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POSITIVE SOLUTIONS OF NONLINEAR M-POINT BOUNDARY-VALUE PROBLEM FOR P-LAPLACIAN DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we study the existence of positive solutions to nonlinear m-point boundary-value problems for a p-Laplacian dynamic equation on time scales. We use fixed point theorems in cones and obtain criteria that generalize and improve known results.

1. Introduction

Recently, there is much attention paid to the existence of positive solutions for three-point boundary-value problems on time scales, see [2, 4, 8, 10, 12] and references therein. However, there are not many results concerning the p-Laplacian problems on time scales.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that $(0, T)$ are points in $\mathbb{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale; that is $(0, T) \cap \mathbb{T}$.

Anderson [2] discussed the dynamic equation on time scales:

$$u^{\triangle\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0, T), \tag{1.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{1.2}$$

He obtained some results for the existence of one positive solution of the problem (1.1) and (1.2) based on the limits $f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}$ as well as existence of at least three positive solutions.

Kaufmann [8] studied the problem (1.1) and (1.2) and obtained existence results of finitely many positive solutions and countably many positive solutions.

Sun and Li [12] considered the existence of positive solutions of the following dynamic equations on time scales

$$u^{\triangle\nabla}(t) + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \tag{1.3}$$

$$\beta u(0) - \gamma u^{\triangle}(0) = 0, \quad \alpha u(\eta) = u(T). \tag{1.4}$$

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They obtained the existence of single and multiple positive solutions of the problem (1.3) and (1.4) by using a fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In this paper concerns the existence of positive solutions of the p-Laplacian dynamic equations on time scales

$$(\phi_p(u^{\Delta}))^{\nabla} + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \tag{1.5}$$

$$\phi_p(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \phi_p(u^{\Delta}(\xi_i)), \quad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i) \tag{1.6}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, and a_i, b_i, a, f satisfy:

- (H1) $a_i,\ b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$, $T\sum_{i=1}^{m-2} b_i \geq \sum_{i=1}^{m-2} b_i\xi_i$;
- (H2) $a(t) \in C_{\rm ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;
- (H3) $f \in C([0, T] \times [0, +\infty), [0, +\infty))$.

We point out that when $\mathbb{T} = \mathbb{R}$ and $p = 2$, (1.5), (1.6) becomes a boundary-value problem of differential equations and is the problem considered in [11]. Our main results extend and include the main results of [11].

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2. Section 3 is devoted to the existence of positive solutions of (1.5), (1.6), the main tool being a fixed point theorem for cone-preserving operators.

2. Preliminaries

For convenience, we list the following definitions which can be found in [1, 3, 4, 5, 7].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$. For $t < \sup\mathbb{T}$ and $r > \inf\mathbb{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\} \in \mathbb{T}.$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\triangle}(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}_k$, the nabla derivative of f at t is the number $f^\nabla(t)$, (provided it exists), with the property that for each $\epsilon>0$, there is a neighborhood U of t such that

$$|f(\rho(t))-f(s)-f^\nabla(t)(\rho(t)-s)|\le\epsilon|\rho(t)-s|,$$

for all $s\in U$.

Definition 2.3. A function f is left-dense continuous (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t)=f(t)$. In this case, it is defined that

$$\int_a^b f(t)\nabla t=F(b)-F(a).$$

If $u^{\triangle\nabla}(t)\le 0$ on $[0,T]$, then we say u is concave on $[0,T]$.

By a positive solution of (1.5), (1.6), we understand a function $u(t)$ which is positive on $(0,T)$, and satisfies (1.5), (1.6).

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear boundary-value problem

$$(\phi_p(u^\Delta))^\nabla+h(t)=0,\quad t\in(0,T), \tag{2.1}$$

$$\phi_p(u^\Delta(0))=\sum_{i=1}^{m-2}a_i\phi_p(u^\Delta(\xi_i)),\quad u(T)=\sum_{i=1}^{m-2}b_iu(\xi_i) \tag{2.2}$$

Lemma 2.4. *For $h\in C_{\rm ld}[0,T]$ the BVP (2.1)–(2.2) has the unique solution*

$$u(t)=-\int_0^t\phi_q\Big(\int_0^s h(\tau)\nabla\tau-A\Big)\Delta s+B, \tag{2.3}$$

where

$$A=-\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i},$$

$$B=\frac{\int_0^T\phi_q\Big(\int_0^s h(\tau)\nabla\tau-A\Big)\Delta s-\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}\phi_q\Big(\int_0^s h(\tau)\nabla\tau-A\Big)\Delta s}{1-\sum_{i=1}^{m-2}b_i}$$

Proof. Let u be as in (2.3). By [3, Theorem 2.10(iii)], taking the delta derivative of (2.3), we have

$$u^\Delta(t)=-\phi_q\Big(\int_0^t h(\tau)\nabla\tau-A\Big),$$

moreover, we get

$$\phi_p(u^\Delta)=-\Big(\int_0^t h(\tau)\nabla\tau-A\Big),$$

taking the nabla derivative of this expression yields $(\phi_p(u^\Delta))^\nabla=-h(t)$. And routine calculation verify that u satisfies the boundary value conditions in (2.2), So that u given in (2.3) is a solution of (2.1) and (2.2).

It is easy to see that the BVP

$$(\phi_p(u^\Delta))^\nabla=0,\quad \phi_p(u^\Delta(0))=\sum_{i=1}^{m-2}a_i\phi_p(u^\Delta(\xi_i)),\quad u(T)=\sum_{i=1}^{m-2}b_iu(\xi_i)$$

has only the trivial solution. Thus u in (2.3) is the unique solution of (2.1), (2.2). The proof is complete. $\square$

Lemma 2.5. *Assume (H1) holds, For $h \in C_{\rm ld}[0,T]$ and $h \ge 0$, then the unique solution u of (2.1)–(2.2) satisfies $u(t) \ge 0$, for $t \in [0,T]$.*

Proof. Let

$$\varphi_0(s) = \phi_q\Big(\int_0^s h(\tau)\nabla\tau - A\Big).$$

Since

$$\int_0^s h(\tau)\nabla\tau - A = \int_0^s h(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i} \ge 0,$$

it follows that $\varphi_0(s) \ge 0$. According to Lemma 2.4, we get

$$\begin{aligned}
u(0) &= B\\
&= \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i}\\
&= \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \Big(\int_0^T \varphi_0(s)\Delta s - \int_{\xi_i}^T \varphi_0(s)\Delta s\Big)}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^T \varphi_0(s)\Delta s + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \ge 0.
\end{aligned}$$

and

$$\begin{aligned}
u(T) &= -\int_0^T \varphi_0(s)\Delta s + B\\
&= -\int_0^T \varphi_0(s)\Delta s + \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i}\\
&= \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \ge 0.
\end{aligned}$$

If $t \in (0,T)$, we have

$$\begin{aligned}
u(t) &= -\int_0^t \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big[\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\Big]\\
&\ge -\int_0^T \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big[\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\Big]\\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big[-\big(1-\sum_{i=1}^{m-2} b_i\big)\int_0^T \varphi_0(s)\Delta s + \int_0^T \varphi_0(s)\Delta s\\
&\quad - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\Big]\\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i} \sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s \ge 0.
\end{aligned}$$

So $u(t) \geq 0$, $t \in [0,T]$. The proof is complete. $\square$

Lemma 2.6. *Assume (H1) holds, if $h \in C_{\rm ld}[0,T]$ and $h \geq 0$, then the unique solution u of* (2.1)–(2.2) *satisfies*

$$\inf_{t\in[0,T]} u(t) \geq \gamma\|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{T-\sum_{i=1}^{m-2} b_i\xi_i}, \quad \|u\| = \sup_{t\in[0,T]} |u(t)|.$$

Proof. It is easy to check that $u^{\Delta}(t) = -\varphi(t) \leq 0$, this implies

$$\|u\| = u(0), \quad \min_{t\in[0,T]} u(t) = u(T).$$

It is easy to see that $u^{\Delta}(t_2) \leq u^{\Delta}(t_1)$ for any $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Hence $u^{\Delta}(t)$ is a decreasing function on $[0,T]$. This means that the graph of $u^{\Delta}(t)$ is concave down on $(0,T)$. For each $i \in \{1,2,\dots,m-2\}$, we have

$$\frac{u(T)-u(0)}{T-0} \geq \frac{u(T)-u(\xi_i)}{T-\xi_i},$$

i.e., $Tu(\xi_i) - \xi_i u(T) \geq (T-\xi_i)u(0)$, so that

$$T\sum_{i=1}^{m-2} b_i u(\xi_i) - \sum_{i=1}^{m-2} b_i\xi_i u(T) \geq \sum_{i=1}^{m-2} b_i(T-\xi_i)u(0).$$

With the boundary condition $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, we have

$$u(T) \geq \frac{\sum_{i=1}^{m-2} b_i(T-\xi_i)}{T-\sum_{i=1}^{m-2} b_i\xi_i} u(0).$$

This completes the proof. $\square$

Let the norm on $C_{\rm ld}[0,T]$ be the sup norm. Then $C_{\rm ld}[0,T]$ is a Banach space. It is easy to see that (1.5)–(1.6) has a solution $u = u(t)$ if and only if u is a fixed point of the operator

$$(Au)(t) = -\int_0^t \phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\Big)\Delta s + \tilde{B}, \tag{2.4}$$

where

$$\tilde{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i},$$

$$\begin{aligned}\tilde{B} = \Big[&\int_0^T \phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\Big)\Delta s \\ &- \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\Big)\Delta s\Big]\frac{1}{1-\sum_{i=1}^{m-2} b_i}.\end{aligned}$$

Denote

$$K = \big\{u : u \in C_{\rm ld}[0,T], u(t) \geq 0, \inf_{t\in[0,T]} u(t) \geq \gamma\|u\|\big\},$$

where γ is the same as in Lemma 2.6. It is obvious that K is a cone in $C_{\rm ld}[0,T]$. By Lemma 2.6, $A(K) \subset K$. It is easy to see that $A : K \to K$ is completely continuous.

Lemma 2.7. *Let*
$$\varphi(s)=\phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\tilde{A}\Big).$$
For $\xi_i,(i=1,\dots,m-2)$, *then*
$$\int_0^{\xi_i}\varphi(s)\Delta s\le\frac{\xi_i}{T}\int_0^T\varphi(s)\Delta s.$$

Proof. Since
$$\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau-\tilde{A}=\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}$$
which greater than or equal to zero, we have $\varphi(s)\ge 0$. Now, for all $t\in(0,T]$, we have
$$\Big(\frac{\int_0^t\varphi(s)\Delta s}{t}\Big)^{\Delta}=\frac{t\varphi(t)-\int_0^t\varphi(s)\Delta s}{t\sigma(t)}\ge 0.$$
In fact, Let $\psi(t)=t\varphi(t)-\int_0^t\varphi(s)\Delta s$, taking the delta derivative of the above expression, we have
$$\psi^{\Delta}(t)=t\varphi^{\Delta}(t)\ge 0.$$
Hence, $\psi(t)$ is a nondecreasing function on $[0,T]$. i.e. $\psi(t)\ge 0$. For all $t\in(0,T]$,
$$\frac{\int_0^t\varphi(s)\Delta s}{t}\le\frac{\int_0^T\varphi(s)\Delta s}{T}. \tag{2.5}$$
By (2.4), for $\xi_i,(i=1,\dots,m-2)$, we have
$$\int_0^{\xi_i}\varphi(s)\Delta s\le\frac{\xi_i}{T}\int_0^T\varphi(s)\Delta s.$$
The proof is complete. □

The following well-known result of the fixed point theorems is needed in our arguments.

Lemma 2.8 ([6])**.** *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K=D\cap K\ne\phi$ and $\overline{D_K}\ne K$. Assume that $A:\overline{D_K}\to K$ is a compact map such that $x\ne AK$ for $x\in\partial D_K$. Then the following results hold:*

1. *If $\|Ax\|\le\|x\|$ for $x\in\partial D_K$, then $i(A,D_K,K)=1$;*
2. *If there exists $x_0\in K\backslash\{\theta\}$ such that $x\ne Ax+\lambda x_0$, for all $x\in\partial D_K$ and all $x>0$, then $i(A,D_K,K)=0$;*
3. *Let U be an open set in X such that $\overline{U}\subset D_K$. If $i(A,U,K)=1$ and $i(A,D_K,K)=0$, then A has a fixed point in $D_K\backslash\overline{U}_K$. The same results holds, if $i(A,U,K)=0$ and $i(A,D_K,K)=1$.*

We define
$$K_\rho=\{u(t)\in K:\|u\|<\rho\},\quad \Omega_\rho=\{u(t)\in K:\min_{\xi_{m-2}\le t\le T}u(t)<\gamma\rho\}.$$

Lemma 2.9 ([9])**.** *The set Ω_ρ defined above has the following properties:*

(a) *$K_{\gamma\rho}\subset\Omega_\rho\subset K_\rho$;*
(b) *Ω_ρ is open relative to K;*
(c) *$X\in\partial\Omega_\rho$ if and only if $\min_{\xi_{m-2}\le t\le T}x(t)=\gamma\rho$;*
(d) *If $x\in\partial\Omega_\rho$, then $\gamma\rho\le x(t)\le\rho$ for $t\in[\xi_{m-2},T]$.*

For our convenience, we introduce the following notation:

$$f_{\gamma\rho}^{\rho} = \min\big\{ \min_{\xi_{m-2}\le t\le T} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho,\rho]\big\},$$

$$f_0^{\rho} = \max\big\{ \max_{0\le t\le T} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0,\rho]\big\},$$

$$f^{\alpha} = \lim_{u\to\alpha}\sup \max_{0\le t\le T} \frac{f(t,u)}{\phi_p(u)}, \quad f_{\alpha} = \lim_{u\to\alpha}\inf \max_{\xi_{m-2}\le t\le T} \frac{f(t,u)}{\phi_p(u)}, \quad (\alpha := \infty \text{ or } 0^+),$$

$$m = \Big\{ \frac{1}{1-\sum_{i=1}^{m-2} b_i} \int_0^T \phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big]\Delta s\Big\}^{-1}, \tag{2.6}$$

$$M = \Big\{ \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1-\sum_{i=1}^{m-2} b_i)} \int_0^T \phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big]\Delta s\Big\}^{-1} \tag{2.7}$$

Lemma 2.10. *If f satisfies the conditions*

$$f_0^{\rho} \le \phi_p(m) \quad \text{and} \quad u \ne Au \tag{2.8}$$

for $u \in \partial K_\rho$, then $i(A, K_\rho, K) = 1$.

Proof. By (2.6) and (2.8), for all $u \in \partial K_\rho$, we have

$$\begin{aligned}
&\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\\
&= \int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\\
&\le \Phi_p(\rho)\phi_p(m)\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big],
\end{aligned}$$

so that

$$\begin{aligned}
\varphi(s) &= \Phi_q\Big(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\Big)\\
&\le \rho m \Phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big].
\end{aligned}$$

Therefore, by (2.4), we have

$$\begin{aligned}
\|Au\| \le \tilde{B} &= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big(\int_0^T \varphi(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\Delta s\Big)\\
&\le \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \varphi(s)\Delta s\\
&\le \rho m \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big]\Delta s\\
&= \rho = \|u\|.
\end{aligned}$$

This implies $\|Au\| \le \|u\|$ for $u \in \partial\ K_\rho$. By Lemma 2.8(1), we have $i(A, K_\rho, K) = 1$. □

Lemma 2.11. *If f satisfies the conditions*

$$f_{\gamma\rho}^{\rho} \ge \Phi_p(M\gamma) \quad \text{and} \quad u \ne Au \tag{2.9}$$

for $u \in \partial\Omega_\rho$, then $i(A, \Omega_\rho, K) = 0$.

Proof. Let $e(t) \equiv 1$, for $t \in [0, T]$; then $e \in \partial K_1$. We claim that $u \ne Au + \lambda e$ for $u \in \partial\ \Omega_\rho$, and $\lambda > 0$. In fact, if not, there exist $u_0 \in \partial\Omega$, and $\lambda_0 > 0$ such that $u_0 = Au_0 + \lambda_0 e$. By (2.7) and (2.9), for $t \in [0, T]$, we have

$$\begin{aligned}
&\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \tilde{A} \\
&= \int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i} \\
&\ge \Phi_p(\rho)\phi_p(M\gamma)\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\Big],
\end{aligned}$$

so that

$$\begin{aligned}
\varphi(s) &= \Phi_q\Big(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \tilde{A}\Big) \\
&\ge \rho M\gamma\Phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\Big].
\end{aligned}$$

Applying (2.4) and Lemma 2.7, it follows that

$$\begin{aligned}
u_0(t) &= Au_0(t) + \lambda_0 e(t) \\
&\ge -\int_0^T \varphi(s)\Delta s + \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\Big(\int_0^T \varphi(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\Delta s\Big) + \lambda_0 \\
&= \frac{\sum_{i=1}^{m-2} b_i}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \varphi(s)\Delta s - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\Delta s}{1 - \sum_{i=1}^{m-2} b_i} + \lambda_0 \\
&\ge \frac{\sum_{i=1}^{m-2} b_i}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \varphi(s)\Delta s - \frac{\sum_{i=1}^{m-2} b_i\xi_i}{T(1 - \sum_{i=1}^{m-2} b_i)}\int_0^T \varphi(s)\Delta s + \lambda_0 \\
&= \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1 - \sum_{i=1}^{m-2} b_i)}\int_0^T \varphi(s)\Delta s + \lambda_0 \\
&\ge \gamma\rho M\frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i\xi_i}{T(1 - \sum_{i=1}^{m-2} b_i)} \\
&\quad\times \int_0^T \phi_q\Big[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\Big]\Delta s + \lambda_0 \\
&= \gamma\rho + \lambda_0
\end{aligned}$$

This implies $\gamma\rho \ge \gamma\rho + \lambda_0$, a contradiction. Hence, by Lemma 2.8 (2), it follows that $i(A, \Omega_\rho, K) = 0$. ☐

3. Existence of Positive Solutions

We now present our results on the existence of positive solutions for (1.5)–(1.6) under the assumptions:

(H4) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ such that
$$f_0^{\rho_1} \leq \phi_p(m), f_{\gamma\rho_2}^{\rho_2} \geq \phi_p(M\gamma);$$
(H5) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that
$$f_0^{\rho_2} \leq \phi_p(m), f_{\gamma\rho_1}^{\rho_1} \geq \phi_p(M\gamma).$$

Theorem 3.1. *Assume that (H1)–(H3) and either (H4) or (H5) hold. Then* (1.5)–(1.6) *has a positive solution.*

Proof. Assume that (H4) holds. We show that A has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. By Lemma 2.10, we have
$$i(A, K_{\rho_1}, K) = 1.$$
By Lemma 2.11, we have
$$i(A, K_{\rho_2}, K) = 0.$$
By Lemma 2.9 (a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.8(3) that A has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$, The proof is similar when H_5 holds, and we omit it here. The proof is complete. □

As a special case of Theorem 3.1, we obtain the following result, under assumptions

(H6) $0 \leq f^0 < \phi_p(m)$ and $\phi_p(M) < f_\infty \leq \infty$;
(H7) $0 \leq f^\infty < \phi_p(m)$ and $\phi_p(M) < f_0 \leq \infty$.

Corollary 3.2. *Assume that (H1)–(H3) and either (H6) or (H7) hold. Then* (1.5)–(1.6) *has a positive solution.*

For the next result we use the following assumptions:

(H8) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that
$$f_0^{\rho_1} \leq \phi_p(m), f_{\gamma\rho_2}^{\rho_2} \geq \phi_p(M\gamma), u \neq Au, \forall\ u \in \partial\Omega_{\rho_2} \quad \text{and} \quad f_0^{\rho_3} \leq \phi_p(m);$$
(H9) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that
$$f_0^{\rho_2} \leq \phi_p(m), f_{\gamma\rho_1}^{\rho_1} \geq \phi_p(M\gamma), u \neq Au, \forall\ u \in \partial K_{\rho_2}, \quad \text{and} \quad f_{\gamma\rho_3}^{\rho_3} \geq \phi_p(M\gamma).$$

Theorem 3.3. *Assume that (H1)–(H3) and either (H8) or (H9) hold. Then* (1.5)–(1.6) *has two positive solutions. Moreover, if in (H8), $f_0^{\rho_1} \leq \phi_p(m)$ is replaced by $f_0^{\rho_1} < \phi_p(m)$, then* (1.5)–(1.6) *has a third positive solution $u_3 \in K_{\rho_1}$.*

Proof. Assume that (H8) holds. We show that either A has a fixed point u_1 in ∂K_{ρ_1} or in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. If $u \neq Au$ for $u \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$, then by Lemmas 2.10 and 2.11, we have
$$i(A, K_{\rho_1}, K) = 1, \quad i(A, K_{\rho_3}, K) = 1, \quad i(A, K_{\rho_2}, K) = 0.$$
By Lemma 2.9 (a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.8 (3) that A has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. Similarly, A has a fixed point in $K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$. The proof is similar when (H9) holds and we omit it here. The proof is complete. □

As a special case of Theorem 3.3, we obtain the following result, using the assumptions:

(H10) $0 \le f^0 < \phi_p(m)$, $f^{\rho}_{\gamma\rho} \ge \phi_p(M\gamma)$, $u \ne Au$, for all $u \in \partial\Omega_\rho$ and $0 \le f^\infty < \phi_p(m)$;

(H11) $\phi_p(m) < f_0 \le \infty$, $f^{\rho}_{0} \le \phi_p(m)$, $u \ne Au$, for all $u \in \partial K_\rho$ and $\phi_p(M) < f_\infty \le \infty$.

Corollary 3.4. *Assume (H1)–(H3). If there exist $\rho > 0$ such that either (H10) or (H11) hold, then* (1.5)–(1.6) *has two positive solutions.*

Note that when $\mathbb{T} = \mathbb{R}$, $(0, T) = (0, 1)$, and $p = 2$, Theorems 3.1 and 3.3 here improve [11, Theorem 3.1].

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