Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 59, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLE STATE OPTIMAL DESIGN PROBLEMS WITH RANDOM PERTURBATION

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Abstract. A multiple state optimal design problem with presence of uncertainty on the right-hand side is considered, in the context of stationary diffusion with two isotropic phases. A similar problem with one state equation has already been considered by Buttazzo and Maestre (2011). We shall address the question of relaxation by the homogenization method and necessary conditions of optimality. The case of discrete probability space leads to another multiple state problem (possibly with an infinite number of states), which could be treated by similar techniques to those presented in Allaire (2002) and Vrdoljak (2010). The relaxation can be expressed in a simpler form for problems with spherical symmetry in the case of minimization (or maximization) of averaged energy, and we present an example which can be solved explicitly.

1. Introduction

For a measurable matrix function $\mathbf{A}$ which is bounded and uniformly elliptic (almost everywhere on an open and bounded set $\Omega \subseteq \mathbb{R}^d$) and $f \in H^{-1}(\Omega)$ there exists a unique solution of the boundary value problem for stationary diffusion equation

$$\begin{aligned} -\operatorname{div}(\mathbf{A}\nabla u) &= f \\ u &\in \mathrm{H}^1_0(\Omega)\,. \end{aligned} \tag{1.1}$$

The homogenization theory allows one to introduce a topology on the appropriate set of coefficients such that the mapping $f \mapsto u$ is continuous, in a reasonable pair of topologies. Historically, these topologies were first introduced (with full mathematical rigour) by Spagnolo [8] through the concept of G-convergence. The notion of H-convergence was also originally introduced for the stationary diffusion equation [6] (it is also known under the name strong G-convergence [14]).

More precisely, Murat and Tartar [6] introduced the set of admissible conductivity matrix functions

$$\mathcal{M}(\alpha,\beta;\Omega) = \Big\{ \mathbf{A} \in L^\infty(\Omega; M_d(\mathbb{R})) : \mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \alpha|\boldsymbol{\xi}|^2\,, \\ \mathbf{A}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq \frac{1}{\beta}|\mathbf{A}(\mathbf{x})\boldsymbol{\xi}|^2 \Big\}\,.$$

2010 *Mathematics Subject Classification.* 49K35, 49K20, 49J20, 80M40.
Key words and phrases. Stationary diffusion; optimal design; homogenization; random perturbation; optimality conditions.

Submitted April 28, 2017. Published March 2, 2018.

A sequence of matrix functions $(\mathbf{A}_n)$ in $\mathcal{M}(\alpha,\beta;\Omega)$ is said to H-converge to $\mathbf{A} \in \mathcal{M}(\alpha',\beta';\Omega)$ if for any $f \in H^{-1}(\Omega)$ the sequence (u_n) of solutions of (1.1), with $\mathbf{A}_n$ instead of $\mathbf{A}$, satisfies

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega)\,, \tag{1.2}$$

$$\mathbf{A}_n \nabla u_n \rightharpoonup \mathbf{A}\nabla u \quad \text{in } L^2(\Omega;\mathbb{R}^d)\,, \tag{1.3}$$

where u is given by (1.1). If we additionally assume that conductivity matrices are symmetric, the above definition coincides with the notion of G-convergence, imposing only the convergence (1.2). The question whether such H-converging sequence exists is simply answered by the compactness theorem: the bounds in the definition of set $\mathcal{M}(\alpha,\beta;\Omega)$ are well chosen in such a way that it is compact with respect to $H-$convergence. In other words, in the definition of H-convergence one can put $\alpha' = \alpha$ and $\beta' = \beta$.

In multiple state optimal design problems, one is trying to find the best arrangement of given materials, such that the obtained body has some optimal properties regarding m different regimes. We shall study the simplest case of two isotropic constituents, with conductivities α and β. Therefore, the conductivity can be written as $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where χ is the characteristic function of the first material. The optimality of a distribution is measured by an objective function, which is usually an integral functional depending on the distribution of materials and the state function, obtained as a solution of the associated boundary value problem. If χ denotes a characteristic function of the first material on a bounded open set $\Omega \subseteq \mathbb{R}^d$, this functional can be written in the general form

$$J(\chi) = \int_\Omega \Big(\chi(\mathbf{x})\, g_\alpha(\mathbf{x},\mathsf{u}) + (1-\chi(\mathbf{x}))g_\beta(\mathbf{x},\mathsf{u})\Big)\, d\mathbf{x}\,,$$

where $\mathsf{u} = (u_1,\dots,u_m)$ is the state function determined by

$$\begin{gathered} -\operatorname{div}(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{gathered} \tag{1.4}$$

for $i = 1,\dots,m$, with $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$. The functions $f_1,\dots,f_m$, as well as g_α and g_β are supposed to be given. The case $m=1$ is studied in [7, 9, 10, 1] and the general case in [1, 2, 3]. As it is common in optimal design problems, due to the lack of existence of a classical optimal design, one can use the relaxation by the homogenization method, introducing generalized designs that correspond to fine mixtures of original phases. Under some growth conditions on g_α and g_β (see [1, Section 3.1.3]) the relaxed problem reads

$$\begin{gathered} J(\theta,\mathbf{A}) = \int_\Omega \Big(\theta(\mathbf{x})\, g_\alpha(\mathbf{x},\mathsf{u}(\mathbf{x})) + (1-\theta(\mathbf{x}))g_\beta(\mathbf{x},\mathsf{u}(\mathbf{x}))\Big)\, d\mathbf{x} \to \min \\ \theta \in \mathrm{L}^\infty(\Omega;[0,1])\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e. on } \Omega\,,\ \mathsf{u} \text{ solves (1.4)}\,. \end{gathered}$$

The purpose of this paper is to study uncertainty perturbation of the right-hand side of state equations, contrary to the above deterministic case. To model such problem we take into consideration a probability space $(S,\mathcal{M},\mu)$. Suppose that $f_i \in L^1(S,H^{-1}(\Omega))$, and denote $\overline{f}_i := \int_S f_i\, d\mu$. In other words we consider $s \in S$ to be a parameter in the boundary value problem

$$\begin{gathered} -\operatorname{div}(\mathbf{A}\nabla u_i) = f_i(s,\cdot) \\ u_i \in \mathrm{H}_0^1(\Omega) \end{gathered} \tag{1.5}$$

for $i = 1 \dots m$.

We consider the following optimal design problem: Given $f_i \in L^1(S, H^{-1}(\Omega))$, $i = 1 \dots m$, one seeks for a characteristic function χ on Ω that minimizes the following functional

$$J(\chi) = \int_S \int_\Omega \Big(\chi(\mathbf{x}) g_\alpha(s, \mathbf{x}, \mathsf{u}(s, \mathbf{x})) + (1 - \chi(\mathbf{x}))\, g_\beta(s, \mathbf{x}, \mathsf{u}(s, \mathbf{x})) \Big)\, d\mathbf{x}\, d\mu\,,$$

where u_i is determined by (1.5) with $\mathbf{A} = \chi\alpha\mathbf{I} + (1 - \chi)\beta\mathbf{I}$, and $\mathsf{u} = (u_1, \dots, u_m)$. Moreover, we assume that the quantity of the first material is given: $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$. This problem could be understood as the averaged version of the multiple state optimal design problem described above.

The case $l = 1$ has been studied in [4]. If S consists of just one point, the problem reduces to the (deterministic) multiple state optimal design problem which we started from. Therefore, the nonexistence of a solution usually occurs, so the proper relaxation should be introduced. Furthermore, in the case of a discrete probability space it is easily seen that an averaged multiple state optimal design problem can be written as a deterministic multiple state optimal design problem, although the number of state equations significantly increases, even to infinity in the case of an infinite probability space.

The content of the paper is as follows. Section 2 is devoted to description of the main issues regarding the proper relaxation of the problem. The relaxed problem has a solution, which can be achieved as a (weak) limit of a minimizing sequence of the original problem, and vice versa: each minimizing sequence of the original problem has a subsequence converging to a solution of the relaxed problem [7]. In Section 3 we further analyse the necessary conditions of optimality for the relaxed problem.

Section 4 deals with the special type of cost functional: we consider the energy functional and maximise its average. An example is presented, where it is even possible to calculate the optimal design explicitly by the technique presented in [12]. It shows that the averaged optimal control can differ significantly from the optimal control obtained by averaged data, but this is due to the nonuniqueness of the averaged control.

2. Proper relaxation

If $\mathbf{A} \in \mathcal{M}(\alpha, \beta; \Omega)$ and $\mathcal{D}u := -\operatorname{div}(\mathbf{A}\nabla u)$, then $\mathcal{D}$ is an isomorphism between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ by the Lax-Milgram lemma. As an easy consequence of the following result $\mathcal{D}$ is also an isomorphism between $L^p(S; H^1_0(\Omega))$ and $L^p(S; H^{-1}(\Omega))$, for any $p \in [1, +\infty]$.

Lemma 2.1. *If X and Y are two Banach spaces and T is a continuous linear operator from X to Y, then mapping $\mathcal{T}$ defined by*

$$(\mathcal{T}\mathsf{f})(s) := T(\mathsf{f}(s))$$

is a continuous linear operator from $L^p(S; X)$ to $L^p(S; Y)$.

The proof of Lemma 2.1 follows from the construction of the Bôchner integral: the case $p = 1$ is considered in [13, Section V.5] or [5, Section I.2], while the general case can be treated in a similar vein.

For a sequence $(\mathbf{A}_n)$ in $\mathcal{M}(\alpha, \beta; \Omega)$, the corresponding sequence of isomorphisms is introduced by $\mathcal{D}_n u := -\operatorname{div}(\mathbf{A}_n\nabla u)$. Following an analogous approach for our original problem we say that a sequence $(\mathbf{A}_n)$ in $\mathcal{M}(\alpha,\beta;\Omega)$ converges to $\mathbf{A} \in \mathcal{M}(\alpha',\beta';\Omega)$ if for any $f \in L^p(S;H^{-1}(\Omega))$ and $u := \mathcal{D}^{-1}f$, $u_n := \mathcal{D}_n^{-1}f$ we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } L^p(S;H_0^1(\Omega)) \\ \mathbf{A}_n\nabla u_n \rightharpoonup \mathbf{A}\nabla u \quad \text{in } L^p(S;L^2(\Omega;\mathbb{R}^d)) \,. \end{gathered}$$

The following Theorem establishes that this convergence actually coincides with classical H-convergence for stationary diffusion equation and with the notion of H-convergence introduced in [4] (part 2 of Theorem 2.2). The proof is straightforward.

Theorem 2.2. *Let $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2, \dots$ be a sequence in $\mathcal{M}(\alpha,\beta;\Omega)$ and $p \in [1,+\infty\rangle$. The following statements are equivalent:*

1. *$\mathbf{A}_n$ H-converges to $\mathbf{A}$.*
2. *For any $f \in L^p(S;H^{-1}(\Omega))$ and $u := \mathcal{D}^{-1}f$, $u_n := \mathcal{D}_n^{-1}f$ we have*
$$\begin{gathered} u_n(s,\cdot) \rightharpoonup u(s,\cdot) \quad \textit{in } H_0^1(\Omega) \\ \mathbf{A}_n\nabla u_n(s,\cdot) \rightharpoonup \mathbf{A}\nabla u(s,\cdot) \quad \textit{in } L^2(\Omega;\mathbb{R}^d)\,, \quad \text{for a.e. } s \in S\,. \end{gathered}$$
3. *For any $f \in L^p(S;H^{-1}(\Omega))$ and $u := \mathcal{D}^{-1}f$, $u_n := \mathcal{D}_n^{-1}f$ we have*
$$\begin{gathered} u_n \rightharpoonup u \quad \textit{in } L^p(S;H_0^1(\Omega)) \\ \mathbf{A}_n\nabla u_n \rightharpoonup \mathbf{A}\nabla u \quad \textit{in } L^p(S;L^2(\Omega;\mathbb{R}^d)) \,. \end{gathered}$$

According to the theory of deterministic multiple state optimal design problems [10, 1], the relaxation of the original optimal design problem via homogenization theory consists in extending the original cost functional to

$$J(\theta,\mathbf{A}) = \int_S\int_\Omega \Big(\chi(\mathbf{x})\, g_\alpha(s,\mathbf{x},\mathsf{u}(s,\mathbf{x})) + (1-\chi(\mathbf{x}))\, g_\beta(s,\mathbf{x},\mathsf{u}(s,\mathbf{x})\Big)\, d\mathbf{x}\, d\mu\,,$$

where u is the unique solution of (1.5). The first step is to propose conditions which ensure the continuity of J on $L^\infty(\Omega)\times\mathcal{M}(\alpha,\beta;\Omega)$ in a reasonable topology.

To this end let us assume that $\mathsf{f} = (f_1,\dots,f_m) \in L^p(S;H^{-1}(\Omega)^m)$ and g_α and g_β to be Carathéodory functions (measurable in $(s,\mathbf{x})$ on product space $S\times\Omega$ and continuous in u) satisfying the growth condition:

$$|g_\gamma(s,\mathbf{x},\mathsf{u})| \le \varphi_\gamma(\mathbf{x},s) + \psi_\gamma(\mathbf{x},s)|\mathsf{u}|^q \quad \text{for } \gamma = \alpha,\beta, \tag{2.1}$$

with $\varphi_\gamma \in L^1(S\times\Omega)$, $\psi_\gamma \in L^{p'}(S;L^{q'}(\Omega))$, where p' denotes the conjugate index to p: $\frac{1}{p}+\frac{1}{p'} = 1$, and $q \in [1,q^*\rangle$, with

$$q^* = \begin{cases} +\infty\,, & d \le 2 \\ \frac{2d}{d-2}\,, & d > 2\,. \end{cases}$$

Theorem 2.3. *Let the growth conditions* (2.1) *be satisfied for some $p \ge 1$. Then for any $\mathsf{f} \in L^p(S;H^{-1}(\Omega)^m)$ the functional J is well defined and continuous on $L^\infty(\Omega)\times\mathcal{M}(\alpha,\beta;\Omega)$ with respect to the weak-$*$ topology for θ and the H-topology for $\mathbf{A}$.*

Proof. Let $f \in L^p(S;H^{-1}(\Omega)^m)$. If a sequence $(\mathbf{A}_n)$ in $\mathcal{M}(\alpha,\beta;\Omega)$ H-converges to $\mathbf{A} \in \mathcal{M}(\alpha,\beta;\Omega)$, then by Theorem 2.2 for the sequence of state functions $\mathsf{u}_n^i := \mathcal{D}_n^{-1}f_i$ and $\mathsf{u}^i := \mathcal{D}^{-1}f_i$, $i = 1,\dots,m$, we have the weak convergence

$$u_n^i(s,\cdot) \rightharpoonup u^i(s,\cdot) \quad \text{in } H_0^1(\Omega)\,, \quad \text{for a.e. } s \in S\,. \tag{2.2}$$

By the Sobolev imbedding theorem, the convergence is actually strong in $L^q(\Omega)$, for any $q \in [1, q^*\rangle$ with q^* given above. Since g_α and g_β are Carathéodory functions, this implies the convergence (up to a subsequence)

$$g_\gamma(s, \mathbf{x}, \mathsf{u}_n(s, \mathbf{x})) \to g_\gamma(s, \mathbf{x}, \mathsf{u}(s, \mathbf{x})$$

almost everywhere on $S \times \Omega$. By growth conditions (2.1) and the Lebesgue dominated convergence [5, Section II.2] we conclude that $g_\gamma(\cdot, \cdot, \mathsf{u}_n)$ converges to $g_\gamma(\cdot, \cdot, \mathsf{u})$ strongly in $L^1(S \times \Omega)$, for $\gamma \in \{\alpha, \beta\}$. Now if $\theta_n \rightharpoonup^* \theta$ in $L^\infty(\Omega)$ we conclude $J(\theta_n, \mathbf{A}_n) \to J(\theta, \mathbf{A})$. Since the considered topology on $L^\infty(\Omega) \times \mathcal{M}(\alpha, \beta; \Omega)$ is metrizable on bounded sets, this means that J is continuous. $\square$

The proper relaxation by homogenization theory consists in introducing the set of generalized designs

$$\mathcal{A} := \Big\{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0, 1] \times \mathrm{Sym_d}) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha, \ \mathbf{A}(\mathbf{x}) \in \mathcal{K}(\theta(\mathbf{x})) \text{ a.e. on } \Omega\Big\},$$

where $\mathcal{K}(\theta)$ stands for G-closure of the original set of conductivities $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $\chi \in L^\infty(\Omega; \{0, 1\})$, with given local fraction θ of the first phase. The set $\mathcal{A}$ is compact with respect to the above considered topology, and it is the closure of the set of original designs. Therefore, the following result is straightforward.

Theorem 2.4. *The proper relaxation of the original problem reads*

$$\begin{gathered} J(\theta, \mathbf{A}) = \int_S \int_\Omega \Big(\theta \, g_\alpha(\cdot, \cdot, \mathsf{u}) + (1-\theta) \, g_\beta(\cdot, \cdot, \mathsf{u})\Big) \, d\mathbf{x} \, d\mu \to \min \\ (\theta, \mathbf{A}) \in \mathcal{A}, \quad \mathsf{u} = (u_1, \dots, u_m) \text{ solves } (1.5). \end{gathered} \tag{2.3}$$

Remark 2.5. If we start from a conic sum of energies for each state equation and take its average over S:

$$J(\theta, \mathbf{A}) = \sum_{i=1}^m \lambda_i \int_S {}_{H^{-1}(\Omega)}\langle f(s), u_i(s)\rangle_{H_0^1(\Omega)} \, d\mu,$$

it is sufficient to assume $f \in L^2(S; H^{-1}(\Omega)) = \big(L^2(S; H_0^1(\Omega))\big)'$ in order to obtain the continuity of J. In particular, the relaxation problem has a form as written in the previous Theorem.

3. Necessary conditions of optimality

Let $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$ denote a solution of the relaxed problem (2.3) with corresponding state u^* and let $\varepsilon \mapsto (\theta^\varepsilon, \mathbf{A}^\varepsilon) \in \mathcal{A}$ be a smooth path in $\mathcal{A}$ passing through $(\theta^*, \mathbf{A}^*)$ for $\varepsilon = 0$. By u_i^ε we denote the corresponding state function, the unique solution of

$$\begin{gathered} -\operatorname{div}(\mathbf{A}^\varepsilon \nabla u_i) = f_i(s, \cdot) \\ u_i \in L^p(S; H_0^1(\Omega)) \end{gathered} \tag{3.1}$$

for $i = 1 \dots m$.

After denoting $\delta\theta = \frac{d}{d\varepsilon}\theta^\varepsilon\big|_{\varepsilon=0}$ and $\delta\mathbf{A} = \frac{d}{d\varepsilon}\mathbf{A}^\varepsilon\big|_{\varepsilon=0}$, we would like to calculate the variation of J in terms of $\delta\theta$ and $\delta\mathbf{A}$, more precisely, we look for $\delta J :=$

$\frac{d}{d\varepsilon}J(\theta^\varepsilon, \mathbf{A}^\varepsilon)\big|_{\varepsilon=0}$:

$$\begin{aligned}
\delta J = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[&\int_S\int_\Omega \theta^\varepsilon(\mathbf{x})g_\alpha(s,\mathbf{x},\mathsf{u}^\varepsilon(s,\mathbf{x})) - \theta^*(\mathbf{x})g_\alpha(s,\mathbf{x},\mathsf{u}^*(s,\mathbf{x}))\,d\mathbf{x}\,d\mu\\
&+\int_S\int_\Omega (1-\theta^\varepsilon(\mathbf{x}))g_\beta(s,\mathbf{x},\mathsf{u}^\varepsilon(s,\mathbf{x})) - (1-\theta^*(\mathbf{x}))\\
&\times g_\beta(s,\mathbf{x},\mathsf{u}^*(s,\mathbf{x}))\,d\mathbf{x}\,d\mu\Big]\\
= \lim_{\varepsilon\to 0}\Big[&\int_S\int_\Omega \frac{\theta^\varepsilon-\theta^*}{\varepsilon}(g_\alpha(\cdot,\cdot,\mathsf{u}^\varepsilon) - g_\beta(\cdot,\cdot,\mathsf{u}^\varepsilon))\,d\mathbf{x}\,d\mu\\
&+\int_S\int_\Omega \theta^*\sum_{i=1}^m \frac{u_i^\varepsilon-u_i^*}{\varepsilon}\int_0^1\frac{\partial g_\alpha}{\partial u_i}(\cdot,\cdot,\mathsf{u}_\tau^\varepsilon)\,d\tau\,d\mathbf{x}\,d\mu\\
&+\int_S\int_\Omega (1-\theta^*)\sum_{i=1}^m \frac{u_i^\varepsilon-u_i^*}{\varepsilon}\int_0^1\frac{\partial g_\beta}{\partial u_i}(\cdot,\cdot,\mathsf{u}_\tau^\varepsilon))\,d\tau\ \,d\mathbf{x}\,d\mu\Big]\,.
\end{aligned} \tag{3.2}$$

Here, $\mathsf{u}_\tau^\varepsilon$ denotes $\mathsf{u}^* + \tau(\mathsf{u}^\varepsilon - \mathsf{u}^*)$, and it is assumed that $\frac{\partial g_\alpha}{\partial u_i}$ and $\frac{\partial g_\beta}{\partial u_i}$ are Carathéodory functions (measurable in $(s,\mathbf{x})$ and continuous in u). In order to pass to the limit above we additionally assume the following growth conditions

$$\Big|\frac{\partial g_\gamma}{\partial u_i}(s,\mathbf{x},\mathsf{u})\Big| \le \widetilde{\varphi}_\gamma(s,\mathbf{x}) + \widetilde{\psi}_\gamma(s,\mathbf{x})|\mathsf{u}|^r \quad \text{for } \gamma=\alpha,\beta;\ i=1,\dots,m\,,$$

where $\widetilde{\varphi}_\gamma \in L^{p'}(S;L^{q'}(\Omega))$, $\widetilde{\psi}_\gamma \in L^{p_r}(S;L^{q_r}(\Omega))$, with q being the same as in the previous section, $p_r = \frac{p}{p-r-1}$ and $q_r = \frac{q}{q-r-1}$ for some $r \ge 0$ such that $r \le p-1$ and $r \le q-1$.

By Theorem 2.2, since L^∞ convergence implies $H-$convergence, as in the proof of Theorem 2.3 we conclude that $\mathsf{u}^\varepsilon \to \mathsf{u}^*$ in $L^q(\Omega)$ for any $q \in [1,q^*\rangle$, almost everywhere on S, as well as $\mathsf{u}_\tau^\varepsilon$, for any $\tau \in [0,1]$. Now, an application of the Lebesgue dominated convergence theorem implies that the inner integrals over τ converge to $\int_0^1 \frac{\partial g_\gamma}{\partial u_i}(\cdot,\cdot,\mathsf{u}^*)\,d\tau$ in $L^{p'}(S;L^{q'}(\Omega))$, for $\gamma=\alpha,\beta$. If $\delta\mathsf{u}$ denotes $\frac{d}{d\varepsilon}\mathsf{u}^\varepsilon\big|_{\varepsilon=0}$, one concludes

$$\begin{aligned}
\delta J = \int_S\int_\Omega &\delta\theta\big[g_\alpha(\cdot,\cdot,\mathsf{u}^*) - g_\beta(\cdot,\cdot,\mathsf{u}^*)\big]\\
&+\sum_{i=1}^m \delta\mathsf{u}_i\Big(\theta^*\frac{\partial g_\alpha}{\partial u_i}(\cdot,\cdot,\mathsf{u}^*) + (1-\theta^*)\frac{\partial g_\beta}{\partial u_i}(\cdot,\cdot,\mathsf{u}^*)\Big)\,d\mathbf{x}\,d\mu\,.
\end{aligned}$$

Here, $\delta\mathsf{u}$ solves

$$\begin{gathered}
\operatorname{div}(\mathbf{A}^*\nabla\delta u_i) = \operatorname{div}(\delta\mathbf{A}\nabla u_i^*)\\
\delta u_i \in L^p(S;H_0^1(\Omega))
\end{gathered}$$

for $i=1,\dots,m$. We introduce adjoint states, as commonly, in order to eliminate those derivatives from the expression for δJ. Since $\theta\frac{\partial g_\gamma}{\partial u_i}$ belongs to $L^{p'}(S;L^{q'}(\Omega)) \hookrightarrow L^{p'}(S;H^{-1}(\Omega))$, for $\gamma=\alpha,\beta$, the following boundary value problems have unique solutions

$$\begin{gathered}
-\operatorname{div}(\mathbf{A}^*\nabla p_i) = \theta\frac{\partial g_\alpha}{\partial u_i}(\cdot,\cdot,\mathsf{u}^*) + (1-\theta)\frac{\partial g_\beta}{\partial u_i}(\cdot,\cdot,\mathsf{u}^*)\\
p_i \in L^{p'}(S;H_0^1(\Omega))
\end{gathered}$$

for $i = 1, \dots, m$. Now, one concludes that

$$\delta J = \int_S \int_\Omega \delta\theta \big[g_\alpha(\cdot, \cdot, \mathsf{u}^*) - g_\beta(\cdot, \cdot, \mathsf{u}^*)\big]\, d\mathbf{x}\, d\mu - \int_S \int_\Omega \sum_{i=1}^m \delta \mathbf{A} \nabla u_i^* \cdot \nabla p_i^*\, d\mathbf{x}\, d\mu\,. \quad (3.3)$$

Remark 3.1 (Conic sum of energies)**.** In the case of energy functional the calculation of variation δJ is straightforward, with the same assumption as in Remark 2.5: $f \in L^2(S; H^{-1}(\Omega))$. The formula for the variation is the same, with $p^* = \mathsf{u}^*$ in the case of minimization, and with $p^* = -\mathsf{u}^*$ for maximization problem.

Remark 3.2 (Discrete probability space)**.** If S is a finite set, the original random problem is easily seen as a (deterministic) multiple state optimal design problem. The similar calculation holds for infinite, but discrete case: $S := \{s_1, s_2, \dots\}$, with probabilities $\mu(s_j) = \mu_j \geq 0$, $\sum_j \mu_j = 1$.

Each $f_i \in L^p(S; H^{-1}(\Omega))$ determines the sequence of functionals $f_i^j := f(s_j, \cdot) \in H^{-1}(\Omega)$. Denoting $u_i^j := \mathcal{D}^{-1}(f_i^j) = (\mathcal{D}^{-1} f_i)(s_j, \cdot)$, $\mathsf{u}^j := (u_1^j, \dots u_m^j)$ and $g_\gamma^j(\mathbf{x}, \mathsf{v}) = \mathsf{g}_\gamma(\mathsf{s_j}, \mathbf{x}, \mathsf{v})$, for $\gamma = \alpha, \beta$, we have

$$J(\theta, \mathbf{A}) \quad = \textstyle\int_\Omega \theta \left(\sum_j \mu_j g_\alpha^j(\cdot, \mathsf{u}^j) \right) + (1-\theta) \left(\sum_j \mu_j g_\beta^j(\cdot, \mathsf{u}^j) \right) d\mathbf{x}\,.$$

Here, the order of summation and integration can be interchanged by the Fubini theorem, due to assumptions (2.1).

Denoting $\mathsf{U} = (\mathsf{u}^1, \mathsf{u}^2, \dots)$ and $h_\gamma(\mathbf{x}, \mathsf{U}) = \sum_{\mathsf{j}} \mu_{\mathsf{j}} \mathsf{g}_\gamma^{\mathsf{j}}(\mathbf{x}, \mathsf{u}^{\mathsf{j}})$ (for $\gamma = \alpha, \beta$) we finally arrive at a multiple state optimal design problem (with infinite number of state equations):

$$J(\theta, \mathbf{A}) = \int_\Omega \Big(\theta h_\alpha(\cdot, \mathsf{U}) + (1-\theta) \mathsf{h}_\beta(\cdot, \mathsf{U}) \Big)\, \mathsf{dx} \to \min$$
$$(\theta, \mathbf{A}) \in \mathcal{A}\,.$$

4. Example

Consider an energy maximization problem: take Ω to be a ball $B(\mathbf{0}, 1) \subseteq \mathbb{R}^2$, $m = 1$, $q_\alpha := 0.8|\Omega|$, $S = \{1, 2\}$ with $\mu_1 = \mu_2 = 1/2$. The right-hand side is given by $f_1 := f(1, \cdot) = \chi_A + \varepsilon \chi_B$ and $f_2 := f(2, \cdot) = \chi_A - \varepsilon \chi_B$, where $A := B(\mathbf{0}, 1/2)^c$ and $B := B(\mathbf{0}, 1/5)$, as depicted in Figure 1.

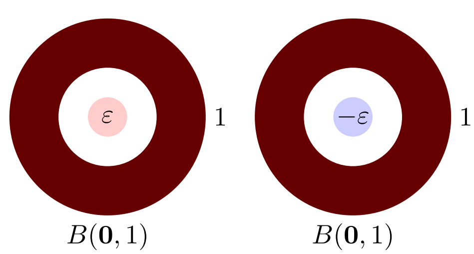

Figure 1. Right-hand sides are small perturbations of a constant heat source on annulus $B(\mathbf{0}, 1) \setminus B(\mathbf{0}, 1/2)$

The average right-hand side $\overline{f}$ is simply χ_A, and for a small ε it would be interesting to compare solutions for the perturbed (right-hand sides are f_1 and

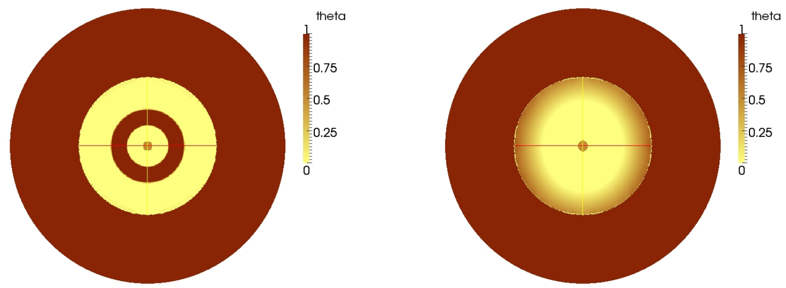

FIGURE 2. a. Numerical solution, $\varepsilon = 0.01$; b. Numerical solution, $\varepsilon = 0$

f_2) and the unperturbed problem (right-hand side is $\overline{f}$). By using the optimality criteria method [1] we obtain numerical solutions presented in Figure 2.

For this example, due to spherical symmetry, it is possible to calculate the exact solution [12] for any $\varepsilon > 0$ and examine what happens for $\varepsilon \to 0$. The method is based on the study of necessary conditions of optimality [7]. We shall just sketch the final result here.

Our problem (because of its spherical symmetry) is equivalent to a simpler relaxation problem written only in terms of local fraction θ [12]:

$$I(\theta) = \sum_{j=1}^{2} \mu_j \int_\Omega f_j u_j \, d\mathbf{x} \to \max \tag{4.1}$$

where $\theta \in L^\infty(\Omega; [0,1])$, $\int_\Omega \theta \, d\mathbf{x} = q_\alpha$, and u_1, u_2 are determined uniquely by

$$\begin{gathered} -\operatorname{div}(\lambda_-(\theta)\nabla u_j) = f_j \\ u_j \in \mathrm{H}_0^1(\Omega) \end{gathered}$$

for $j = 1, 2$. To be more precise, for any solution $(\theta^*, \mathbf{A}^*)$ of proper relaxation (2.3), θ^* is a solution of (4.1), and for any solution $\widetilde{\theta}$ of (4.1) we can construct a solution $\widetilde{\theta}, \widetilde{\mathbf{A}}$ of (2.3) by taking simple laminate $\widetilde{\mathbf{A}}$ with local fraction $\widetilde{\theta}$ and layers orthogonal to the radial direction, at almost any point of domain [12, Theorem 3.2].

The necessary (and sufficient) conditions of optimality for (4.1) state that there exists unique functions $\boldsymbol{\sigma}_i^*$ in $L^2(\Omega; \mathbb{R}^2)$ satisfying $-\operatorname{div} \boldsymbol{\sigma}_i^* = f_i$ and a Lagrange multiplier $c \geq 0$ such that

$$\begin{aligned} &\sum_{j=1}^{2} \mu_i |\boldsymbol{\sigma}_j^*|^2 > c \Rightarrow \theta^* = 1\,, \\ &\sum_{j=1}^{2} \mu_i |\boldsymbol{\sigma}_j^*|^2 < c \Rightarrow \theta^* = 0\,. \end{aligned} \tag{4.2}$$

From the spherical symmetry one can show that $\boldsymbol{\sigma}_i^*$ are radial functions, and if Ω is a ball they can be uniquely determined by solving $-\operatorname{div} \boldsymbol{\sigma}_j^* = f_j$.

Here, we can calculate explicitly $\psi = \frac{1}{2}(|\boldsymbol{\sigma}_1^*|^2 + |\boldsymbol{\sigma}_2^*|^2)$

$$\psi(r) = \begin{cases} \varepsilon^2 r^2/4, & 0 \le r \le 1/5 \\ \frac{\varepsilon^2}{2500 r^2}, & 1/5 < r \le r1/2 \\ \frac{r^2}{4} + \left(\frac{1}{64} + \frac{\varepsilon^2}{2500}\right)\frac{1}{r^2} - \frac{1}{8}, & 1/2 < r \le 1 \end{cases}$$

The graph of function ψ is presented in Figure 3. The optimality conditions (4.2) can be used now to determine the unique optimal design, as it is depicted in Figure 3.

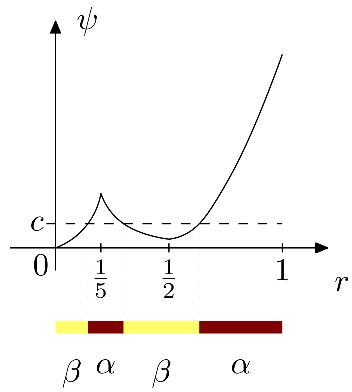

FIGURE 3. The graph of function $\psi = \frac{1}{2}(|\boldsymbol{\sigma}_1^*|^2 + |\boldsymbol{\sigma}_2^*|^2)$ and the geometric representation of optimality conditions.

However, the limiting case $\varepsilon = 0$ exhibits different behaviour: a solution is not unique [12]: it is only important to set α on $B(\mathbf{0}, \frac{1}{2})^c$, and to satisfy the constraint on the amount of the first phase.

Acknowledgments. This work has been supported by Croatian Science Foundation under the project 9780 WeConMApp.

The author wishes to thank the anonymous referee for the comments and suggestions.

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