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INTEGRODIFFERENTIAL EQUATIONS OF MIXED TYPE ON TIME SCALES WITH Δ-HK AND Δ-HKP INTEGRALS

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Abstract. In this article we prove the existence of solutions to the integrodifferential equation of mixed type

$$x^{\Delta}(t) = f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big),$$
$$x(0) = x_0, \quad x_0 \in E,\ t \in I_a = [0,a] \cap \mathbb{T},\ a > 0,$$

where $\mathbb{T}$ denotes a time scale (nonempty closed subset of real numbers $\mathbb{R}$), I_a is a time scale interval. In the first part of this paper functions f, g, h are Carathéodory functions with values in a Banach space E and integrals are taken in the sense of Henstock-Kurzweil delta integrals, which generalizes the Henstock-Kurzweil integrals. In the second part f, g, h, x are weakly-weakly sequentially continuous functions and integrals are taken in the sense of Henstock-Kurzweil-Pettis delta integrals. Additionally, functions f, g, h satisfy some boundary conditions and conditions expressed in terms of measures of noncompactness.

1. Introduction

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. Thus, $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ and the Cantor set are the examples of time scales, while $\mathbb{Q}$ and $(0;1)$ are not time scales. Time scales (or a measure chain) was introduced by Hilger in his Ph.D. thesis in 1988, [25]. It was met with big interest of scientists from various disciplines, including pure and applied mathematics, biology, economics, engineering and physics. It is applicable in any field that requires simultaneous modeling of discrete and continuous data. For example standard economic models are continuous models (described by differential equations) and discrete models (described by difference equations). These two types of models require different techniques of proofs. An example might be the Ramsey model-macroeconomic growth model that explores the relationship between consumption and capital. In a discrete model, a consumer receives some income in a time period and decides how much to consume and save during the

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same period. All decisions are assumed to be made at evenly spaced intervals. Discrete model is

$$\sum_{t=0}^{T-1}(1+p)^t U(C_t) \to \max, \quad C_t = W_t - \frac{W_{t+1}}{1+r},$$

where C_t is consumption, $p(t)$ is the discount rate, W_t is production function, U_t is instantaneous product. The corresponding continuous model can be written in the form

$$\int_0^T e^{-pt}U(C(t))dt \to \max, \quad C(t) = rW(t) - W'(t)$$

But there are economic situations, which these models do not include. For example, if the income is not regular (consumer receives income sometimes once a month, sometimes once a quarter or more per week). The introduction of the time scale $\mathbb{R}$ enables description these three situations with a single model [7]. The most important advantage of a time scale is that it provides not only a unified approach to study the discrete intervals with uniform step size the lattice hZ, continuous intervals and discrete intervals with non-uniform (variable) step size (for instance X-numbers), but also, more interestingly, it gives an opportunity to extend the approach to study the combination of continuous and discrete intervals. Therefore, the concept of time scale can be build bridges between the continuous, discrete and X-discrete analysis. While some of the results for difference equations move quite easily to the corresponding results for differential equations, others seem to be completely different for their continuous counterparts. We consider the differential equation

$$y''(x) + y(x) = 0$$

whose solution has a form $y(x) = c_1 \cos x + c_2 \sin x$. Each specific solution is a continuous periodic function, so bounded.

The corresponding difference equation is

$$\Delta^2 y_n + y_n = 0$$

whose solution is a class of sequences

$$y_n = c_1 2^{n/2} \cos(n\pi/4 + c_2).$$

Putting $c_1 = -1$, $c_2 = \pi/2$, we obtain the specific solution

$$y_n = 2^{n/2} \cos(n\pi/4).$$

For $n = 8k+1$ we have $y_{8k+1} = 2^k \to \infty$, if $k \to \infty$. This solution is not periodic and is not bounded.

In a study of dynamic equations on time scale we do not have to prove some theorems twice, once for differential equations, and again for difference equations. The general idea is to prove results for the dynamic equation, where the domain of unknown function is called time scale, i.e. any nonempty closed subset of real numbers. The dynamic equations on time scale have numerous applications in many fields of science, such as mechanics, electrical engineering, neural networks and combinatorics [15, 27]. In particular, dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population [9]. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [49] discusses several possible applications. Since then several authors have expounded on various aspects of this new theory [10]. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [9] summarizes and organizes much of time scale calculus.

This article is divided into two main sections. In Section 1, we prove some existence theorem for the integrodifferential equation of mixed type

$$\begin{gathered} x^{\Delta}(t) = f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big), \\ x(0) = x_0, \quad x_0 \in E,\ t \in I_a = [0,a] \cap \mathbb{T},\ a > 0, \end{gathered} \tag{1.1}$$

where E is a Banach space with the norm $\|\cdot\|$, $\mathbb{T}$ denotes a time scale, $0 \in \mathbb{T}$, I_a denotes a time scale interval and integrals are taken in the sense of HL Δ integral. Moreover f, g, h, x are functions with values in a Banach space E, and k_j, $j = 1, 2$ are real-valued functions.

In Section 2, we prove some existence theorems for problem (1.1), where f, g, h, x are functions with values in a Banach space E, weakly-weakly sequentially continuous, and k_j, $j = 1, 2$ are real-valued functions. The integrals are taken in the sense of Henstock-Kurzweil-Pettis Δ-integrals.

As it is know, ordinary integrodifferential equations, an extreme case of integrodifferential equations on time scales, find many applications in various mathematical problems: see Corduneanu's book [15] and references therein for details. In addition, the existence of extremal solutions of ordinary integrodifferential equations and impulsive integrodifferential equations have been studied extensively in [4], [17]-[21], [35]-[37], [40, 41, 45, 46, 48],[53]-[56]. In [56] the authors extended such results to the integrodifferential equations on time scales and therefore obtained corresponding criteria which can be employed to study the difference equation of Volterra type [28, 54], q difference equations of Volterra type, etc. In [57] the authors proved a new comparison result and developed the monotone iterative technique to show the existence of extremal solutions of the periodic boundary value problems of nonlinear integrodifferential equation on time scales. In [45] authors prove the existence theorem of solution for integrodifferential equation

$$\begin{gathered} x'(t) = f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))ds, \int_0^a k_2(t,s)h(s,x(s))ds\Big), \\ x(0) = x_0, x_0 \in E, t \in I_a = [0,a], a > 0, \end{gathered}$$

with Henstock-Kurzweil type of integrals which encompasses the Newton, Riemann and Lebesgue integrals [24, 34]. Additionally functions satisfy some conditions expressed in terms of the measure of noncompactness.

In this article we extend this result proving some existence theorem for this problem on time scales. In this paper we will use a new type of integrals on time scales (the Henstock-Kurzweil delta integral, HL delta integral, Henstock-Kurzweil-Pettis delta integral), which lets us consider the wider class of the function than so far. The Henstock-Kurzweil delta integral contains the Riemann delta, the Lebesgue delta and the Bochner delta integrals as special cases. These integrals will enable time scale researchers to study more general dynamic equations. Petterson and Thomson [42] showed that there are highly oscillatory functions that are not delta integrable on a time scale, but are the Henstock-Kurzweil delta integrable.

Let us remark that the existence of the Henstock-Kurzweil integral over $[a, b]$ implies the existence of such integrals over all subintervals of $[a, b]$ but not for all measurable subsets of this interval, so the theory of such integrals on $\mathbb{T}$ does not follow from general theory on $\mathbb{R}$.

Cichoń [13] introduced a definition of the Henstock-Kurzweil delta integral (Δ-HK integral) and HL delta integral (Δ-HL integral) on Banach spaces for checking the existence of solutions of differential (or: dynamic) equations in Banach spaces. He presented also a new definition of the Henstock-Kurzweil-Pettis delta integral on time scales. The study for weak solutions of Cauchy differential equations in Banach spaces was initiated by Szep [50] and theorems on the existence of weak solutions of this problem were proved by Cramer et al. [16], Kubiaczyk [30], Kubiaczyk and Szufla [31], Mitchell and Smith [38], Szufla [52], and Cichoń and Kubiaczyk [14]. There are also some existence theorems for the Volterra and Urysohn integral equations [32] on time scales. Similar methods for solving existence problems for difference equations in Banach spaces equipped with its weak topology were studied, for instance, in [3]. We will unify both cases and using the weak topology, we will obtain the result for pseudosolutions of an integrodifferential dynamic problem. (This is new also for q-difference equations). Our result extends the existence of pseudosolutions not only to the discrete intervals with uniform step size (hZ) but also to the discrete intervals with nonuniform step size (K_q).

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be the dual space. Denote, by $C(I_a, E)$, the set of all continuous bounded functions from I_a to E endowed with the topology of almost uniform convergence (i.e. uniform convergence on each closed bounded subsets of I_a). Moreover, let $(C(I_a, E), \omega)$ denote the space of all continuous functions from I_a to E endowed with the topology $\sigma(C(I_a, E), C(I_a, E)^*)$ and by $C_{rd}(I_a, E)$ denote the space of all rd-continuous functions from the time scale interval I_a to E. By μ_Δ we denote the Lebesgue measure on $\mathbb{T}$. For a precise definition and basic properties of this measure we refer the reader to [11]. This part is divided into three sections.

(I) To let the reader understand the so-called dynamic equations and follow this paper easily, we present some preliminary definitions and notations of time scales which are very common in the literature (see [1, 2, 9, 10], [22]-[26], [33, 34, 44] and references therein). A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. By an interval we mean the time scale interval

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

We put $\inf \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called right dense, right scattered, left dense, left scatered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) <= t$, $\rho(t) = t = \sigma(t)$, $\rho(t) < t < \sigma(t)$ respectively.

Definition 2.2. We say that is right-dense continuous (rd - continuous) if k is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s\to t^-} k(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. Next, we define the so-called Δ-derivative.

Definition 2.3. *Fix $t \in \mathbb{T}$. Let $f : I_a \to E$. Then we define Δ-derivative of f by*

$$\lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

The function f is called Δ-differentiable on $\mathbb{T}$, if for each $t \in \mathbb{T}$ there exists $f^{\Delta}(t)$. Note that

(1) *$f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,*
(2) *$f^{\Delta} = \Delta f$, is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$,*
(3) *$f^{\Delta} = D_q f$ is the q-derivative if $\mathbb{T} = q^{N_0} = \{q^t : t \in N_0, 0 < q \le 1\}$.*

Hence, the time scale allows us to consider the unification of differential, difference and q-difference equations as particular cases (but our results hold also for more exotic time scales which appear in mathematical biology or economics cf. [9, 10], for instance).

(II) As in classical case ([12] and [47] for real valued functions), we need to introduce of vector valued Henstock-Kurzweil Δ-integrals and HL Δ-integrals. Definitions and basic properties of non absolute integrals (HK Δ-integral and HL Δ-integral) were presented in [13]. We will use the notation $\eta(t) = \sigma(t) - t(t)$ where η is called the graininess function and $v(t) = t - \rho(t)$, where v is called the left - graininess function.

We say that $\delta = (\delta_L, \delta_R)$ is a Δ-gauge for time scale interval $[a, b]$ provided $\delta_L(t) > 0$ on $(a, b], \delta_R(t) > 0$ on $[a, b)$, $\delta_L(t) \ge 0$, $\delta_R(t) \ge 0$ and $\delta_R(t) \ge \eta(t)$ for all $t \in [a, b)$.

We say that a partition D for a time scale interval $[a, b]$ given by

$$D = \{a = t_0 \le \xi_1 \le t_1 \le \cdots \le t_{n-1} \le \xi_n \le t_n = b\}$$

with $t_i > t_{i-1}$, for $1 \le i \le n$ and $t_i, \xi_i \in \mathbb{T}$ is δ-fine if $\xi_i - \delta_L(\xi_i) \le t_{i-1} < t_i \le \xi_i + \delta_R(\xi_i)$, for $1 \le i \le n$.

Definition 2.4 ([13])**.** A function $f \in [a, b] \to E$ is the Henstock-Kurzweil Δ-integrable on $[a, b]$ (HK Δ-integrable in short) if there exists a function $F \in [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $D = \{[u, v], \xi\}$ is δ-fine division of a $[a, b]$, we have

$$\| \sum_{D} f(\xi)(v - u) - (F(v) - F(u))\| < \epsilon$$

Definition 2.5 ([13])**.** A function $f \in [a, b] \to E$ is the Henstock-Lebesgue Δ-integrable on $[a, b]$ (HL Δ-integrable in short) if there exists a function $F \in [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $D = \{[u, v], \xi\}$ is δ-fine division of a $[a, b]$, we have

$$\sum_{D} \|f(\xi)(v - u) - (F(v) - F(u))\| < \epsilon$$

Definition 2.6. The function $f \in [a, b] \to E$ is Henstock-Kurzweil-Pettis Δ-integrable (HKP Δ-integrable for short) if

(1) $\forall x^* \in E^*, x^*f$ is Henstock-Kurzweil Δ-integrable on I_a,
(2) $\forall t \in I_a \forall x^* \in E^*, x^*g(t) = (\Delta - HK)\int_0^t x^*f(s)\Delta s$.

The function g will be called a primitive of f and by $g(t) = (\Delta - HK)\int_0^t f(s)\Delta s$ we will denote the Henstock-Kurzweil-Pettis Δ-integral of f on the interval I_a.

In [14] the author give examples of Henstock-Kurzweil-Pettis Δ-integrable functions which are not integrable in the sense of Pettis and Henstock-Kurzweil on time scales.

Remark 2.7. We note that by the triangle inequality if f is HL δ-integrable, it is also HK δ- integrable. In general, the converse is not true. For real-valued functions the two integrals are equivalent. It is well known that Henstock's Lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach space valued functions, Cao [12] pointed out that Henstock's Lemma holds for the case of finite dimension, but it does not always hold for the case of infinite dimension. In this paper we will use the definition of HL δ-integral which satisfies Henstock's Lemma.

Theorem 2.8 (Henstock's Lemma). *If f is the Henstock-Kurzweil δ-integrable on $[a,b]$ with primitive F, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-fine division of $[a,b]$ we have*

$$\sum_D |f(\xi)(v-u) - (F(v) - F(u))| < \epsilon.$$

Theorem (2.8) says that in the definition of the Henstock-Kurzweil delta integral for real valued functions [47], we may put the absolute value sign $|\cdot|$ inside the summation sign $\sum$. We know from [12] that this is no longer true if we replace $|\cdot|$ with $\|\cdot\|$, i.e., Henstock's Lemma is not satisfied by Henstock-Kurzweil integrable Banach valued functions. By the definition of HL integral, an HL integrable function with primitive F satisfies Henstock's Lemma with $|\cdot|$ replaced with $\|\cdot\|$.

Theorem 2.9 ([13]). *If $f : [a,b] \to E$ is HL Δ-integrable, then function $F(t) = (\Delta - HL)\int_0^t f(s)\Delta s$ is continuous at each point $t \in \mathbb{T}$. Moreover, for every point t of the continuity of f we have $F^\Delta(t) = f(t)$.*

Theorem 2.10 ([13]). *Suppose that $f_n : [a,b] \to E$, $n = 1, 2, \dots$ is a sequence of HL Δ-integrable functions satisfying the following conditions:*

(1) *$f_n(x) \to f(x)\mu_\Delta$ almost everywhere in $[a,b]$, as $n \to \infty$;*
(2) *the set of primitives $\{F_n(t)\}$, of f_n, where $F_n(t) = \int_a^t f_n(s)\Delta s$ is uniformly ACG_* in n;*
(3) *the primitives F_n are equicontinuous on $[a,b]$;*

then, f is HL Δ-integrable on $[a,b]$ and $\int_a^t f_n \to \int_a^t f$, μ_Δ uniformly on $[a,b]$, as $n \to \infty$.

The proof is similar to that of [34, Theorem 7.6], see also [47, Theorem 4].

Theorem 2.11. *Suppose that $f, f_n : [a,b] \to E, n = 1, 2, \dots$ are HKP Δ-integrable functions. Let F_n be a primitive of f_n. If one assumes that:*

(1) *$\forall x^* \in E^*, x^*f_n(x) \to x^*f(x)$ μ_Δ almost everywhere on I_a*

(2) *$\forall x^* \in E^*$ the family $G = \{x^*F_n : n = 1, 2, \dots\}$ is uniformly ACG^* on I_a (i.e., weakly uniformly ACG^* on I_a),*
(3) *$\forall x^* \in E^*$ the set G is equicontinuous on I_a*

then f is Δ-HKP integrable on I_a and $\int_0^t f_n(s)\Delta s$ tends weakly in E to $\int_0^t f(s)\Delta s$ for each $t \in I_a$.

Theorem 2.12 (Mean Value Theorem [13])**.** *For each Δ-subinterval $[c, d] \subset [a, b]$, if the integral $\int_c^d y(s)\Delta s$, exists, then we have*

$$\int_c^d y(s)\Delta s \in \mu_\Delta([c, d])\overline{\operatorname{conv}} y([c, d]),$$

where $\overline{\operatorname{conv}} y([c, d])$ denotes the close convex hull of the set $y([c, d])$, and integral is taken in the sense of Δ-HK or Δ-HKP.

Theorem 2.13 (Gronwall's inequality [5])**.** *Suppose that $u, g, h \in C_{rd}(I_a, E)$ and $h \geq 0$. Then*

$$u(t) \leq g(t) + \int_0^t h(\tau)u(\tau)\Delta\tau, \quad \text{for each } t \in I_a$$

implies

$$u(t) \leq (g(t) + \int_0^t h(\tau)u(\tau)\Delta\tau) \exp\Big(\int_0^t h(\tau)\Delta\tau\Big), \quad \text{for each } t \in I_a$$

(III) Our fundamental tools are the Kuratowski measure of noncompactness $\alpha(A)$ and the deBlasi measure of weak noncompactness $\beta(A)$.

For any bounded subset A of E, we denote by $\alpha(A)$ the Kuratowski measure of noncompactness of A, that is, the infimum of all $\epsilon > 0$, such that there exists a finite covering of A by sets of diameter smaller than ϵ.

The deBlasi measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\}$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

The properties of the measure of noncompactness $\alpha(A)$ are as follows:

(i) if $A \subset B$ then $\alpha(A) \leq \alpha(B)$;
(ii) $\alpha(A) = \alpha(A)$, where A denotes the closure of A;
(iii) $\alpha(A) = 0$ if and only if A is relatively compact;
(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $(\lambda \in R)$;
(vi) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
(vii) $\alpha(\operatorname{conv}(A)) = \alpha(A)$, where $\operatorname{conv}(A)$ denotes the convex extension of A.

The properties of the weak measure of noncompactness β are analogous to the properties of the measure of noncompactness $\alpha(A)$ [8]. We now gather some well-known definitions and results from the literature, which we will use throughout this paper. The lemma below is an adaptation of the corresponding result of Ambrosetti [6]

Theorem 2.14 ([29])**.** *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let $H(t) = \{h(t) \in E, h \in H\}$, for $t \in I_a$ and $H(I_a) = \cup_{t \in I_a} H(t)$. Then*

$$\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a)),$$

where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$, and the function $t \mapsto \alpha(H(t))$ is continuous.

Definition 2.15. A function $f : I_a \times E \to E$, where E is a Banach space, is L^1-Carathéodory, if the following conditions hold:

(i) the map $s \to f(s, x)$ is μ_Δ-measurable for all $x \in E$;
(ii) the map $x \to f(s, x)$ is continuous for almost all $s \in I_a$.

Definition 2.16. A function $f : I_a \times E \times E \times E \to E$, where E is a Banach space, is L^1- Carathéodory, if the following conditions hold:

(i) the map $s \to f(s, x, y.z)$ is μ_Δ-measurable on I_a for all $(x, y, z) \in E^3$;
(ii) the map $(x, y, z) \to f(s, x, y, z)$ is continuous for almost all $s \in I_a$.

Definition 2.17. A function $f : I_a \to E$ is said to be weakly continuous if it is continuous from I_a to E endowed with its weak topology. A function $g : E \to E_1$ where E and E_1 are Banach spaces, is said to be weakly-weakly sequentially continuous if for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we will write $x_n \to^\omega x_0$.

Definition 2.18 ([23])**.** A family F of functions F is said to be uniformly absolutely continuous in the restricted sense on A or in short uniformy $AC_*(A)$, if for every $\epsilon > 0$ there is $\eta > 0$, such that for every F in F and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in A$, and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \epsilon$ where ω denotes the oscillation of F over $[a_i, b_i]$.

A family F of functions F is said to be uniformly generalized absolutely continuous in the restricted sense on $[a, b]$ or uniformly $ACG_*([a, b])$ if is the union of a sequence of closed sets A_i such that on each A_i the function F is uniformy $AC_*(A_i)$. In the proof of the main theorem we will apply the following fixed point theorem.

Theorem 2.19 ([39])**.** *Let D be a closed convex subset of E, and let F be a continuous map from D into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively compact,} \tag{2.1}$$

holds for every countable subset V of D, then F has a fixed point.

In Section 2 we will apply the following theorem.

Theorem 2.20 ([35])**.** *Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a weakly-weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication that*

$$\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively weakly compact,} \tag{2.2}$$

holds for every subset V of D, then F has a fixed point.

3. An existence result for integrodifferential equations

Now we will consider the equivalently integral problem

$$\begin{aligned} x(t) &\\ &= x_0 + \int_0^t f(z, x(z), \int_0^z k_1(z,s) g(s, x(s))\Delta s, \int_0^a k_2(z,s) h(s, x(s))\Delta s)\Delta z, \end{aligned} \tag{3.1}$$

for $t \in I_a$, where $f : I_a \times E \times E \times E \to E, h, g : I_a \times E \to E$, k_j, $j = 1, 2$ are real-valued functions, $\mathbb{T}$ denotes a time scale (nonempty closed subset of real numbers $\mathbb{R}$), $0 \in \mathbb{T}$, I_a denotes a time scale interval, $(E, \|\cdot\|)$ is a Banach space and integrals are taken in the sense of HL Δ-integrals. To obtain the existence result it is necessary to define a notion of a solution.

Definition 3.1. An ACG_* function $x : I_a \to E$ is said to be a Carathéodory solution of the problem (1.1) if it satisfies the following conditions:

(i) $x(0) = x_0$
(ii) $x^\Delta(t) = f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big)$ for μ_Δ a.e. $t \in I_a$

Definition 3.2. *A continuous function $x : I_a \to E$ is said to be a solution of problem* (3.1) *if it satisfies*

$$x(t) = x_0 + \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z,$$

for every $t \in I_a$.

Now we prove that, each solution of the problem (1.1) is equivalent to the solutions of the problem (3.1). Let x be a continuous solution of (1.1). By definition, x is an ACG_* function and $x(0) = x_0$. Since, for μ_Δ a.e. $t \in I_a$,

$$x^\Delta(t) = f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big)$$

and integrals are in the sense of HL Δ-integrals, so it is differentiable μ_Δ a.e. Moreover

$$\begin{aligned}
&\int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z \\
&= \int_0^t x^\Delta(s)\Delta s = x(t) - x(0)
\end{aligned}$$

Thus (3.1) is satisfied.

Now assume that y is an ACG_* function and it is clear that $y(0) = x_0$. By the definition of HL Δ-integrals, there exists an ACG_* function G such that $G(0) = x_0$ and

$$G^\Delta(t) = f\Big(t, y(t), \int_0^t k_1(t,s)g(s,y(s))\Delta s, \int_0^a k_2(t,s)h(s,y(s))\Delta s\Big),$$

for μ_Δ a.e. $t \in I_a$. Hence

$$\begin{aligned}
y(t) &= x_0 + \int_0^t f\Big(z, y(z), \int_0^z k_1(z,s)g(s,y(s))\Delta s, \int_0^a k_2(z,s)h(s,y(s))\Delta s\Big)\Delta z \\
&= x_0 + \int_0^t G^\Delta(s)\Delta s = x_0 + G(t) - G(0) = G(t)
\end{aligned}$$

We obtain $y = G$ and then

$$y^\Delta(t) = f\Big(t, y(t), \int_0^t k_1(t,s)g(s,y(s))\Delta s, \int_0^a k_2(t,s)h(s,y(s))\Delta s\Big).$$

For $x \in C(I_a, E)$, we define the norm of x by: $\|x\|_C = \sup\{\|x(t), \ t \in I_a\}$. Let $B(p) = \{x \in C(I_a, E) : \|x\|_C \le \|x_0\|_C + p, p > 0\}$. Note that these sets are closed and convex. Define the operator $F : C(I_a, E) \to C(I_a, E)$ by

$$F(x)(t) = \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z,$$

where $t \in I_a$, $x \in B(p)$ and integrals are in the sense of HL Δ-integrals. Let

$$\Gamma(p) = \{F(x) \in C(I_a, E) : x \in B(p)\}, \text{ for each } p > 0.$$

Let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^c k(t,s)u(s)\Delta s,$$

where the kernel $k \in C(I_a \times I_a, \mathbb{R})$, $u \in C(I_a; E)$ and c denotes any fixed value in I_a, $a > 0$.

Theorem 3.3. *Assume that for each uniformly ACG_* function $x : I_a \to E$, functions $g(\cdot, x(\cdot))$,$h(\cdot, x(\cdot))$, $f(\cdot, x(\cdot), \int_0^{(\cdot)} k_1(\cdot, s)g(s, x(s))\Delta s, \int_0^a k_2(\cdot, s)h(s, x(s))\Delta s)$ are HL Δ-integrable, f, g, h are Carathéodory functions. Let $k_1, k_2 : I_a \times I_a \to \mathbb{R}$ be measurable functions such that $k_1(t, \cdot), k_2(t, \cdot)$ are continuous. Assume that there exist $p_0 > 0$ and positive constants L, L_1, d_1 such that*

$$\alpha(g(I, X)) \le L\alpha(X), \quad \text{for } I \subset I_a \text{ and every } X \subset B(p_0), \tag{3.2}$$

$$\alpha(h(I, X)) \le L_1\alpha(X), \quad \text{for } I \subset I_a \text{ and every } X \subset B(p_0), \tag{3.3}$$

$$\begin{gathered}\alpha(f(t, A, C, D)) \le d_1 \max\{\alpha(A), \alpha(C), \alpha(D)\}, \\ \text{for every } A, C, D \subset B(p_0), t \in I_a,\end{gathered} \tag{3.4}$$

where

$$\begin{gathered}g(I, X) = \{g(t, x(t)), t \in I, x \in X\}, \\ h(I, X) = \{h(t, x(t)), t \in I, x \in X\}, \\ f(t, A, C, D) = \{f(t, x_1, x_2, x_3) : (x_1, x_2, x_3) \in A \times C \times D\}\end{gathered}$$

and α denotes the Kuratowski measure of noncompactness. Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly ACG_* on I_a. Then, there exist at least one solution of the problem (1.1) *on I_c for some $0 < c \le a$, such that $d_1cLr(K) < 1$ and $d_1c < 1$.*

Proof. Fix an arbitrary $p \ge 0$. Put $B(p) = \{x \in C(I_c, E) : \|x\|_C \le \|x_0\|_C + p, p > 0\}$ where c will be given below. Recall that a set $B(p)$ of continuous functions $F(x) \in \Gamma(p)$, defined on a time scale interval I_a is equicontinuous on I_a if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|F(x)(t) - F(x)(\tau)\| < \epsilon, \text{ for all } x \in B(p),$$

whenever $|t - \tau| < \delta$, for each $F(x) \in \Gamma(p)$. Thus, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|\int_\tau^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\| < \epsilon,$$

for all $x \in B(p)$, whenever $|t-\tau| < \delta$ and $t, \tau \in I_a$. As a result, there exists a number c, $0 < c \leq a$ such that

$$\| \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^c k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\| \leq p_0,$$

where $t \in I_c, x \in B(p)$.

Now, we show that, the operator F is well defined and maps $B(p)$ into $B(p)$.

$$\begin{aligned}
&\|F(x)(t)\| \\
&= \|x_0 + \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\| \\
&\leq \|x_0\| + \| \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\| \\
&\leq \|x_0\| + p.
\end{aligned}$$

We will show, that the operator F is continuous. Let $x_n \to x$ in $B(p)$. Then

$$\begin{aligned}
&\|F(x_n) - F(x)\| \\
&= \sup_{t\in I_a} \| \int_0^t f\Big(z, x_n(z), \int_0^z k_1(z,s)g(s,x_n(s))\Delta s, \int_0^c k_2(z,s)h(s,x_n(s))\Delta s\Big)\Delta z \\
&\quad - \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^c k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\| \\
&= \sup_{t\in I_a} \| \int_0^t \Big(f\Big(z, x_n(z), \int_0^z k_1(z,s)g(s,x_n(s))\Delta s, \int_0^c k_2(z,s)h(s,x_n(s))\Delta s\Big) \\
&\quad - f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^c k_2(z,s)h(s,x(s))\Delta s\Big)\Big)\Delta z\|
\end{aligned}$$

Since g and h are Carathéodory functions in $B(p)$ we have that $g(s, x_n(s)) \to g(s, x(s))\mu_\Delta$ a.e. on I_a, and $h(s, x_n(s)) \to h(s, x(s))$ μ_Δ a.e. on I_a. Using theorem (2.10) we obtain

$$\begin{gathered}
\int_0^z k_1(t,s)g(s,x_n(s))\Delta s \to \int_0^z k_1(t,s)g(s,x(s))\Delta s \quad \mu_\Delta \text{ a.e. on } I_a, \\
\int_0^z k_2(t,s)h(s,x_n(s))\Delta s \to \int_0^z k_2(t,s)h(s,x(s))\Delta s \quad \mu_\Delta \quad \text{a.e. on } I_a.
\end{gathered}$$

Moreover, because f is the Caretheodory function in $B(p)$

$$\begin{aligned}
&f\Big(z, x_n(z), \int_0^z k_1(z,s)g(s,x_n(s))\Delta s, \int_0^c k_2(z,s)h(s,x_n(s))\Delta s\Big) \\
&\to f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^c k_2(z,s)h(s,x(s))\Delta s\Big)
\end{aligned}$$

μ_Δ a.e. on I_a. Thus theorem (2.10) implies $\|F(x_n) - F(x)\| \to 0$.

Suppose that $V \subset B(p_0)$ satisfies the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$. We will prove that V is relatively compact and so (2.1) is satisfied. Since $V \subset B(p)$, $F(V) \subset \Gamma(p)$, then $V \subset \overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ is equicontinuous. By theorem (2.14), $t \mapsto v(t) = \alpha(V(t))$ is continuous on I_c.

For fixed $t \in I_c$ we divide the interval $[0,t]$ into m parts in following way:

$$t_0 = 0, \quad t_1 = \sup_{t\in I_a}\{s : s \geq t_0, s - t_0 < \delta\},$$

$$t_2 = \sup_{t\in I_a}\{s : s \geq t_1, s - t_1 < \delta\}, \dots, t_n = \sup_{t\in I_a}\{s : s \geq t_{n-1}, s - t_{n-1} < \delta\}.$$

Since $\mathbb{T}$ is closed, it follows that $t_i \in I_a$. If some $t_{i+1} = t_i$, then $t_{i+2} = \inf\{t \in \mathbb{T} : t > t_{i+1}\}$.

Let $V([t_i, t_{i+1}] = \{u(s) : u \in V, t_i \leq s \leq t_{i+1}, i = 0, 1, \dots m-1\}$. By theorem (2.14) and the continuity of v there exists $s_i \in I_i = [t_i, t_{i+1}]$, such that

$$\alpha(V([t_i, t_{i+1}]) = \sup\{\alpha(V(s)) : t_i \leq s \leq t_{i+1}\} = v(s_i)$$

For fixed $z \in [0, t]$ we divide the interval $[0, z]$ into m parts as follows

$$z_0 = 0, \quad z_1 = \sup_{t\in[0,t]}\{s : s \geq z_0, s - z_0 < \delta\},$$

$$z_2 = \sup_{t\in[0,t]}\{s : s \geq z_1, s - z_1 < \delta\}, \dots, z_n = \sup_{t\in[0,t]}\{s : s \geq z_{n-1}, s - z_{n-1} < \delta\},$$

such that $\mu_\Delta(I_j) = jz/m, j = 0, 1, \dots, m, I_j = [z_j, z_{j+1}]$.

Let $V([z_j, z_{j+1}] = \{u(s) : u \in V, z_j \leq s \leq z_{j+1}, j = 0, 1, \dots m-1\}$. By theorem (2.14) and the continuity of v there exists $s_j \in I_j = [z_j, z_{j+1}]$, such that

$$\alpha(V([z_j, z_{j+1}])) = \sup\{\alpha(V(s)) : z_j \leq s \leq z_{j+1}\} = v(s_j).$$

Furthermore, we divide the interval $[0, c]$ into m parts:

$$r_0 = 0, \quad r_1 = \sup_{t\in[0,c]}\{s : s \geq r_0, s - r_0 < \delta\},$$

$$r_2 = \sup_{t\in[0,c]}\{s : s \geq r_1, s - r_1 < \delta\}, \dots, r_n = \sup_{t\in[0,c]}\{s : s \geq r_{n-1}, s - r_{n-1} < \delta\}$$

such that $\mu_\Delta(I_j) = kr/m, k = 0, 1, \dots, m, I_k = [r_k, r_{k+1}]$.

Let $V([r_k, r_{k+1}] = \{u(s) : u \in V, r_k \leq s \leq r_{k+1}, k = 0, 1, \dots m-1\}$. By theorem (2.14) and the continuity of v there exists $s_k \in I_k = [r_k, z_{k+1}]$, such that

$$\alpha(V([r_k, r_{k+1}]) = \sup\{\alpha(V(s)) : r_k \leq s \leq r_{k+1}\} = v(s_k)$$

By theorem (2.12) and the properties of HL Δ-integral, for $x \in V$, we have

$$\begin{aligned}
F(x) &= x_0 + \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} f\Big(z, x(z), \sum_{j=0}^{m-1}\int_{z_j}^{z_{j+1}} k_1(z,s)g(s,x(s))\Delta s, \\
&\quad \sum_{r=0}^{m-1}\int_{r_k}^{r_{k+1}} k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z \\
&\in x_0 + \sum_{i=0}^{m-1}\mu_\Delta(I_i)\overline{\text{conv}} f\Big(I_i, V(I_i), \sum_{j=0}^{m-1}\mu_\Delta(I_j)\overline{\text{conv}}\big(k_1(I_i,I_j)g(I_j,V(I_j))\big), \\
&\quad \sum_{k=0}^{m-1}\mu_\Delta(I_k)\overline{\text{conv}}(k_2(I_i,I_k)g(I_k,V(I_k)))\Big),
\end{aligned}$$

where $k(I, J) = \{k(t,s) : t \in I, s \in J\}$, $g(I, V(I)) = \{g(t, x(t)) : t \in I, x \in V\}$. Using (3.2), (3.3), (3.4), and properties of the measure of noncompactness we obtain

$$\begin{aligned}
\alpha(F(V)) &\leq \sum_{i=0}^{m-1}\mu_\Delta(I_i)\overline{\text{conv}}\alpha(f(I_i, V(I_i), \sum_{j=0}^{m-1}\mu_\Delta(I_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j))), \\
&\quad \sum_{k=0}^{m-1}\mu_\Delta(I_k)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))
\end{aligned}$$

$$\leq \sum_{i=0}^{m-1} \mu_\Delta(I_i) d_1 \max\Big(\alpha(V(I_i)), \alpha\Big(\sum_{j=0}^{m-1} \mu_\Delta(I_j)\overline{\mathrm{conv}}(k_1(I_i,I_j)g(I_j,V(I_j))), \alpha\Big(\sum_{k=0}^{m-1} \mu_\Delta(I_k)\overline{\mathrm{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Big)\Big)$$

Let us observe that if

$$\alpha(V(I_i)) = \max Big\{\alpha(V(I_i)), \alpha\Big(\sum_{j=0}^{m-1} \mu_\Delta(I_j)\overline{\mathrm{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))Big), \alpha\Big(\sum_{k=0}^{m-1} \mu_\Delta(I_k)\overline{\mathrm{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Big)\Big\},$$

then

$$\alpha(V(t)) = \alpha(\overline{\mathrm{conv}}(\{x(t)\} \cup F(V(t)))) \leq \alpha(F(V(t))) < d_1 c\alpha(V(t)),$$

$t \in I_c$. Because $d_1 c < 1$, so $\alpha(V(t)) < \alpha(V(t))$ which is a contradiction. If

$$\begin{aligned}
&\alpha\Big(\sum_{j=0}^{m-1} \mu_\Delta(I_j)\overline{\mathrm{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Big) \\
&= \max\Big\{\alpha(V(I_i)), \alpha\Big(\sum_{j=0}^{m-1} \mu_\Delta(I_j)\overline{\mathrm{conv}}(k_1(I_i,I_j)g(I_j,V(I_j)))\Big), \\
&\quad \alpha\Big(\sum_{k=0}^{m-1} \mu_\Delta(I_i)\overline{\mathrm{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Big)\Big\},
\end{aligned}$$

then

$$\begin{aligned}
\alpha(F(V(t))) &\leq \sum_{i=0}^{m-1} \mu_\Delta(I_i) d_1 \sum_{j=0}^{m-1} \mu_\Delta(I_j)(k_1(I_i,I_j)\alpha(g(I_j,V(I_j)))) \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(I_i) d_1 L \sum_{j=0}^{m-1} \mu_\Delta(I_j)(k_1(I_i,I_j)\alpha(V((I_j)))) \\
&\leq d_1 L(c/m) \sum_{j=0}^{m-1} \mu_\Delta(I_j)\alpha(V(I_j)) \sum_{i=0}^{m-1} k_1(I_i,I_j)
\end{aligned}$$

For $j = 0, 1, \dots, m-1$ there exists $q_j, j = 0, 1, \dots, m-1$ such that $k_1(I_i, I_j) \leq k_1(I_{q_j}, I_j)$. So

$$\alpha(F(V(t))) \leq d_1 Lc \sum_{j=0}^{m-1} \mu_\Delta(I_j) k_1(I_{q_j}, I_j) v(s_j), \quad \text{where } s_j \in I_j.$$

Hence

$$\begin{aligned}
\alpha(F(V(t))) &\leq d_1 Lc \sum_{j=0}^{m-1} \mu_\Delta(I_j) k_1(I_{q_j}, I_j)(v(s_j) - v(p_j)) \\
&\quad + d_1 Lc \sum_{j=0}^{m-1} \mu_\Delta(I_j) k_1(I_{q_j}, I_j)(v(p_j))
\end{aligned}$$

By the continuity of v we have $v(s_j)-v(p_j) < \epsilon_1$ and $\epsilon_1 \to 0$ as $m \to \infty$. Therefore,

$$\alpha(F(V)(t)) \le d_1 Lc \int_0^c k_1(t,s)v(s)\Delta s, \quad \text{for } t \in I_c$$

Since $V = \overline{\text{conv}}(x \cup F(V))$, by the property of the measure of noncompactness, we have $\alpha(V(t)) \le \alpha(F(V)(t)) \le d_1 Lc \int_0^c k_1(t,s)v(s)\Delta s$, for $t \in I_c$. Because this inequality holds for every $t \in I_c$ and $Ld_1cr(K) < 1$, so by applying Gronwall's inequality, we conclude that $\alpha(V(t)) = 0$ for $t \in I_c$. Hence Arzela-Ascoli's theorem implies that the set V is relatively compact. Consequently, by Theorem (2.19), F has a fixed point which is a solution of the problem (1.1). Similarly, if

$$\begin{aligned}
&\alpha\Big(\sum_{k=0}^{m-1} \mu_\Delta(I_i)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Big) \\
&= \max\Big\{\alpha(V(I_i)), \alpha\Big(\sum_{j=0}^{m-1} \mu_\Delta(I_j)\overline{\text{conv}}(k_1(I_i,I_j)g(I_j,V(I_j))), \\
&\quad \alpha\Big(\sum_{k=0}^{m-1} \mu_\Delta(I_i)\overline{\text{conv}}(k_2(I_i,I_k)h(I_k,V(I_k)))\Big)\Big)\Big\}
\end{aligned}$$

then we prove that

$$\alpha(F(V)(t)) \le d_1 L_1 c \int_0^c k_2(t,s)v(s)\Delta s$$

and we conclude that the set V is relatively compact. By Theorem (2.19), F has a fixed point which is a solution of problem (1.1). □

Remark 3.4. For discrete time scales the existence of solutions is trivially given without imposing further compactness assumptions on the right-hand side of the equation. If a time scale admits at least one right-dense point, then the continuity assumption is not sufficient for the existence of (rd continuous) solutions of problem (1.1). Nevertheless, we will not distinguish such a discrete case, because some continuity and compactness conditions are necessary to unify the continuous problems and their discretization.

4. An existence result for integrodifferential equations in weak sense

Now we consider the equivalently integral problem

$$\begin{aligned}
&x(t) \\
&= x_0 + \int_0^t f(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s)\Delta z,
\end{aligned} \tag{4.1}$$

for $t \in I_a$, where $f : I_a \times E \times E \times E \to E, h, g : I_a \times E \to E, k_j, j = 1,2$ are real-valued functions, $\mathbb{T}$ denotes a time scale (nonempty closed subset of real numbers $\mathbb{R}$), $0 \in \mathbb{T}$, I_a denotes a time scale interval, $(E, \|\cdot\|)$ is a Banach space and integrals are taken in the sense of HKP Δ-integrals. Fix $x^* \in E^*$ and consider the problem

$$(x^*x)^\Delta(t) = x^*\Big(f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big)\Big) \tag{4.2}$$

Let us introduce a definition.

Definition 4.1. Let $F : I \to E$ and let $A \subset I$. The function $f : A \to E$ is a pseudo Δ -derivative of F on A if for each $x^* \in E^*$ the real-valued function x^*F is Δ-differentiable μ_Δ almost everywhere on A and $(x^*F)^\Delta = x^*f$ μ_Δ almost everywhere on A.

Regarding the above definition it is clear that the left-hand side of (4.2) can be rewritten to the form $x^*(x^\Delta)$, where x^Δ denotes the pseudo Δ-derivative.

To obtain the existence result for our problem it is necessary to define a notion of a solution.

Definition 4.2. A function $x : I_a \to E$ is said to be a pseudosolution of problem (1.1) if it satisfies the following conditions:

(1) $x(\cdot)$ is ACG^* function,
(2) $x(0) = x_0$
(3) for each $x^* \in E^*$ there exists a set $A(x^*)$ with μ_Δ measure zero, such that for each $t \notin A(x^*)$

$$(x^*x)^\Delta(t) = x^*\Big(f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big)\Big).$$

Definition 4.3. A continuous function $x : I_a \to E$ is said to be a solution of problem (4.1) if it satisfies

$$x(t) = x_0 + \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z,$$

Because we consider a new type of integral and a new type of solutions is necessary to prove that each solution x of problem (1.1) is the solution of problem (4.1). Let x is a continuous solution of (4.1). Fix $x^* \in E^*$. By definition, x is ACG^* function and $x(0) = x_(0)$. Since, for each $x^* \in E^*$ and μ_Δ a.e. $t \in I_a$,

$$\begin{aligned}
(x^*x)^\Delta &= x^*(x^\Delta(t)) \\
&= x^*\Big(f\Big(t, x(t), \int_0^t k_1(t,s)g(s,x(s))\Delta s, \int_0^a k_2(t,s)h(s,x(s))\Delta s\Big)\Big)
\end{aligned}$$

and the last are Δ-HK integrable, so is differentiable μ_Δ a.e. Moreover

$$\begin{aligned}
&\int_0^t x^*\Big(f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Big)\Delta z \\
&= \int_0^t (x^*x(s))^\Delta \Delta s = x^*(x(t) - x_0)
\end{aligned}$$

Thus $x(t)$ satisfies (4.1) Now assume that y is ACG^* function and $y(0) = x_0$. By the definition of HKP Δ-integrals, there exists an ACG^* function G such that $G(0) = x_0$ and

$$x^*(G^\Delta(t)) = x^*\Big((f\Big(t, y(t), \int_0^t k_1(t,s)g(s,y(s))\Delta s, \int_0^a k_2(t,s)h(s,y(s))\Delta s\Big)\Big),$$

for μ_Δ a.e. $t \in I_a$. Hence

$$\begin{aligned}
x^*(y(t)) = x_0 + \int_0^t x^*\Big(f\Big(z, y(z), &\int_0^z k_1(z,s)g(s,y(s))\Delta s, \\
&\int_0^a k_2(z,s)h(s,y(s))\Delta s\Big)\Big)\Delta z
\end{aligned}$$

$$
\begin{aligned}
&= x_0 + \int_0^t x^*(G^\Delta(s))\Delta s \\
&= x_0 + x^*(G(t)) - x^*(G(0)) = x^*(G(t)).
\end{aligned}
$$

We obtain $y = G$ and then

$$
y^\Delta(t) = f\Big(t, y(t), \int_0^t k_1(t,s)g(s,y(s))\Delta s, \int_0^a k_2(t,s)h(s,y(s))\Delta s\Big).
$$

Lemma 4.4. *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a$, $H(t) = \{h(t) \in E, h \in H\}$. Then $\beta(H(I_a)) = \sup_{t\in I_a} \beta(H(t))$ and the function $t \mapsto \beta(H(t))$ is continuous.*

Theorem 4.5. *Assume that for each uniformly ACG^* function $x : I_a \to E$, functions $g(\cdot, x(\cdot))$, $h(\cdot, x(\cdot))$, $f(\cdot, x(\cdot), \int_0^{(\cdot)} k_1(\cdot, s)g(s, x(s))\Delta s, \int_0^a k_2(\cdot, s)h(s, x(s))\Delta s)$ are HKP Δ-integrable, f, g, h are weakly-weakly sequentially continuous functions. Let $k_1, k_2 : I_a \times I_a \to \mathbb{R}$ be measurable functions such that $k_1(t, \cdot)$, $k_2(t, \cdot)$ are continuous. Assume that there exist $p_0 > 0$ and positive constants L, L_1, d such that*

$$
\beta(g(I, X)) \le L\beta(X), \quad \text{for } I \subset I_a \text{ and every } X \subset B(p_0), \tag{4.3}
$$

$$
\beta(h(I, X)) \le L_1\beta(X), \quad \text{for } I \subset I_a \text{ every } X \subset B(p_0), \tag{4.4}
$$

$$
\begin{gathered}
\beta(f(t, A, C, D)) \le d \max\{\beta(A), \beta(C), \beta(D)\}, \\
\text{for every } A, C, D \subset B(p_0),\ t \in I_a,
\end{gathered} \tag{4.5}
$$

where

$$
\begin{aligned}
g(I, X) &= \{g(t, x(t)), t \in I, x \in X\}, \\
h(I, X) &= \{h(t, x(t)), t \in I, x \in X\}, \\
f(t, A, C, D) &= \{f(t, x_1, x_2, x_3) : (x_1, x_2, x_3) \in A \times C \times D\}
\end{aligned}
$$

and β denotes the de Blasi measure of noncompactness. Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly ACG^* on I_a. Then, there exists at least one pseudosolution of the problem (1.1) on I_c for some $0 < c \le a$, such that $dcLr(K) < 1$ and $dc < 1$.
Moreover, let $\Gamma(p_0)$ be equicontinuous and uniformly ACG^* on I_a. Then there exists at least one pseudosolution of the problem (1.1) on I_c, for some $0 < c \le a$, such that $dcLr(K) < 1$ and $dc < 1$.

Proof. By equicontinuity of $\Gamma(p_0)$, there exists a number $c, 0 < c \le a$, such that

$$
\| \int_0^t f(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^c k_2(z,s)h(s,x(s))\Delta s)\Delta z\| \le p_0,
$$

for $t \in I_c$ and $x \in B(p)$. Indeed, for any $x^* \in E^*$, such that $\|x^*\| \le 1$ and for any $x \in B(p_0)$, we have

$$
\begin{aligned}
&|x^*F(x)(t)| \\
&= |x^*x_0| + \Big|x^* \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s, \int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\Big| \\
&\le \|x^*\|\|x_0\| + \|x^*\|\| \int_0^t f\Big(z, x(z), \int_0^z k_1(z,s)g(s,x(s))\Delta s,
\end{aligned}
$$

$$\int_0^a k_2(z,s)h(s,x(s))\Delta s\Big)\Delta z\|$$
$$\le \|x_0\|+p.$$

From here

$$\sup\{|x^*F(x)(t)| : x^* \in E^*, \|x^*\| \le 1\} \le \|x_0\|+p, \|F(x)(t)\| \le \|x_0\|+p,$$

so $F(x)(t) \in B(p_0)$. We will show, that the operator F is weakly-weakly sequentially continuous. By [38], a sequence $x_n(\cdot)$ is weakly convergent in $C(I_c,E)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_c$. Because $g(s,\cdot)$ and $h(s,\cdot)$ are weakly- weakly sequentially continuous, so if $x_n \to^\omega x$ in $(C(I_c,E),\omega)$, then $g(s,x_n(s)) \to^\omega g(s,x(s))$ and $h(s,x_n(s)) \to^\omega h(s,x(s))$ in E for $t \in I_c$ and by Theorem (2.11) we have

$$\lim_{n\to\infty}\int_0^z k_1(z,s)g(s,x_n(s))\Delta s = \int_0^z k_1(z,s)g(s,x(s))\Delta s,$$

weakly in E for each $t \in I_c$, and

$$\lim_{n\to\infty}\int_0^z k_2(z,s)h(s,x_n(s))\Delta s = \int_0^z k_2(z,s)h(s,x(s))\Delta s,$$

weakly in E for each $t \in I_c$. Moreover, because f is weakly-weakly sequentially continuous,

$$\int_0^t f(z,x_n(z),\int_0^z k_1(z,s)g(s,x_n(s))\Delta s,\int_0^a k_2(z,s)h(s,x_n(s))\Delta s)\Delta z$$

tends to

$$\int_0^t f(z,x(z),\int_0^z k_1(z,s)g(s,x(s))\Delta s,\int_0^a k_2(z,s)h(s,x(s))\Delta s)\Delta z$$

weakly in E for each $t \in I_c$.

Suppose that $V \subset B(p_0)$ satisfies the condition $\overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V))$. We will prove that V is relatively compact and so (4.2) is satisfied. Since $V \subset B(p), F(V) \subset \Gamma(p)$, then $V \subset \overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V))$ is equicontinuous. By Lemma (4.4) $t \mapsto v(t) = \beta(V(t))$ is continuous on I_c. Therefore, as in Theorem (3.3) we prove that $\beta(V(t)) = 0$, for $t \in I_c$, so that the set V is relatively weakly compact. Consequently, by Theorem (2.20) F has a fixed point which is a pseudosolution of the problem (1.1). $\square$

Remark 4.6. Conditions (3.2), (3.3), (3.4), (4.3), (4.4), and (4.5) in Theorems (3.3), and (4.5) respectively can be also generalized to the Sadovskii condition [43]. Szufla condition [51] and others and α or β can be replaced by some axiomatic measure of noncompactness.

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