Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 272, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

APPROXIMATE CONTROLLABILITY OF FRACTIONAL CONTROL SYSTEMS WITH TIME DELAY USING THE SEQUENCE METHOD

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Abstract. The aim of this article is to establish sufficient conditions for the approximate controllability of fractional control systems with time delay in Hilbert spaces. By the technique of sequential approach, we prove that the fractional control systems with time delay are approximately controllable. Finally, an example is provided to illustrate our main results.

1. Introduction

Let V and U be Hilbert spaces. For $0 < h < b$, let $Z = L^2([0,b];V)$ be the function space corresponding to V and $Y = L^2([0,b];U)$ be the function space associating to U. $C(J,V)$ denotes the Banach space of all continuous functions from the time interval J to V with the norm $\|x\|_C^* = \sup_{t\in J} e^{-\lambda t}\|x(t)\|_V$, where λ is a fixed positive constant which will be fixed in Theorem 3.1. The purpose of this paper is to consider the approximate controllability of the fractional control systems with time delay:

$$\begin{gathered} {}^C D_t^\alpha x(t) = Ax(t) + A_1 x(t-h) + Bu(t) + f(t, x(t-h)), \\ t > 0, \quad \frac{1}{2} < \alpha \le 1, \\ x(t) = \phi(t), \quad -h \le t \le 0. \end{gathered} \tag{1.1}$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order α with the lower limit zero. The state function $x(\cdot)$ takes its values in the space Z and the control function $u(\cdot)$ takes its values in the space Y. Let $\phi \in C([-h,0];V)$. $A : D(A) \subseteq V \to V$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \ge 0)$ on the Hilbert space V. A_1 is a bounded linear operator on V. $B : Y \to Z$ is a bounded linear operator. $f : [0,b] \times V \to V$ is a given function to be specified later.

Fractional differential equations arise in a natural manner as mathematical models of dynamic systems that exhibit such properties as long-term memory and self-similarity. Recently, they have drawn great applications in the mathematical modeling of systems and processes in the fields of physics, aerodynamics, electrodynamics of complex medium, heat conduction, electricity mechanics, blood flow phenomena,

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2010 *Mathematics Subject Classification.* 26A33, 93B05.

Key words and phrases. Approximate controllability; fractional control systems; time delay; sequential approach.

Submitted March 8, 2017. Published November 5, 2017.

fluid flow in porous media, fitting of experimental data, etc. For more details on this topic, we refer to [1, 2, 4, 5, 8, 9, 14, 16] and the references therein.

In recent years, an important number of problems arising in the modeling of physical phenomena from science and engineering lead to the nonlinear systems with time or state delay (cf. [3, 6, 7, 18, 19, 20]). On the other hand, extensive attention has been paid to the approximate controllability of the nonlinear control systems by many authors. To our knowledge, most of them focused on the approximate controllability of the nonlinear control system provided that the corresponding linear system is approximately controllable. For instance, one can see [6, 7, 10, 12, 13, 15, 17] and the references therein. There are also some researchers use other methods, such as sequence method, to study the approximate controllability for nonlinear control systems. For instance, Zhou [22] obtained sufficient conditions for the existence of solution and approximate controllability for a class of semilinear abstract equations without delay using sequence method. By applying similar technique of [22], Kumar et al. [7] obtained some suitable sufficient conditions for approximate controllability of fractional order system; Liu and Li [11] considered the approximate controllability of fractional evolution systems with Riemann-Liouville fractional derivatives. Very recently, Shuklan et al. [18] obtained results on approximate controllability of semilinear system with state delay also by using sequence techniques. However, in best of our knowledge, the suitable sufficient conditions for the approximate controllability of fractional control system with state delay using similar techniques as in [7, 11, 18, 22] is still untreated topics in the literature and this fact is the motivation of the present work. The main contributions of our present paper is to show the approximate controllability of linear and nonlinear system with delay by using a sequence approach with some suitable hypotheses.

This article has five sections. In the next section, we include some basic definitions, notations and results. In section 3, some sufficient conditions are established to guarantee the existence of mild solution of system (1.1). In section 4, we are concerned with the approximate controllability of the fractional control systems with time delay. In the last section, a concrete application of our main results is provided.

2. Preliminaries

Let us recall the following definitions related to fractional differentiation and integration (cf. [5, 14]).

Definition 2.1. The Riemann-Liouville fractional integral of order α with the lower limit zero is defined by

$$I_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,\quad t>0,\quad 0<\alpha<1,$$

where Γ is the gamma function.

Definition 2.2. The fractional derivative of a function $f\in C[0,\infty)$, in the Caputo sense, can be written as

$${}^C D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}[f(s)-f(0)]ds,\quad t>0,\ 0<\alpha<1.$$

Remark 2.3. (i) The Caputo derivative of a constant is equal to zero.
(ii) If the function $f' \in C[0,\infty)$, then we can get

$$ {}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds = I_t^{1-\alpha} f'(t), \quad 0 < \alpha < 1. $$

(iii) If f is an abstract function with values in a Banach space V, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

In view of [23, Lemma 3.1], we can give the following concept.

Definition 2.4. For any given $u \in Y$, a function $x(\cdot) \in C([-h,b];V)$ is said to be a mild solution of the system (1.1), if

$$ x(t) = \begin{cases} \mathcal{P}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} \mathcal{Q}_\alpha(t-s)[A_1 x(s-h) \\ +Bu(s) + f(s, x(s-h))]ds, & t \geq 0, \\ \phi(t), & -h \leq t < 0. \end{cases} \tag{2.1} $$

where

$$ \mathcal{P}_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta, \quad \mathcal{Q}_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta, $$

$$ \xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0, $$

$$ \varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty). $$

It is interesting to notice that the functions $\varpi_\alpha(\theta)$ and $\xi_\alpha(\theta)$ act as a bridge between the fractional and the classical abstract theories. Next results play an important role for the main assertions of this article.

Lemma 2.5 ([23, Lemma 3.2-3.4])**.** *The operators $\mathcal{P}_\alpha(t)$, $\mathcal{Q}_\alpha(t)$ appeared in Definition 2.4 have the following properties:*

(i) For any $t \geq 0$, the operators $\mathcal{P}_\alpha(t)$ and $\mathcal{Q}_\alpha(t)$ are linear. Moreover, if $\sup_{t\geq 0} \|T(t)\| \leq M$, then the operators $\mathcal{P}_\alpha(t)$ and $\mathcal{Q}_\alpha(t)$ are bounded, i.e., for any $x \in V$,

$$ \|\mathcal{P}_\alpha(t)x\| \leq M\|x\|_V, \quad \|\mathcal{Q}_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|_V. $$

(ii) Operators $\mathcal{P}_\alpha(t)(t \geq 0)$ and $\mathcal{Q}_\alpha(t)(t \geq 0)$ are strongly continuous, that is, for all $x \in V$ and $0 \leq t_1 \leq t_2 \leq b$, we have

$$ \|\mathcal{P}_\alpha(t_1)x - \mathcal{P}_\alpha(t_2)x\|_V \to 0, $$
$$ \|\mathcal{Q}_\alpha(t_1)x - \mathcal{Q}_\alpha(t_2)x\|_V \to 0, \quad \text{as } t_1 \to t_2. $$

(iii) For $t > 0$, $\mathcal{P}_\alpha(t)$ and $\mathcal{Q}_\alpha(t)$ are compact operators if $T(t)$ is compact.

3. Existence results

This section is devoted to the study of the existence of mild solution of the fractional control systems with time delay. In the sequel, we make the following hypotheses on the data of our problems:

(H1) The semigroup $T(t)$ generated by A is uniformly bounded on V, i.e., there is a constant $M > 0$ such that $\sup_{t\in[0,\infty)} \|T(t)\| \leq M$.

(H2) The nonlinear function $f(t,x)$ is continuous in t for all $x \in V$ and continuous with respect to x for almost all $t \in [0,b]$ and there exists a positive constant L, such that

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|_V, \quad \text{for all } x, y \in V.$$

Now, we are in a position to present the main result of this section.

Theorem 3.1. *For each control function $u(\cdot) \in Y$, problem* (1.1) *has a unique mild solution on $C([-h,b];V)$ if conditions* (H1) *and* (H2) *are satisfied.*

Proof. Consider the operator $\mathcal{F}$ defined by

$$(\mathcal{F}x)(t) = \begin{cases} \mathcal{P}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[A_1 x(s-h) \\ +Bu(s) + f(s, x(s-h))]ds, & t \ge 0, \\ \phi(t), & -h \le t < 0. \end{cases} \tag{3.1}$$

In view of the definition of $\mathcal{F}$, the problem of finding mild solutions of (1.1) is equivalent to obtaining fixed points of $\mathcal{F}$.

For a given control $u(\cdot) \in Y$, to show that the operator $\mathcal{F}$ has a fixed point on the interval $[-h,b]$, the positive constant λ appearing in the definition of the norm $\|\cdot\|_C^*$ is chosen below

$$\lambda \ge \frac{b^{2\alpha-1}}{(2\alpha-1)}\Big(\frac{M(\|A_1\|+L)}{\Gamma(\alpha)}\Big)^2 > 0,$$

and the radius of the sphere B_R is defined by

$$R \ge \max\Big\{\|\phi\|_C^*,\ 2M\|\phi(0)\| + 2\frac{M}{\Gamma(\alpha)}\Big[\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z + \frac{L_f b^\alpha}{\alpha}\Big]\Big\},$$

where $L_f = \max_{t\in[0,b]}\|f(t,0)\|_V$, $M = \sup_{t\in[0,+\infty)}\|T(t)\|$. Now, to prove that $\mathcal{F}$ has a fixed point, we subdivide the proof in two steps.

Step 1: For the sphere $B_R = \{x(\cdot) \in C([-h,b],X) : \|x\|_C^* \le R\}$, we show $\mathcal{F}(B_R) \subset B_R$.

If $t \in [-h,0)$, it is readily to get that $\|\mathcal{F}x\|_C^* = \|\phi\|_C^* \le R$. For any $x \in B_R$, if $t \in [0,b]$, under the assumption (H2) and by Lemma 2.5 (i), we have

$$\begin{aligned}
&e^{-\lambda t}\|(\mathcal{F}x)(t)\|_V \\
&\le e^{-\lambda t}\|\mathcal{P}_\alpha(t)\phi(0)\| + e^{-\lambda t}\int_0^t (t-s)^{\alpha-1}\|\mathcal{Q}_\alpha(t-s)\| \\
&\quad \times \|A_1x(s-h) + Bu(s) + f(s,x(s-h))\|ds \\
&\le M\|\phi(0)\| + \frac{Me^{-\lambda t}}{\Gamma(\alpha)}\Big[\int_0^t (t-s)^{\alpha-1}\|A_1x(s-h)\|_V ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|Bu(s)\|_Z ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}[\|f(s,x(s-h)) - f(s,0)\|_V + \|f(s,0)\|_V]ds\Big] \\
&\le M\|\phi(0)\| + \frac{Me^{-\lambda t}}{\Gamma(\alpha)}\Big[\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z + (\|A_1\|+L)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t (t-s)^{\alpha-1}\|x(s-h)\|_V ds + \frac{L_f b^\alpha}{\alpha}\Big] \\
&\le \kappa + \frac{M(\|A_1\|+L)e^{-\lambda t}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\lambda(s-h)} e^{-\lambda(s-h)} \|x(s-h)\|_V ds \\
&\le \kappa + \frac{M(\|A_1\|+L)e^{-\lambda t}}{\Gamma(\alpha)} \Big(\int_0^t (t-s)^{2(\alpha-1)} ds\Big)^{\frac12} \Big(\int_0^t e^{2\lambda(s-h)} ds\Big)^{\frac12} \|x\|_C^* \\
&\le \kappa + \frac{M(\|A_1\|+L)e^{-\lambda t}}{\Gamma(\alpha)} \sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}} \sqrt{\frac{e^{-2\lambda h}(e^{2\lambda t}-1)}{2\lambda}} R \\
&\le \kappa + \frac{M(\|A_1\|+L)}{\Gamma(\alpha)} \sqrt{\frac{b^{2\alpha-1}}{2\lambda(2\alpha-1)}} R.
\end{aligned}
$$

where

$$
\kappa = M\|\phi(0)\| + \frac{M}{\Gamma(\alpha)}\Big[\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z + \frac{L_f b^\alpha}{\alpha}\Big].
$$

From the definitions of λ and R, we obtain $\|\mathcal{F}x\|_C^* \le R$, which proves the claim.

Step 2: We show that $\mathcal{F}$ is a contraction operator on $C([-h,b];V)$. If $t \in [-h,0)$, the claim is obviously valid.

If $t \in [0,b]$, for any $x, y \in C([-h,b];V)$ and under the assumption (H2), we know

$$
\begin{aligned}
&\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|_V \\
&\le \int_0^t (t-s)^{\alpha-1} \|\mathcal{Q}_\alpha(t-s)[A_1 x(s-h) - A_1 y(s-h)]\| ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)[f(s,x(s-h)) - f(s,y(s-h))]\| ds \\
&\le \frac{M(\|A_1\|+L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\lambda(s-h)} e^{-\lambda(s-h)} \|x(s-h)-y(s-h)\|_V ds \\
&\le \frac{M(\|A_1\|+L)e^{\lambda t}}{\Gamma(\alpha)} \sqrt{\frac{b^{2\alpha-1}}{2\lambda(2\alpha-1)}} \|x-y\|_C^*.
\end{aligned}
$$

Using the definition of λ, it follows easily that

$$
\|\mathcal{F}x - \mathcal{F}y\|_C^* \le \frac{M(\|A_1\|+L)}{\Gamma(\alpha)} \sqrt{\frac{b^{2\alpha-1}}{2\lambda(2\alpha-1)}} \|x-y\|_C^* \le \frac12 \|x-y\|_C^*.
$$

Hence, $\mathcal{F}$ is a contraction operator on $C([-h,b];V)$. As a consequence of the Banach's fixed point theorem, we can deduce that $\mathcal{F}$ has a unique fixed point $x(\cdot)$ on $C([-h,b];V)$, which is the desired solution of the system (1.1), which completes the proof. □

4. Approximate controllability results

In the remainder of this section, we study the approximate controllability results of the nonlinear control systems driven by fractional-order involving time delay. To prove the approximate controllability results, the following definitions are essential for our work.

Definition 4.1. Let $x(t;u)$ be the state value of system (1.1) at time t corresponding to the control $u(\cdot) \in Y$. The set $K_b(f) = \{x(b;u) : u(\cdot) \in Y\}$ is called the reachable set of system (1.1) at terminal time b ($b > h$). If $f \equiv 0$, then the system (1.1) is called the corresponding linear system and is denoted by $(1.1)^*$. In this case, $K_b(0)$ denotes the reachable set of the linear system $(1.1)^*$.

Definition 4.2. The system (1.1) is said to be approximately controllable at time b ($b > h$) if $\overline{K_b(f)} = V$, where $\overline{K_b(f)}$ denotes the closure of $K_b(f)$. Clearly, the corresponding linear system $(1.1)^*$ is approximately controllable if $\overline{K_b(0)} = V$.

Now, to begin our study on the approximate controllability of the system (1.1), we define a bounded and linear operator $\mathcal{G} : Z \to C([0, b]; V)$ by

$$\mathcal{G}h = \int_0^b (b-s)^{\alpha-1}\mathcal{Q}_\alpha(b-s)h(s)ds, \quad \text{for } h(\cdot) \in Z.$$

From Definition 4.2, we know that system (1.1) is approximately controllable at time b ($b > h$), if for every desired final state $\zeta \in V$ and any $\epsilon > 0$, there exists a control $u_\epsilon(\cdot) \in Y$, such that

$$\|\zeta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_\varepsilon^h - \mathcal{G}Fx_\varepsilon^h - \mathcal{G}Bu_\varepsilon\| < \epsilon,$$

where $(Fx^h)(t) = f(t, x(t-h))$, $A_1 x^h = A_1 x(t-h)$ and $x_\epsilon(t) = x(t; u_\varepsilon)$ is a mild solution of system (1.1) corresponding to $u_\varepsilon(\cdot) \in Y$. Relevant results regarding the approximate controllability can be found in [6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22].

Remark 4.3. In Definition 4.2, we require that $b > h$. This is reasonable since one cannot control the value of $x(s)$ for $[b-h, 0]$ if $b < h$. Indeed, it is easy to see that $x(s) = \phi(s)$ in $[-h, 0]$ is independent of the control $u(t)$.

To prove our main result, we also suppose that:

(H3) For any $\varepsilon > 0$ and $\varphi(\cdot) \in Z$, there exists a $u(\cdot) \in Y$, such that

$$\|\mathcal{G}\varphi - \mathcal{G}Bu\|_V < \varepsilon, \tag{4.1}$$

$$\|Bu(\cdot)\|_Z < \gamma\|\varphi(\cdot)\|_Z, \tag{4.2}$$

where γ is a positive constant which is independent of $\varphi(\cdot) \in Z$ and satisfies

$$\frac{M(\|A_1\| + L)\gamma}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}E_\alpha(M(\|A_1\| + L)b^\alpha) < 1. \tag{4.3}$$

To investigate the approximate controllability of system (1.1), we need the following lemma.

Lemma 4.4. *Suppose that conditions* (H1) *and* (H2) *hold, then any mild solutions of system* (1.1) *satisfy the following inequalities*

$$\|x(\cdot; u)\|_C^* \le \kappa E_\alpha(M(\|A_1\| + L)b^\alpha), \quad \textit{for any } u(\cdot) \in Y,$$
$$\|x_1(\cdot) - x_2(\cdot)\|_C^* \le \rho E_\alpha(M(\|A_1\| + L)b^\alpha)\|Bu_1(\cdot) - Bu_2(\cdot)\|_Z,$$
$$\textit{for any } u_1(\cdot),\ u_2(\cdot) \in Y,$$

where

$$\kappa = M\|\phi(0)\| + \frac{M}{\Gamma(\alpha)}\Big[\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z + \frac{L_f b^\alpha}{\alpha}\Big], \quad \rho = \frac{M}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}},$$

and E_α is the Mittag-Leffler function defined by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}.$$

Proof. If $x(\cdot;u)=x(\cdot)$ is a mild solution of system (1.1) with respect to $u(\cdot)\in Y$, then

$$x(t)=\mathcal{P}_\alpha(t)\phi(0)+\int_0^t(t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[A_1x(s-h)+Bu(s)+f(s,x(s-h))]ds.$$

For $t\in[0,b]$, we obtain

$$\begin{aligned}
&\|x(t)\|_V\\
&\le\|\mathcal{P}_\alpha(t)\phi(0)\|+\int_0^t(t-s)^{\alpha-1}\|\mathcal{Q}_\alpha(t-s)\|\\
&\quad\times\|A_1x(s-h)+Bu(s)+f(s,x(s-h))\|_V ds\\
&\le M\|\phi(0)\|+\frac{M}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}\|A_1x(s-h)\|_V ds+\int_0^t(t-s)^{\alpha-1}\|Bu(s)\|ds\\
&\quad+\int_0^t(t-s)^{\alpha-1}[\|f(s,x(s-h))-f(s,0)\|_V+\|f(s,0)\|_V]ds\Big]\\
&\le M\|\phi(0)\|+\frac{M}{\Gamma(\alpha)}\Big[\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|Bu\|_Z\\
&\quad+(\|A_1\|+L)\int_0^t(t-s)^{\alpha-1}\|x(s-h)\|ds+\frac{L_fb^\alpha}{\alpha}\Big]\\
&\le\kappa+\frac{M(\|A_1\|+L)}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|x(s-h)\|_V ds.
\end{aligned}$$

It follows from Corollary 2 in [21] that

$$\|x\|_C^*=\sup_{t\in[0,b]}e^{-\lambda t}\|x(t)\|_V\le\kappa E_\alpha(M(\|A_1\|+L)b^\alpha).$$

Similarly, we obtain

$$\|x_1(\cdot)-x_2(\cdot)\|_C^*\le\rho E_\alpha(M(\|A_1\|+L)b^\alpha)\|Bu_1(\cdot)-Bu_2(\cdot)\|_Z.$$

This completes the proof. □

Now, we are in a position to present the other main result of this section.

Theorem 4.5. *Under conditions* (H1)–(H3)*, system* (1.1) *is approximately controllable.*

Proof. Since $\overline{D(A)}=V$, it is sufficient to prove that $D(A)\subset\overline{K_b(f)}$, i.e., for any $\epsilon>0$ and $\eta\in D(A)$, there exists a control $u_\epsilon(\cdot)\in Y$, such that

$$\|\eta-\mathcal{P}_\alpha(b)\phi(0)-\mathcal{G}A_1x_\varepsilon^h-\mathcal{G}Fx_\varepsilon^h-\mathcal{G}Bu_\varepsilon\|<\epsilon. \tag{4.4}$$

Firstly, for any given $\eta\in D(A)$ and any $x_0\in V$, there exists a function $\varphi(\cdot)\in Z$, such that $\mathcal{G}\varphi=\eta-\mathcal{P}_\alpha(b)\phi(0)$. Next, we show that there is a control $u_\epsilon(\cdot)\in Y$ such that the inequality (4.4) holds.

Now, we begin to construct a sequence of the control. Let $\epsilon > 0$ and $u_1(\cdot) \in Y$ be arbitrary. It follows from $H(3)$ that there is a $u_2(\cdot) \in Y$, such that

$$\|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_1^h - \mathcal{G}Fx_1^h - \mathcal{G}Bu_2\| < \frac{\epsilon}{2^2},$$

where $x_1(t) = x(t; u_1)$ for $0 \le t \le b$. Denote $x_2(t) = x(t; u_2)$ for $0 \le t \le b$, by $H(3)$ again, we know there exists $w_2(\cdot) \in Y$, such that

$$\|\mathcal{G}[A_1 x_2^h - A_1 x_1^h + Fx_2^h - Fx_1^h] - \mathcal{G}Bw_2\| < \frac{\epsilon}{2^3},$$

and

$$\begin{aligned}
\|Bw_2(\cdot)\|_Z &\le \gamma\|(A_1 x_2^h - A_1 x_1^h) + (Fx_2^h - Fx_1^h)\| \\
&\le \gamma(\|A_1 + L)\|x_2(\cdot) - x_1(\cdot)\| \\
&\le \frac{M(\|A_1 + L)\gamma}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha - 1}} E_\alpha(M(\|A_1 + L)b^\alpha)\|Bu_1(\cdot) - Bu_2(\cdot)\|_Z.
\end{aligned}$$

Now, we define $u_3(t) = u_2(t) - w_2(t)$, $u_3(\cdot) \in Y$, and it is readily to get that

$$\begin{aligned}
&\|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_2^h - \mathcal{G}Fx_2^h - \mathcal{G}Bu_3\| \\
&\le \|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_1^h - \mathcal{G}Fx_1^h - \mathcal{G}Bu_2\| \\
&\quad + \|\mathcal{G}Bw_2 - [A_1 x_2^h - A_1 x_1^h + Fx_2^h - Fx_1^h]\| \\
&\le (\frac{1}{2^2} + \frac{1}{2^3})\epsilon.
\end{aligned}$$

By induction, we know there exists a sequence $\{u_n(\cdot)\} \subset Y$, which follows that

$$\|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_n^h - \mathcal{G}Fx_n^h - \mathcal{G}Bu_{n+1}\| < (\frac{1}{2^2} + \dots + \frac{1}{2^n})\epsilon,$$

where $x_n(\cdot) = x(\cdot; u_n)$ for $0 \le t \le b$, and

$$\begin{aligned}
&\|Bu_{n+1} - Bu_n\|_Z \\
&< \frac{M(\|A_1 + L)\gamma}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha - 1}} E_\alpha(M(\|A_1 + L)b^\alpha)\|Bu_n(\cdot) - Bu_{n-1}(\cdot)\|_Z.
\end{aligned}$$

From (4.3), we know the sequence $\{Bu_n : n = 1, 2, \cdots\}$ is a Cauchy sequences on the Banach space Z. Therefore, there exists a $\psi(\cdot) \in Z$, such that

$$\lim_{n\to\infty} Bu_n(\cdot) = \psi(\cdot) \quad \text{in } Z.$$

Then, for all $\epsilon > 0$, there exists a positive integer number N, such that

$$\|\mathcal{G}Bu_{N+1} - \mathcal{G}Bu_N\| < \frac{\epsilon}{2}.$$

Therefore, we have

$$\begin{aligned}
&\|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_N^h - \mathcal{G}Fx_N^h - \mathcal{G}Bu_N\| \\
&\le \|\eta - \mathcal{P}_\alpha(b)\phi(0) - \mathcal{G}A_1 x_N^h - \mathcal{G}Fx_N^h - \mathcal{G}Bu_{N+1}\| + \|\mathcal{G}Bu_{N+1} - \mathcal{G}Bu_N\| \\
&\le (\frac{1}{2^2} + \dots + \frac{1}{2^n})\epsilon + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Hence, from the above argument, it is easy to get that system (1.1) is approximate controllability. This completes the proof. □

5. Application

In this section, a possible application of Theorem 4.5 on the approximate controllability of control systems with state delay is presented, such as:

$$
\begin{gathered}
D_t^{2/3}x(t,\theta)=\frac{\partial^2}{\partial\theta^2}x(t,\theta)+x(t-h,\theta)+f(t,x(t-h,\theta))+Bu(t,\theta),\\
t\in(0,1],\quad \theta\in[0,\pi],\\
x(t,0)=x(t,\pi)=0,\quad t\in(0,1),\\
x(t,\theta)=\phi(t),\quad t\in[-h,0],\quad \theta\in[0,\pi].
\end{gathered}
\tag{5.1}
$$

Take $V=U=L^2(0,\pi)$ and the operator $A=x''$ with the domain given by

$$
D(A)=\{x\in V: x,x' \text{ are absolutely continuous}, x''\in V,\ x(0)=x(\pi)=0\}.
$$

Then, A can be written as

$$
Ax=-\sum_{n=1}^{\infty}n^2(x,\sigma_n)\sigma_n,\quad x\in D(A),
$$

where $\sigma_n(\theta)=\sqrt{2/\pi}\sin(n\theta)(n=1,2,\cdots)$ is an orthonormal basis of V. It is well known that A is the infinitesimal generator of a compact semigroup $T(t)(t>0)$ in V given by

$$
T(t)x=\sum_{n=1}^{\infty}e^{-n^2t}(x,\sigma_n)\sigma_n,\quad x\in V,\quad\text{and}\quad \|T(t)\|\le 1.
$$

For every $u(\cdot)\in Y=L^2([0,1];U)$, we have

$$
u(t)=\sum_{n=1}^{\infty}u_n(t)\sigma_n,\quad u_n(t)=\langle u(t),\sigma_n\rangle,
$$

Define the operator B by

$$
Bu(t)=\sum_{n=1}^{\infty}\overline{u}_n(t)\sigma_n,
$$

where for $n=1,2,\dots$,

$$
\overline{u}_n(t)=\begin{cases}0, & 0\le t<1-\frac{1}{n^2},\\ u_n(t), & 1-\frac{1}{n^2}\le t\le 1;\end{cases}
$$

then, one can easily obtain that $\|Bu(\cdot)\|\le\|u(\cdot)\|$, which implies that B is a bounded linear operator from Y to $Z=L^2([0,1];V)$.

Firstly, by the definition of the operator B, the corresponding linear system of (5.1) is

$$
\begin{gathered}
D_t^{2/3}x_n(t)+n^2x_n(t)=u_n(t),\quad 1-\frac{1}{n^2}<t<1,\\
x_n(0)=\phi(t),\quad t\in[-h,0],.
\end{gathered}
\tag{5.2}
$$

Next, we will check that the hypotheses of (H3) are satisfied. To check these, let us denote

$$
h=\int_0^1(1-s)^{-1/3}\mathcal{Q}_{2/3}(1-s)g(s)ds=\sum_{n=1}^{\infty}h_n\sigma_n,\quad h_n=\langle h,\sigma_n\rangle,
$$

for every $g(\cdot) \in L^2(J,X)$. In fact, we can choose $\widetilde{u}_n(t)$ which follows from

$$\widetilde{u}_n(t) = \frac{2n^2}{1-e^{-2}} h_n e^{-n^2(1-t)}, \quad 1 - \frac{1}{n^2} \le t \le 1,$$

and

$$h_n = \int_{1-\frac{1}{n^2}}^{1} \int_0^\infty (1-t)^{-1/3} \theta \xi_{2/3}(\theta) e^{-n^2\theta(1-t)^{2/3}} \widetilde{u}_n(t) d\theta dt.$$

For this, we define $u(t) = \sum_{n=1}^\infty u_n(t)\sigma_n$, where for $n = 1, 2, \dots$,

$$u_n(t) = \begin{cases} 0, & 0 \le t < 1 - \frac{1}{n^2}, \\ \widetilde{u}_n(t), & 1 - \frac{1}{n^2} \le t \le 1. \end{cases}$$

Therefore, for any given function $g(\cdot) \in Z$, there exists $u(\cdot) \in Y$, such that

$$\int_0^1 (1-s)^{-1/3} \mathcal{Q}_{2/3}(1-s) Bu(s) = \int_0^1 (1-s)^{-1/3} \mathcal{Q}_{2/3}(1-s) g(s) ds,$$

which implies the condition (4.2) of (H3) is satisfied. Moreover, we can get

$$\begin{aligned} \|Bu(\cdot)\|^2 &= \sum_{n=1}^\infty \int_{1-\frac{1}{n^2}}^1 |\widetilde{u}_n(t)|^2 dt = (1-e^{-2})^{-1} \sum_{n=1}^\infty 2n^2 h_n^2 \\ &= \frac{3}{2}(1-e^{-2})^{-1} \sum_{n=1}^\infty (1-e^{-2n^2}) \int_0^1 |g_n(t)|^2 dt \\ &\le \frac{3}{2}(1-e^{-2})^{-1} |g(\cdot)|^2. \end{aligned}$$

Hence, it can be seen that the conditions of (H3) are satisfied, then system (5.1) is approximately controllable on J, if

$$\frac{3\sqrt{3}(1+L)}{2\Gamma(\frac{2}{3})}(1-e^{-2})^{-1} E_{2/3}(1+L) < 1,$$

is satisfied.

Acknowledgments. This research was supported by the NNSF of China Grants Nos. 11671101, 11661001, by the Special Funds of Guangxi Distinguished Experts Construction Engineering, by the National Science Center of Poland under the Maestro Advanced Project No. DEC-2012/06/A/ST1/00262, and by the the Project of Guangxi Education Department grant No. KY2016YB417.

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