < L \leq (n+1)L_1$. Let $\varphi_n \in E_n = \text{span}\{\sin\frac{\pi x}{L}, \dots, \sin\frac{n\pi x}{L}\}$ such that

$$\varphi_n(x) = \sum_{k=1}^{n} c_k \sin(\frac{k\pi x}{L}),$$

and set $c_1^2 + \dots + c_n^2 = \rho^2$. By (3.3) we have that

$$\alpha_n = \max\{p_k(L) + B : k = 1, \dots, n\} < 0.$$

Let us take a small constant ε, such that $0 < \varepsilon < -\alpha_n/2$.

By assumption (H2), there exists $\delta > 0$ such that if $|u| \le \delta$ then $F(x,u) \le \varepsilon |u|^2$, $x \in [0,L]$. Let us take $\rho, 0 < \rho \le \delta/\sqrt{n}$. Then by

$$|\varphi_n(x)| \le \sum_{k=1}^{n} |c_k| \le \sqrt{n} (\sum_{k=1}^{n} c_k^2)^{1/2} = \sqrt{n}\rho \le \delta$$

it follows that $F(x, \varphi_n(x)) \le \varepsilon |\varphi_n(x)|^2$ and

$$\begin{aligned} \int_0^L F(x,\varphi_n(x)) dx &\le \varepsilon \int_0^L |\varphi_n(x)|^2 dx \\ &= \varepsilon \int_0^L \sum_{k=1}^{n} c_k^2 \sin^2(\frac{k\pi x}{L}) dx \\ &= \varepsilon \frac{L}{2} \sum_{k=1}^{n} c_k^2 = \varepsilon \frac{L}{2}\rho^2. \end{aligned}$$

We have

$$\begin{aligned} I(\varphi_n; L) &= \frac{L}{4} \sum_{k=1}^{n} (p_k(L) + B) c_k^2 + \int_0^L F(x, \varphi_n(x)) dx \\ &\le \frac{L}{4} \alpha_n \rho^2 + \varepsilon \frac{L}{2} \rho^2 \\ &= \frac{L}{2} \rho^2 (\frac{1}{2}\alpha_n + \varepsilon) < 0, \end{aligned} \tag{3.6}$$

if $0 < \rho \le \delta/\sqrt{n}$. The functional I has a local linking at 0. Indeed, by (3.6), for sufficiently small $\rho > 0$ we have

$$I(u; L) \le 0, \quad u \in E_n, \|u\| < \rho.$$

Let $u \in E_n^{\perp}$ and $\|u\| \le \rho$. It follows that $p_{n+1}(L) + B \ge 0$ if $nL_1 < L \le (n+1)L_1$ by (3.2). Since $p_{n+1}(L) < p_{n+2}(L) < \dots$,by assumption (H1) there exists $C(L) > 0$ such that

$$\begin{aligned} I(u; L) &\ge \frac{1}{2} \min((p_k(L) + B) : k \ge n+1) \|u\|_{L^2}^2 + C(L) \|u\|_{L^2(0,L)}^p \\ &\ge \frac{1}{2} (p_{n+1}(L) + B) \|u\|_{L^2}^2 + C(L) \|u\|_{L^2(0,L)}^p \ge 0, \end{aligned}$$

if $u \in E_n^{\perp}$. The functional I satisfies the (PS) condition. In view of Theorem 1.1, for $L > L_1$ the functional I has at least two nontrivial critical points.
(ii). By Lemma 3.1, (b) $p_k(L) + B < 0$ iff $L > kL_1$. If $L > L_1$ there exists a natural number n such that $nL_1 < L \le (n+1)L_1$ and

$$\begin{gathered} p_k(L) + B < 0, \quad k = 1, \dots, n \\ p_k(L) + B \ge 0, \quad k \ge n+1. \end{gathered}$$

In this case the proof is finished exactly as in the proof in the Case (i).
Step 1. Nontrivial solutions in the case $B > 0$. Let $\Delta_n =]nL_1, nM_1[$. Observe that for a fixed $L \in \Delta_n$ there exist finite number of intervals Δ_j numbered as

$\Delta_m, \Delta_{m+1}, \dots, \Delta_{m+k}$ such that $L \in \Delta_j \cup \Delta_n$ if and only if $j \in S := \{m, m+1, \dots, m+k\}$ and

$$\begin{aligned} p_j(L) + B < 0, &\quad j \in S, \\ p_j(L) + B \geq 0, &\quad j \notin S. \end{aligned}$$

Let

$$E_{k+1} := \operatorname{span}\big\{\sin\big(\frac{m\pi x}{L}\big), \sin\big(\frac{(m+1)\pi x}{L}\big), \dots, \sin\big(\frac{(m+k)\pi x}{L}\big)\big\}.$$

With a computation similar to the one in the proof of Theorem 1.3 we observe that

$$I(u,L) < 0, \quad u \in E_{k+1},\ 0 < \|u\| \leq r,$$

if r is sufficiently small and

$$I(u,L) \geq 0, \quad u \in E_{k+1}^{\perp},$$

which implies that I has a local linking at 0. Then I has at least two nontrivial critical points. $\square$

Proof of Theorem 1.3. By Lemma 2.1, for $u \in X(L)$ we have:

$$\begin{gathered} u = \sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{L}, \\ I(u;L) = \frac{L}{4}\sum_{k=1}^{\infty} c_k^2 P\big(\frac{k\pi}{L}\big) + \int_0^L F(x,u)dx, \\ \mathcal{B}(u,u) := \langle I'(u;L), u\rangle = \frac{L}{2}\sum_{k=1}^{\infty} c_k^2 P\big(\frac{k\pi}{L}\big) + \int_0^L f(x,u)u\,dx, \end{gathered}$$

where $P(\xi) = \xi^4 - A\xi^2 + B = p(\xi) + B$ is the symbol of the linear differential operator

$$\mathcal{L}(u) := u^{iv} + Au'' + Bu.$$

Case (i). Let $B < 0$ and $0 < L \leq L_1$. We have seen that $p_1(L) + B \geq 0$. As $P(\frac{k\pi}{L}) \geq p_1(L) + B \geq 0$ we infer that $\mathcal{B}(u,u) > 0$ if $u \neq 0$ which means that the functional I has only the trivial critical point. If $B \geq 0$ the same argument applies for every $L > 0$.
Case (ii). We consider the solvability of the inequality

$$q(L) + B \geq 0, \tag{3.7}$$

where $q(L) = \inf\{p_n(L) : n \in \mathbb{N}\}$. Let $0 < B \leq (\frac{4}{25})A^2 = h_1$ and

$$T_1 := \begin{cases}]0, L_1], & B < h_1 \\ ]0, L_1] \cup \{l_1\}, & B = h_1. \end{cases} \tag{3.8}$$

By Lemma 3.1, (c) if $B \leq h_1$, inequality (3.7) holds if and only if $L \in T_1$. Let $l_0 = 0, h_n < B < h_{n+1}$ and

$$D_{n+1} =]0, L_1] \cup [M_1, 2L_1] \cup \dots \cup [nM_1, (n+1)L_1].$$

Let

$$T_{n+1} := \begin{cases} D_{n+1}, & h_n < B < h_{n+1}, \\ D_{n+1} \cup \{l_{n+1}\}, & B = h_{n+1}. \end{cases} \tag{3.9}$$

By Lemma 3.1, (c) if $h_n < B \le h_{n+1}$ the inequality (3.7) is satisfied if and only if $L \in T_{n+1}$.

We estimate the quadratic term in $\mathcal{B}(u,u)$ as

$$\frac{L}{2}\sum_{k=1}^{\infty} c_k^2 P(\frac{k\pi}{L}) \ge (q(L)+B)\|u\|_{L^2}^2 \ge 0,$$

and conclude that that $\mathcal{B}(u,u) > 0$ if $u \ne 0$. Then the functional I has only the trivial critical point which completes the proof of Theorem 1.3. □

4. Bounds for the minimizer

Let us consider the problem

$$u^{iv} + Au'' + Bu + u^3 = 0, \quad 0 < x < L,$$
$$u(0) = u(L) = u''(0) = u''(L) = 0,$$

and the corresponding energy functional

$$J(u;L) = \frac{1}{L}\Big\{\frac{1}{2}\int_0^L (u''^2 - Au'^2 + Bu^2)dx + \frac{1}{4}\int_0^L u^4 dx\Big\}.$$

For $u \in X(L)$, let

$$|u|^2 := \frac{1}{L}\int_0^L u^2(x)dx,$$

and

$$J(u;L) = \frac{1}{2}J_1(u;L) + \frac{1}{4}J_2(u;L) \tag{4.1}$$

where

$$J_1(u;L) := \frac{1}{L}\int_0^L (u''^2 - Au'^2 + Bu^2)dx,$$
$$J_2(u;L) := \frac{1}{L}\int_0^L u^4 dx.$$

Let $P(\xi) = \xi^4 - A\xi^2 + B$ be the symbol of the linear operator $\mathcal{L}u = u^{iv} + Au'' + Bu$ and

$$P_n(L) = P(\frac{n\pi}{L}) = p(\frac{n\pi}{L}) + B.$$

We have that $u \in X(L)$

$$u = \sum_{j=1}^{\infty} c_j \sin(\frac{j\pi x}{L}),$$
$$J(u;L) = \frac{1}{4}\sum_{j=1}^{\infty} c_j^2 P_j(L) + \frac{1}{4L}\int_0^L u^4 dx.$$

If u_0 is the minimizer of J, then

$$\frac{d}{dt}J(tu_0;L)\Big|_{t=1} = J_1(u_0;L) + J_2(u_0;L) = 0. \tag{4.2}$$

By the proof of Theorem 1.2, if $L \in \Delta_n$, where Δ_n is an interval which is the set of solutions of the inequality $P_n(L) < 0$, (1.4) has at least two nontrivial solutions. Moreover, if $L \in \Delta_n$, there exist natural numbers $m, m+1, \dots, m+k, \quad m \ge$

$1, k \geq 0$ depending on L such that $P_j(L) < 0$ if $j \in S = \{m, m+1, \dots, m+k\}$ and $P_j(L) \geq 0$ if $j \notin S$.

Let $E_{k+1}(L)$ be the finite dimensional subspace of $X(L)$

$$E_{k+1}(L) = \operatorname{span}\{\sin(\frac{m\pi x}{L}), \dots, \sin(\frac{(m+k)\pi x}{L})\},$$

and for $u \in X(L),\ u = \overline{u} + \widetilde{u} \quad \overline{u} \in E_{k+1},\ \widetilde{u} \in E_{k+1}^{\perp}$ be the orthogonal decomposition of u. We have

$$J_1(\overline{u}; L) = \sum_{j \in S} c_j^2 P_j(L) \leq 0, \tag{4.3}$$

$$J_1(\widetilde{u}; L) = \sum_{j \notin S} c_j^2 P_j(L) \geq 0. \tag{4.4}$$

Denote

$$p_n = P_{m_n}(L) = \min\{P_j(L) : j \in S\} < 0,$$

where $m_n \in S$ depends on n and L.

Proof of Theorem 1.4. We have

$$J(u; L) = \frac{1}{4} \sum_{j=1}^{\infty} c_j^2 P_j(L) + \frac{1}{4L} \int_0^L u^4 dx \geq \frac{1}{2} p_n |u|^2 + \frac{1}{4} |u|^4 \geq -\frac{1}{4} p_n^2. \tag{4.5}$$

The proof of the estimates (i)-(iv) is based upon a selection of a suitable test function. We set

$$E_n(L) = sp\{\sin(\frac{m_n \pi x}{L})\} \subset E(L), u_1(x) = c \sin(\frac{m_n \pi x}{L}) \in E_n(L),$$

where c will be chosen later. We have

$$J(u_0; L) \leq J(u_1; L) = \frac{1}{4} c^2 p_n + \frac{3c^4}{32}.$$

Taking $c^2 = c_0^2 = -\frac{4}{3} p_n = \frac{4}{3} |p_n|$, we obtain

$$J(u_0; L) \leq J(u_1; L) = -\frac{1}{6} p_n^2, \tag{4.6}$$

which together with (4.5) proves (i).

We have by (4.1), (4.2), and (4.6)

$$-\frac{1}{6} p_n^2 \geq J(u_0; L) = \frac{1}{4} J_1(u_0; L) \geq \frac{1}{4} p_n |\overline{u}_0|^2$$

which implies

$$|\overline{u}_0|^2 \geq -\frac{2}{3} p_n = \frac{2}{3} |p_n| = |u_1|^2. \tag{4.7}$$

However by (4.2) and (4.4)

$$\begin{aligned} |u_0|^4 &\leq \frac{1}{L} \int_0^L u_0^4 dx = -J_1(u_0; L) \\ &= -J_1(\overline{u}_0; L) - J_1(\widetilde{u}_0; L) \\ &\leq -J_1(\widetilde{u}_0; L) + |p_n| |\overline{u}_0|^2 \\ &\leq |p_n| |\overline{u}_0|^2 \leq |p_n| |u_0|^2 \end{aligned} \tag{4.8}$$

and

$$|u_0|^2 \leq |p_n|, \tag{4.9}$$

which together with (4.7) proves (ii).

Denote $p=|\overline{u}_0|^2$ and $q=|\widetilde{u}_0|^2$. Then we have $|u_0|^2=p+q$ and by (4.8),

$$(p+q)^2=|u_0|^4\le -J_1(\widetilde{u}_0;L)+|p_n|p$$

which is equivalent to

$$p^2+(2q-|p_n|)p+(q^2+J_1(\widetilde{u}_0;L))\le 0. \tag{4.10}$$

Then

$$(2q-|p_n|)^2-4(q^2+J_1(\widetilde{u}_0;L))\ge 0$$

which implies

$$D_n^1:=|p_n|^2-4|p_n|q-4J_1(\widetilde{u}_0;L)\ge 0.$$

It follows by (4.10) that

$$\frac{1}{2}(|p_n|-2q-\sqrt{D_n^1})\le p\le \frac{1}{2}(|p_n|-2q+\sqrt{D_n^1}).$$

By (i), we have $\frac{2}{3}|p_n|\le p$ and then

$$\frac{2}{3}|p_n|\le \frac{1}{2}(|p_n|-2q+\sqrt{D_n^1})$$

or

$$\frac{1}{6}|p_n|+q\le \frac{1}{2}\sqrt{D_n^1}.$$

Then

$$\frac{1}{36}|p_n|^2+\frac{1}{3}|p_n|q+q^2\le \frac{1}{4}(|p_n|^2-4|p_n|q-4J_1(\widetilde{u}_0;L)),$$

or

$$q^2+\frac{4}{3}|p_n|q+(J_1(\widetilde{u}_0;L)-\frac{2}{9}|p_n|^2)\le 0.$$

Then, we have

$$D_n^2=\frac{4}{9}|p_n|^2-\big(J_1(\widetilde{u}_0;L)-\frac{2}{9}|p_n|^2\big)\ge 0$$

or

$$\frac{2}{3}|p_n|^2-J_1(\widetilde{u}_0;L)\ge 0.$$

Moreover

$$-\frac{2}{3}|p_n|-\sqrt{D_n^2}\le q\le -\frac{2}{3}|p_n|+\sqrt{D_n^2}$$

and since $J_1(\widetilde{u};L)\ge 0$,

$$0\le q\le \Big(-\frac{2}{3}+\sqrt{\frac{2}{3}}\Big)|p_n|,$$

which proves (iii). Observe that from

$$0\le -\frac{2}{3}|p_n|+\sqrt{\frac{2}{3}|p_n|^2-J_1(\widetilde{u}_0;L)}$$

it follows $J_1(\widetilde{u}_0;L)\le \frac{2}{9}|p_n|^2$, which proves (iv). $\square$

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