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ON THE PROPERTIES OF ∞-HARMONIC FUNCTIONS AND AN APPLICATION TO CAPACITARY CONVEX RINGS

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Abstract. We study positive ∞-harmonic functions in bounded domains. We use the theory of viscosity solutions in this work. We prove a boundary Harnack inequality and a comparison result for such functions near a flat portion of the boundary where they vanish. We also study ∞-capacitary functions on convex rings. We show that the gradient satisfies a global maximum principle, it is nonvanishing outside a set of measure zero and the level sets are star-shaped.

1. Introduction

This is a continuation of the work in [4] and, while we derive a chain of results for ∞-harmonic functions, our primary effort in this work will be to prove two sets of results. The first would be for nonnegative ∞-harmonic functions, which vanish on a flat portion of the boundary of the set in which they are defined, and the second will be for ∞-capacitary functions in convex rings. More precisely, the first result discusses the behaviour of nonnegative ∞-harmonic functions near flat boundaries, on which they vanish, and we prove that any two such ∞-harmonic functions vanish at the same rate. In the second set of results, we show the non-vanishing of the gradient of ∞-capacitary functions on convex rings and the star-shapedness of the level sets of such functions. Clearly, the results are quite different in nature, however, the techniques used have a lot in common. A more detailed discussion follows in Section 2. We now comment on the approach used in this work. Our work utilizes the notion of a viscosity solution in this context and relies on techniques developed in [3,4,7,11,12,13,17,19-22,25,30]. While our results are motivated by the results in [15,18,27,30-35], which are about the weak solutions of the analogous problems with the p-Laplacian, for finite p, we do not work with approximating weak solutions as has been done in [5,16,18,23,29,30,31,33,34]. The idea in these works was to take the limit as $p \to \infty$ to capture properties and estimates for the ∞-harmonic functions. Instead our approach is closer to the works in [3,4,12,13,21,25,30]. Our intention is to use the framework of viscosity solutions to provide simpler and direct proofs. In this context we also refer the reader to the works in [3,4,13,22,24]. There is some overlap between our current

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work and [13]. This latter work contains, at times, finer and more detailed versions of some of the results proven here.

2. Notation and statements of the main results

We now introduce the notations we will be using in this work. These will be employed faithfully throughout this work with perhaps minor modifications for local use. By Ω, we will always mean a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\bar{A}$ will stand for the closure of a set A in $\mathbb{R}^n$. The letter $O = (0, 0, \dots, 0)$ will always stand for the origin in $\mathbb{R}^n$; for a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, define $\xi = \xi(x) = (x_1, x_2, \dots, x_{n-1})$ and $x_n(x) = x_n$, then $x = (\xi(x), x_n)$. Also $|\xi(x)| = \sqrt{x_1^2 + x_2^2 + \dots + x_{n-1}^2}$. We will sometimes use the notation $y = (0, a)$ to mean $y_1 = y_2 = \dots = y_{n-1} = 0$ and $y_n = a$. In this context, we will often think of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. We will be working with cylinders in $\mathbb{R}^n$: $A_r(P) = \{x : |\xi(x-P)| < r,\ P_n < x_n < P_n + 2r\} = \{y \in \mathbb{R}^{n-1} : |y - \xi(P)| < r\} \times (P_n, P_n + 2r)$ is the cylinder with axis parallel to the x_n-axis, radius r, length $2r$ with P being the center of the bottom face. Let $\lambda > 0$, then $A_{\lambda r}(P) = \{y \in \mathbb{R}^{n-1} : |y - \xi(P)| < \lambda r\} \times (P_n, P_n + 2\lambda r)$ is a λ scaling of $A_r(P)$ with the bottom faces situated at the same height. Also $F_r(P) = \{x : |\xi(x-P)| < r,\ x_n = P_n\} = \{y \in \mathbb{R}^{n-1} : |y - \xi(P)| < r\} \times \{P_n\}$ will denote the bottom face. We also describe a cylinder using the point half way on its axis. Let $K_r(P) = \{x : |\xi(x-P)| < r,\ P_n - r < x_n < P_n + r\} = \{y \in \mathbb{R}^{n-1} : |y - \xi(P)| < r\} \times (P_n - r, P_n + r)$; then $K_r(P)$ and $K_{\lambda r}(P) = \{y \in \mathbb{R}^{n-1} : |y - \xi(P)| < \lambda r\} \times (P_n - \lambda r, P_n + \lambda r)$ are concentric cylinders with center P. By $B_r(P)$, we will always mean the ball of radius r, centered at P. For ease of notation, we take $A_r = A_r(O)$, $K_r = K_r(O)$ and $B_r = B_r(O)$. Their use will be clear from the context. The sets $B_r^+(P) = \{x \in B_r(P) : x_n > P_n\}$ and $B_r^-(P)$ defined analogously, denote the half-balls. If A and B are two points, with $A \neq B$, then AB stands for the straight segment joining A to B.

The ∞-Laplacian operator Δ_∞ is defined as $\Delta_\infty u = \sum_{i,j=1}^{n} D_i u D_j u D_{ij} u$, where $D_i u = \partial u / \partial x_i$ and $D_{ij} u = \partial^2 u / \partial x_i \partial x_j$. This operator is elliptic but highly degenerate. In this work, we study viscosity solutions of solutions of

$$\Delta_\infty u = 0, \quad \text{in } \Omega. \tag{2.1}$$

We provide a definition in this context. We say that u is a viscosity subsolution (or ∞-subharmonic) of above equation, in Ω, if u is upper-semicontinuous in Ω, and whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$\phi(x_0) = u(x_0), \quad \text{and} \quad \phi(x) < u(x), \quad \text{for } x \neq x_0,$$

then $\Delta_\infty \phi(x_0) \geq 0$. Analogously, we may define a viscosity supersolution (or ∞-superharmonic) of (2.1), by requiring that u be lower-semicontinuous and $\Delta\phi(x_0) \leq 0$, whenever $u(x) - \phi(x)$ has a local minimum at x_0. We say u is a viscosity solution (or ∞-harmonic) of (2.1) if it is both a subsolution and a supersolution. It is well known that, if u is ∞-harmonic then u is locally Lipschitz continuous in Ω [4,13,19,21,29]. We must point out a key property, we will often exploit here, is that if a function u has cone comparison property then it is ∞-harmonic, a fact proven in [13]. Also see [4]. We now state the first result.

Theorem 2.1 (Comparison). *Let $u_i(x) > 0$, $i = 1, 2$, be ∞-harmonic in $A_8(O)$. If u_1 and u_2 vanish continuously on $F_8(O)$, then there exist positive constants M_1, M_2 and M_3 independent of u_i, such that for $x \in A_1(O)$,*

(i) $u_i(x) \le M_1 u_i(z)$, $i = 1, 2$, *and*

(ii) $M_2 \dfrac{u_1(z)}{u_2(z)} \le \dfrac{u_1(x)}{u_2(x)} \le M_3 \dfrac{u_1(z)}{u_2(z)}$, *where* $z = (0, 2)$.

One may think of part (i) as a boundary Harnack inequality and plays an important role in the proof of part (ii). This type of comparison result, near a flat portion of the boundary, is well known in the theory of both divergence and non-divergence type elliptic partial differential equations. We refer the reader to the works in [2,6,8,9,15,32] and the references therein. However, the work in [32], done in connection with a Fatou theorem, is the earliest work which proves a result of this type for p-harmonic functions, for $1 < p < 3 + 2/(n-2)$. Among other things, the work used associated kernel functions and is based on the earlier fundamental works [6], for nondivergence type equations, and [9]. The result for the entire range of $1 < p < \infty$ was later proven in [15], by carrying out a detailed refinement of the works in [6] and [33]. These works are quite nontrivial in nature. To the best of our knowledge, no attempt has been made yet to study the behaviour of the constants M_2 and M_3 as $p \to \infty$, in the context of the p-Laplacian. For the case $p = \infty$, however, we do not utilize the notion of a kernel as employed in these aforementioned works. But it needs to be mentioned that while we use ideas from [4,13] and work directly with (2.1), the work in [9] continues to be very useful, even in this context. Our approach is as follows. We first prove that the oscillation $\nu(r) = \operatorname{osc}_{K_r} u$ satisfies $\nu(2r) \ge C\nu(r)$, for some $C > 1$. A version of the Harnack inequality (see Lemma 3.2 and [29]), part (a) of Lemma 3.6 and the comparison principle permits us to apply the device in Theorem 1.1 [9]. This leads to a proof of part (i) and implies that the solutions are well behaved near F_2; away from F_2, the solution can be controlled by the Harnack inequality. Putting these together yields the result.

The second set of results are concerned with ∞-capacitary functions. We now introduce notations for this set up. Let C_1 and C_2, with $C_2 \subset C_1$, be bounded domains in $\mathbb{R}^n$, $n \ge 2$. Let $\Gamma = \Gamma(C_1, C_2) = C_1 \setminus \bar{C}_2$, denote the annular domain. We take C_1 and C_2 to be convex C^2 domains and we will also assume that the origin O lies in C_2. We will refer to Γ as a convex ring and $\partial\Gamma = \partial C_1 \cup \partial C_2$. If $Q \in \partial C_2$, then the line $L = L(Q)$ will often denote the straight ray normal to ∂C_2, at Q, directed towards ∂C_1. If $\nu = \nu(Q)$ is the unit outer normal to ∂C_2 at Q (relative to C_2), then the hyperplane $\langle x - Q, \nu(Q)\rangle = 0$ will be denoted by T_Q. Since C_2 is convex, it lies on one side of T_Q and $L \perp T_Q$ at Q. We may also define analogously the hyperplane T_P at a point $P \in \partial C_1$. The hyperplane T_Q generates two disjoint half-spaces

$$H_Q^+ = \{x \in \mathbb{R}^n : \langle x - Q, \nu(Q)\rangle < 0\} \quad \text{and} \quad H_Q^- = \{x \in \mathbb{R}^n : \langle x - Q, \nu(Q)\rangle > 0\}.$$

Clearly $H_Q^+ \supset C_2$. For $P \in \partial C_1$, we will again take H_P^+ to be the half-space that contains C_1. We will be studying the problem

$$\Delta_\infty u = 0, \quad \text{in } \Gamma, \quad u \in C(\bar{\Gamma}) \quad \text{with} \quad u|_{\partial C_1} = 1 \quad \text{and} \quad u|_{\partial C_2} = 0.$$

We again interpret this in the viscosity sense; see [11]. We call u an ∞-capacitary function. Invoking the Harnack inequality [4,29], we see that $0 \le u \le 1$. As a matter of fact if P is a point of an interior minimum of u then $u - u(P) \geq 0$ in Γ and since $u - u(P) > 0$ somewhere in Γ, being connected this would mean $u - u(P) > 0$ everywhere. This contradiction implies that u has no interior minimum (nor maximum for that matter) and so $0 < u < 1$. We will derive better bounds for u. By Γ_t, we mean the set $\{x \in \Gamma : u(x) < t\}$. This part of our work has been motivated by the prior works in [18,27,31,33,34].

Theorem 2.2. *Let u be an ∞-capacitary function in a bounded convex ring $\Gamma \subset \mathbb{R}^n$. Then*
(A) the level sets $\{x \in \Gamma : u(x) = t\}$, $0 < t < 1$, are star-shaped and satisfy a cone condition;
(B) there exists a positive number λ, depending only on the geometry of the domain, such that for any $x \in \Gamma$, there is a direction $\vec{e} = e(\vec{x})$, such that

$$|u(x + t\vec{e}) - u(x)| \geq \lambda t, \quad \forall\, 0 < t < t_0 = t_0(x).$$

It clearly follows that $|Du(x)| \geq \lambda > 0$, a.e. in Ω. It is not known yet whether u is better than Lipschitz in regularity and hence we are unable to assert the existence of $|Du|$ everwhere. See [12,25] for a discussion regarding this issue. The results in Theorem 2.2 were proven in [33,34] for the ∞-Laplacian, by utilizing the approximating procedure involving the p-Laplacian, for finite p; also see [18,31]. The works [33,34] also deal with star-shaped regions and contain interesting results. However, for convex rings, the result for the p-Laplacian, for finite p, was originally done in [27]. In this context also see [18,31]. Our approach will be to work directly with the viscosity solutions as discussed before. Our proof utilizes scaling and estimates near the boundaries, proven by employing auxilliary functions as in [27]. While a great many of the comparison type results used in this work may be worked in fairly elementary fashion as in [4,13], the comparison principle employed to compare u to its scaled version requires the application of a stronger result. More general versions of a comparison principle for such functions, originally proven in [19], may be found in [3,7,21]. Also see [17,20,23] for related works. Our approach also utilizes a property of nonnegative ∞-harmonic functions first alluded to in [4, see Remark 6] which follows from cone comparsion. See (3.1) and part (a) of Lemma 3.6 in Section 3. This is used in the proof of the existence of normal derivatives of u at the boundaries and also in the proof of a general bound for the gradients. We must point out that at this time we do not have a proof of the convexity of level sets uitilizing the viscosity framework. This fact was proven in [27] for the p-Laplacian, for finite p, and also holds for $p = \infty$ and appears in [33,34]. We make some remarks about this issue in Section 6.

We have divided our work as follows. Section 3 contains preliminary results needed for our work and Section 4 contains the proof of Theorem 2.1. Section 5 contains results applicable to the context of convex rings and the proof of Theorem 2.2 appears in Section 6. Appendix contains (i) the proof of the fact that odd reflections of ∞-harmonic functions are also ∞-harmonic, and (ii) the proof of Theorem 1.1 in [9].

We thank Michael Crandall for showing us a short proof of a sharper version of the Harnack inequality (see Lemma 3.2) and also for showing us some elegant proofs of results related to those in [4]. We also thank Juan Manfredi for several discussions in connection with this work and also for pointing out the work in [31,32]. We are also indebted to the referee whose comments have greatly improved the presentation of this work and also for pointing out the reference [18]. This research was partially supported by a grant from NSERC.

3. Preliminary results

In this section we will state and prove a sequence of results which lead to the proofs of Theorems 2.1 and 2.2. To achieve our end we will require somewhat more refined versions of the Hopf principle and the Harnack inequality. The proofs rely on the comparison principle and some auxilliary functions. A general version of the comparison principle is proven in [3; also see 7,19,21], however simpler arguments, such as those used in [4,13], will also suffice in many instances.

We will first recall Remark 6 in [4]. Also see Lemme 3.6. Let $u > 0$ be an ∞-harmonic function in a domain Ω and $B_r(O) \subset\subset \Omega$. We set $d(x) = \operatorname{dist}(x, \partial B_r(O)) = r - |x|$, $x \in B_r(O)$. Part (i) of Lemma 2 [4], then states

$$\frac{u(x)}{d(x)} \geq \frac{u(O)}{d(O)} = \frac{u(O)}{r}, \quad \forall x \in B_r(O).$$

Utilizing this, we showed, in Remark 6 [4], that if $\vec{e}$ is a unit vector, $x = s\vec{e}$ and $y = t\vec{e}$, where $0 < s$, $t < r$ (clearly, x, $y \in B_r(O)$, $d(x) = r - s$ and $d(y) = r - t$), then

$$\frac{u(x)}{d(x)} = \frac{u(x)}{r-s} \leq \frac{u(y)}{r-t} = \frac{u(y)}{d(y)}, \quad \forall\, 0 < s < t < r. \tag{3.1}$$

In other words, $u(x)/(r - |x|)$ is monotonic along radial lines through O and is increasing as $x \to \partial B_r(O)$ along $\vec{e}$. One notes that this may prove useful especially when $u(r\vec{e}) = 0$, i. e., u vanishes at some boundary point. This observation leads to viscosity proofs of some well-known results and proves important in our work. We prove (3.1) in Lemma 3.6. We first start with a more general version of the Hopf boundary point lemma; in this context also see [4,31,33,34].

Lemma 3.1 (Hopf boundary point lemma)**.** *Let Ω be a C^2 domain and $u \in C(\bar{\Omega})$ be ∞-harmonic in Ω. Suppose $S \in \partial\Omega$ is such that there is a ball $B_r(P) \subset \Omega$ with $\partial B_r(P) \cap \partial\Omega \ni S$. Let $x \in B_r(P)$, I be the fixed straight line segment containing x and S, $\vec{a} = (P - S)/r$ and $\vec{b} = (x - S)/|x - S|$. If $u(S) = max_{x \in B_r(P)} u(x)$, then*

$$\limsup_{x \to s,\, x \in I} \frac{u(x) - u(S)}{|x - S|} \leq \langle \vec{a}, \vec{b} \rangle \frac{u(P) - u(S)}{r} < 0.$$

Proof: We use the result in [4]. By comparison it follows that $\forall x \in B_r(P)$,

$$u(x) - u(S) \leq (u(P) - u(S))\Big(1 - \frac{|x - P|}{r}\Big). \tag{3.2}$$

Writing $x - P = (x - S) - (P - S)$, $\epsilon = |x - S|/r$, we see that

$$\begin{aligned} 1 - \frac{|x-P|}{r} &= 1 - |\vec{a} - \epsilon\vec{b}| = 1 - \{1 - (2\langle \vec{a}, \vec{b} \rangle - \epsilon)\epsilon\}^{1/2} \\ &= \frac{(2\langle \vec{a}, \vec{b} \rangle - \epsilon)\epsilon}{1 + \{1 - (2\langle \vec{a}, \vec{b} \rangle - \epsilon)\epsilon\}^{1/2}} \geq 0. \end{aligned} \tag{3.3}$$

Letting $\epsilon \to 0$, (3.2) and (3.3) yield

$$\limsup_{x \to s,\, x \in I} \frac{u(x) - u(S)}{|x - S|} \leq \langle \vec{a}, \vec{b} \rangle \frac{u(P) - u(S)}{r} < 0.$$

$\square$

We now present a proof of a sharper version of the Harnack inequality for non-negative ∞-harmonic functions. This version was first proven in [29; also see 21] using approximating p-harmonic functions. The proof below uses the notion of viscosity solutions and uses the estimates proven in [4]. Note that it uses none of the differentiation theory utilized in [29]. This proof was pointed out to us by Michael Crandall.

Lemma 3.2 (The Harnack inequality)**.** *Let $u > 0$ be ∞-harmonic in Ω, and $\delta > 0$ be such that the set $\Omega_\delta = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) \geq \delta\} \neq \emptyset$. Suppose A and B are points, in Ω_δ, such that the segment $AB \subset \Omega_\delta$. Then*

$$u(B) \geq e^{-\frac{|A-B|}{\delta}} u(A).$$

Proof: We note that, by employing the comparison principle, if $y \in \Omega_\delta$ then

$$u(y)\Big(1 - \frac{|x-y|}{\delta}\Big) \leq u(x), \quad \forall x \in B_\delta(y). \tag{3.4}$$

Let $x_0, x_1, x_2, \ldots, x_m$ be points on the segment AB such that $x_0 = A, x_m = B$ and $|x_i - x_{i+1}| = |A-B|/m, \ \forall i = 0, 1, \ldots, \ m-1$. Choose m large so that $|A-B|/m \leq \delta/2$. Since $x_{i+1} \in B_\delta(x_i)$, applying (3.4), we find that

$$u(x_{i+1}) \geq u(x_i)\Big(1 - \frac{|A-B|}{m\delta}\Big), \quad \forall i = 0, 1, \ldots, \ m-1.$$

Thus

$$u(B) \geq u(A) \quad (1 - \frac{|A-B|}{m\delta} \quad)^m. \tag{3.5}$$

The lemma follows by letting $m \to \infty$ in (3.5). □

Remark 3.3. It is clear that above estimate can be extended very easily to ∞-superharmonic functions and to polygonal paths joining two points in Ω_δ.

We now prove a result about the oscillation of u which will prove important in our proof of Theorem 2.1. Calling $w(r) = \max_{B_r(O)} |u(x) - u(y)| = \operatorname{osc}_{B_r(O)} u$, we show that $w(r)$ is convex and in particular $w(2r) \geq 2w(r)$. This fact together with Theorem 1.1 in [9] will lead to a proof of Theorem 2.1. Note in Lemma 3.4, we do not assume that $u > 0$.

Lemma 3.4 (Convexity of oscillation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and u be ∞-harmonic in Ω. Let $B_R(O) \subset\subset \Omega$, be the ball of radius R, centered at O. Suppose that $M(r) = \sup_{B_r(O)} u(x)$, and $m(r) = \inf_{B_r(O)} u(x)$. Then for $0 \leq r \leq R$,*
(i) $w(r) = \operatorname{osc}_{B_r(O)} u(x)$ is convex and
(ii) $\dfrac{w(r)}{r} = \dfrac{M(r) - m(r)}{r} \downarrow$ as $r \downarrow 0$.

Proof: Let $0 < \delta < R$; set

$$W_\delta(x) = \frac{M(R) - M(\delta)}{R - \delta}|x| + \frac{RM(\delta) - \delta M(R)}{R - \delta},$$
$$w_\delta(x) = \frac{m(R) - m(\delta)}{R - \delta}|x| + \frac{Rm(\delta) - \delta m(R)}{R - \delta},$$

for all $x \in B_R(O) \backslash B_\delta(O)$. It is easily checked that $W_\delta(x) \geq u(x)$ and $w_\delta(x) \leq u(x)$, when $|x| = R$ and $|x| = \delta$. Since W_δ and w_δ are both ∞-harmonic, it follows that $w_\delta \leq u(x) \leq W_\delta(x), \ \delta \leq |x| \leq R$. Set $r = |x|$, it then follows that

$$M(r) \leq \frac{M(R) - M(\delta)}{R - \delta} r + \frac{RM(\delta) - \delta M(R)}{R - \delta} = \frac{r - \delta}{R - \delta} M(R) + \frac{R - r}{R - \delta} M(\delta),$$

and

$$m(r) \geq \frac{m(R)-m(\delta)}{R-\delta}r + \frac{Rm(\delta)-\delta m(R)}{R-\delta} = \frac{r-\delta}{R-\delta}m(R) + \frac{R-r}{R-\delta}m(\delta).$$

It is clear that $M(r)$ is convex and increasing while $m(r)$ is concave and decreasing. It is clear from above that $M(r)-m(r)$ is convex. Letting $\delta \to 0$ and noting that $M(0) = m(0) = u(0)$, we find that $(M(r)-m(r))/r \leq (M(R)-m(R))/R$. This proves (i) and (ii). □

Remark 3.5. From Lemma 3.4 (ii), it follows that for $t > 1$, $w(tr) \geq tw(r)$ and in particular, $w(2r) \geq 2w(r)$. Now let $K_r = K_r(O) = \{x : |\xi(x)| < r,\ |x_n| < r\} = \{x : |\xi(x)| < r, \} \times (-r, r)$ be the cylinder of length $2r$, radius r and centered at O; let $\nu(r) = \operatorname{osc}_{K_r} u$. Then $K_{2r} \supset B_{2r} \supset B_{\sqrt{2}r} \supset K_r$ (see Section 2). Thus

$$\nu(2r) \geq w(2r) \geq \sqrt{2}w(\sqrt{2}r) \geq \sqrt{2}\nu(r). \tag{3.6}$$

We now work a proof of Lemma 3.6; this will imply (3.1). We introduce some additional notation. Let u be ∞-harmonic in Ω, $B_r(z) \subset\subset \Omega$, $M = \sup_{x\in B_r(z)} u(x)$ and $m = \inf_{x\in B_r(z)} u(x)$. By the maximum principle, these are attained on the boundary $\partial B_r(z)$. Let $P_M,\ P_m \in \partial B_r(z)$ be such that $u(P_M) = M$ and $u(P_m) = m$. We define the following difference quotients on the segments zP_m and zP_M;

$$\forall x \in zP_m,\ \ D_1(x, P_m) = \frac{u(x)-m}{|x-P_m|} \quad \text{and} \quad \forall x \in zP_M,\ D_2(x, P_M) = \frac{M-u(x)}{|x-P_M|}.$$

Note that $D_1(x, P_m) \geq 0$ and $D_2(x, P_M) \geq 0$; also define

$$D(m) = \lim_{x\to P_m} D_1(x, P_m),\ x \in zP_m \quad \text{and} \quad D(M) = \lim_{x\to P_M} D_2(x, P_M),\ x \in zP_M.$$

whenever these exist. For any $x \in B_r(z)$, set $d(x) = \operatorname{dist}(x, \partial B_r(z)) = r - |x-z|$, and for $x \neq z$, take $\vec{e} = \frac{z-x}{|x-z|}$ and $y = x - (z - r\vec{e})$. Note that $(z - r\vec{e}) \in \partial B_r(z)$ and $|y| = r - |x-z| = d(x)$. We will also take $u(y)$ to stand for the value $u(x)$.

Lemma 3.6 (Monotonicity)**.** *Let u be ∞-harmonic in Ω and $B_r(z) \subset\subset \Omega$.*

(a) If $u > 0$ in Ω and $\vec{\eta}$ is such that $|\vec{\eta}| = 1$, then for $0 \leq a \leq r$,

$$\frac{u(z+a\vec{\eta})}{r-a} = \frac{u(z+a\vec{\eta})}{r-|a\vec{\eta}|} \geq \frac{u(z+b\vec{\eta})}{r-|b\vec{\eta}|} = \frac{u(z+b\vec{\eta})}{r-b},\ \ \forall\, 0 \leq b \leq a \leq r.$$

In other words, $\forall\ x \in B_r(z)$, $u(x)/d(x)$, is increasing as $x \to \partial B_r(z)$ along a radial line. We also note that by taking $b = 0$, $u(z+a\vec{\eta})/(r-a) \geq u(z)/r$, $\forall 0 \leq a \leq r$.
(b) Under the assumptions in (a), $x \neq z$, $\vec{e}$ and y as defined above, we have

$$u(ty) \leq tu(y), \quad \textit{whenever} \quad t \geq 1 \quad \textit{and} \quad 0 < t|y| \leq r.$$

Furthermore, if $Du(x)$ exists then $\langle Du(x), \vec{e}\rangle \leq \dfrac{u(x)}{d(x)}$.
(c) Moreover, if $u > 0$ and $|Du|(z)$ exists then

$$(i)\ \ |Du|(z) \leq \frac{u(z)}{\operatorname{dist}(z, \partial B_r(z))}; \quad \textit{and} \quad (ii)\ \ |Du|(z) \leq \frac{u(z)}{\operatorname{dist}(z, \partial\Omega)}.$$

(d) Regardless of the sign of u, $D(M)$ and $D(m)$ exist on $B_r(z)$ and $|Du|(z) \leq \min(D(m),\ D(M))$.

Proof: The proof follows by an application of part (i) Lemma 2 in [4]; see (3.1).
Part (a): For $0 \le b \le a \le r$, set $x = z + a\vec{\eta}$ and $v = z + b\vec{\eta}$; then $d(x) = \mathrm{dist}(x, \partial B_r(z)) = r - a$, $d(v) = \mathrm{dist}(v, \partial B_r(z)) = r - b$ and $d(v) \ge d(x)$. Clearly, x lies in the ball $B_{d(v)}(v)$; applying Lemma 2 in [4] (also see discussion preceding (3.1)), we see that $u(x) \ge u(v)d(x)/d(v)$. This proves part (a). The rest of the assertions are consequences of this fact.
Part (b): We reinterpret part (a). One notes that for $x \in B_r(z)$, the ray $z - s\vec{e}$, $s \ge 0$, cuts $\partial B_r(z)$ at $z - r\vec{e}$. Thus by part (a) we find that $u(x)/(r - |x-z|) = u(y)/|y|$ increases as $x \to z - r\vec{e}$, i.e., as $|y| = d(x) \downarrow 0$. If $t > 1$ is such that $0 < t|y| \le r$, then

$$\frac{u(ty)}{|ty|} \le \frac{u(y)}{|y|} \quad \Rightarrow \quad u(ty) \le tu(y).$$

If $x_1,\ x_2 \in B_r(z)$ such that $x_i = z - s_i\vec{e},\ i = 1, 2$, for some $\vec{e} \in \mathbb{R}^n$ with $|\vec{e}| = 1$, $0 \le s_2 < s_1 \le r$, and $\theta = s_1 - s_2$, then $x_2 = x_1 + \theta\vec{e}$, $d(x_i) = r - s_i$ and $d(x_1) < d(x_2)$. Using part (a), i.e., $u(x_1)/d(x_1) \ge u(x_2)/d(x_2)$,

$$\frac{u(x_1 + \theta\vec{e}) - u(x_1)}{\theta} = \frac{u(x_2) - u(x_1)}{|x_1 - x_2|} \le \begin{cases} \frac{u(x_1)}{d(x_1)} = \frac{u(x_1)}{r - |x_1 - z|}, \\ \frac{u(x_2)}{d(x_2)} = \frac{u(x_2)}{r - |x_2 - z|}. \end{cases} \tag{3.7}$$

Note that $s_2 = |x_2 - z| < |x_1 - z| = s_1$. Suppose $Du(x_1)$ exists; letting $x_2 \to x_1$ in (3.7) yields the conclusion. Recall that the solution u is Lipschitz continuous hence such directional derivatives exist a. e. on these rays.
Part (c): We use part (3.7) with $x_2 = z$, $\vec{e}$ any unit vector and $x_1 = z + s\vec{e},\ s > 0$. Then

$$\frac{u(z) - u(z + s(\vec{e}))}{s} \le \frac{u(z)}{r} \quad \Rightarrow \langle Du(z), -\vec{e}\rangle \le \frac{u(z)}{r}, \quad \forall \vec{e} \in \mathbb{R}^n,$$

if $Du(z)$ exists. If $Du(z) \ne 0$, we may take $\vec{e} = -Du(z)/|Du(z)|$ and part (i) follows. To prove (ii), let $R = \mathrm{dist}(z, \partial\Omega)$, then the ball $B_R(z) \subset \Omega$, and (i) continues to hold by considering an increasing sequence of balls.
Part(d): All the results discussed above require that $u > 0$. Now we drop this requirement. With m and M as above, part (c) holds for $u - m$ and $M - u$, i. e.,

$$|Du(z)| \le \min\Big(\frac{M - u(z)}{r},\ \frac{u(z) - m}{r}\Big).$$

Applying part (a) to $M - u$, it follows that $(M - u(z))/r \le (M - u(x))/|x - P_M| \le D(M), \forall x \in zP_M$, if $D(M)$ exists. To see this, note

$$\forall x \in OP_M,\ 0 < D_2(x, P_M) = \frac{M - u(x)}{|x - P_M|} = \frac{u(P_M) - u(x)}{|x - P_M|} \ \uparrow \ \text{as } x \to P_M.$$

The case of the minimum follows analogously. Clearly the directional derivatives of u at P_M along OP_M, and at P_m along OP_m, either exist or blow up. Since P_m and P_M are points in the interior of Ω, the local Lipschitz regularity of u thus implies that $D_1(x, P_m)$ and $D_2(x, P_M)$ are all uniformly bounded. Hence the normal derivatives $D(m)$ and $D(M)$ exist and are strictly positive and finite. Thus

$$|Du(z)| \le \min(D(m), D(M)).$$

4. Proof of Theorem 2.1

We follow the outline of Theorem 1.1 in [9] (see Appendix) and use Lemmas 3.2, 3.4 and 3.6. We point out that part (i) of Theorem 2.1 (also see (4.3)) turns out to be crucial for the proof of part (ii). One may think of (i) as a boundary Harnack inequality. For notations, see Section 2. Recall the expression $x = (\xi(x), x_n)$, for all $x \in \mathbb{R}^n$ and $A_r = \{x : \ |\xi(x)| < r, \text{ and } 0 < x_n < 2r\}$. We proceed as follows. Assume that $u_i > 0, \ i = 1, 2$; by scaling if necessary, we may take u_i's to be ∞-harmonic in A_8 and vanishing continuously on the face $F_8 = \{x : |\xi(x)| < 8, \ x_n = 0\} \subset \{x_n = 0\}$. We suppress the subscript and work with a general u that satisfies the requirements of the theorem. The constants M, C are positive constants, that are independent of u, but may depend on the geometry. We will often write $x = (\xi(x), x_n)$ and set $z = (0, 2)$. We achieve our proof in five steps.

Step 1. We first show that

$$\frac{u(x)}{u(z)} \geq \frac{x_n}{4}, \quad \forall x \in A_1. \tag{4.1}$$

For $x \in A_1$, write $x = (\xi(x), x_n)$ and set $P = (\xi(x), 2)$; then P lies in the hyperplane $x_n = 2$ and $x \in B_2(P) \subset A_8$. Applying Lemma 3.6 (a), in $B_2(P)$, we see

$$\frac{u(x)}{x_n} \geq \frac{u(P)}{2}.$$

Again $P \in B_2(z)$, and applying once more Lemma 3.6 (a) to $B_2(z)$ and noting that $d(P) = \operatorname{dist}(P, \partial B_2(z)) \geq 1$, we have

$$u(P) \geq \frac{u(P)}{d(P)} \geq \frac{u(z)}{2}.$$

Combining these observations yields (4.1).

Step 2. We now make a few remarks which again follow from Lemmas 3.2 and 3.4. For $x \in A_2$ with $0 < x_n < 3/2$, an application of Lemma 3.2, to the points $(\xi(x), x_n)$ and $(\xi(x), 2x_n)$, implies

$$u(\xi(x), x_n) \leq e^{(|2x_n - x_n|/x_n)} u(\xi(x), 2x_n) = e^1 u(\xi(x), 2x_n).$$

Now for $x \in A_2$ with $1 < x_n < 3$, we see that $\operatorname{dist}(x, F_2) \geq 1$ and $|z - x| \leq \sqrt{5}$. For these x's, Lemma 3.2 again implies $u(\xi(x), x_n) \leq e^{\sqrt{5}} u(z)$. To recap, with $M = e^{\sqrt{5}}$,

$$u(x) = u(\xi(x), x_n) \leq \begin{cases} Mu(\xi(x), 2x_n) : & |\xi(x)| \leq 2, \ 0 < x_n < 3/2, \\ Mu(z) : & |\xi(x)| \leq 2, \ 1 < x_n < 3. \end{cases} \tag{4.2}$$

We also recall, with the notations of Remark 3.5, that if K_r and $K_{r/2}$ are concentric cylinders and $\nu(r) = \operatorname{osc}_{K_r} u$, then

$$\nu(r) \geq \sqrt{2}\nu(r/2).$$

This together with (4.1), the fact that odd reflection of u about $x_n = 0$ continues to be ∞-harmonic (see Appendix) and Theorem 1.1 [9] implies that there is a universal constant C such that

$$\sup_{A_1} u(x) \leq Cu(z) \ \Rightarrow \ \sup_{A_1} u_i(x) \leq Cu_i(z), \ i = 1, 2. \tag{4.3}$$

This achieves the proof of part (i) of Theorem 2.1.

Step 3. Our next goal is to prove that for some universal constant C,

$$u(x) \le Cu(z)x_n, \quad \forall x \in A_1. \tag{4.4}$$

We first describe a proof of (4.4) when $\xi(x) = 0$ and $0 < x_n \le 2$, i.e., when x is on the segment Oz. It is clear that $A_1 \subset B^+_{\sqrt{5}}(O) \subset A_8$. Let $M_O = \sup_{B^+_{\sqrt{5}}(O)} u > 0$. By the maximum principle, $M_O = \sup_S u$, where $S = \partial B^+_{\sqrt{5}}(O)$, since $u = 0$ on F_8. By comparison, $u(x) \le M_O|x|/\sqrt{5}$, $\forall x \in B^+_{\sqrt{5}}(O) \supset A_1$. In particular,

$$u(x) \le \frac{M_O x_n}{\sqrt{5}}, \quad \forall x = (0, x_n),\ 0 < x_n \le 2. \tag{4.5}$$

Our next task now will be to estimate M_O in terms of $u(z)$. We do this as follows. If the maximum M_O occurs near F_8, then it can be controlled first by u at a point away from F_8 by an application of (4.3). This in turn can be estimated by $u(z)$ by the Harnack inequality. Note that a direct application of the Harnack inequality is not possible since the constants degenerate near F_8. If the maximum occurs away from F_8, then the Harnack inequality suffices to achieve our end. We set $T = F_8 \cap \partial B_{\sqrt{5}}(O)$. For $P \in T$, $x_n(P) = 0$, $|\xi(P)| = \sqrt{5}$ and the cylinder $A_4(P) \subset A_8$. Thus by (4.3) and scaling

$$u(x) \le C_1 u(\bar{P}),\ \ \forall x \in A_{1/2}(P), \tag{4.6}$$

where $\bar{P} = (\xi(P), 1)$ and C_1 is the constant in (4.3). Clearly, (4.6) holds in $I = \cup_{P\in T} A_{1/2}(P)$. Let $E = \cup_{P\in T}\{\bar{P}\} = \{x : |\xi(x)| = \sqrt{5},\ x_n = 1\}$. Observe that $\operatorname{dist}(\bar{P}, F_8) = \operatorname{dist}(\bar{P}, \partial A_8) = 1$, $\forall \bar{P} \in E$. Employing Lemma 3.2, (4.6) and recalling that $z = (0, 2)$, we have

$$u(\bar{P}) \le u(z)e^{|\bar{P}-z|} \le u(z)e^{\sqrt{6}} \ \ \Rightarrow\ \ u(x) \le C_2 u(z),\ \ \forall x \in I. \tag{4.7}$$

Clearly, this also holds on $\partial B^+_{\sqrt{5}}(O) \cap I$. If the maximum M_O occurs in I, (4.7) applies. Now for $x \in \partial B^+_{\sqrt{5}}(O) \setminus I$, $\operatorname{dist}(x, \partial A_8) = \operatorname{dist}(x, F_8) \ge 1$ and $|x - z| \le \sqrt{6}$; we may apply Lemma 3.2, as done above, to conclude that

$$u(x) \le e^{\sqrt{6}} u(z),\ \ \forall x \in \partial B_{\sqrt{5}}(O) \setminus I.$$

This together with (4.7) implies that $M_O \le C_3 u(z)$, where C_3 is again a universal constant. The inequality in (4) now implies for some appropriate constant C,

$$u(x) \le Cu(z)x_n, \quad \forall x = (0, x_n), \ \ 0 < x_n \le 2. \tag{4.8}$$

Step 4. We now show that (4.4) holds in all of A_1. Let $x \in A_1$, with $\xi(x) \ne 0$; define $L = (\xi(x), 0)$ and $\bar{L} = (\xi(x), 2)$. Then L and $\bar{L}$ belong to ∂A_1; the cylinders $A_1(L)$, $A_4(L)$ all lie in $A_8(O)$ and the half-ball $B^+_{\sqrt{5}}(L) \subset A_8(O)$. Furthermore, $\forall N \in F_8 \cap \partial B_{\sqrt{5}}(L)$, $A_4(N) \subset A_8(O)$. Now working with $A_{1/2}(N)$, we may now apply Step 3 to conclude that for all $\bar{L} \in \partial A_1$, with $x_n(\bar{L}) = 2$, (4.8) holds i.e.,

$$u(x) \le Cu(\bar{L})x_n,\ \ \forall x \in A_1,\ \ \text{with}\ \ \xi(x) = \xi(\bar{L}),\ 0 < x_n \le 2.$$

All that remains now is to relate $u(\bar{L})$ to $u(z)$ and this is achieved by Lemma 3.2. Clearly, $u(\bar{L}) \le e^1 u(z)$ and hence we see that

$$u(x) \le Cu(z)x_n, \quad x \in A_1. \tag{4.9}$$

Step 5. We combine (4.1) and (4.9) to deduce that for some universal $C > 0$ and $\bar{C} > 0$

$$\bar{C}\frac{u_1(z)}{u_2(z)} \leq \frac{u_1(x)}{u_2(x)} \leq C\frac{u_1(z)}{u_2(z)}, \quad \forall x \in A_1.$$

This finishes the proof of part (ii) of Theorem 2.1.

5. ∞-Capacitary functions in convex rings

Our effort, in Sections 5 and 6, is to prove Theorem 2.2 (see Section 2 for notation). The main ideas used here are similar to those in Section 3. Our strategy will be to prove bounds for u, show strict monotonicity by using scaling, a global maximum principle for $|Du|$ and make remarks about a global lower bound.

We start with bounds for u in Γ. Our approach is to use appropriate barrier functions and comparison and while these will suffice for our purposes, an approach based on Lemma 3.6 can also be worked out. We make comments along this direction later in this work. The function $u \in C(\bar{\Gamma})$, from hereon, is an ∞-capacitary function with $u|_{\partial C_1} = 1$, $u|_{\partial C_2} = 0$, and, as observed in Section 2, $0 < u < 1$ in Γ. We take the origin O to lie in C_2. We also remind the reader that, for $Q \in \partial C_2$, $L = L(Q)$ will be the straight ray originating from Q, normal to ∂C_2, directed toward ∂C_1.

Lemma 5.1 (Lower bound)**.** *Let $Q \in \partial C_2$ and L be as described above. Let $P = P(Q) = L \cap \partial C_1$ and $R = |P - Q|$, then*

$$u(x) \geq 1 - \frac{|x - P|}{|Q - P|} = 1 - \frac{|x - P|}{R} > 0, \quad \forall x \in B_R(P) \cap \Gamma.$$

Proof: Set $w = w_{P(Q)}(x) = 1 - (|x - P|/R)$; then w is (i) ∞-harmonic in $B_R(P)\backslash\{P\}$, (ii) $w = 0$ on $\partial B_R(P)$ and (iii) $0 < w(x) < 1$ in $B_R(P)\backslash\{P\}$ with $w(P) = 1$. We compare w to u in $B_R(P) \cap \Gamma$ and conclude that $u \geq w$ on $\partial(B_R(P) \cap \Gamma) = (\partial B_R(P) \cap \Gamma) \cup (B_R(P) \cap \partial\Gamma)$. Both being ∞-harmonic in $B_R(P) \cap \Gamma$, comparison yields $u \geq w$ in $B_R(P) \cap \Gamma$ and

$$u(x) \geq w_{P(Q)} = 1 - \frac{|x - P|}{R} > 0, \quad x \in B_R(P) \cap \Gamma.$$

Note that the function $\tilde{w}(x) = \sup_{Q \in \partial C_2} w_{P(Q)}(x)$ is ∞-subharmonic and $u(x) \geq \tilde{w}(x)$. □

Before we prove an upper bound for u, we note the following. Being a C^2 domain, C_1 satisfies an interior ball condition at every point on ∂C_1. For $\eta > 0$, let $C_{1,\eta} = \{x \in \Gamma : \operatorname{dist}(x, \partial C_1) \leq \eta\}$. Since ∂C_1 is C^2, for every $A \in \partial C_1$, there is a $\delta_A > 0$ and an $H_A \in \Gamma$ with the property that $B_{\delta_A}(H_A) \subset \Gamma$ and $B_{\delta_A}(H_A) \cap \partial C_1 \ni A$. We take δ_A to be the largest such number, and if $\delta_0 = \inf_A\{\delta_A\}$, then $\delta_0 > 0$. Let $l = \operatorname{dist}(\partial C_1, \partial C_2)/2$ and $\delta = \min(\delta_0, l)$. This choice is made for technical reasons. For notational ease, define $\delta(x) = \operatorname{dist}(x, \partial C_1)$. By Δ, we denote the diameter of C_1. We should point out that while Lemma 5.2, as stated below, provides a bound only for points near ∂C_1, its derivation requires the calculation of upper bounds in the rest of Γ.

Lemma 5.2 (Upper bound)**.** *Let δ, $\delta(x)$, Δ, l and $C_{1,\delta}$ be as described above. If $x \in C_{1,\delta}$, i.e., $\delta(x) \leq \delta$, then*

$$u(x) \leq 1 - \frac{e^{-(\Delta/\delta)}}{2\delta}\delta(x).$$

Proof: We do this in three steps.
Step 1 (Upper bound near ∂C_2) For every $Q \in \partial C_2$, the ball $B_{2l}(Q) \subset C_1$. Fix Q and set $v = v(x) = |x-Q|/(2l)$. Observe that (i) v is ∞-harmonic in $B_{2l}(Q) \cap \Gamma$, (ii) $v = 1$ on $\partial B_{2l}(Q) \cap \Gamma$ and $0 \le v \le 1$ on $\partial C_2 \cap B_{2l}(Q)$. Since $u \le v$ on $\partial(B_{2l}(Q) \cap \Gamma)$, comparison implies

$$u(x) \le \frac{|x-Q|}{2l} \le 1, \quad x \in B_{2l}(Q) \cap \Gamma. \tag{5.1}$$

Clearly $u \le 1/2$ in $B_l(Q)$, and so defining $E_l = \cup_{Q \in \partial C_2}(B_l(Q) \cap \Gamma) = \{x \in \Gamma : \operatorname{dist}(x, \partial C_2 < l)\}$, (5.1) yields

$$u(x) \le 1/2, \quad \forall x \in \bar{E}_l. \tag{5.2}$$

Step 2 (Upper bound away from ∂C_2 and ∂C_1) Let $Q_1 \in (\partial E_l \setminus \partial C_2)$ and $P_1 \in (\partial C_{1,\delta} \setminus \partial C_1)$; hence by (5.2)

$$u(Q_1) \le 1/2. \tag{5.3}$$

Now Q_1 and P_1 are in $\Gamma_\delta = \{x \in \Gamma : \operatorname{dist}(x, \partial\Gamma) \ge \delta\}$. We now apply Lemma 3.2 to the function $1 - u(x) \ge 0$ in Γ, along the segment P_1Q_1, to conclude that, $\forall x \in P_1Q_1$,

$$1 - u(x) \ge (1 - u(Q_1))e^{-(|x-Q_1|/\delta)}.$$

Noting (5.3), it follows that

$$u(x) \le 1 - \frac{e^{-(|Q_1-x|/\delta)}}{2}, \quad \forall x \in P_1Q_1.$$

Taking $x = P_1$, and observing that $|P_1 - Q_1| \le \Delta$, we see

$$u(P_1) \le 1 - \frac{e^{-(|Q_1-P_1|/\delta)}}{2} \le 1 - \frac{e^{-(\Delta/\delta)}}{2} \;\Rightarrow\; 1 - u(P_1) \ge \frac{e^{-(\Delta/\delta)}}{2}. \tag{5.4}$$

Step 3 (Bound near ∂C_1) To get our target estimate for every $x \in C_{1,\delta}$, we proceed as follows. For a fixed $x \in C_{1,\delta}$, let $H \in \partial C_1$ be such that $|x - H| = \delta(x)$. Then the straight line R containing the segment xH is perpendicular to ∂C_1 at H. Set $P_1 = R \cap (\partial C_{1,\delta} \setminus \partial C_1)$ and $Q_1 \in \bar{E}_l$ be such that $\operatorname{dist}(P_1, E_l) = |Q_1 - P_1|$. We now employ Lemma 3.1 as follows. Observe that $B_\delta(P_1) \subset \Gamma$, $\partial B_\delta(P_1) \cap \partial C_1 \ni H$, $u(H) = 1 = \max_{x \in B_\delta(P_1)} u(x)$ and $\delta(x) = \delta - |x - P_1|$. Then (3.2) implies

$$\begin{aligned} u(x) - 1 &\le (u(P_1) - 1)\left(1 - \frac{|x-P_1|}{\delta}\right) = (u(P_1) - 1)\frac{\delta(x)}{\delta} \\ \Rightarrow\; u(x) &\le 1 - (1 - u(P_1))\frac{\delta(x)}{\delta}, \quad \forall x \in P_1H. \end{aligned} \tag{5.5}$$

Clearly our estimate in (5.3) now holds for $u(Q_1)$ and $|P_1 - Q_1| \le \Delta$; an application of Lemma 3.2, (5.4) and (5.5) implies

$$u(x) \le 1 - \Big(\frac{e^{-(\Delta/\delta)}}{2\delta}\Big)\delta(x), \quad \forall x \in C_{1,\delta}. \tag{5.6}$$

□

Remark 5.3. The boundaries of Γ being C^2, the distance functions $d_i(x) = \operatorname{dist}(x, \partial C_i)$, $i = 1, 2$, is C^2 in a neighborhood of ∂C_i. Note that $d_2(x)$ is C^2, $\forall x \in \mathbb{R}^n \setminus C_2$. Thus $\Delta_\infty d_2 = 0$, $\forall x \in \Gamma$, while $\Delta_\infty d_1 = 0$ only for x near ∂C_1. One may also use these functions as barriers.

Our next step is to prove strict monotonicity of u along rays emanating from points in C_2. This lemma relies on the comparison principle proven in [3,7,19,21]. A stronger result is proven in Theorem 2.2 but this weaker result will prove adequate for what is to follow. We use scaling as was done in [27]. We introduce the following notations. For $t > 0$, define $C_i^t = tC_i = \{tx,\ \forall x \in C_i\}$, and $\partial C_i^t = t\partial C_i = \{tx : x \in \partial C_i\}$, $i = 1, 2$. If $0 < t < 1$ then $C_i^t \subset C_i$. Also set $u_t(y) = u(y/t)$. We will take t to be close to 1 and assume that $C_2 \subset C_1^t \subset C_1$.

Lemma 5.4 (Strict monotonicity)**.** *Let T denote a straight ray originating from $O \in C_2$ and $Q = T \cap \partial C_2$. Then u is strictly increasing along $T \cap \Gamma$, in the direction of ∂C_1.*

Proof: We employ scaling. Let P_0 and P_1 be on $T \cap \Gamma$ with $0 < |P_0| < |P_1|$. Set $t = |P_0|/|P_1| < 1$, $y = tx$ and $u_t(y) = u(x) = u(y/t)$. Clearly, $tP_1 = P_0$, $u_t(P_0) = u_t(tP_1) = u(P_1)$ and u_t is ∞-harmonic in $\Gamma^t = \Gamma(C_1^t, C_2^t)$. Notice that $0 < u_t < 1$ in Γ^t, $u_t = 0$ on ∂C_2^t, $u_t = 1$ on ∂C_1^t and $\partial(\Gamma \cap \Gamma^t) = \partial C_2 \cup \partial C_1^t$. Now

$$\Delta_\infty u = \Delta_\infty u_t = 0, \text{ in } \Gamma \cap \Gamma^t, \ u < u_t = 1 \text{ on } \partial C_1^t \text{ and } u = 0 < u_t, \text{ on } \partial C_2.$$

Applying the comparison principle, we obtain

$$u_t(y) \geq u(y),\ \forall y \in \Gamma^t \cap \Gamma, \ \Rightarrow \ u(P_1) = u_t(tP_1) = u_t(P_0) \geq u(P_0).$$

Since Δ_∞ is translation and rotation invariant, we may move O around in C_2 and conclude that u is nondecreasing along the cone of rays passing through Q (call it K_Q) and in particular if $P \in \Gamma$ then there is a cone K_P (with apex at P) of rays through P, opening towards ∂C_1, along which u is nondecreasing. To see this just join O to P and move O in C_2. Fix $Q \in \partial C_2$; we now show that u is strictly increasing along $T \cap \Gamma$. Let P_0 and $P_1 \in T$ as before and K_{P_0} be the cone as described above. Clearly, $u(x) \geq u(P_0)$ in K_{P_0}. Recall that $u(P_0) < 1$, and if $S \in T \subset K_{P_0}$, close enough to ∂C_1, then Lemma 5.1 ensures that $u(S) > u(P_0)$. To see this, let $L \ni S$ be a straight line such that $L \perp \partial C_2$ and take $P = L \cap \partial C_1$. Thus $u(S) - u(P_0) > 0$ and $u(x) - u(P_0) \geq 0$ in K_{P_0}. The Harnack inequality implies that $u(P_1) > u(P_0)$. Clearly u is strictly increasing along rays in a strict sub-cone of K_{P_0}. $\square$

One of our goals is to prove the strict positivity of $|Du|$, whenever it exists, in Γ. To do this, we will need to derive bounds on u, at points near $\partial\Gamma$, which in turn require estimates of distances.

Lemma 5.5 (Distance estimate)**.** *Let $0 < t < 1$, t close to 1, be fixed. For $i = 1, 2$, let $\partial C_i^t = t\partial C_i$ be the t scaling of ∂C_i, and $l_i = \operatorname{dist}(O, \partial C_i)$. Then*

$$\operatorname{dist}(\partial C_i, \partial C_i^t) \geq l_i(1 - t),\ i = 1, 2.$$

Proof: Let $A_1 \in \partial C_1$ and $A_2 \in \partial C_1^t$ be such that $\operatorname{dist}(\partial C_1^t, \partial C_1) = |A_1 - A_2|$. Since ∂C_1^t, ∂C_1 are boundaries of C^2 convex sets, it follows that the supporting hyperplanes T_1 (at A_1) and T_2 (at A_2) are parallel and the segment A_1A_2 is orthogonal to both T_1 and T_2. We show that T_2 is obtained form T_1 by the t scaling. Let R be the line containing O and A_1; this intersects ∂C_1^t at B. Clearly $|B| = t|A_1|$ and the supporting hyperplane T at B is parallel to T_1 since T is the t scaling of T_1. Then $T \parallel T_2$ and by convexity $T_2 = T$, which in turn implies that T_2 is the t scaling of T_1. If C_1 is strictly convex this would also imply $B = A_2$. Let L be the straight line containing O and perpendicular to T_1 and T_2. Let the intersection of L with ∂C_1^t be C, with T_2 be D and with T_1 be E. Then $|E - D| = |A_1 - A_2| = \operatorname{dist}(\partial C_1^t, \partial C_1)$. Clearly $|D| = t|E|$ and $tl_1 \leq |D|$. The latter follows since D lies on the supporting hyperplane T_2 and $D \notin C_1^t$. Thus $|E-D| = (1-t)|D|/t \geq (1-t)l_1$. A similar argument now proves the statement when $i=2$. $\square$

Define, for $\eta > 0$, $C_{i,\eta} = \{x \in \Gamma : \operatorname{dist}(x, \partial C_i) \leq \eta\}$, $i = 1, 2$. For $Q \in \partial C_2$, let $L = L(Q)$ be the line $\perp$ to ∂C_2, $P = P(Q) = L \cap \partial C_1$. Set $\eta_0 = \delta/10$, where δ is the number in Lemma 5.2, and $\Delta =$diameter(C_1). For every such Q, let $\bar{Q} = \bar{Q}(Q,t) \in \partial C_2^t$ be such that $|Q - \bar{Q}| = \operatorname{dist}(Q, \partial C_2^t)$. We select t such that $\sup_{Q \in \partial C_2} |Q - \bar{Q}| \leq \eta_0$. Also take u_t as in Lemma 5.4.

Lemma 5.6 (Bounds on u and u_t). *Let $t \in (0,1)$ be such that $\partial C_i^t \subset C_{i,\eta_0}$, $i = 1, 2$. Then there exist positive constants η_1 and η_2, depending only on the geometry of Γ, such that*

(i) *$u(x) \leq 1 - \eta_1(1-t)$, for all $x \in \partial C_1^t$ and*
(ii) *$u_t(x) \geq \eta_2(1-t)$, for all $x \in \partial C_2$.*

Proof: To prove (i), we use Lemma 5.1. Let $Q \in \partial C_2$ and select $\bar{Q} \in \partial C_2^t$ closest to Q. The line L containing $\bar{Q}$ and Q is orthogonal to ∂C_2^t. Let $P = L \cap \partial C_1$, $P_t = L \cap \partial C_1^t$; set $R = |Q-P|$ and $R_t = |\bar{Q} - P_t|$. Then $R_t = R + |\bar{Q} - Q| - |P - P_t| \leq R + \eta_0 \leq \Delta + \eta_0$. Applying Lemma 5.1 to $u_t(Q)$, we see

$$u_t(y) \geq 1 - (|Q - P_t|/R_t) = (R_t - |Q - P_t|)/R_t = |Q - \bar{Q}|/R_t.$$

Since Q lies on ∂C_2, by Lemma 5.5, $|Q - \bar{Q}| \geq (1-t)l_2$, and we have

$$u_t(y) \geq \frac{(1-t)l_2}{\Delta + \eta_0} \quad \text{on } \partial C_2.$$

To prove (ii) we use Lemma 5.2. For $x \in \partial C_1^t$, $\delta(x) \geq \operatorname{dist}(\partial C_1, \partial C_1^t) \geq l_1(1-t)$. We use (5.6) to conclude

$$u(x) \leq 1 - \frac{C(\delta, \eta_0)\delta(x)}{2\delta},$$

and this in turn yields, $u(x) + C(\delta)(1-t)l_1 \leq 1$ on ∂C_1^t. $\square$

We now begin our study of the boundary behaviour of ∞-capacitary functions in convex rings. We will utilize the observation in (3.1) in Section 2 and Lemma 3.6. We recall and introduce some notations. For $Q \in \partial C_2$, let $\nu(Q)$ denote the unit outer normal to C_2, and for $A \in \partial C_1$, let $\nu(A)$ stand for the unit outer normal to C_1. For $Q \in \partial C_2$, let $L = L(Q) \ni Q$ be the straight line with $L \perp \partial C_2$. Call $P = P(Q) = L \cap \partial C_1$; for $x \in L \cap \Gamma$, define $d(x) = d(x,Q) = |x - Q|$. For $A \in \partial C_1$, let $B_r(H) = B_r(H, A) \subset \Gamma$ be the interior ball at A, centered at H, and for x on the segment formed by HA, set $\delta(x) = \delta(x, A) = |x - A|$. Note that HA is directed along $\nu(A)$. The following notation is set up for directional derivatives of u along $\nu(Q)$ and along $\nu(A)$. For $Z \in \partial\Gamma$, set

$$\Delta(x, Z) = \Delta(x, \nu(Z)) = \frac{du(x + \theta\nu(Z))}{d\theta}\Big|_{\theta=0}, \quad x \in \Gamma;$$

and when $x = Z$, we write $\Delta(Z) = \Delta(Z, Z) = \Delta(Z, \nu(Z))$, where

$$\Delta(Z) = \begin{cases} \lim_{\theta \to 0^+} \frac{u(Z + \theta\nu(Z)) - u(Z)}{\theta} : & Z \in \partial C_2, \\ \lim_{\theta \to 0^-} \frac{u(Z + \theta\nu(Z)) - u(Z)}{\theta} : & Z \in \partial C_1, \end{cases}$$

whenever they exist. We make an observation before we start. Suppose $A \in \partial C_1$ and T_A is the supporting hyperplane at A. Let $J \in \partial C_2$ be such that the supporting hyperplane $T_J \parallel T_A$, i.e., $\nu(J) = \nu(A)$. This is possible since C_1 and C_2 are both C^2. Recall the definitions of the hyperplanes H_A^+ and H_J^-; see Section 2. Set $G =$

$H_J^- \cap H_A^+$ and define $w_A(x) = 1 + \langle x - A, \nu(A)\rangle/R$, where $R = R(A) = \operatorname{dist}(T_A, T_J)$. Note that convexity of C_1 implies $\langle x - A, \nu(A)\rangle \le 0$, $\forall x \in H_A^+$; it is easily seen that $w_A|_{T_J} = 0$, $w_A|_{T_A} = 1$ and $\Delta_\infty w_A = 0$ in G. Now $u \ge w_A = 0$ on $T_J \cap \Gamma$ and $w_A \le u = 1$ on ∂C_1. Comparison in $G \cap \Gamma$ yields that

$$u(x) \ge w_A(x) = 1 + \frac{\langle x - A, \nu(A)\rangle}{R}, \quad x \in G \cap \Gamma. \tag{5.7}$$

Thus, for $x \in G \cap \Gamma$ with $(x - A) \parallel \nu(A)$, i.e., $x = A - t\nu(A)$, for some $t > 0$, we have

$$1 - u(x) \le \frac{\delta(x)}{R} \ \Rightarrow\ \frac{1 - u(x)}{\delta(x)} \le \frac{1}{R}. \tag{5.8}$$

Theorem 5.7 (A global maximum principle for $|Du|$)**.** *Let u be the ∞-capacitary function in Γ; for $Q \in \partial C_2$, let $d(x)$, $\delta(x)$, $L = L(Q)$ and $P = P(Q)$ be as described above. Then the following are true.*
(a) The normal derivatives of u exist on $\partial\Gamma$, i.e., $\forall A \in \partial C_1$ and $\forall Q \in \partial C_2$,

$$\Delta(A) > 0, \ \ \Delta(Q) > 0, \ \ \textit{and} \ \ \max(\sup_Q \Delta(Q), \ \sup_A \Delta(A)) < \infty.$$

(b) Let $x \in \Gamma$ and $Q \in \partial C_2$ be such that $|x - Q| = \operatorname{dist}(x, \partial C_2)$. If $x_1,\ x_2 \in L(Q)$ are such that $d(x_1) \le d(x_2)$, then

$$0 < \frac{u(x_2) - u(x_1)}{|x_1 - x_2|} \le \frac{u(x_2) - u(Q)}{d(x_2)} \le \frac{u(x_1) - u(Q)}{d(x_1)} \le \Delta(Q).$$

In particular,

$$u(y) < u(ty) \le tu(y), \ \ \forall t \ \textit{ with } \ 1 \le t \le |Q - P|/|x - Q|,$$

where $y = x - Q$ and $u(y)$ stands for the value $u(x)$. Moreover, if the directional derivative of u along L exists at x, then $0 < \Delta(x, Q) \le \Delta(Q)$.
(c) Suppose $A \in \partial C_1$ and $B_r(H)$ is the interior ball at A. Let $x_1,\ x_2 \in HA$, with $x_1 \ne x_2$, and $\delta(x_1) \le \delta(x_2)$. Then

$$\frac{u(x_1) - u(x_2)}{|x_2 - x_1|} \le \frac{u(A) - u(x_1)}{\delta(x_1)} \le \frac{u(A) - u(x_2)}{\delta(x_2)} \le \Delta(A).$$

In particular, if the directional derivative of u along HA exists at x, then $0 < \Delta(x, A) \le \Delta(A)$.
(d) Finally, we have

$$||Du||_{L^\infty(\Gamma)} \le \max\Big(\sup_{Q \in \partial C_2} \Delta(Q), \ \sup_{P \in \partial C_1} \Delta(P)\Big).$$

Proof. *Part (a):* Let $A \in \partial C_1$ and $Q \in \partial C_2$. We first note that $u(Q) = 0$ and $u(A) = 1$. Since C_2 is C^2 and convex, we may find an outer ball $B_r(S) \subset \Gamma$ at Q; note that $(S - Q)/|S - Q| = \nu(Q)$. Recall that $u > 0$ in $B_r(S)$ and so an application of part (a) of Lemma 3.6, yields that for $x \in SQ$,

$$0 < \frac{u(x)}{d(x)} = \frac{u(x) - u(Q)}{|x - Q|} \uparrow \ \text{ as } \ d(x) \downarrow 0, \text{ i.e., } x \to Q. \tag{5.9}$$

Recalling Lemma 5.2, in particular (5.1), we know that $u(x) \le |x - Q|/d$ for an appropriate $D = D(Q)$. Thus

$$0 < \frac{u(S)}{r} \le \frac{u(x)}{d(x)} = \frac{u(x) - u(Q)}{|x - Q|} \le \frac{1}{D}, \ \ \forall x = Q + \theta\nu(Q), \ 0 < \theta \le \min(r, D).$$

Letting $\theta \to 0^+$ yields the result for $\Delta(Q)$. To see the result for $\Delta(A)$, note that $1-u(x) > 0$ in Γ and $1-u(A) = 0$. Let $B_r(H) \subset \Gamma$ be the interior ball at A. Then $(A-H)/|A-H| = \nu(A)$. An application of Lemma 3.6 (a) to $1-u(x)$ in $B_r(H)$, and (5.8) implies that, for $x \in HA$,

$$\begin{gathered} \frac{1-u(x)}{\delta(x)} \uparrow \text{ as } x \to A, \text{ and} \\ 0 < \frac{1-u(H)}{r} \le \frac{1-u(x)}{\delta(x)} = \frac{u(A)-u(x)}{|x-A|} \le \frac{1}{R(A)}. \end{gathered} \tag{5.10}$$

The result for $\Delta(A)$ now follows. Note that $\Delta(Q) \le 1/D(Q)$ and $\Delta(A) \le 1/R(A)$. An inspection of (5.1) and (5.8) shows that the supremum of each of these quantities is also finite.

Part (b): Let Q, P and L be as above; set $r = |P-Q|$ and $w(x) = 1 - |x-P|/r$ in $B_r(P)$. From Lemma 5.1,

$$u(x) \ge w(x) = 1 - \frac{|x-P|}{r} = \frac{|x-Q|}{r} = \frac{d(x)}{r}, \quad x \in L \cap B_r(P) \cap \Gamma.$$

Now let $x_1,\ x_2 \in L \cap \Gamma$ such that $d(x_1) \le d(x_2)$. Then the ball $B = B_{d(x_2)}(x_2) \subset B_r(P)$ and $B \ni x_1$. If we fix x_2 then

$$v(z) = u(x_2)\left(1 - \frac{|z-x_2|}{d(x_2)}\right) \le u(z),\ \forall z \in B \cap \Gamma.$$

To see this, note that (i) $u(z) \ge v(z) = 0$ on $\partial B \cap \Gamma$, (ii) $u(x_2) = v(x_2)$, and (iii) $u(z) = 1 > u(x_2) \ge v(z)$ on $B \cap \partial C_1$; now apply comparison. For z on the segment x_1x_2, $d(z) = d(x_2) - |z-x_2|$; taking $z = x_1$ yields that $u(x_1)/d(x_1) \ge u(x_2)/d(x_2)$. Thus for all $x \in L$, $u(x)/d(x) \uparrow$ as $d(x) \downarrow 0$, i.e., as $x \to Q$. This implies the assertion about scaling. Also

$$\frac{u(x_1)}{d(x_1)} \ge \frac{u(x_2)}{d(x_2)} \ge \frac{u(P)}{R} = \frac{1}{R} \Rightarrow 0 < \frac{u(x_2)-u(x_1)}{|x_2-x_1|} \le \frac{u(x_2)}{d(x_2)} \le \frac{u(x_1)}{d(x_1)}.$$

The positivity follows from Lemma 5.4. By considering x close to Q, applying (5.9) and part (a), we obtain the complete assertion in part (b).

Part (c): We work with $v(x) = 1-u(x)$ and use (5.10) much the way we did in part (b).

Part (d): Let $x \in \Gamma$ and $\mu(x) = \operatorname{dist}(x, \partial\Gamma) = \min(\operatorname{dist}(x,\partial C_1), \operatorname{dist}(x,\partial C_2))$. Thus ball $B_{\mu(x)}(x)$ either touches ∂C_1 or ∂C_2 or both. In the first case, calling the point of tangency as P, $u(P) = 1 = \sup_\Gamma u$. By Lemma 3.6 (d) and part (c) above,

$$|Du|(x) \le D(M) = \Delta(P), \text{ where } M = 1.$$

An analogous situation arises if the second case happens; calling the point of tangency to be Q, we see that from part (b)

$$|Du|(x) \le D(m) = \Delta(Q), \text{ where } m = 0.$$

Clearly, the statement follows from part (a). $\square$

Remark 5.8. The assertion in Theorem 5.7 (b) yields some type of concavity of u along L. Note also that if we take $Q = O$ then, along L, $u(P) = u(|P|x/|x|) \le u(x)|P|/|x|$, implying thereby $u(x) \ge |x|/|P|$. This fact has been derived in Lemma

5.1. The inequalities in Theorem 5.7 (a), clearly imply

$$\begin{aligned} u(x_2) &\leq u(x_1) + \Delta(Q)|x_2 - x_1| \ \Rightarrow\ u(x) \leq \Delta(Q)|x - Q|, \text{ (take } x_1 = Q\text{), and} \\ u(x) &\geq 1 - \Delta(Q)|x - P|, \text{ (take } x_2 = P\text{)}, \ \forall x \in QP. \end{aligned}$$

The latter is similar to Lemma 5.2.

We now prove a global lower bound for u in Γ. This consists in taking the supremum (call it w) of affine functions that lie below u; see (5.7) and (5.8). It turns out that this lower bound is a solution in many special cases. It is not clear whether this is actually a solution in more general situations. We adopt the notations used in (5.7) and (5.8); also see Section 2. Let $P \in \partial C_1$, $R(P)$ and $H_P^{\pm}$, as before. Suppose T_P is the supporting hyperplane at P and $Q \in \partial C_2$ is such that the supporting hyperplane T_Q, at Q, is parallel to T_P. Let $H_Q^{\pm}$ be corresponding the half-spaces. Set $G = H_P^+ \cap H_Q^-$ and $\nu(P)$ is the unit outer normal to ∂C_1 at P.

Lemma 5.9 (Universal lower bound for u)**.** *Let $P \in \partial C_1$, $Q \in \partial C_2$, G and $\nu(P)$ as described above. Set*

$$w_P(x) = 1 + \frac{\langle x - P, \nu(P)\rangle}{\delta(Q)}, \ \ \forall x \in \Gamma \ \text{ and } \ w(x) = \sup_{P \in \partial C_1} w_P(x).$$

Then $w|_{\partial C_1} = 1$, $w|_{\partial C_2} = 0$, w is ∞-subharmonic and $u(x) \geq w(x)$, $x \in \Gamma$.

Proof: From (5.7), we see that $w_P(x) \leq u(x)$. Clearly then $w(x) \leq u(x)$ and it is well known that $w(x)$ is ∞-subharmonic. Note that $w(x) = w_P(x) = 1 - |x - P|/|P - Q|$ along the segment PQ. It is also to be noted that in case Γ is a spherical annulus or more generally if the geometry of Γ is such that $C_1 = \cup_{x \in C_2} B_r(x) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, C_2) < r\}$, for some $r > 0$, then $u(x) = w(x)$. As a matter of fact $u(x) = \operatorname{dist}(x, \partial C_2)/r$. □

Remark 5.10. Let $P \in \partial C_1$ and $Q \in \partial C_2$ be such that $|P - Q| = \operatorname{dist}(\partial C_1, \partial C_2) = \delta$. Clearly the smallest ball in Γ, that touches both ∂C_1 and ∂C_2, has radius $\delta/2$. From Lemma 5.1 and (5.1) ($2l = \delta$), it follows that $u(x) \leq |x - Q|/\delta$ and $u(x) \geq 1 - (|x - P|/\delta)$. Note that the segment PQ is orthogonal to both ∂C_1 and ∂C_2. It then follows that u is linear on PQ and $u(x) = |x - Q|/\delta$, $\forall x \in PQ$.

6. Proof of Theorem 2.2

(a) Proof of part A of Theorem 2.2: Star-shapedness of level sets $\{u = t\}$ and cone condition.

For $0 < t < 1$, let $\Gamma_t = \{x \in \Gamma : u(x) < t\}$, then $\partial\Gamma_t = \partial C_2 \cup \{u(x) = t\}$. This follows from Lemma 5.4, since any $x \in \Gamma$, with $u(x) = t$, may be approached by points where $u(x) < t$. This is seen by considering the straight line containing x and a point in C_2. Also at x, there are two cones with the apex at x such that $u > t$ in one and $u < t$ in the other. Thus $\{x \in \Gamma : u(x) = t\}$ satisfies an interior and an exterior cone condition. The shape of the cones depend on the geometry of Γ. This also implies the set $\{u(x) = t\}$ is locally Lipschitz. It is also clear that Γ_t is star-shaped with respect to any point in C_2.

(b) Proof of part B of Theorem 2.2: Strict positivity of the difference quotient and $|Du|$.

We employ the idea of Lemma 5.4 again. The selection of $O \in C_2$ will influence the lower bound λ, but it will stay positive. We recall the notations and the scaling in Lemma 5.4. We consider P_0 and P_1, in Γ, such that $|P_0| = |P_0 - O| < |P_1 - O| = |P_1|$

and $t = |P_0|/|P_1| < 1$. We select t close to 1. Note that $u_t(y)$ is ∞-harmonic in $\Gamma^t = \Gamma(C_1^t, C_2^t)$ and

$$C_i^t \subset C_i,\ i = 1, 2,\ \ u_t|_{\partial C_1^t} = 1 \ \text{ and } \ u_t|_{\partial C_2^t} = 0.$$

From Lemma 5.6 there exists a positive η, depending only on the geometry, such that

$$u|_{\partial C_1^t} + \eta(1-t) \le 1 = u_t|_{\partial C_1^t} \ \text{ and } \ u|_{\partial C_2} = 0 \le \eta(1-t) + u|_{\partial C_2} \le u_t|_{\partial C_2}.$$

Thus $u_t \ge u + \eta(1-t)$ on $\partial(\Gamma^t \cap \Gamma)$. Thus by comparison $u_t(y) \ge u(y) + \eta(1-t)$ in $\Gamma^t \cap \Gamma$. Recalling that $P_0 = tP_1$ and $u_t(y) = u(y/t)$, it is seen that

$$\frac{u_t(P_0) - u(P_0)}{|P_1 - P_0|} = \frac{u(P_1) - u(P_0)}{|P_1 - P_0|} \ge \frac{\eta(1-t)}{(1-t)|P_1|} = \frac{\eta}{|P_1|} \ge \frac{\eta}{\Delta} > 0,$$

where Δ is the diameter of C_1. The result follows.

7. Appendix

In this section we put together results needed in the proofs of the theorems. We will show that odd reflections of ∞-harmonic functions stay ∞-harmonic and also include the proof of Theorem 1.1 [9].

In the proof of Theorem 2.1, we required the the following result which we now prove. Let F be the $n-1$ dimensional hyperplane given by $x_n = 0$. Let us write $x = (\xi, x_n)$, where $\xi = \xi(x) = (x_1, \dots, x_{n-1})$; also take

$$B^+ = B_R^+(O) = \{x \in B_R(O) :\ x_n > 0\},\ B^- = B_R^-(O) = \{x \in B_R(O) :\ x_n < 0\}$$

and $F_R = F \cap B_R(O)$.

Proposition 7.1 (Odd reflection of u)**.** *Let F and O be as above and u be ∞-harmonic in B^+; also assume that u vanishes continuously on F. Define*

$$v(x) = v(\xi(x), x_n) = \begin{cases} u(\xi(x), x_n) : & x_n \ge 0 \\ -u(\xi(x), -x_n) : & x_n \le 0. \end{cases}$$

Then v is ∞-harmonic in $B_R(O)$.

Proof: We will show that $\Delta_\infty v = 0$ in the viscosity sense. Let $\psi \in C^2$ and $P \in B_R(O)$ be such that $v - \psi$ attains a local minimum at P. We show that $\Delta_\infty \psi(P) \le 0$. We will concern ourselves with the cases when $P \in B^-$ and when $P \in F_R$.

Case A ($\mathbf{P \in B^-}$): Since $v(x) - \psi(x) \ge v(P) - \psi(P)$, it follows that $u(\xi, -x_n) - \phi(\xi, -x_n) \le u(\xi(P), -P_n) - \phi(\xi(P), -P_n)$, where $\phi(\xi, -x_n) = -\psi(\xi, x_n)$. Clearly $\Delta_\infty \phi(\xi(P), -P_n) \ge 0$, since $(\xi(P), -P_n) \in B^+$ is a point of local maximum of $u - \phi$. Clearly then $\Delta_\infty \psi(P) \le 0$.

Case B ($\mathbf{P \in F_R}$): Note $v(P) = v(\xi, 0) = 0$. Thus $v(x) \ge \psi(x) - \psi(P)$, which in turn implies

$$v(x) \ge \langle D\psi(P), x - p\rangle + \frac{1}{2}\langle D^2\psi(P)(x-P), x-P\rangle + o(|x-P|^2),\ \ x \to P. \quad (7.1)$$

We study various situations. Suppose that $x \in F_R$, i.e., $x_n = x_n - P_n = 0$. Since $v(\xi, 0) = 0$, (7.1) implies

$$0 \ge \sum_{i=1}^{i=n-1} D_i\psi(P)(x-P)_i + \frac{1}{2}\sum_{i,j=1}^{n-1} D_{ij}\psi(P)(x-P)_i(x-P)_j + o(|\xi(x-P)|^2),$$

$x \to P$. Now select x such that $(x-P)_i = t$ and $(x-P)_j = 0$, $j = 1, \dots, n-1$, $j \neq i$. Then

$$0 \geq t D_i\psi(P) + \frac{t^2}{2} D_{ii}\psi(P) + o(t^2), \quad t \to 0.$$

Since this holds for all $t \in (-\varepsilon, \varepsilon)$ for small $\varepsilon > 0$, it follows that $D_i\psi(P) = 0$, $i = 1, \dots, n-1$. From (7.1), it follows that

$$v(x) = v(\xi, x_n) \geq D_n\psi(P)(x-P)_n + \frac{1}{2}\sum_{i,j=1}^{n} D_{ij}\psi(P)(x-P)_i(x-P)_j + o(|x-P|^2),$$

$x \to P$, and $\Delta_\infty\psi(P) = (D_n\psi(P))^2 D_{nn}\psi(P)$. To prove this is non-positive, we consider x's such that $\xi(x) = \xi(P)$ (i.e., $x_i = P_i$, $i = 1, \dots, n-1$) and $x_n = \pm t$, for small t. Let $t > 0$, then the above inequality for ψ yields

$$v(\xi(P), t) = u(\xi(P), t) \geq t D_n\psi(P) + \frac{t^2}{2} D_{nn}\psi(P) + o(t^2),$$
$$v(\xi(P), -t) = -u(\xi(P), t) \geq -t D_n\psi(P) + \frac{t^2}{2} D_{nn}\psi(P) + o(t^2),$$

as $t \to 0$. Adding the two inequalities, dividing by t^2 and letting $t \to 0$, we obtain that $D_{nn}\psi(P) \leq 0$. Thus $\Delta_\infty\psi(P) \leq 0$. The case of local maximum may be handled analogously. □

Proof of Theorem 1.1 [9] For easy reference, we now include the proof of Theorem 1.1 in [9], as applied to our situation. This is essentially a repetition of the proof in [9], nonetheless we provide details.

Theorem 7.2 (Boundary Harnack Principle)**.** *Let $A_8 = \{x : |\xi(x)| < 8,\ 0 < x_n < 16\}$, $A_1 = \{x : |\xi(x)| < 1,\ 0 < x_n < 2\}$ and $X_0 = (0, 1)$. Let $u > 0$ be ∞-harmonic in A_8. Then there exists a constant C, independent of u but depending on the geometry, such that $\sup_{A_1} \leq Cu(X_0)$.*

Proof. Recall (4.2) from the proof of Theorem 2.1 and (3.6). Let us continue to call u the extended function obtained by the odd reflection about F_8. Let us note that Lemma 3.4 continues to apply to this extended function. Clearly then u is Lipschitz continuous in any sub-cylinder of A_8. Our selection of X_0 is different from z. This means the Harnack constant M will need modification (see (4.2)). For x with $1 < x_n < 3$ and $|\xi(x)| \leq 2$, $\operatorname{dist}(x, F_8) \geq 1$ and $\operatorname{dist}(x, X_0) \leq 2\sqrt{2}$. This implies $u(x) \leq e^{2\sqrt{2}/1}u(X_0)$ by Lemma 3.2. Take $M = e^{2\sqrt{2}}$ in (4.2). The letters l, m and k denote positive integers. Rewritten

$$u(x) = u(\xi(x), x_n) \leq \begin{cases} Mu(\xi(x), 2x_n) : & |\xi(x)| \leq 2,\ 0 < x_n < 3/2, \\ Mu(X_0) : & |\xi(x)| \leq 2,\ 1 < x_n < 3. \end{cases} \tag{7.2}$$

We argue by contradiction. Suppose that there is a $Y_0 \in A_1$ such that $u(Y_0) \geq M^{l+2}u(X_0)$, where l is large and its value will be determined later in the proof. We now make an observation which will be used repeatedly in the proof. If $x \in \bar{A}_1$ is such that $\operatorname{dist}(x, F_8) = x_n \geq 2^{-l}$ then $u(x) \leq M^{l+1}u(X_0)$. This follows by an application of (7.2). If $x_n \geq 1$ then (7.2) implies the result. If $0 < x_n < 1$ and s is the smallest integer such that $2^s x_n \geq 1$, then (7.2) implies

$$u(x) = u(\xi(x), x_n) \leq Mu(\xi(x), 2x_n) \leq \dots \leq M^s u(\xi(x), 2^s x_n) \leq M^{s+1}u(X_0). \tag{7.3}$$

Since $l \geq s$, $u(x) \leq M^{l+1}u(X_0)$. It follows from (7.3) that

$$\operatorname{dist}(Y_0, F_8) \leq 2^{-l}.$$

Let $K(r, Z) = K_r(Z)$ be the cylinder of dimension r and center Z (see Section 2) and $\nu(Z, r) = \operatorname{osc}_{K(r,Z)} u$. Recall from Remark 3.5 that $\nu(Z, r) \geq C\nu(Z, r/2)$. We consider $K(r, Y_0)$. Clearly $\bar{K}(Y_0, 2^{-l}) \cap F_8 \neq \emptyset$ and $\nu(Y_0, 2^{-l}) \geq u(Y_0)$. It follows then

$$\nu(Y_0, 2^{-l+m}) \geq C^m \nu(Y_0, 2^{-l}) \geq C^m M^{l+2} u(X_0).$$

Choose m so that $C^m \geq 2M^2$. Thus $\nu(Y_0, 2^{-l+m}) \geq 2M^{l+4}u(X_0)$. Noting u has been extended as an odd function about F_8, there is a Y_1 such that

$$Y_1 \in K(Y_0, 2^{-l+m}) \cap \{x_n > 0\} \text{ and } u(Y_1) \geq M^{l+4}u(X_0).$$

Thus $\operatorname{dist}(Y_1, F_8) \leq 2^{-l-2}$ (if not, an argument along the lines of (7.3) will imply $u(Y_0) \leq M^{l+3}$) and $\nu(Y_1, 2^{-l-2+m}) \geq C^m \nu(Y_1, 2^{-l-2}) \geq C^m u(Y_1) \geq 2M^{l+6}u(X_0)$. Once again there exists a

$$Y_2 \in K(Y_1, 2^{-l-2+m}) \cap \{x_n > 0\} \quad \text{and} \quad u(Y_2) \geq M^{l+6}u(X_0).$$

Again $\operatorname{dist}(Y_2, F_8) \leq 2^{-l-4}$ and $\nu(Y_2, 2^{-l-4+m}) \geq C^m \nu(Y_2, 2^{-l-4}) \geq 2M^{l+8}u(X_0)$. We obtain by induction a sequence of points $\{Y_k\}$ such that

$$\begin{gathered} \operatorname{dist}(Y_k, F_8) \leq 2^{-l-2k}, \ \ Y_k \in K(Y_{k-1}, 2^{-l-2(k-1)+m}) \cap \{x_n > 0\}, \\ u(Y_k) \geq M^{l+2(k+1)}u(X_0). \end{gathered} \tag{7.4}$$

Recalling that $K(Y_{k-1}, 2^{-l-2(k-1)+m})$ is a cylinder with center Y_{k-1} with radius $2^{-l-2(k-1)+m}$ and long axis $2(2^{-l-2(k-1)+m})$,

$$|Y_k| \leq |Y_{k-1} - Y_k| + |Y_{k-1}| \leq 2^{-l-2(k-1)+m+1} + |Y_{k-1}|.$$

Thus $|Y_k| \leq 2^{-l+m+1}\sum_{j=1}^{k} 2^{-2(j-1)} + |Y_0| \leq 2^{-l+m+1}\sum_{j=0}^{k} 2^{-2j} + 1$. Now choose l so large that $|Y_k| \leq 3/2$, $\forall k$. Thus if $Y \in K(Y_k, 2^{-l-2k+m})$ then $|Y| \leq 3$. Thus each $K(Y_k, 2^{-l-2k+m})$ lies in a fixed sub-cylinder of A_8. Letting $k \to \infty$ results in a contradiction.

The above proves that for some constant $C > 0$, $u(x) \leq Cu(X_0) \leq CMu(z)$. This completes the proof.

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