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STANDING WAVES TO CHERN-SIMONS-SCHRÖDINGER SYSTEMS WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. In this article we study the existence of standing waves to nonlinear Chern-Simons-Schrödinger systems with critical exponential growth.

1. Introduction and main result

We study the existence of ground state to the Chern-Simons-Schrödinger system (CSS system) involving a nonlinearity $f(u)$ in the case of critical exponential growth

$$\begin{gathered}
-\Delta u+u+A_0u+\sum_{j=1}^{2}A_j^2u=f(u),\\
\partial_1A_0=A_2|u|^2,\quad \partial_2A_0=-A_1|u|^2,\\
\partial_1A_2-\partial_2A_1=-\frac{1}{2}u^2,\quad \partial_1A_1+\partial_2A_2=0,
\end{gathered}\tag{1.1}$$

where $A_\mu\in\mathbb{R}$, $\mu=0,1,2$, is vector potential of the gauge fields, $\partial_0=\frac{\partial}{\partial t}$, $\partial_1=\frac{\partial}{\partial x_1}$, $\partial_2=\frac{\partial}{\partial x_2}$. This system arises in the study of the standing wave of Chern-Simons-Schrödinger system that describes the dynamics of large number of particles in a electromagnetic field. Chern-Simons terms in CSS system are necessary ingredients in various anyon models describing many fermimion systems such as electron paring in the high-temperature superconductor, fractional quantum Hall effect and Aharovnov-Bohm scattering, see [28, 29] and references therein.

Since the gauge field A_μ is coupled to complex field $\phi\in\mathbb{C}$, the Euler-Lagrange equations of the energy which are given by

$$\begin{gathered}
iD_0\phi+(D_1D_1+D_2D_2)\phi=f(\phi),\\
\partial_0A_1-\partial_1A_0=-\operatorname{Im}(\bar\phi D_2\phi),\\
\partial_0A_2-\partial_2A_0=\operatorname{Im}(\bar\phi D_1\phi),\\
\partial_1A_2-\partial_2A_1=-\frac{1}{2}|\phi|^2.
\end{gathered}\tag{1.2}$$

Here $D_\mu\phi=(\partial_\mu+iA_\mu)\phi$, $\mu=0,1,2$. The CSS system (1.2) is invariant under the following gauge transformation $\phi\to\phi e^{i\chi}$, $\quad A_\mu\to A_\mu-\partial_\mu\chi$ where $\chi:\mathbb{R}^{1+2}\to\mathbb{R}$ is an arbitrary C^∞ function. We assume that the gauge field satisfies the Coulomb

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gauge condition $\partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 = 0$. Then the standing wave $\psi(x,t) = e^{i\omega t}\, u$ satisfies

$$\begin{gathered} -\Delta u + \omega u + A_0 u + A_1^2 u + A_2^2 u = f(u), \\ \partial_1 A_0 = A_2 u^2, \quad \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0. \end{gathered} \tag{1.3}$$

We say that $f(s)$ has subcritical growth at $+\infty$ if for all $\alpha > 0$,

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = 0 \tag{1.4}$$

and $f(s)$ has critical growth at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases} \tag{1.5}$$

We assume $f(u)$ satisfies the following conditions:

(A1) $f \in C(\mathbb{R},\mathbb{R})$ and $f(0) = 0$, $\lim_{s\to 0} F(s)/s^2 = 0$;

(A2) There exist $\theta > 6$ and $s_1 > 0$ such that for all $|s| \geq s_1$

$$0 < \theta F(s) := \theta \int_0^s f(t)\, dt \leq s f(s);$$

(A3) There exists $\beta_0 > 0$ such that

$$\lim_{s\to+\infty} s f(s) e^{-\alpha_0 s^2} \geq \beta_0.$$

We remark that the condition (A2) can be replaced by

$$0 < F(s) \leq M_0 f(s), \quad \text{if } |s| \geq R_0,$$

for some constants R_0, $M_0 > 0$.

The standing waves of (1.2) have been investigated by Byeon, Huh and Seok [2]. They were seeking the radial solutions when $f(u) = \lambda |u|^{p-1} u$, $\lambda > 0$ and $p > 2$ by variational methods, see also [11, 12]. A series of existence and nonexistence results of solitary waves has been established in [4, 5, 17, 24, 25, 26, 30]. We studied the existence, non-existence, and multiplicity of standing waves to the nonlinear CSS systems with an external potential $V(x)$ without the Ambrosetti-Rabinowitz condition in [27]. Sign-changing solutions and Nodal standing waves to a gauged nonlinear Schrödinger equation have been established by [7, 18, 19, 20]. Sign-changing multi-bump solutions were found in [3].

Moreover, we have shown the existence of nontrivial solutions to Chern-Simons-Schrödinger systems (1.1) by using the concentration compactness principle with $V(x)$ is a constant and the argument of global compactness with $V \in C(\mathbb{R}^2)$ and $0 < V_0 < V(x) < V_\infty$ under the condition $p > 4$ in [28]. We also have obtained the concentration behavior of the solutions to system (1.1) with $p > 6$ in [29]. The main characteristic of system (1.1) is that the non-local term A_μ, $\mu = 0, 1, 2$ depends on u and there is a lack of compactness in $\mathbb{R}^2$. By using the variational method we can obtain the following result.

Theorem 1.1. *If $f(s)$ is critical growth and* (A1)–(A3) *hold, then Problem* (1.1) *has a solution.*

We mention that Zhang and Wan also proved that if $f(s)$ is subcritical growth then Problem (1.1) has a solution in [33]. On the other hand, radial solutions for the Chern-Simons-Schrödinger equation with exponential growth can be found in [16]. To demonstrate the desired result, we employ the approach which was developed by do Ó, Medeiros and Severo [8]. Here we mention that Pan, Li, Tang [23] studied CSS system with critical growth; see also [6, 21]. Sign-changing solutions have been found for the nonlinear Chern-Simons-Schrödinger equations in [31] Normalized solutions of Chern-Simons-Schrödinger system are studied by [10, 22, 32].

This article is organized as follows. In Section 2 we introduce the framework and prove some technical lemmas. In Section 3 we prove Theorem 1.1.

2. Mathematical framework

In this section, we outline the variational framework for a future study. We consider the functions which belong to the usual Sobolev space $H^1(\mathbb{R}^2)$ with

$$\|u\| = \Big(\int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2 \, dx \Big)^{1/2}.$$

Define the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + |u|^2 + A_1^2 |u|^2 + A_2^2 |u|^2 \Big) \, dx - \int_{\mathbb{R}^2} F(u) \, dx, \tag{2.1}$$

where $F(u) = \int_0^u f(s) \, ds$. We have the derivative of J in $H^1(\mathbb{R}^2)$ as follows

$$\begin{aligned} \langle J'(u), \eta \rangle &= \int_{\mathbb{R}^2} \Big(\nabla u \nabla \eta + u\eta - f(u)\eta + (A_1^2(u) + A_2^2(u)) u\eta + A_0 u \eta \Big) \, dx \\ &\quad + 2 \int_{\mathbb{R}^2} A_1 u^2 \int_{\mathbb{R}^2} K_2(x,y) u(y) \eta(y) \, dy \, dx \\ &\quad + 2 \int_{\mathbb{R}^2} A_2 u^2 \int_{\mathbb{R}^2} -K_1(x,y) u(y) \eta(y) \, dy \, dx, \end{aligned} \tag{2.2}$$

for all $\eta \in C_0^\infty(\mathbb{R}^2)$. Especially, from (2.4), we have

$$\langle J'(u), \, u \rangle = \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + |u|^2 + 3\big(A_1^2(u) + A_2^2(u)\big) |u|^2 - f(u)u \Big) \, dx. \tag{2.3}$$

Substituting $\partial_1 A_0 = A_2 u^2$, $\partial_2 A_0 = -A_1 u^2$ in the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$, we obtain

$$\begin{aligned} 0 &= \partial_2 \partial_1 A_0 - \partial_1 \partial_2 A_0 \\ &= \partial_2 (A_2 u^2) + \partial_1 (A_1 u^2) \\ &= 2u(A_1 \partial_1 u + A_2 \partial_2 u) + u^2 (\partial_1 A_1 + \partial_2 A_2). \end{aligned}$$

This implies

$$\sum_{j=1}^{2} A_j \partial_j u = 0.$$

This also implies the imaginary part of the CSS system vanishes.

Again we can derive from $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}u^2$ that

$$\begin{aligned}\int_{\mathbb{R}^2} A_0|u|^2\,dx &= -2\int_{\mathbb{R}^2} A_0(\partial_1 A_2 - \partial_2 A_1)\,dx\\ &= 2\int_{\mathbb{R}^2}(A_2\partial_1 A_0 - A_1\partial_2 A_0)\,dx\\ &= 2\int_{\mathbb{R}^2}(A_1^2 + A_2^2)|u|^2\,dx.\end{aligned} \tag{2.4}$$

Combining the equation $\partial_1 A_2 - \partial_2 A_1 = -u^2/2$ and the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$ provides that the components A_j can be determined from u by solving elliptic system

$$\Delta A_1 = \partial_2(\frac{|u|^2}{2}), \quad \Delta A_2 = -\partial_1(\frac{|u|^2}{2}).$$

That are equivalent to

$$\mathcal{F}(A_1) = \frac{-\xi_2}{|\xi|^2}\mathcal{F}(\frac{|u|^2}{2}), \quad \mathcal{F}(A_2) = \frac{\xi_1}{|\xi|^2}\mathcal{F}(\frac{|u|^2}{2})$$

where $\mathcal{F}$ denotes the Fourier transform of an integrable function.

Then we have the following representation of (A_1, A_2),

$$A_1 = A_1(u) = K_2 * (\frac{|u|^2}{2}) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2 - y_2}{|x-y|^2}\frac{|u|^2(y)}{2}\,dy, \tag{2.5}$$

$$A_2 = A_2(u) = -K_1 * (\frac{|u|^2}{2}) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_1 - y_1}{|x-y|^2}\frac{|u|^2(y)}{2}\,dy, \tag{2.6}$$

where $K_j = \frac{-x_j}{2\pi|x|^2}$ for $j = 1, 2$ and $*$ denotes the convolution. Moreover, the system $\partial_1 A_0 = A_2 u^2, \partial_2 A_0 = -A_1 u^2$ implies that

$$\Delta A_0 = \partial_1(A_2|u|^2) - \partial_2(A_1|u|^2),$$

which yields the following representation

$$\begin{aligned}A_0 = A_0(u) &= K_1 * (A_1|u|^2) - K_2 * (A_2|u|^2)\\ &= K_1 * \big(|u|^2 K_2 * \frac{|u|^2}{2}\big) + K_2 * \big(|u|^2 K_1 * \frac{|u|^2}{2}\big).\end{aligned} \tag{2.7}$$

We know that J is well defined in $H^1(\mathbb{R}^2)$, $J \in C^1(H^1(\mathbb{R}^2))$, and the weak solution of (1.1) is the critical point of the functional J from the following properties, which we refer to [28, 29].

Proposition 2.1. *Let $1 < s < 2$ and $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$.*

(i) *There is a constant C depending only on s and q such that*

$$\Big(\int_{\mathbb{R}^2}\big|Tu(x)\big|^q\,dx\Big)^{1/q} \le C\Big(\int_{\mathbb{R}^2}|u(x)|^s\,dx\Big)^{1/s},$$

where the integral operator T is defined as

$$Tu(x) := \int_{\mathbb{R}^2}\frac{u(y)}{|x-y|}\,dy.$$

(ii) *If $u \in H^1(\mathbb{R}^2)$, then we for $j = 1, 2$,*

$$\|A_j^2(u)\|_{L^q(\mathbb{R}^2)} \le C\|u\|^2_{L^{2s}(\mathbb{R}^2)},$$
$$\|A_0(u)\|_{L^q(\mathbb{R}^2)} \le C\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)}.$$

(iii) *For $q' = \frac{q}{q-1}$ and $j = 1, 2$, we have*

$$\|A_j(u)u\|_{L^2(\mathbb{R}^2)} \le \||A_j(u)|^2\|_{L^q(\mathbb{R}^2)}\|u\|^2_{L^{2q'}(\mathbb{R}^2)}.$$

We will need the following properties of the convergence for A_j, see [29].

Proposition 2.2. *Suppose that u_n converges to u a.e. in $\mathbb{R}^2$ and u_n converges weakly to u in $H^1(\mathbb{R}^2)$. Let $A_{\mu,n} := A_\mu(u_n(x))$, $\mu = 0, 1, 2$. Then*
(i) *$A_{\mu,n}$ converges to $A_\mu(u(x))$ a.e. in $\mathbb{R}^2$.*
(ii) *$\int_{\mathbb{R}^2} A^2_{j,n} u_n u\,dx$, $\int_{\mathbb{R}^2} A^2_{j,n}|u|^2\,dx$, and $\int_{\mathbb{R}^2} A^2_{j,n}|u_n|^2\,dx$ converge to $\int_{\mathbb{R}^2} A_j^2|u|^2\,dx$, for $j = 1, 2$; $\int_{\mathbb{R}^2} A_{0,n}u_n u\,dx$ and $\int_{\mathbb{R}^2} A_{0,n}|u_n|^2\,dx$ converge to $\int_{\mathbb{R}^2} A_0|u|^2\,dx$.*
(iii) *$\int_{\mathbb{R}^2} |A_j(u_n - u)|^2|u_n - u|^2\,dx = \int_{\mathbb{R}^2} |A_j(u_n)|^2|u_n|^2\,dx - \int_{\mathbb{R}^2} |A_j(u)|^2|u|^2\,dx + o_n(1)$, for $j = 1, 2$.*

To prove the mountain pass construction, we need the following results from [8].

Proposition 2.3. (i) *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx < \infty.$$

Moreover, if $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le M < \infty$ and $\alpha < 4\pi$ then there exists a constant $C = C(M, \alpha)$, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx < C(M, \alpha).$$

(ii) *Let $\{w_n\}$ in $H^1(\mathbb{R}^2)$ satisfy $\|w_n\| = 1$. Suppose that w_n weakly converges to w_0 in $H^1(\mathbb{R}^2)$ with $\|w_0\| < 1$. Then for all $0 < \beta < \frac{4\pi}{1 - \|w_0\|^2}$,*

$$\sup_n \int_{\mathbb{R}^2} \left(e^{\beta|w_n|^2} - 1\right) dx < \infty.$$

(iii) *Let $\beta > 0$ and $r > 1$. Then for each $\alpha > r$ there exists a positive constant $C = C(\alpha)$ such that for all $s \in \mathbb{R}$,*

$$\left(e^{\beta s^2} - 1\right)^r \le C\left(e^{\alpha\beta s^2} - 1\right).$$

In particular, if $u \in H^1(\mathbb{R}^2)$ then $(e^{\beta u^2} - 1)^r$ belongs to $L^1(\mathbb{R}^2)$.
(iv) *If $v \in H^1(\mathbb{R}^2)$, $\beta > 0$, $q > 0$ and $\|v\| \le M$ with $\beta M^2 < 4\pi$, then there exists $C = C(\beta, M, q) > 0$ such that*

$$\int_{\mathbb{R}^2} \left(e^{\beta v^2} - 1\right)|v|^q\,dx \le C\|v\|^q. \tag{2.8}$$

Next, we prove that the energy functional J has the mountain pass structure.

Lemma 2.4. *Assume* (A1), (A2), *and* (1.5) *hold. Then there exists $\rho > 0$ such that $J(u) > 0$ if $\|u\| = \rho$.*

Proof. From (A1), (A2), and (1.5), there exists $\epsilon < \lambda/2$, where λ is the best constant of $L^2(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2)$, such that

$$|F(s)| \le \epsilon|s|^2 + C_1|s|^q(e^{\alpha s^2} - 1), \tag{2.9}$$

for all $s \in \mathbb{R}$ and $q > 2$. By (iv) of Proposition 2.3 and the Sobolev embeddings, we obtain

$$J(u) \geq (\frac{1}{2} - \frac{\epsilon}{\lambda})\|u\|^2 - C_1\|u\|^q. \tag{2.10}$$

Consequently, by using $\epsilon < \frac{1}{2}\lambda$ and $q > 2$, we can choose $\rho > 0$ such that for $\|u\| = \rho$

$$J(u) \geq \|u\|[(\frac{1}{2} - \frac{\epsilon}{\lambda})\|u\| - c\|u\|^{q-1}] > 0.$$

□

Lemma 2.5. *Assume that f satisfies* (A2). *Then there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < \inf_{\|u\|=\rho} I(u)$.*

Proof. Let $u \in H^1(\mathbb{R}^2)$ such that $u \equiv s_1$ in B_1, $u \equiv 0$ in B_2^c and $u \geq 0$. Denoting $k = supp(u)$. From (A2), for all $s \in \mathbb{R}$ we have

$$F(s) \geq C_1|s|^\theta - C_2. \tag{2.11}$$

Then, for $t > 1$ we have

$$I(tu) \leq \frac{t^2}{2}\|u\|^2 + ct^6\|u\|^6 - ct^\theta \int_{\{x:\, t|u(x)| \geq s_1\}} u^\theta \, dx + C_1|k|.$$

Since $\theta > 6$, we obtain $I(tu) \to -\infty$ as $t \to +\infty$. Setting $e = tu$ with t large enough, the proof is complete. □

We need the following result to prove the (PS) condition.

Lemma 2.6. *Assume* (A2) *and* (1.5). *Let (u_n) in E such that $J(u_n) \to c$ and $J'(u_n) \to 0$. Then, $\|u_n\| \leq c_0$, $\int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq c_0$, and $\int_{\mathbb{R}^2} F(u_n) \, dx \leq c_0$.*

Proof. First, we prove that $\|u_n\| \leq c_0$. We have

$$\frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \left(A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2\right) dx - \int_{\mathbb{R}^2} F(u_n) \, dx = c + o_n(1)$$

and for any $\varphi \in E$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\nabla u_n \nabla \varphi + u_n \varphi\right) dx + \int_{\mathbb{R}^2} \left(A_{1,n}^2 + A_{2,n}^2 + A_{0,n}\right) u_n \varphi \, dx - \int_{\mathbb{R}^2} f(u_n)\varphi \, dx \\ &= o_n(\|\varphi\|). \end{aligned}$$

From (A2) and $\theta > 6$, we obtain

$$\begin{aligned} \theta c + \varepsilon_n\|u_n\| &\geq (\frac{\theta}{2} - 1)\|u_n\|^2 + (\frac{\theta}{2} - 3)\int_{\mathbb{R}^2} \left(A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2\right) dx \\ &\quad - \int_{\mathbb{R}^2} \left(\theta F(u_n) - f(u_n)u_n\right) dx \\ &\geq (\frac{\theta}{2} - 1)\|u_n\|^2 - \int_{\{x:|u_n(x)|<s_1\}} \left(\theta F(u_n) - f(u_n)u_n\right) dx, \end{aligned}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Using that $|f(s)s - F(s)| \leq c_1|s|$ for all $|s| \leq s_1$, we obtain

$$\theta c + \varepsilon_n\|u_n\| \geq (\frac{\theta}{2} - 1)\|u_n\|^2 - c_1\|u_n\|,$$

which implies $\|u_n\| \le c_0$. Next, we show $\int_{\mathbb{R}^2} f(u_n)u_n\,dx \le c_0$ and $\int_{\mathbb{R}^2} F(u_n)\,dx \le c_0$. In fact, since $\|u_n\| \le c_0$, $J(u_n) \to c$, and $J'(u_n) \to 0$, we have

$$\int_{\mathbb{R}^2} F(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \Big(A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2\Big)\,dx - c + o_n(1),$$
$$\int_{\mathbb{R}^2} f(u_n)u_n\,dx = \|u_n\|^2 + 3\int_{\mathbb{R}^2} (A_{1,n}^2 + A_{2,n}^2)u_n^2\,dx - \varepsilon_n\|u_n\|,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. By Proposition 2.1 and Sobolev embedding theorem, we obtain

$$\int_{\mathbb{R}^2} F(u_n) \le \frac{1}{2}\|u_n\|^2 + C\|u_n\|^4 - c + o_n(1),$$
$$\int_{\mathbb{R}^2} f(u_n)u_n\,dx = \|u_n\|^2 + C\|u_n\|^4 - \varepsilon_n\|u_n\|.$$

From $\|u_n\| \le c_0$, we obtain $\int_{\mathbb{R}^2} f(u_n)u_n\,dx \le c_0$ and $\int_{\mathbb{R}^2} F(u_n)\,dx \le c_0$. $\square$

3. Proof of main results

First we need prove the Palais-Smale condition. Using Moser's function sequences, we can obtain the minimax level of the mountain pass solution. Let

$$\tilde{\psi}_n(x) := \frac{1}{\sqrt{2\pi}}\begin{cases} (\log n)^{1/2} & \text{if } |x| \le \frac{r_0}{n},\\ \frac{\log(r_0/|x|)}{(\log n)^{1/2}} & \text{if } r_0/|n| \le |x| \le r_0,\\ 0 & \text{if } |x| > r_0. \end{cases}$$

Notice that $\tilde{\psi}_n \in H^1(\mathbb{R}^2)$, $\operatorname{supp}\tilde{\psi}_n \subset \overline{B}_{r_0}$, for a fixed r_0. By using the fact

$$\int_{\{a_0<|x|<1\}} \nabla\log|x|\,dx = 2\pi\int_{a_0}^1 |\nabla\log r|^2 r\,dr = 2\pi\int_{a_0}^1 \frac{1}{r}dr = -2\pi\ln a_0,$$

we can prove that $\int_{\mathbb{R}^2} |\nabla\tilde{\psi}_n|^2\,dx = 1$. Moreover,

$$\int_{\mathbb{R}^2} |\tilde{\psi}_n|^2\,dx = O(\frac{1}{\log n}), \quad \text{as } n \to \infty.$$

Thus, we can conclude that $\|\tilde{\psi}_n\| \to 1$ as $n \to \infty$.

Considering $\psi_n = \tilde{\psi}_n/\|\tilde{\psi}_n\|$, we can rewrite

$$\psi_n^2(x) = (2\pi)^{-1}\log n + d_n, \quad \text{for all } |x| \le \frac{r_0}{n},$$

where $d_n = (2\pi)^{-1}(\|\tilde{\psi}_n\|^{-1} - 1)\log n$. Consequently

$$\frac{d_n}{\log n} \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

On the other hand, we know that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} |\psi_n|^2\,dx = 0.$$

By the Hölder inequality, for $2\theta + q_1(1-\theta) = 4$ we have

$$\|\psi_n\|_{L^4(\mathbb{R}^2)}^4 \le \|\psi_n\|_{L^2(\mathbb{R}^2)}^{2\theta}\|\psi_n\|_{L^{q_1}(\mathbb{R}^2)}^{(1-\theta)q_1}.$$

Then we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}|\psi_n|^{q_1}\,dx=0\quad\text{for } q_1\geq 2,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^2}A_j(\psi_n)^2\psi_n^2\,dx=0.$$

Proposition 3.1. *Assume that* (A2)–(A4), *hold. Then there exists* $n\in\mathbb{N}$ *such that*

$$\max_{t\geq 0}\Big[\frac{t^2}{2}+\frac{t^6}{2}\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)|\psi_n|^2+A_2^2(\psi_n)|\psi_n|^2\big)\,dx-\int_{\mathbb{R}^2}F(t\psi_n)\,dx\Big]<\frac{2\pi}{\alpha_0}.$$

Proof. Let us choose $r_0>0$ such that

$$\beta_0>\frac{2}{r_0^2\alpha_0},\tag{3.2}$$

where β_0 has been fixed in (A3). Suppose by contradiction that for all n

$$\frac{t^2}{2}+\frac{t^6}{2}\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)|+A_2^2(\psi_n)\big)|\psi_n|^2\,dx-\int_{\mathbb{R}^2}F(t\psi_n)\,dx\geq\frac{2\pi}{\alpha_0}.\tag{3.3}$$

From (A2), there exist positive constants C_1, C_1 such that $F(s)\geq C_1e^{\frac{|s|}{M_0}}-C_2$. Consequently, if $t>0$ is sufficiently large and $m>2$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}F(t\psi_n)\,dx&\geq -C_1+\int_{\{t\psi_n\geq s_1\}}e^{t\psi_n/M_0}\,dx\\&\geq -C_1+C_3\int_{\{t\psi_n\geq s_1\}}(\psi_n)^m\,dx\\&\geq -C_1+C_3t^m\int_{\{\psi_n\geq s_1\}}(\psi_n)^m\,dx.\end{aligned}$$

Hence, for each n there exists unique maximum point t_n such that

$$\frac{t_n^2}{2}+\frac{t_n^6}{2}\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx-\int_{\mathbb{R}^2}F(t_n\psi_n)\,dx=\max_{t>0}J(t\psi_n)$$

and

$$\frac{d}{dt}J(t\psi_n)\big|_{t=t_n}=0.$$

From which it follows that

$$t_n^2+3t_n^6\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx-\int_{\mathbb{R}^2}t_n\psi_nf(t_n\psi_n)\,dx=0.\tag{3.4}$$

By (A3) for each $\varepsilon>0$ there exists $R_\varepsilon>0$ such that

$$\psi_nf(\psi_n)\geq(\beta_0-\varepsilon)\exp(\alpha_0\psi_n^2)\tag{3.5}$$

for all $\psi_n\geq R_\varepsilon$ and $|x|\leq r_0$. From (3.4) and (3.5), we have

$$\begin{aligned}&t_n^2+3t_n^6\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx\\&\geq(\beta_0-\epsilon)\pi\big(\frac{r_0}{n}\big)^2\exp(\frac{\alpha_0}{2\pi}t_n^2\log n+2\alpha_0t_n^2d_n).\end{aligned}\tag{3.6}$$

That is,

$$1+3t_n^4\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx$$

$$\geq (\beta_0-\epsilon)\pi r_0^2 \exp(\frac{\alpha_0}{2\pi}t_n^2\log n + 2\alpha_0 t_n^2 d_n - 2\log t_n - 2\log n).$$

Since $\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx \to 0$, as $n\to\infty$, we obtain that $\{t_n\}$ is bounded.

We claim that

$$t_n^2 \to \frac{4\pi}{\alpha_0}, \quad \text{as } n\to\infty. \tag{3.7}$$

In fact, by (3.3), (3.4), and (A2), we have

$$\frac{t_n^2}{2} + \frac{t_n^6}{2}\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx \geq \frac{2\pi}{\alpha_0} + \int_{\{t_n\psi_n\leq s_1\}} F(t_n\psi_n)\,dx$$

Since $\{t_n\}$ is bounded, by (2.9) and $\|\tilde\psi_n\|^2\to 1$ as $n\to\infty$, we obtain

$$\Big|\int_{\{t_n\psi_n\leq s_1\}} F(t_n\psi_n)\,dx\Big| \leq C\int_{\mathbb{R}^2}\psi_n^2\,dx = C\frac{1}{\|\tilde\psi_n\|^2}\int_{\mathbb{R}^2}\tilde\psi_n^2\,dx \to 0.$$

Note that $\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx \to 0$ as $n\to\infty$. Consequently,

$$t_n^2 \geq \frac{4\pi}{\alpha_0} + o_n(1), \quad \text{as } n\to\infty.$$

Suppose by contradiction that $\lim_{n\to\infty} t_n^2 > \frac{4\pi}{\alpha_0}$. From (3.6), we have

$$\begin{aligned}
&t_n^2 + 3t_n^6\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx\\
&\geq (\beta_0-\epsilon)\pi r_0^2\exp\Big((\frac{\alpha_0}{4\pi}t_n^2-1)2\log n + 2\alpha_0 t_n^2 d_n\Big).
\end{aligned}$$

Since (3.1), the last inequality contradicts the boundedness of $\{t_n\}$ and the claim holds.

Let us denote

$$\Omega_{1,n} := \{x\in B_{r_0} : t_n\psi_n \geq R_\epsilon\}, \text{ and } \Omega_{2,n} := B_{r_0}\setminus\Omega_{1,n}.$$

By (3.4) and (3.5), we obtain

$$\begin{aligned}
&t_n^2 + 3t_n^6\int_{\mathbb{R}^2}\big(A_1^2(\psi_n)+A_2^2(\psi_n)\big)|\psi_n|^2\,dx\\
&\geq (\beta_0-\epsilon)\int_{B_{r_0}} e^{\alpha_0 t_n^2\psi_n^2} + \int_{\Omega_{2,n}} t_n\psi_n f(t_n\psi_n) - (\beta_0-\epsilon)\int_{\Omega_{2,n}} e^{\alpha_0 t_n^2\psi_n^2}.
\end{aligned} \tag{3.8}$$

Since $\psi_n(x)\to 0$ as $n\to\infty$ and the characteristic functions $\chi_{\Omega_{2,n}}\to 1$ for almost every x such that $|x|\leq r$. By the Lebesgue dominated convergence theorem, we have

$$\int_{\Omega_{2,n}} t_n\psi_n f(t_n\psi_n)\,dx \to 0 \quad \text{and} \quad \int_{\Omega_{2,n}} e^{\alpha_0 t_n^2\psi_n^2}\,dx \to \pi r_0^2 \quad \text{as } n\to\infty.$$

By $t_n^2 \geq \frac{4\pi}{\alpha_0}$, we obtain

$$\begin{aligned}
\int_{\{|x|\leq r_0\}} e^{\alpha_0 t_n^2\psi_n^2}\,dx &\geq \int_{\{|x|\leq r_0\}} e^{4\pi\psi_n^2}\,dx\\
&= \int_{\{|x|\leq \frac{r_0}{n}\}} e^{4\pi\psi_n^2}\,dx + \int_{\{\frac{r_0}{n}\leq|x|\leq r_0\}} e^{4\pi\psi_n^2}\,dx.
\end{aligned} \tag{3.9}$$

A direct computation gives

$$\begin{aligned}\lim_{n\to\infty}\int_{\{|x|\le\frac{r_0}{n}\}} e^{4\pi\psi_n^2}\,dx &= \lim_{n\to\infty}\int_{\{|x|\le\frac{r_0}{n}\}} e^{2\log n+4\pi d_n}\,dx\\ &= \lim_{n\to\infty}\pi\frac{r_0^2}{n^2}n^{2+4\pi(\log n)^{-1}d_n} = \pi r_0^2.\end{aligned}$$

Set $t=\log(\frac{r_0}{|x|})/(\xi_n\log n)$, where $\xi_n=\|\tilde{\psi}_n\|>1$. We have

$$\int_{\{\frac{r_0}{n}\le|x|\le r_0\}} e^{4\pi\psi_n^2}\,dx = 2\pi r_0^2\xi_n\log n\int_0^{\xi_n^{-1}} e^{2\log n(t^2-\xi_n t)}\,dt.$$

Since

$$t^2-\xi_n t\ge\begin{cases}-\xi_n t & \text{if } 0\le t\le\frac{\xi_n^{-1}}{2},\\ (2\xi_n^{-1}-\xi_n)(t-\xi_n^{-1})+(\xi_n^{-2}-1) & \text{if } \frac{\xi_n^{-1}}{2}\le t\le\xi_n^{-1},\end{cases}$$

we obtain

$$\lim_{n\to\infty}\int_{\{r_0/n\le|x|\le r_0\}} e^{4\pi\psi_n^2}\,dx\ge 2\pi r_0^2.$$

Taking $n\to\infty$ in (3.8) and using (3.7), we obtain

$$\frac{4\pi}{\alpha_0}\ge(\beta_0-\varepsilon)2\pi r_0^2,$$

which gives $\beta_0\le\frac{2}{\alpha_0 r_0^2}$. This contradicts (3.2). The proof is complete. □

Assuming that

$$\liminf_{n\to\infty}\|u_n\|^2<\frac{4\pi}{\alpha_0},$$

then there exists a subsequence of $\{u_n\}$ which converges to u_0 in $H^1(\mathbb{R}^2)$.

Proposition 3.2. $J(u_0)=c$.

Proof. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$, there exists a subsequence denoted again by $\{u_n\}$ such that

$$\begin{gathered}u_n\rightharpoonup u_0\quad\text{in } H^1(\mathbb{R}^2),\\ u_n\to u_0\quad\text{in } L^q_{\rm loc}(\mathbb{R}^2),\ q\ge1,\\ u_n(x)\to u_0(x)\quad\text{a.e. in }\mathbb{R}^2.\end{gathered}$$

Moreover, for any $R>0$,

$$\lim_{n\to\infty}\int_{B_R}\big(F(u_n)-F(u_0)\big)\,dx=0.$$

It is known that for $u\in L^2(\mathbb{R}^2)$, the Schwartz symmetrization of u satisfies

$$|u^*|\le\frac{\|u^*\|_{L^2(\mathbb{R}^2)}}{\sqrt{\pi}|x|}.$$

Since

$$\int_{B_R^c}F(u_n)\le C_1\int_{B_R^c}|u_n|^2+C_2\int_{B_R^c}\big(|u_n|e^{\alpha|u_n|^2}-1\big)\,dx,$$

where $\alpha \geq \alpha_0$, and

$$\int_{B_R^c} \left(|u_n| e^{\alpha |u_n|^2} - 1\right) dx \leq \int_{B_R^c} \sum_{l=1}^{\infty} \frac{|u_n^*|^{2l+1}}{l!} \leq \frac{C}{R},$$

for each $\delta > 0$, there exists $R > 0$ such that

$$\max\{\int_{B_R^c} F(u_n)\,dx, \int_{B_R^c} F(u_0)\,dx, \int_{B_R} \left(F(u_n) - F(u_0)\right) dx\} \leq \frac{\delta}{3},$$

from which it follows that

$$\int_{\mathbb{R}^2} \left(F(u_n) - F(u_0)\right) dx < \delta.$$

Hence by using $J(u_n) \to c$ we conclude that

$$\frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_n) + A_2^2(u_n)\right)|u_n|^2\,dx = c + \int_{\mathbb{R}^2} F(u_0)\,dx + o_n(1).$$

We observe that $\lim_{n\to\infty} \|u_n\| \geq \|u_0\| > 0$, so that we define

$$w_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad w_0 = \frac{u_0}{\lim_{n\to\infty} \|u_n\|}.$$

Then $\|w_n\| = 1$ and $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^2)$. Suppose that $\|w_0\| < 1$. By Proposition 3.1, we see that $\alpha_0 < \frac{2\pi}{c - J(u_0)}$. Let us choose $\beta > 1$ sufficiently close to 1 and $\delta > 0$ such that

$$\begin{aligned}
\beta\alpha_0\|u_n\|^2 \leq& \frac{2\pi\|u_n\|^2}{c - J(u_0)} - \delta \\
\leq& 4\pi \frac{c + \int_{\mathbb{R}^2} F(u_0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_0) + A_2^2(u_0)\right)|u_0|^2\,dx + o_n(1)}{c - J(u_0)} - \delta.
\end{aligned}$$

On the other hand, by using the formula for $J(u_0)$ and $J(u_0) < c$ we deduce that

$$\begin{aligned}
&\left(1 - \|w_0\|^2\right)\left(c + \int_{\mathbb{R}^2} F(u_0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_0) + A_2^2(u_0)\right)|u_0|^2\,dx + o_n(1)\right) \\
&\leq c + \int_{\mathbb{R}^2} F(u_0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_0) + A_2^2(u_0)\right)|u_0|^2\,dx \\
&\quad - \|w_0\|^2\Big(\int_{\mathbb{R}^2} F(u_0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_0) + A_2^2(u_0)\right)|u_0|^2\,dx + o_n(1)\Big) \\
&= c + (-J(u_0) + \frac{1}{2}\|u_0\|^2) \\
&\quad - \|w_0\|^2\Big(c + \int_{\mathbb{R}^2} F(u_0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2(u_0) + A_2^2(u_0)\right)|u_0|^2\,dx + o_n(1)\Big) \\
&\leq c - J(u_0).
\end{aligned}$$

Therefore, there exists $\delta > 0$ such that

$$\beta\alpha_0\|u_n\|^2 \leq \frac{4\pi}{1 - \|w_0\|^2} - \delta.$$

Thus, $(\beta + \epsilon)\alpha_0\|u_n\|^2 \leq \frac{4\pi}{1 - \|w_0\|^2}$, which implies by (ii) of Proposition 2.3 that

$$\int_{\mathbb{R}^2} \left(e^{(\beta+\epsilon)\alpha_0\|u_n\|^2 w_0^2} - 1\right) dx \leq C.$$

We observe that

$$|\int_{\mathbb{R}^2} f(u_n)(u_n-u_0)\,dx| \le |\int_{\mathbb{R}^2} |(u_n-u_0)|e^{\alpha_0\|u_n\|^2 w_n^2}\,dx| \le C\int_{\mathbb{R}^2} |u_n-u_0|^{\frac{q}{q-1}}\,dx.$$

Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} \nabla u_0\nabla(u_n-u_0)+u_0(u_n-u_0)\,dx=0.$$

Hence, $\{u_n\}$ converges to u_0 in $H^1(\mathbb{R}^2)$. □

Proof of Theorem 1.1. Let $\{u_n\}$ satisfying $J(u_n)\to c_0$ and $J'(u_n)\to 0$ as $n\to\infty$. By Lemma 2.6, $\{u_n\}$ is bounded, up to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, $u_n\to u_0$ in $L^q_{\rm loc}(\mathbb{R}^2)$ for all $q\ge 2$ and $u_n\to u_0$ almost everywhere in $\mathbb{R}^2$, as $n\to\infty$. Then, if $f(s)$ satisfies (1.5), we have for each $\alpha>\alpha_0$ there exist $b_1,b_2>0$ such that for all $s\in\mathbb{R}$, for all $\alpha>0$,

$$|f(s)|\le b_1|s|+b_2\big(e^{\alpha s^2}-1\big). \tag{3.10}$$

If the vanishing case occurs, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}|u_n|^2\,dx=0. \tag{3.11}$$

Consequently, by (3.10), (3.11), Hölder' inequality, and Proposition 2.3, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^2}|f(u_n)u_n|\,dx\\
&\le \lim_{n\to\infty}\int_{\mathbb{R}^2}\Big(b_1|u_n|^2+b_2|u_n|\big(e^{\alpha|u_n|^2}-1\big)\Big)\,dx\\
&\le b_1\lim_{n\to\infty}\int_{\mathbb{R}^2}|u_n|^2\,dx\\
&\quad +b_2\Big(\lim_{n\to\infty}\int_{\mathbb{R}^2}|u_n|^2\,dx\Big)^{1/2}\Big(\lim_{n\to\infty}\int_{\mathbb{R}^2}\big(e^{\alpha|u_n|^2}-1\big)^2\,dx\Big)^{1/2}=0.
\end{aligned} \tag{3.12}$$

By Proposition 2.2, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\big(A_1^2(u_n)+A_2^2(u_n)\big)u_n^2\,dx=0. \tag{3.13}$$

From (2.3), (3.12), (3.13), and that $\{u_n\}$ is bounded, we have

$$\begin{aligned}
&\|u_n\|^2\\
&=\langle J'(u_n),u_n\rangle-3\int_{\mathbb{R}^2}\big(\big(A_1^2(u_n)+A_2^2(u_n)\big)u_n^2\,dx+\int_{\mathbb{R}^2}f(u_n)u_n\,dx\to 0,
\end{aligned} \tag{3.14}$$

as $n\to\infty$. By (2.9), (3.12), (3.14), and Hölder's inequality, we have

$$\begin{aligned}
&\lim_{n\to\infty}|\int_{\mathbb{R}^2}F(u_n)\,dx|\\
&\le \lim_{n\to\infty}\big(\epsilon\int_{\mathbb{R}^2}|u_n|^2\,dx+C_1\int_{\mathbb{R}^2}|u_n|^q(e^{\alpha u_n^2}-1)\,dx\big)=0.
\end{aligned} \tag{3.15}$$

This implies that $0<J(u_n)\to 0$ as $n\to\infty$, which means that vanishing is impossible.

Hence only the nonvanishing case happens. Since

$$\int_{\mathbb{R}^2}u^2(x)\langle A_0'(u),\eta\rangle\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} u^2(x)\Big(\int_{\mathbb{R}^2} \frac{x_1-y_1}{2\pi|x-y|^2} u(y)\eta(y)(y)A_2(u(y))\,dy \\
&\quad - \int_{\mathbb{R}^2} \frac{x_2-y_2}{2\pi|x-y|^2} u(y)\eta(y)A_1(u(y))\,dy\Big)dx \\
&= \int_{\mathbb{R}^2} A_2(u(y))u(y)\eta(y)\int_{\mathbb{R}^2} \frac{x_1-y_1}{2\pi|x-y|^2} u(x)^2\,dxdy \\
&\quad - \int_{\mathbb{R}^2} A_1(u(y))u(y)\eta(y)\int_{\mathbb{R}^2} \frac{x_2-y_2}{2\pi|x-y|^2} u^2(x)\,dx\,dy \\
&= \int_{\mathbb{R}^2} |A_2(u(y))|^2 u(y)\eta(y) + |A_1(u(y))|^2 u(y)\eta(y)\,dy,
\end{aligned}
$$

For each $\eta \in C_0^\infty(\mathbb{R}^2)$, we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \langle J'(u_n), \eta\rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^2} \Big(\nabla u_n \nabla \eta + u_n\eta + (A_1^2(u_n) + A_2^2(u_n))u_n\eta + A_0(u_n)u_n\eta - f(u_n)\eta\Big)\,dx \\
&= \langle J'(u_0), \eta\rangle.
\end{aligned}
$$

Hence u_0 is a week solution of Problem (1.1). □

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