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CONTROLLABILITY FOR THE WAVE EQUATION WITH MOVING BOUNDARY

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Abstract. In this article, we study the boundary controllability for a one-dimensional string equation on a domain with time-dependent boundary. This equation models small vibrations of a string with one of its endpoint fixed and other moving with speed $k(t)$. We use an inverse inequality to obtain a controllability result. We consider a linear wave equation.

1. Introduction

1.1. **Statement of the problem and main result.** The controllability of linear and non-linear PDEs has been the subject of much work in the last decades. Theoretical aspects and their connection to applications have been considered by many authors and a lot of advances can be fortunately mentioned. Initially, the concept of *hierarchic control* was introduced by J.-L. Lions (see [17, 18]), where some techniques are presented. Also we mention the papers [15, 16, 28, 29, 9, 7, 8, 6] where the authors combine the concepts of multi-criteria optimization and controllability.

As in [3], given $T > 0$, we consider the non-cylindrical domain defined by

$$\widehat{Q} = \{(x,t) \in \mathbb{R}^2;\ 0 < x < \alpha_k(t),\ t \in (0,T)\},$$

where $\alpha_k(t) = 1 + kt$, $0 < k < 1$. Its lateral boundary is defined by $\widehat{\Sigma} = \widehat{\Sigma}_0 \cup \widehat{\Sigma}_0^*$, with

$$\widehat{\Sigma}_0 = \{(0,t);\ t \in (0,T)\}, \quad \widehat{\Sigma}_0^* = \widehat{\Sigma} \backslash \widehat{\Sigma}_0 = \{(\alpha_k(t),t);\ t \in (0,T)\}.$$

We also represent by Ω_t and Ω_0 the intervals $(0, \alpha_k(t))$ and $(0,1)$, respectively.

Motivated by the arguments contained in the work of J.-L. Lions [19], we consider the following wave equation in the non-cylindrical domain $\widehat{Q}$:

$$\begin{gathered} u'' - u_{xx} = 0 \quad \text{in } \widehat{Q}, \\ u(x,t) = \begin{cases} \widetilde{w}(t) & \text{on } \widehat{\Sigma}_0, \\ 0 & \text{on } \widehat{\Sigma}_0^*, \end{cases} \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega_0, \end{gathered} \tag{1.1}$$

where u is the state variable, $\widetilde{w}$ is the control variable and $(u_0(x), u_1(x)) \in L^2(0,1) \times H^{-1}(0,1)$. By $u' = u'(x,t)$ we represent the derivative $\frac{\partial u}{\partial t}$ and by $u_{xx} = u_{xx}(x,t)$

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the second order partial derivative $\frac{\partial^2 u}{\partial x^2}$. Equation (1.1) models the motion of a string with a fixed endpoint and a moving one. The constant k is called the speed of the moving endpoint.

The novelty of this paper is the consideration of a domain with moving boundary. Indeed, instead of transforming the problem (1.1) from a non-cylindrical domain into a cylindrical domain, we study the controllability problem directly in a non-cylindrical domain, when the control is put on the fixed point. For this, we use an inverse inequality.

Now, let $\alpha(t) = t + \alpha_k(t)$, $\beta(t) = t - \alpha_k(t)$ and $\gamma = \alpha \circ \beta^{-1}$. We assume that

$$T > \gamma(0), \tag{1.2}$$

$$0 < k < 1. \tag{1.3}$$

The main result of this paper reads as follows.

Theorem 1.1. *Assume that* (1.2) *and* (1.3) *hold. Let us consider* $\widetilde{w}_1 \in L^2(\widehat{\Sigma}_1)$ *and* $\widetilde{w}_2$ *a Nash equilibrium in the sense* (1.10)*. Then the pair*

$$(u(T), u'(T)) = (u(., T, \widetilde{w}_1, \widetilde{w}_2), u'(., T, \widetilde{w}_1, \widetilde{w}_2)),$$

where u solves the system (2.8)*, generates a dense subset of* $L^2(\Omega_T) \times H^{-1}(\Omega_T)$*.*

As in [19], we divide $\widehat{\Sigma}_0$ into two parts

$$\widehat{\Sigma}_0 = \widehat{\Sigma}_1 \cup \widehat{\Sigma}_2, \tag{1.4}$$

and consider

$$\widetilde{w} = \{\widetilde{w}_1, \widetilde{w}_2\}, \quad \widetilde{w}_i = \text{control function in } L^2(\widehat{\Sigma}_i),\ i = 1, 2. \tag{1.5}$$

We can also write

$$\widetilde{w} = \widetilde{w}_1 + \widetilde{w}_2, \quad \text{with } \widehat{\Sigma}_0 = \widehat{\Sigma}_1 = \widehat{\Sigma}_2. \tag{1.6}$$

Thus, we observe that system (1.1) can be rewritten as follows:

$$\begin{gathered} u'' - u_{xx} = 0 \quad \text{in } \widehat{Q}, \\ u(x,t) = \begin{cases} \widetilde{w}_1(t) & \text{on } \widehat{\Sigma}_1, \\ \widetilde{w}_2(t) & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma} \backslash \widehat{\Sigma}_0, \end{cases} \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega_0. \end{gathered} \tag{1.7}$$

In decomposition (1.4), (1.5) we establish a hierarchy. We think of $\widetilde{w}_1$ as being the "main" control, the leader, and we think of $\widetilde{w}_2$ as the follower, in Stackelberg terminology.

Associated with the solution $u = u(x,t)$ of (1.7), we will consider the (secondary) functional

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) = \frac{1}{2} \int\int_{\widehat{Q}} (u(\widetilde{w}_1, \widetilde{w}_2) - \widetilde{u}_2)^2 \, dx \, dt + \frac{\widetilde{\sigma}}{2} \int_{\widehat{\Sigma}_2} \widetilde{w}_2^2 \, d\widehat{\Sigma}, \tag{1.8}$$

and the (main) functional

$$\widetilde{J}(\widetilde{w}_1) = \frac{1}{2} \int_{\widehat{\Sigma}_1} \widetilde{w}_1^2 \, d\widehat{\Sigma}, \tag{1.9}$$

where $\widetilde{\sigma} > 0$ is a constant and $\widetilde{u}_2$ is a given function in $L^2(\widehat{Q})$.

Now, let us describe the Stackelberg-Nash strategy. Thus, for each choice of the leader $\widetilde{w}_1$, we try to find a Nash equilibrium for the cost functional $\widetilde{J}_2$, that is, we look for a control $\widetilde{w}_2 = \mathfrak{F}(\widetilde{w}_1)$, depending on $\widetilde{w}_1$, satisfying

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) = \inf_{\widehat{w}_2 \in L^2(\widehat{\Sigma}_2)} \widetilde{J}_2(\widetilde{w}_1, \widehat{w}_2). \tag{1.10}$$

After this, we consider the state $u\left(\widetilde{w}_1, \mathfrak{F}(\widetilde{w}_1)\right)$ given by the solution of

$$\begin{gathered} u'' - u_{xx} = 0 \quad \text{in } \widehat{Q}, \\ u(x,t) = \begin{cases} \widetilde{w}_1 & \text{on } \widehat{\Sigma}_1, \\ \mathfrak{F}(\widetilde{w}_1) & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma} \backslash \widehat{\Sigma}_0, \end{cases} \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega_0. \end{gathered} \tag{1.11}$$

We will look for any optimal control $\widetilde{w_1}$ such that

$$\widetilde{J}(\widetilde{w}_1, \mathfrak{F}(\widetilde{w}_1)) = \inf_{\overline{w}_1 \in L^2(\widehat{\Sigma}_1)} \widetilde{J}(\overline{w}_1, \mathfrak{F}(\widetilde{w}_1)), \tag{1.12}$$

subject to the following restriction of the approximate controllability type

$$\begin{aligned} &(u(x,T;\widetilde{w}_1, \mathfrak{F}(\widetilde{w}_1)), u'(x,T;\widetilde{w}_1, \mathfrak{F}(\widetilde{w}_1))) \\ &\in B_{L^2(\Omega_T)}(u^0, \rho_0) \times B_{H^{-1}(\Omega_T)}(u^1, \rho_1), \end{aligned} \tag{1.13}$$

where $B_X(C,r)$ denotes the ball in X with center C and radius r.

The control problems to be studied in this paper are described as follows:

Problem 1 Fixed any leader control $\widetilde{w}_1$, find the follower control $\widetilde{w}_2 = \mathfrak{F}(\widetilde{w}_1)$ (depending on $\widetilde{w}_1$) and the associated state u, solution of (1.7) satisfying the condition (1.10) (Nash equilibrium) related to $\widetilde{J}_2$, defined in (1.8).

Problem 2 Assuming that the existence and uniqueness of the Nash equilibrium $\widetilde{w}_2$ was proved, then when $\widetilde{w}_1$ varies in $L^2(\widehat{\Sigma}_1)$, prove that the solutions $(u(x,t;\widetilde{w}_1,\widetilde{w}_2), u'(x,t;\widetilde{w}_1,\widetilde{w}_2))$ of the state equation (1.7), evaluated at $t=T$, that is, $(u(x,T;\widetilde{w}_1,\widetilde{w}_2), u'(x,T;\widetilde{w}_1,\widetilde{w}_2))$, generate a dense subset of $L^2(\Omega_T) \times H^{-1}(\Omega_T)$.

Remark 1.2. By the linearity of system (1.11), without loss of generality we may assume that $u_0 = 0 = u_1$.

1.2. **Related problems.** Controllability of system (1.1) has been extensively studied in the recent past years; most of the papers in this direction dealt with the case of one moving endpoint with boundary conditions of the form

$$u(0,t) = 0, \quad u(1+kt,t) = \widetilde{w}(t), \quad k \in (0,1), \quad t \in (0,\infty).$$

In [24], it has been shown that exact controllability holds at any time

$$T > \frac{e^{\frac{2k(1+k)}{1-k}} - 1}{2}.$$

The same authors came back in [25] and improved the latter result to

$$T > \frac{e^{\frac{2k(1+k)}{(1-k)^3}} - 1}{2}.$$

Later, in [11], the controllability time has been improved to be $T > \frac{2}{1-k}$. In these papers, only a sufficient condition is provided for the exact controllability.

Concerning the two moving endpoints case, the boundary functions considered in [1] are of the form

$$\alpha(t) = -kt, \quad \beta(t) = 1 + rt, \quad t \in (0, \infty), \quad k, r \in [0, 1) \quad \text{with } r + k > 0.$$

It has been shown that exact controllability holds if, and only if $T \geq \frac{2}{(1-k)(1-r)}$. More general boundary functions are considered in [3] with boundary conditions

$$u(0, t) = 0, \quad u(s(t), t) = \widetilde{w}(t), \quad t \in (0, \infty),$$

where $s : [0, \infty) \to (0, \infty)$ is assumed to be C^1 function with $\|s'\|_{L^\infty(0,\infty)} < 1$. Furthermore, it has been assumed that s must be in some admissible class of curves (see [3] for more details). Under these assumptions, the authors proved that exact controllability holds if, and only if $T \geq s^+ \circ (s^-)^{-1}(0)$, where $s^\pm(t) = t \pm s(t)$. Also, they provided a controllability result when the control is located on the non-moving part of the boundary. By considering the boundary conditions

$$u(0, t) = \widetilde{w}(t), \quad u(s(t), t) = 0, \quad t \in (0, \infty),$$

they proved that exact controllability holds if, and only if $T \geq (s^-)^{-1}(1)$. In all the cited works, the proofs rely on the multipliers technique or the non-harmonic Fourier analysis.

Recently, in [2], a new Carleman estimate has been established for the wave equation in time-dependent domain in a more general setting. The existence of solutions of the initial boundary value problem for the nonlinear wave equation in non-cylindrical domains has been studied in [5, 20]. The controllability problem for a multi-dimensional wave equation in a non-cylindrical domain has been investigated in [4, 26, 27]. Also, about the one-dimension cases, there have been extensive study of the controllability problem in a non-cylindrical domain. We refer the reader to [12, 14, 21, 22, 23]. Finally, we can mention also the paper by Yang and Feng [31], where the authors present the approximate controllability of Euler-Bernoulli viscoelastic systems.

The content of this article is organized as follows. Section 2 is devoted to establish the optimality system for the follower control. In Section 3, we investigate the approximate controllability proving the density Theorem 1.1. Finally, we present the optimality system for the leader control in Section 4.

2. Optimal system for the follower control

In this section, fixed any leader control $\widetilde{w}_1 \in L^2(\widehat{\Sigma}_1)$ we determine the existence and uniqueness of solutions to the problem

$$\inf_{\widetilde{w}_2 \in L^2(\widehat{\Sigma}_2)} \widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2), \tag{2.1}$$

and a characterization of this solution in terms of an adjoint system.

For this, we consider $\mathcal{U}_{ad} = \{(u, \widetilde{w}_2) \in L^2(Q) \times L^2(\widehat{\Sigma}_2); u \text{ solution of } (1.7)\}$, and $\widetilde{J}_2 : \mathcal{U}_{ad} \to \mathbb{R}$ defined by (1.8). Note that $\mathcal{U}_{ad}$ is a nonempty closed convex subset of $L^2(Q) \times L^2(\Sigma_2)$, and $\widetilde{J}_2$ is weakly coercive, weakly sequentially lower semicontinuous and strictly convex. Therefore, there exists a unique solution $\widetilde{w}_2$ of (2.1), i.e.,

$$\widetilde{J}_2(\widetilde{w}_1, \widetilde{w}_2) = \inf_{\widehat{w}_2 \in L^2(\widehat{\Sigma}_2)} \widetilde{J}_2(\widetilde{w}_1, \widehat{w}_2).$$

The Euler-Lagrange equation for problem (2.1) is

$$\int_0^T \int_{\Omega_t} (u - \widetilde{u}_2)\widehat{u}\, dx\, dt + \widetilde{\sigma} \int_{\widehat{\Sigma}_2} \widetilde{w}_2 \widehat{w}_2 d\widehat{\Sigma} = 0, \quad \forall \widehat{w}_2 \in L^2(\widehat{\Sigma}_2), \tag{2.2}$$

where $\widehat{u}$ is solution of the system

$$\begin{gathered} \widehat{u}'' - \widehat{u}_{xx} = 0 \quad \text{in } \widehat{Q}, \\ \widehat{u} = \begin{cases} 0 & \text{on } \widehat{\Sigma}_1, \\ \widehat{w}_2 & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma} \backslash (\widehat{\Sigma}_1 \cup \widehat{\Sigma}_2), \end{cases} \\ \widehat{u}(x,0) = 0, \quad \widehat{u'}(x,0) = 0, \quad x \text{ in } \Omega_0. \end{gathered} \tag{2.3}$$

To express (2.2) in a convenient form, we introduce the adjoint state defined by

$$\begin{gathered} p'' - p_{xx} = u - \widetilde{u}_2 \quad \text{in } \widehat{Q}, \\ p(T) = p'(T) = 0, \quad x \text{ in } \Omega_T, \\ p = 0 \quad \text{on } \widehat{\Sigma}. \end{gathered} \tag{2.4}$$

Multiplying (2.4) by $\widehat{u}$ and integrating by parts, we find that

$$\int_0^T \int_{\Omega_t} (u - \widetilde{u}_2)\widehat{u}\, dx\, dt + \int_{\widehat{\Sigma}_2} p_x\, \widehat{w}_2\, d\widehat{\Sigma} = 0, \tag{2.5}$$

so that (2.2) becomes

$$p_x = \widetilde{\sigma}\, \widetilde{w}_2 \quad \text{on } \widehat{\Sigma}_2. \tag{2.6}$$

We summarize these results in the following theorem.

Theorem 2.1. *For each $\widetilde{w}_1 \in L^2(\Sigma_1)$ there exists a unique Nash equilibrium $\widetilde{w}_2$ in the sense of* (1.10)*. Moreover, the follower $\widetilde{w}_2$ is given by*

$$\widetilde{w}_2 = \mathfrak{F}(\widetilde{w}_1) = \frac{1}{\widetilde{\sigma}} p_x \quad \text{on } \widehat{\Sigma}_2, \tag{2.7}$$

where $\{v, p\}$ is the unique solution of (the optimality system)

$$\begin{gathered} u'' - u_{xx} = 0 \quad \text{in } \widehat{Q}, \\ p'' - p_{xx} = u - \widetilde{u}_2 \quad \text{in } \widehat{Q}, \\ u = \begin{cases} \widetilde{w}_1 & \text{on } \widehat{\Sigma}_1, \\ \frac{1}{\widetilde{\sigma}} p_x & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma} \backslash \widehat{\Sigma}_0, \end{cases} \\ p = 0 \quad \text{on } \widehat{\Sigma}, \\ u(0) = u'(0) = 0, \quad x \text{ in } \Omega_0, \\ p(T) = p'(T) = 0, \quad x \text{ in } \Omega_T. \end{gathered} \tag{2.8}$$

Of course, $\{u, p\}$ depends on $\widetilde{w}_1$:

$$\{u, p\} = \{u(\widetilde{w}_1), p(\widetilde{w}_1)\}. \tag{2.9}$$

3. Proof of Theorem 1.1

Since we have proved the existence, uniqueness and characterization of the follower $\widetilde{w}_2$, the leader $\widetilde{w}_1$ now wants that solutions u and u', evaluated at time $t = T$, to be as close as possible to (u^0, u^1). This will be possible if the system (2.8) is approximately controllable. We are looking for

$$\inf \frac{1}{2} \int_{\widehat{\Sigma}_1} \widetilde{w}_1^2 \, d\widehat{\Sigma}, \tag{3.1}$$

where $\widetilde{w}_1$ is subject to

$$(u(T; \widetilde{w}_1), u'(T; \widetilde{w}_1)) \in B_{L^2(\Omega_T)}(u^0, \rho_0) \times B_{H^{-1}(\Omega_T)}(u^1, \rho_1), \tag{3.2}$$

assuming that $\widetilde{w}_1$ exists, ρ_0 and ρ_1 being positive numbers arbitrarily small and $\{u^0, u^1\} \in L^2(\Omega_T) \times H^{-1}(\Omega_T)$.

Lemma 3.1 (Inverse Inequality). *If $(g^0, g^1) \in H_0^1(\Omega_0) \times L^2(\Omega_0)$, then there exists $\gamma(0) > 0$ such that for $T > \gamma(0)$, the weak solution of problem*

$$\begin{gathered} z'' - z_{xx} = 0 \quad \text{in } \widehat{Q}, \\ z = 0 \quad \text{on } \widehat{\Sigma}, \\ z(0) = g^0, \quad z'(0) = g^1 \quad \text{in } \Omega_0 \end{gathered} \tag{3.3}$$

satisfies

$$\int_0^{\gamma(0)} |z_x(0,t)|^2 \, dt \geq C(|g^0|^2 + |g^1|^2), \tag{3.4}$$

where C is given in [3].

Proof. We construct a solution of the form

$$z(x,t) = \sum_{n \in \mathbb{Z}} A_n (e^{2\xi i n \varphi(t+x)} - e^{2\xi i n \varphi(t-x)}), \tag{3.5}$$

where $\varphi \in C^2$ is a solution to the functional equation

$$\varphi(t + \alpha_k(t)) - \varphi(t - \alpha_k(t)) = 1.$$

We let $F(x) = \int_0^x g^1(s) ds$ and

$$h(x) = \begin{cases} \frac{1}{2}\big(g^0(x) + F(x)\big) & \text{for } 0 \leq x \leq 1, \\ \frac{1}{2}\big(-g^0(-x) + F(-x)\big) & \text{for } -1 \leq x \leq 0. \end{cases}$$

We note that

$$A_n = \int_{-1}^{1} h(x) e^{2\xi i n \varphi(x)} \varphi'(x) \, dx.$$

Then $z \in C^2$ and their derivatives can be calculated term by term. Let us define some often appearing values:

$$\begin{gathered} m(t) = \min\{\varphi'(x) : x \in [t - \alpha_k(t), t + \alpha_k(t)]\}, \\ M(t) = \max\{\varphi'(x) : x \in [t - \alpha_k(t), t + \alpha_k(t)]\}. \end{gathered}$$

Now, we adapt the proof of [3, Theorem 2.1], with $0 < k < 1$. Indeed, we consider $\beta(t) = t - \alpha_k(t)$. Then $\beta'(t) = 1 - k > 0$. Therefore, $\beta(t)$ is strictly increasing and since $\beta(0) = -1 < 0$, there exists a unique t_0 such that $\beta(t_0) = 0$. Let $\tau_0 = t_0 + \alpha_k(t_0) = \gamma(0)$.

Differentiating z term by term in (3.5), and evaluating at $x=0$, for all $\tau>0$ we obtain

$$\begin{aligned} |z_x(0,t)|^2_{L^2(0,\tau_0)} &\geq m(t_0)|z_x(0,t)|^2_{L^2(0,\tau_0,\frac{1}{\varphi'(t)})} \\ &= m(t_0)16\xi^2\sum_{n\in\mathbb{Z}} n^2|A_n|^2 \\ &\geq C(|g^0|^2+|g^1|^2), \end{aligned} \tag{3.6}$$

where the last inequality in (3.6) is obtained from [3, Proposition 1.4]. □

Remark 3.2. The proof of Lemma 3.1 also holds for $\alpha_k(t)=1+kT-kt$.

Now as in the case (1.6), we conclude this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. We decompose the solution (u,p) of (2.8) by setting

$$\begin{gathered} u=\vartheta_0+g, \\ p=p_0+q, \end{gathered} \tag{3.7}$$

where ϑ_0, p_0 is given by

$$\begin{gathered} \vartheta_0''-(\vartheta_0)_{xx}=0 \quad \text{in } \widehat{Q}, \\ \vartheta_0=\begin{cases} 0 & \text{on } \widehat{\Sigma}_1, \\ \frac{1}{\sigma}\,(p_0)_x & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma}\backslash\widehat{\Sigma}_0, \end{cases} \\ \vartheta_0(0)=\vartheta_0'(0)=0, \quad x \text{ in } \Omega_0, \end{gathered} \tag{3.8}$$

and

$$\begin{gathered} p_0''-(p_0)_{xx}=u_0-\widetilde{u}_2 \quad \text{in } \widehat{Q}, \\ p_0=0 \quad \text{on } \widehat{\Sigma}, \\ p_0(T)=p_0'(T)=0, \quad x \text{ in } \Omega_T; \end{gathered} \tag{3.9}$$

And $\{g,q\}$ is given by

$$\begin{gathered} g''-g_{xx}=0 \quad \text{in } \widehat{Q}, \\ g=\begin{cases} \widetilde{w}_1 & \text{on } \widehat{\Sigma}_1, \\ \frac{1}{\sigma}\,q_x & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma}\backslash\widehat{\Sigma}_0, \end{cases} \\ g(0)=g'(0)=0, \quad x \text{ in } \Omega_0, \end{gathered} \tag{3.10}$$

and

$$\begin{gathered} q''-q_{xx}=g \quad \text{in } \widehat{Q}, \\ q=0 \quad \text{on } \widehat{\Sigma}, \\ q(T)=q'(T)=0, \quad x \text{ in } \Omega_T. \end{gathered} \tag{3.11}$$

We next set $A: L^2(\widehat{\Sigma}_1)\to H^{-1}(\Omega_T)\times L^2(\Omega_T)$ as

$$A\widetilde{w}_1=\big\{g'(T;\widetilde{w}_1),\ -g(T;\widetilde{w}_1)\big\}, \tag{3.12}$$

which defines

$$A\in\mathcal{L}\big(L^2(\widehat{\Sigma}_1);\ H^{-1}(\Omega_T)\times L^2(\Omega_T)\big).$$

Using (3.7) and (3.12), we can rewrite (3.2) as

$$A\widetilde{w}_1\in\{-\vartheta_0'(T)+B_{H^{-1}(\Omega_T)}(u^1,\rho_1),\ -\vartheta_0(T)+B_{L^2(\Omega_T)}(u^0,\rho_0)\}. \tag{3.13}$$

We will show that $A\widetilde{w}_1$ generates a dense subspace of $H^{-1}(\Omega_T)\times L^2(\Omega_T)$. For this, let $\{f^0,f^1\}\in H_0^1(\Omega_T)\times L^2(\Omega_T)$ and consider the following systems ("adjoint states"):

$$\begin{gathered}\varphi''-\varphi_{xx}=\psi \quad \text{in } \widehat{Q},\\ \varphi=0 \quad \text{on } \widehat{\Sigma},\\ \varphi(T)=f^0,\quad \varphi'(T)=f^1,\quad x \text{ in } \Omega_T,\end{gathered} \tag{3.14}$$

and

$$\begin{gathered}\psi''-\psi_{xx}=0 \quad \text{in } \widehat{Q},\\ \psi=\begin{cases}0 & \text{on } \widehat{\Sigma}_1,\\ \frac{1}{\widetilde{\sigma}}\varphi_x & \text{on } \widehat{\Sigma}_2,\\ 0 & \text{on } \quad \widehat{\Sigma}\backslash\widehat{\Sigma}_0,\end{cases}\\ \psi(0)=\psi'(0)=0,\quad x \text{ in } \Omega_0.\end{gathered} \tag{3.15}$$

Multiplying $(3.15)_1$ by q, $(3.14)_1$ by g, where q, g solve (3.11) and (3.10), respectively, and integrating in $\widehat{Q}$ we obtain

$$\int_0^T\int_{\Omega_t} g\,\psi\,dx\,dt=-\frac{1}{\widetilde{\sigma}}\int_{\widehat{\Sigma}_2} q_x\varphi_x d\widehat{\Sigma}, \tag{3.16}$$

$$\langle g'(T),f^0\rangle_{H^{-1}(\Omega_T)\times H_0^1(\Omega_T)}-\big(g(T),f^1\big)=-\int_{\widehat{\Sigma}_1}\varphi_x\,\widetilde{w}_1\,d\widehat{\Sigma}. \tag{3.17}$$

Considering the left-hand side of this equation as the inner product of $\{g'(T),-g(T)\}$ and $\{f^0,f^1\}$ in $H^{-1}(\Omega_T)\times L^2(\Omega_T)$ and $H_0^1(\Omega_T)\times L^2(\Omega_T)$, we obtain

$$\langle\langle A\widetilde{w}_1,f\rangle\rangle=-\int_{\widehat{\Sigma}_1}\varphi_x\,\widetilde{w}_1\,d\widehat{\Sigma},$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ represent the duality pairing between $H^{-1}(\Omega_T)\times L^2(\Omega_T)$ and $H_0^1(\Omega_T)\times L^2(\Omega_T)$. Therefore, if

$$\langle\langle A\,\widetilde{w}_1,f\rangle\rangle=0,$$

for all $\widetilde{w}_1\in L^2(\widehat{\Sigma}_1)$, then

$$\varphi_x=0 \quad \text{on } \widehat{\Sigma}_1. \tag{3.18}$$

Hence, in case (1.6),

$$\psi=0 \quad \text{on } \widehat{\Sigma}, \quad \text{so that } \psi\equiv 0. \tag{3.19}$$

Therefore,

$$\varphi''-\varphi_{xx}=0,\quad \varphi=0 \quad \text{on } \widehat{\Sigma}, \tag{3.20}$$

and satisfies (3.18). Therefore, by Lemma 3.1 and by Remark 3.2, with $z(x,t)=\varphi(x,T-t)$, $g^0=f^0$, and $g^1=f^1$ we have that $f^0=0$, $f^1=0$. This completes the proof. □

4. Optimal system for the leader control

In the previous sections, we have seen that no matter what strategy, the leader assumes that the follower make their choice $\widetilde{w}_2$ satisfying the Nash equilibrium. The goal of this section is to obtain a optimal system for the leader control. More precisely, we obtain the following result.

Theorem 4.1. *Assume that the hypotheses* (1.2), (1.3) *and* (1.6) *are satisfied. Then for* $\{f^0, f^1\}$ *in* $H_0^1(\Omega_T) \times L^2(\Omega_T)$ *we uniquely define* $\{\varphi, \psi, u, p\}$ *by*

$$
\begin{gathered}
\varphi'' - \varphi_{xx} = \psi \quad \text{in } \widehat{Q}, \\
\psi'' - \psi_{xx} = 0 \quad \text{in } \widehat{Q}, \\
u'' - u_{xx} = 0 \quad \text{in } \widehat{Q}, \\
p'' - p_{xx} = u - \widetilde{u}_2 \quad \text{in } \widehat{Q}, \\
\varphi = 0 \quad \text{on } \widehat{\Sigma}, \\
\psi = \begin{cases} 0 & \text{on } \widehat{\Sigma}_1, \\ \frac{1}{\sigma}\,\varphi_x & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma}\backslash\widehat{\Sigma}_0, \end{cases} \\
u = \begin{cases} -\varphi_x & \text{on } \widehat{\Sigma}_1, \\ \frac{1}{\sigma} p_x & \text{on } \widehat{\Sigma}_2, \\ 0 & \text{on } \widehat{\Sigma}\backslash\widehat{\Sigma}_0, \end{cases} \\
p = 0 \quad \text{on } \widehat{\Sigma}, \\
\varphi(\cdot, T) = f^0, \quad \varphi'(\cdot, T) = f^1 \quad \text{in } \Omega_T, \\
u(0) = u'(0) = 0 \quad \text{in } \Omega_0, \\
p(T) = p'(T) = 0 \quad \text{in } \Omega_T.
\end{gathered}
\tag{4.1}
$$

We uniquely define $\{f^0, f^1\}$ *as the solution of the variational inequality*

$$
\begin{aligned}
&\big\langle u'(T,f) - u^1, \widehat{f}^0 - f^0 \big\rangle_{H^{-1}(\Omega_T)\times H_0^1(\Omega_T)} - \big(u(T,f) - u^0, \widehat{f}^1 - f^1\big) \\
&+ \rho_1\big(\|\widehat{f}^0\| - \|f^0\|\big) + \rho_0\big(|\widehat{f}^1| - |f^1|\big) \geq 0, \quad \forall \widehat{f} \in H_0^1(\Omega_T) \times L^2(\Omega_T).
\end{aligned}
\tag{4.2}
$$

Then the optimal leader is

$$
\widetilde{w}_1 = -\varphi_x \quad \text{on } \widehat{\Sigma}_1,
$$

where φ *corresponds to the solution of* (4.1).

Proof. Let A be the continuous linear operator defined by (3.12) and we introduce the following two convex proper functions: $F_1 : L^2(\widehat{\Sigma}_1) \to \mathbb{R} \cup \{\infty\}$ by

$$
F_1(\widetilde{w}_1) = \frac{1}{2} \int_{\widehat{\Sigma}_1} \widetilde{w}_1^2 \, d\widehat{\Sigma} \tag{4.3}
$$

and $F_2 : H^{-1}(\Omega_T) \times L^2(\Omega_T) \to \mathbb{R} \cup \{\infty\}$ by

$$
F_2(\xi, \mu) = \begin{cases} 0, & \text{if } (\xi, \mu) \in u^1 - \vartheta_0'(T) + \rho_1 \, B_{H^{-1}(\Omega_T)} \\ & \qquad -u^0 + \vartheta_0(T) - \rho_0 B_{L^2(\Omega_T)}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.4}
$$

With these notation, problems (3.1)–(3.2) become equivalent to

$$
\inf_{\widetilde{w}_1 \in L^2(\widehat{\Sigma}_1)} \big[F_1(\widetilde{w}_1) + F_2(A\widetilde{w}_1)\big] \tag{4.5}
$$

provided that we prove that the range of A is dense in $H^{-1}(\Omega_T) \times L^2(\Omega_T)$, under conditions (1.2) and (1.3).

By the Duality Theorem of Fenchel and Rockafellar [30] (see also [10, 13]), we have

$$
\begin{aligned}
&\inf_{\widetilde{w}_1 \in L^2(\widehat{\Sigma}_1)} [F_1(\widetilde{w}_1) + F_2(A\widetilde{w}_1)] \\
&= - \inf_{(\widehat{f}^0, \widehat{f}^1) \in H_0^1(\Omega_T) \times L^2(\Omega_T)} [F_1^*\big(A^*\{\widehat{f}^0, \widehat{f}^1\}\big) + F_2^*\{-\widehat{f}^0, -\widehat{f}^1\}],
\end{aligned} \tag{4.6}
$$

where F_i^* is the conjugate function of F_i $(i = 1, 2)$ and A^* the adjoint of A.

We have $A^* : H_0^1(\Omega_T) \times L^2(\Omega_T) \to L^2(\widehat{\Sigma}_1)$ as

$$
(f^0, f^1) \mapsto A^* f = -\varphi_x, \tag{4.7}
$$

where φ is given in (3.14).

We see easily that

$$
F_1^*(\widetilde{w}_1) = F_1(\widetilde{w}_1) \tag{4.8}
$$

and

$$
\begin{aligned}
F_2^*(\{\widehat{f}^0, \widehat{f}^1\}) &= \langle u^1 - \vartheta_0'(T), \widehat{f}^0 \rangle_{H^{-1}(\Omega_T) \times H_0^1(\Omega_T)} \\
&\quad + \big(\vartheta_0(T) - u^0, \widehat{f}^1\big) + \rho_1 \|\widehat{f}^0\| + \rho_0 |\widehat{f}^1|.
\end{aligned} \tag{4.9}
$$

Therefore the (opposite of) right-hand side of (4.6) is given by

$$
\begin{aligned}
&- \inf_{\widehat{f} \in H_0^1(\Omega_T) \times L^2(\Omega_T)} \Big\{ \frac{1}{2} \int_{\widehat{\Sigma}_1} \varphi_x^2 d\widehat{\Sigma} + \big(u^0 - \vartheta_0(T), \widehat{f}^1\big) \\
&- \langle u^1 - \vartheta_0'(T), \widehat{f}^0 \rangle_{H^{-1}(\Omega_T) \times H_0^1(\Omega_T)} + \rho_1 \|\widehat{f}^0\| + \rho_0 |\widehat{f}^1| \Big\}.
\end{aligned}
$$

This is the dual problem of (3.1) and (3.2). Hence, we can use the primal or the dual problem to derive the optimality system for the leader control. □

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In response to a reader's comments, the first author wants to indicate that the statements from page 3 line 25 to page 4 line 18 are quoted from the article

> Mokhtari Yacine; *Boundary controllability and boundary time-varying feedback stabilization of the 1D wave equation in non-cylindrical domains*, Evolution Equations & Control Theory, doi: 10.3934/eect.2021004.

End of addendum.

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