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POSITIVE SOLUTIONS FOR SECOND-ORDER M-POINT BOUNDARY-VALUE PROBLEMS WITH NONLINEARITY DEPENDING ON THE FIRST DERIVATIVE

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Abstract. We consider multiplicity of positive solutions for second-order m-point boundary-value problems, with the first order derivative involved in the nonlinear term. Using a fixed point theorem, we show the existence of at least three positive solutions. By giving an example we illustrate the main result of the article.

1. Introduction

Multi-point boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary-value problem,many problems in the theory of elastic stability can be handled as multi-point boundary-value problems too.Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have received a great deal of attentions.To identify a few,we refer the reader to [1, 5, 10, 11, 12] and references therein. Ma and Wang [13] obtained the existence of one positive solution for more general three-point boundary-value problem

$$u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \quad 0 < \eta < 1, \tag{1.2}$$

under the assumption that f is either suplinear or sublinear, and that the following conditions are satisfied:

- (H1) $f \in C([0, +\infty), [0, +\infty))$
- (H2) $h \in C([0, 1], [0, +\infty))$ and there exists $x_0 \in (0, 1)$ such that $h(x_0) > 0$
- (H3) $a \in C[0, 1]$, $b \in C([0, 1], (-\infty, 0])$
- (H4) $0 < \alpha\phi_1(\eta) < 1$, where ϕ_1 is the unique solution of the linear problem

$$\phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, \quad t \in (0, 1), \tag{1.3}$$

$$\phi_1(0) = 0, \quad \phi_1(1) = 1. \tag{1.4}$$

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In [13], the authors used a fixed point theorem for a mapping defined on Banach spaces with cones, by Guo and Krasnosel'skii [6]. However all the above works about positive solutions were done under the assumption that the first order derivative x' is not involved in the nonlinear term. On the other hand, to the best of our knowledge, there are very few work considering the multiplicity of positive solutions with dependence on derivatives.

In this paper, we consider the existence of at least three positive solutions for the equation

$$x''(t) + a(t)x'(t) + b(t)x(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \tag{1.5}$$

subject to the boundary conditions

$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{1.6}$$

or to the boundary conditions

$$x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{1.7}$$

where $\xi_i \in (0, 1)$, $\alpha_i > 0$, $i = 1, 2, \dots, m-2$ are given constants.

The interest in triple solutions evolved from the Leggett-Williams fixed point theorem [9]. When x' does not appear in nonlinear term there are results about several nonlinear ordinary differential equations, obtained by the Leggett-Williams fixed point theorem; see [7, 8]. Recently Avery and Peterson [2], Bai and Ge [3] generalized the fixed point theorem of Leggett-Williams by using theorem of fixed point index and Dugundji extension theorem. As applications of the results in [3, 4], it has been obtained the existence of triple positive solutions of the boundary-value problem

$$x''(t) + a(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{1.8}$$

$$x(0) = x(1) = 0, \quad \text{or} \quad x(0) = x'(1) = 0. \tag{1.9}$$

By using the main results of [3, 13], we give some simple criteria for the existence of multiple positive solutions for problem (1.5) subject to (1.6) or (1.7).

2. Background definitions and preliminaries

For the convenience of the reader,we present here the necessary definitions from cone theory in Banach spaces. This definitions can be found in the literature.

Definitions. Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P \subset E$ is said to be a cone provided that (i) $au \in P$, for all $u \in P$, $a \geq 0$, and (ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

The map α is said to be a nonnegative continuous convex functional on cone P of a real Banach space E provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y), \quad \forall x, y \in P \ t \in [0, 1].$$

The map β is said to be a nonnegative continuous concave functional on cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and $\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Suppose $\theta, \gamma : P \to [0, +\infty)$ are two nonnegative continuous convex functionals satisfying

$$\|x\| \leq k \max\{\theta(x), \gamma(x)\} \quad \text{for } x \in P, \tag{2.1}$$

where k is a positive constant,and

$$\Omega = \{x \in P | \theta(x) < r, \gamma(x) < L\} \neq \emptyset, \quad r > 0,\ L > 0. \tag{2.2}$$

Let $r > a > 0, L > 0$ be given, $\gamma, \theta : P \to [0, +\infty)$ be nonnegative continuous convex functionals satisfying (2.1) and (2.2), α be a nonnegative continuous concave functional on P. Define the following convex sets:

$$\begin{gathered}
P(\gamma, L; \theta, r) = \{x \in P | \gamma(x) < L, \theta(x) < r\}, \\
\overline{P}(\gamma, L; \theta, r) = \{x \in P | \gamma(x) \leq L, \theta(x) \leq r\}, \\
P(\gamma, L; \theta, r; \alpha, a) = \{x \in P | \gamma(x) < L, \theta(x) < r, \alpha(x) > a\}, \\
\overline{P}(\gamma, L; \theta, r; \alpha, a) = \{x \in P | \gamma(x) \leq L, \theta(x) \leq r, \alpha(x) \geq a\}.
\end{gathered}$$

Lemma 2.1. *Let E be a Banach space,$P \subset E$ be a cone and $r_2 \geq c > b > r_1 > 0$, $L_2 \geq L_1 > 0$ be given. Assume that γ, θ are nonnegative continuous convex functionals on P such that* (2.1), (2.2) *are satisfied. α is a nonnegative continuous concave functional on P such that $\alpha(x) \leq \theta(x)$ for all $x \in \overline{P}(\gamma, L_2; \theta, r_2)$ and let $T : \overline{P}(\gamma, L_2; \theta, r_2) \to \overline{P}(\gamma, L_2; \theta, r_2)$ be a completely continuous operator. Suppose that*

- (S1) *The set $\{x \in \overline{P}(\gamma, L_2; \theta, c; \alpha, b) : \alpha(x) > b\}$ is not empty, and $\alpha(Tx) > b$ for x in $\overline{P}(\gamma, L_2; \theta, c; \alpha, b)$;*
- (S2) *$\gamma(Tx) < L_1$, $\theta(Tx) < r_1$ for all $x \in \overline{P}(\gamma, L_1; \theta, r_1)$;*
- (S3) *$\alpha(Tx) > b$, for all $x \in \overline{P}(\gamma, L_2; \theta, r_2; \alpha, b)$ with $\theta(Tx) > c$.*

Then T has at least three fixed points x_1, x_2, x_3 in $\overline{P}(\gamma, L_2; \theta, r_2)$. Further,

$$\begin{gathered}
x_1 \in P(\gamma, L_1; \theta, r_1); \quad x_2 \in \{\overline{P}(\gamma, L_2; \theta, r_2; \alpha, b) : \alpha(x) > b\}, \\
x_3 \in \overline{P}(\gamma, L_2; \theta, r_2) \setminus (\overline{P}(\gamma, L_2; \theta, r_2; \alpha, b) \cup \overline{P}(\gamma, L_1; \theta, r_1)).
\end{gathered}$$

3. Positive solutions of (1.5), (1.6)

To state the main results of this section,we need the following lemma, which was established by Ma and Wang [13].

Lemma 3.1. *Assume that (H3) holds. Let ϕ_1, ϕ_2 be solutions of* (1.3), (1.4), *and*

$$\phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) = 0, \quad t \in (0, 1), \tag{3.1}$$

$$\phi_2(0) = 1, \quad \phi_2(1) = 0. \tag{3.2}$$

Then ϕ_1 is strictly increasing and ϕ_2 is strictly decreasing on [0,1].

Inspired by [13], we state following lemma which can be regard as a natural extension.

Lemma 3.2. *Suppose (H3) and*

$$0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i) < 1. \tag{3.3}$$

Then the problem

$$x''(t) + a(t)x'(t) + b(t)x(t) + y(t) = 0, \quad t \in (0,1) \tag{3.4}$$

$$x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{3.5}$$

is equivalent to integral equation

$$x(t) = \int_0^1 G(t,s)p(s)y(s)ds + A\phi_1(t), \tag{3.6}$$

where

$$A = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i} \int_0^1 G(\xi_i, s)p(s)y(s)ds, \tag{3.7}$$

$$p(t) = \exp\Big(\int_0^t a(s)ds\Big), \quad \rho = \phi_1'(0), \tag{3.8}$$

$$G(t,s) = \frac{1}{\rho}\begin{cases} \phi_1(t)\phi_2(s) & s \geq t \\ \phi_1(s)\phi_2(t) & s \leq t, \end{cases}$$

$$u(t) \geq 0 \quad \text{if } y(t) \geq 0.$$

The proof of this lemma is very similar to a proof in [13], so we omit it here. Let

$$q(t) = \min\{\frac{\phi_1(t)}{|\phi_1|_0}, \frac{\phi_2(t)}{|\phi_2|_0}\}, \quad t \in [0,1]$$

where $|y(t)|_0 = \max |y(t)|, t \in [0,1]$.

The following Lemma was also established by Ma and Wang.

Lemma 3.3. *Suppose (H3) and* (3.3) *are satisfied,* $y \in C[0,1], y \geq 0$*, then the solution of* (3.4)-(3.5) *satisfies*

$$u(t) \geq |u|_0 q(t), t \in [0,1]. \tag{3.9}$$

Thus, for any $\delta \in [0, 1/2]$, there exists λ such that

$$u(t) \geq \lambda |u|_0, t \in [\delta, 1-\delta], \tag{3.10}$$

where $\lambda = \min\{q(t) : t \in [\delta, 1-\delta]\}$. Let

$$M = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)p(s)ds + \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)} \int_0^1 G(\xi_i, s)p(s)ds;$$

$$N = \max_{0 \leq t \leq 1} \Big| \int_0^1 \frac{\partial G(t,s)}{\partial t} p(s)ds + \frac{\sum_{i=1}^{m-2} \alpha_i \phi_1'(t)}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)} \int_0^1 G(\xi_i, s)p(s)ds \Big|;$$

$$m = \min_{\delta \leq t \leq 1-\delta} \int_\delta^{1-\delta} G(t,s)p(s)ds + \frac{\sum_{i=1}^{m-2} \alpha_i \phi_1(\delta)}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\xi_i)} \int_\delta^{1-\delta} G(\xi_i, s)p(s)ds.$$

To present our main results, we assume there exist constants $r_2 \geq \frac{b}{\lambda} > b > r_1 > 0$, $L_2 \geq L_1 > 0$ such that $\frac{b}{m} < \min\{\frac{r_2}{M}, \frac{L_2}{N}\}$ and the following assumptions hold:

- (A1) $f(t,u,v) \in C([0,1] \times [0,+\infty) \times R, [0,+\infty))$;
- (A2) $f(t,u,v) < \min\{r_1/M, L_1/N\}, (t,u,v) \in [0,1] \times [0,r_1] \times [-L_1, L_1]$;
- (A3) $f(t,u,v) > b/m, (t,u,v) \in [\delta, 1-\delta] \times [b, b/\lambda] \times [-L_2, L_2]$;
- (A4) $f(t,u,v) \leq \min\{r_2/M, L_2/N\}, (t,u,v) \in [0,1] \times [0,r_2] \times [-L_2, L_2]$.

Theorem 3.4. *Under assumption* (A1)-(A4), (H3), (3.3), *Problem* (1.5)-(1.6) *has at least three positive solutions* x_1, x_2, x_3 *satisfying*

$$\begin{gathered}\max_{0\le t\le 1} x_1(t) \le r_1,\ \max_{0\le t\le 1} |x_1'(t)| \le L_1;\\ b < \min_{\delta\le t\le 1-\delta} x_2(t) \le \max_{0\le t\le 1} x_2(t) \le r_2,\ \max_{0\le t\le 1} |x_2'(t)| \le L_2;\\ \max_{0\le t\le 1} x_3(t) \le \frac{b}{\lambda},\ \max_{0\le t\le 1} |x_3'(t)| \le L_2.\end{gathered} \tag{3.11}$$

Proof. Problem (1.5)-(1.6) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = \int_0^1 G(t,s)p(s)f(s,x(s),x'(s))ds + A\phi_1(t) = (Tx)(t), \quad 0 < t < 1.$$

Let $X = C^1[0,1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]$, and the maximum norm

$$\|x\| = \max\big\{\max_{0\le t\le 1} |x(t)|, \max_{0\le t\le 1} |x'(t)|\big\}.$$

Define the cone $P \subset X$ by

$$P = \{x \in X | x(t) \ge 0, \min_{\delta\le t\le 1-\delta} x(t) \ge \lambda |x(t)|_0, t \in [0,1]\}.$$

Define functionals

$$\begin{gathered}\gamma(x) = \max_{0\le t\le 1} |x'(t)|, \quad \theta(x) = \max_{0\le t\le 1} |x(t)|,\\ \alpha(x) = \min_{\delta\le t\le 1-\delta} |x(t)|, for\ x \in X.\end{gathered}$$

Then $\gamma, \theta : P \to [0, +\infty)$ are nonnegative continuous convex functionals satisfying (2.1) and (2.2); α is nonnegative continuous concave functional with $\alpha(x) \le \theta(x)$ for all $x \in X$.

Now we verify that all the conditions of Lemma 2.1 are satisfied. If $x \in \overline{P}(\gamma, L_2; \theta, r_2)$, then $\gamma(x) \le L_2, \theta(x) \le r_2$ and assumption (A_4) implies

$$f(t,x(t),x'(t)) \le \min\{\frac{L_2}{N}, \frac{r_2}{M}\},$$

consequently

$$\begin{aligned}\theta(Tx) &= \max_{0\le t\le 1}[\int_0^1 G(t,s)p(s)f(s,x(s),x'(s))ds\\ &\quad + \frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)f(s,x(s),x'(s))ds]\\ &\le \frac{r_2}{M}\max_{0\le t\le 1}[\int_0^1 G(t,s)p(s)ds + \frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)ds]\\ &\le \frac{r_2}{M}[\max_{0\le t\le 1}\int_0^1 G(t,s)p(s)ds + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)ds]\\ &\le \frac{r_2}{M}M = r_2.\end{aligned}$$

Also

$$\gamma(Tx) = \max_{0\le t\le 1}|\int_0^1 \frac{\partial G(t,s)}{\partial t}p(s)f(s,x(s),x'(s))ds$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_1'(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)f(s,x(s),x'(s))ds|\\
&\le \frac{L_2}{N}\max_{0\le t\le 1}|\int_0^1 \frac{\partial G(t,s)}{\partial t}p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_1'(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)ds|\\
&\le \frac{L_2}{N}N=L_2.
\end{aligned}
$$

Hence,$T:\overline{P}(\gamma,L_2;\theta,r_2)\to\overline{P}(\gamma,L_2;\theta,r_2)$ and T is completely continuous on $[0,1]$. In the same way, if $x\in\overline{P}(\gamma,L_1;\theta,r_1)$, then assumption (A2) yields

$$
f(t,x(t),x'(t))<\min\{\frac{L_1}{N},\frac{r_1}{M}\},\quad 0\le t\le 1.
$$

As in the argument above,we can obtain that $T:\overline{P}(\gamma,L_1;\theta,r_1)\to P(\gamma,L_1;\theta,r_1)$. Therefore, condition (S2) of Lemma 2.1 is satisfied.

To check condition (S1) of Lemma 2.1, we choose $x(t)=\frac{b}{\lambda}=c$. It's easy to see $x(t)=\frac{b}{\lambda}\in\overline{P}(\gamma,L_2;\theta,c;\alpha,b)$ and $\alpha(\frac{b}{\lambda})>b$. So $\{x\in\overline{P}(\gamma,L_2;\theta,c;\alpha,b)|\alpha(x)>b)\}\ne\emptyset$. If $x\in P(\gamma,L_2;\theta,c;\alpha,b)$, we have $b\le x(t)\le\frac{b}{\lambda},|x'(t)|<L_2$ for $\delta\le t\le 1-\delta$. From the assumption (A2), we have

$$
f(t,x(t),x'(t))>\frac{b}{m}.
$$

By the definition of α and the cone P,

$$
\begin{aligned}
\alpha(Tx)&=\min_{\delta\le t\le 1-\delta}[\int_0^1 G(t,s)p(s)f(s,x(s),x'(s))ds\\
&\quad+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_0^1 G(\xi_i,s)p(s)f(s,x(s),x'(s))ds]\\
&>\frac{b}{m}\min_{0\le t\le 1}[\int_\delta^{1-\delta} G(t,s)p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_\delta^{1-\delta} G(\xi_i,s)p(s)ds]\\
&\le\frac{b}{m}[\min_{0\le t\le 1}\int_\delta^{1-\delta} G(t,s)p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(\delta)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\xi_i)}\int_\delta^{1-\delta} G(\xi_i,s)p(s)ds]\\
&\ge\frac{b}{m}m=b.
\end{aligned}
$$

Then $\alpha(Tx)>b$, for all $x\in\overline{P}(\gamma,L_2;\theta,c;\alpha,b)$. This shows that condition (S1) of lemma 2.1 is also satisfied. Finally we show (S3) holds too. Suppose $x\in\overline{P}(\gamma,L_2;\theta,r_2;\alpha,b)$ with $\theta(Tx)>\frac{b}{\lambda}$. Then,by the definition of α and $Tx\in P$, we have

$$
\alpha(Tx)=\min_{\delta\le t\le 1-\delta}|(Tx)(t)|\ge\lambda\cdot\max_{0\le t\le 1}|(Tx)(t)|=\lambda\cdot\theta(Tx)=\lambda\cdot\frac{b}{\lambda}=b.
$$

So condition (S3) of lemma 2.1 is satisfied. Therefore, Lemma 2.1 yields that problem (1.5)-(1.6) has at least three positive solutions x_1,x_2,x_3 in $\overline{P}(\gamma,L_2;\theta,r_2)$ and (3.11) is satisfied. □

Remark 3.5. In Lemma 2.1, we need only $T:\overline{P}(\gamma,L_2;\theta,r_2)\to\overline{P}(\gamma,L_2;\theta,r_2)$; therefore, condition (A1) can be substituted with the weaker condition

(C1) $f\in C([0,1]\times[0,r_2]\times[-L_2,L_2],[0,+\infty))$.

From the proof of Theorem 3.4, it is easy to see that, if conditions like (A1)-(A4) are appropriate combined, we can obtained an arbitrary number of positive solutions for this problem.

Corollary 3.6. *Suppose condition (A1) is satisfied and there exist constants* $0 < r_1 < b_1 < b_1/\lambda \leq r_2 < b_2 < b_2/\lambda \cdots \leq r_n, 0 < L_1 \leq L_2 \leq \cdots \leq L_{n-1}, n \in N$, *such that*

$$b_i/m \leq \min\{\frac{r_{i+1}}{M}, \frac{L_{i+1}}{N}\}$$

If the following two conditions are satisfied then problem (1.5)-(1.6) *admits at least* $2n-1$ *positive solutions.*

(E1) $f(t,u,v) < \min\{\frac{r_i}{M}, \frac{L_i}{N}\}$, $(t,u,v) \in [0,1] \times [0,r_i] \times [-L_i, L_i]$, $1 \leq i \leq n$;

(E2) $f(t,u,v) > \frac{b_i}{m}$, $(t,u,v) \in [\delta, 1-\delta] \times [b_i, \frac{b_i}{\lambda}] \times [-L_{i+1}, -L_{i+1}], 1 \leq i \leq n-1$.

Proof. When $n = 1$, it follows from condition (E1) that

$$T : \overline{P}(\gamma, L_1; \theta, r_1) \to P(\gamma, L_1; \theta, r_1) \subseteq \overline{P}(\gamma, L_1; \theta, r_1).$$

Then by Schauder's fixed-point theorem, T has at least one fixed point x_1 in $P(\gamma, L_1; \theta, r_1)$. When $n = 2$, it is clear that Theorem 3.4 holds. Then we can obtain at least three positive solutions x_2, x_3, x_4. Along this way, we can complete the proof by the induction method. $\square$

4. Positive solutions of (1.5), (1.7)

In this section we study problem (1.5), (1.7). The method and existence results are remarkable analogous to those in section 3. First, we give some Lemmas. Suppose ϕ_3 is the unique solution of the linear boundary-value problem

$$\phi_3''(t) + a(t)\phi_3'(t) + b(t)\phi_1(t) = 0, \quad t \in (0,1), \tag{4.1}$$

$$\phi_3'(0) = 0, \quad \phi_3(1) = 1. \tag{4.2}$$

satisfying

$$0 < \sum_{i=1}^{m-2} \alpha_i \phi_3(\xi_i) < 1. \tag{4.3}$$

Then problem

$$x''(t) + a(t)x'(t) + b(t)x(t) + y(t) = 0, t \in (0,1) \tag{4.4}$$

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{4.5}$$

is equivalent to integral equation

$$x(t) = \int_0^1 G_1(t,s)p(s)y(s)ds + A_1\phi_3(t), \tag{4.6}$$

where

$$A_1 = \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_3(\xi_i)} \int_0^1 G_1(\xi_i, s)p(s)y(s)ds, \tag{4.7}$$

$$p(t) = \exp(\int_0^t a(s)ds), \quad \rho_1 = -\phi_3(0)\phi_2'(0), \tag{4.8}$$

$$G_1(t,s)=\frac{1}{\rho_1}\begin{cases}\phi_3(t)\phi_2(s) & 1\geq s\geq t\geq 0\\ \phi_3(s)\phi_2(t) & 0\leq s\leq t\leq 1,\end{cases}$$
$$x(t)\geq 0 \quad \text{if } y(t)\geq 0.$$

Let (H3), (4.3) be satisfied, substituting $\phi_1(t)$ with $\phi_3(t)$, we can obtain a similar result as in lemma 3.3. Let

$$M_1=\max_{0\leq t\leq 1}\int_0^1 G_1(t,s)p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_3(\xi_i)}\int_0^1 G_1(\xi_i,s)p(s)ds;$$
$$N_1=\max_{0\leq t\leq 1}|\int_0^1 \frac{\partial G_1(t,s)}{\partial t}p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_3'(t)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_3(\xi_i)}\int_0^1 G_1(\xi_i,s)p(s)ds|;$$
$$m_1=\min_{\delta\leq t\leq 1-\delta}\int_\delta^{1-\delta} G_1(t,s)p(s)ds+\frac{\sum_{i=1}^{m-2}\alpha_i\phi_3(\delta)}{1-\sum_{i=1}^{m-2}\alpha_i\phi_3(\xi_i)}\int_\delta^{1-\delta} G_1(\xi_i,s)p(s)ds.$$

Analogous to Theorem 3.4, using results established above, it is not difficult to show that problem (1.5),(1.7) has at least three positive solutions.

Theorem 4.1. *Suppose conditions (H3),* (4.3)*, (C1) are satisfied and there exist constants $r_2\geq \frac{b}{\lambda_1}>b>r_1>0$, $L_2\geq L_1>0$ such that $\frac{b}{m_1}<\min\{\frac{r_2}{M_1},\frac{L_2}{N_1}\}$ and the following assumptions hold:*

- (A5) $f(t,u,v)<\min\{r_1/M_1,L_1/N_1\},(t,u,v)\in[0,1]\times[0,r_1]\times[-L_1,L_1]$*;*
- (A6) $f(t,u,v)>b/m_1,(t,u,v)\in[\delta,1-\delta]\times[b,b/\lambda_1]\times[-L_2,L_2]$*;*
- (A7) $f(t,u,v)\leq\min\{r_2/M_1,L_2/N_1\},(t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2]$.

Then problem (1.5), (1.7) *has at least three positive solutions x_1,x_2,x_3 satisfying*

$$\begin{gathered}\max_{0\leq t\leq 1}x_1(t)\leq r_1,\quad \max_{0\leq t\leq 1}|x_1'(t)|\leq L_1;\\ b<\min_{\delta\leq t\leq 1-\delta}x_2(t)\leq\max_{0\leq t\leq 1}x_2(t)\leq r_2,\quad \max_{0\leq t\leq 1}|x_2'(t)|\leq L_2;\\ \max_{0\leq t\leq 1}x_3(t)\leq\frac{b}{\lambda_1},\quad \max_{0\leq t\leq 1}|x_3'(t)|\leq L_2.\end{gathered}\tag{4.9}$$

Further we can establish following multiplicity results of problem (1.5), (1.7).

Corollary 4.2. *Suppose condition (A1) is satisfied and there exist constants $0<r_1<b_1<b_1/\lambda\leq r_2<b_2<b_2/\lambda\dots\leq r_n$, $0<L_1\leq L_2\leq\dots\leq L_{n-1}$, $n\in N$, such that*

$$b_i/m\leq\min\{\frac{r_{i+1}}{M},\frac{L_{i+1}}{N}\}$$

If the following two conditions are satisfied then problem (1.5), (1.7) *admits at least* $2n-1$ *positive solutions:*

- (E3) $f(t,u,v)<\min\{\frac{r_i}{M_1},\frac{L_i}{N_1}\}$, $(t,u,v)\in[0,1]\times[0,r_i]\times[-L_i,L_i]$, $1\leq i\leq n$*;*
- (E4) $f(t,u,v)>\frac{b_i}{m_1}$, $(t,u,v)\in[\delta,1-\delta]\times[b_i,\frac{b_i}{\lambda_1}]\times[-L_{i+1},-L_{i+1}]$, $1\leq i\leq n-1$

5. Examples

In this section we present an example to illustrate our main results. Consider the boundary-value problem

$$\begin{gathered}x''(t)-x(t)+f(t,x(t),x'(t))=0,\quad 0<t<1\\ x(0)=0,x(1)=e^{1/2}x(1/2),\end{gathered}\tag{5.1}$$

where

$$f(t,u,v)=\frac{1}{5}\begin{cases} e^t+u^4+(\frac{v}{1000})^3 & 0\le u\le 5\\ e^t+625+(\frac{v}{1000})^3 & u>5\end{cases}$$

Considering lemma 3.1, 3.2, we obtain

$$\phi_1(t)=\frac{e^{1+t}-e^{1-t}}{e^2-1},\quad \phi_1'(0)=\frac{2e}{e^2-1},$$

$$\phi_2(t)=\frac{e^{2-t}-e^t}{e^2-1},\quad p(t)=1,\quad \delta=\frac{1}{4},\quad \lambda=\frac{e^{\frac{5}{4}}-e^{\frac{3}{4}}}{e^2-1},$$

$$G(t,s)=\frac{1}{2e(e^2-1)}\begin{cases}(e^{1+t}-e^{1-t})(e^{2-s}-e^s) & s\ge t\\ (e^{1+s}-e^{1-s})(e^{2-t}-e^t) & s\le t,\end{cases}$$

$$\begin{aligned} M&=\max_{0\le t\le 1}\int_0^1 G(t,s)ds+\frac{e^{1/2}}{1-e^{1/2}\phi_1(1/2)}\int_0^1 G(1/2,s)ds\\ &=\frac{(e+1-2e^{1/2})(1+e^{1/2}+e^{3/2})}{e+1}.\end{aligned}$$

$$\begin{aligned} m&=\min_{\frac14\le t\le\frac34}\int_{\frac14}^{\frac34} G(t,s)ds+\frac{e^{1/2}\phi_1(1/4)}{1-e^{1/2}\phi_1(1/2)}\int_{\frac14}^{\frac34} G(1/2,s)ds\\ &=\frac{1}{2}-e^{1/2}+\frac{e^{7/4}-e^{5/4}}{e-1}.\end{aligned}$$

$$\begin{aligned} N&=\max_{0\le t\le 1}\Big|\int_0^1\frac{\partial G(t,s)}{\partial t}ds+\frac{e^{1/2}\phi_1'(t)}{1-e^{1/2}\phi_1(1/2)}\int_0^1 G(1/2,s)ds\Big|\\ &=\frac{(e^{1/2}-1)(e^2+1)}{e-1}.\end{aligned}$$

Choose $r_1=1$, $r_2=1000$, $b=4$, $L_1=10, L_2=1000$, then $\min\{\frac{r_1}{M},\frac{L_1}{N}\}=\frac{1}{M}$, $\min\{\frac{r_2}{M},\frac{L_2}{N}\}=\frac{1000}{N}$. We can check that conditions (C1), (H3), (3.3) are satisfied and that $f(t,u,v)$ satisfies

$$f(t,u,v)<\frac{1}{M},\quad \text{for } (t,u,v)\in[0,1]\times[0,1]\times[-10,10];$$

$$f(t,u,v)>\frac{4}{m},\quad \text{for } (t,u,v)\in[\frac14,\frac34]\times[4,\frac{4}{\lambda}]\times[-1000,1000];$$

$$f(t,u,v)<\frac{1000}{N},\quad \text{for } (t,u,v)\in[0,1]\times[0,1000]\times[-1000,1000].$$

Then all assumptions of Theorem 3.4 hold. Thus, (5.1) has at least three positive solutions x_1,x_2,x_3 satisfying

$$\max_{0\le t\le1}x_1(t)\le 1,\quad \max_{0\le t\le 1}|x_1'(t)|\le 10;$$

$$4<\min_{\frac14\le t\le\frac34}x_2(t)\le\max_{0\le t\le1}x_2(t)\le 1000,\quad \max_{0\le t\le1}|x_2'(t)|\le 1000;$$

$$\max_{0\le t\le 1}x_3(t)\le\frac{4}{\lambda},\quad \max_{0\le t\le1}|x_3'(t)|\le 1000\,.$$

Remark 5.1. We see that the nonlinear term involves the first order derivative and can it change sign. The early results for multiplicity of positive solutions, to the author's best knowledge, are not applicable to the problem above. Meanwhile, as the nonlinear term does not satisfy the suplinear or sublinear condition even if the nonlinear term is $f(t,u,v) = f(u)$, we can not obtain even one positive solution of this problem from [13].

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