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LOCALIZED NODAL SOLUTIONS FOR PARAMETER-DEPENDENT QUASILINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. In this article, we apply a new variational perturbation method to study the existence of localized nodal solutions for parameter-dependent semiclassical quasilinear Schrödinger equations, under a certain parametric conditions.

1. INTRODUCTION

In this article, we study the existence of localized nodal solutions for the parameter-dependent semiclassical quasilinear Schrödinger equation

$$\begin{gathered}\varepsilon^2 \sum_{i,j=1}^{N} \big(D_j(b_{ij}(v)D_i v) - \frac{1}{2}D_z b_{ij}(v) D_i v D_j v\big) - V(x)v + \lambda |v|^{q-2}v = 0, \\ v(x) \to 0 \quad \text{as } |x| \to \infty,\end{gathered} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $\varepsilon > 0$ is a small parameter, $D_i v = \frac{\partial v}{\partial x_i}$, $D_z b_{ij}(z) = \frac{d}{dz} b_{ij}(z)$, $2 < q < 4$, $N \geq 3$, $\lambda > 0$, and V is the potential function.

We assume the following conditions on b_{ij} and V:

(A1) $b_{ij} \in C^{1,1}(\mathbb{R}, \mathbb{R}), b_{ij} = b_{ji}$, $i, j = 1, \dots, N$ and there exists $c_0 > 0$ such that
$$|D_z b_{ij}(z) - D_z b_{ij}(w)| \leq c_0 |z - w| \quad \text{for } z, w \in \mathbb{R};$$

(A2) there exist $c_+, c_- > 0$ such that
$$c_-(1 + z^2)|\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(z)\xi_i\xi_j \leq c_+(1 + z^2)|\xi|^2 \quad \text{for } z \in \mathbb{R},\ \xi = (\xi_i) \in \mathbb{R}^N;$$

(A3) there exists $\delta > 0$ such that
$$\delta \sum_{i,j=1}^{N} b_{ij}(z)\xi_i\xi_j \leq \sum_{i,j}^{N} \Big(b_{ij}(z) + \frac{1}{2} z D_z b_{ij}(z)\Big)\xi_i\xi_j \leq q\Big(\frac{1}{2} - \delta\Big) \sum_{i,j=1}^{N} b_{ij}(z)\xi_i\xi_j$$
for $z \in \mathbb{R}$, $\xi = (\xi_i) \in \mathbb{R}^N$;

(A4) $b_{ij}(z)$ is even in z;

(A5) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and there exists $c_0 > 0$ such that
$$c_0 \leq V(x) \leq c_0^{-1}, \quad \text{for } x \in \mathbb{R}^N;$$

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(A6) there exists a bounded domain $M \subset \mathbb{R}^N$ with smooth boundary ∂M such that
$$\langle \nabla V(x), n(x)\rangle > 0, \quad \text{for } x \in \partial M,$$
where $n(x)$ is the outer normal of ∂M at the point $x \in \partial M$.

Without loss of generality we assume $0 \in M$. Under assumption (A6), the critical set $\mathcal{A}$ of V contained M is a nonempty closed set:
$$\mathcal{A} = \{x \in M | \nabla V(x) = 0\}.$$
For a set $B \subset \mathbb{R}^N$ and $\delta > 0$ we denote
$$B^\delta = \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, B) = \inf_{y\in B} |x - y| < \delta\big\},$$
$$B_\delta = \big\{x \in \mathbb{R}^N : \delta x \in B\big\}.$$

Our main result reads as follows.

Theorem 1.1. *Assume* $2 < q < 4$, (A1)–(A6). *Then for any positive integer* k *there exist* $\Lambda_k > 0$ *and* $\varepsilon_k > 0$ *such that if* $\lambda \geq \Lambda_k$, $0 < \varepsilon < \varepsilon_k$, *then* (1.1) *has* k *pairs of sign-changing solutions* $\pm v_{j,\epsilon}, j = 1, \dots, k$. *Moreover, for any* $\delta > 0$ *there exist* $\alpha > 0$, $c = c_k > 0$ *and* $\varepsilon_k(\delta) > 0$ *such that if* $0 < \varepsilon < \varepsilon_k(\delta)$, *then*
$$|v_{j,\varepsilon}(x)| \leq c \exp\big\{ - \frac{\alpha}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\big\}, \quad \text{for } x \in \mathbb{R}^N, j = 1, \dots, k.$$

For small ε and $4 < q < 2\cdot 2^*$, the authors in [6] established the existence of a sequence of localized nodal solutions concentrating near a given local minimum point of the potential function V, by developing new variational perturbation method to treat this class of non-smooth variational problems. There are few results for the case $2 < q < 4$. Motivated by their work, we will use the variational perturbation developed in [6] to deal with the existence and multiplicity of localized nodal solutions of (1.1), for the case $2 < q < 4$. Next we outline the approach.

First, we denote $u(x) = v(\varepsilon x)$. Then equation (1.1) is equivalent to
$$\begin{gathered}\sum_{i,j=1}^N \Big(D_j(b_{ij}(u)D_i u) - \frac{1}{2} D_z b_{ij}(u) D_i u D_j u\Big) - V(\varepsilon x)u + \lambda |u|^{q-2}u = 0, \\ u(x) \to 0 \quad \text{as } |x| \to \infty.\end{gathered} \tag{1.2}$$
We are looking for weak solutions to (1.2), namely a function $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying
$$\begin{aligned}&\int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(b_{ij}(u) D_i u D_j \varphi + \frac{1}{2} D_z b_{ij}(u) D_i u D_j u \varphi\Big)\, dx + \int_{\mathbb{R}^N} V(\varepsilon x) u\varphi \, dx \\ &= \lambda \int_{\mathbb{R}^N} |u|^{q-2} u\varphi \, dx\end{aligned}$$
for $\varphi \in C_0^\infty(\mathbb{R}^N)$. Formally problem (1.2) has a variational structure, given by the functional
$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N b_{ij}(u) D_i u D_j u \, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u^2 \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q \, dx, \quad u \in Y,$$
where
$$Y = \big\{u : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx < +\infty\big\}.$$

Now we define a truncation function and a perturbed functional. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi(s) = 1$ for $|s| \le 1; \varphi(s) = 0$ for $|s| \ge 2; |\varphi'(s)| \le 2, \varphi$ is even and decreasing in the interval [1,2]. For $\mu \in (0,1], x \in \mathbb{R}^N, z \in \mathbb{R}$ define

$$b_\mu(x,z) = \varphi(\mu \exp\{\operatorname{dist}(\mu x, M)\} z), \quad m_\mu(x,z) = \int_0^z b_\mu(x,\tau) d\tau. \tag{1.3}$$

Assume $x = 0 \in M$. For $x = 0$ we simply use the notation

$$b_\mu(z) = b_\mu(0,z) = \varphi(\mu z), \quad m_\mu(z) = m_\mu(0,z) = \int_0^z b_\mu(\tau) d\tau\,. \tag{1.4}$$

Let

$$\beta_{ij}(z) = b_{ij}(z) - \sigma(1+z^2)\delta_{ij}, \quad i,j = 1,\dots,N,$$

where $\sigma > 0$ is a fixed small positive constant so that $\beta_{ij}, i,j = 1,\dots,N$ also satisfy the assumptions (A1)-(A3) (with possibly different constants c_0 and δ). Now we define the perturbed functional $I_{\mu,\varepsilon}$ by

$$\begin{aligned} I_{\mu,\varepsilon}(u) =& \frac{1}{2}\sigma \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-2} |\nabla u|^2\, dx + \frac{1}{2}\sigma \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-4} u^2 |\nabla u|^2\, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u^2\, dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\, dx \end{aligned}$$

for $\mu \in (0,1]$, $u \in X = W^{1,m}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, where $m > 4$. Here we introduce one additional coercive term for perturbation because the problem on unbounded domain $\mathbb{R}^N$ and the imbedding from $W^{1,m}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is not compact. Moreover, we use the penalization method due to[1, 2, 3] to localize the solutions. For more results on standing waves, sign-changing solutions, ground state solutions and asymptotic behavior of solutions to quasilinear Schrödinger equations, we refer the reader to [1, 5, 10, 11].

Let $\zeta \in C_0^\infty(\mathbb{R})$ be such that $\zeta(t) = 0$ for $t \le 0$, $\zeta(t) = 1$ for $t \ge 1$, and $0 \le \zeta'(t) \le 2$. We define

$$\chi_\varepsilon(x) = \varepsilon^{-6}\zeta(\operatorname{dist}(x, M_\varepsilon)).$$

Let $E(x) = V(x) - \sigma$ and define

$$\begin{aligned} &\Gamma_{\mu,\varepsilon}(u) \\ &= \frac{1}{2}\sigma \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-2} |\nabla u|^2\, dx + \frac{1}{2}\sigma \int_{\mathbb{R}^N} \Big(\frac{u}{m_\varepsilon(x,u)}\Big)^{m-2} u^2\, dx \\ &\quad + \frac{1}{2}\sigma \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-4} u^2 |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \beta_{ij}(u) D_i u D_j u\, dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} E(\varepsilon x) u^2\, dx + \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)_+^\beta - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\, dx \end{aligned} \tag{1.5}$$

for $u \in X_\varepsilon = W_\varepsilon^{1,m}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $2 < \beta < q$, and

$$W_\varepsilon^{1,m}(\mathbb{R}^N) = W^{1,m}(\mathbb{R}^N) \cap L_\varepsilon^m(\mathbb{R}^N),$$

where $L_\varepsilon^m(\mathbb{R}^N)$ is a weighted L^m-spaces

$$L_\varepsilon^m(\mathbb{R}^N) = \big\{u \in L^m(\mathbb{R}^N), \int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\} |u|^m\, dx < +\infty\big\}$$

endowed with the norm

$$\|u\|_{L^m_\varepsilon(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u|^m \, dx \Big)^{\frac{1}{m}}$$

with a coercive weight. Then we know the space $W^{1,m}_\varepsilon(\mathbb{R}^N)$ is compactly imbedded to $L^p(\mathbb{R}^N)$ for $m \le p < m^* = \frac{Nm}{N-m}$, in particular the imbedding into $L^q(\mathbb{R}^N)$ is compact. If

$$|u(x)| \le \frac{1}{\varepsilon} \exp\{-\operatorname{dist}(\varepsilon x, M)\} \quad \text{for } x \in \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2 \, dx < 1,$$

then $\Gamma_{\mu,\varepsilon}(u) = I_{\mu,\varepsilon}(u)$. And if $|\nabla u(x)| \le \frac{1}{\mu}$ for $x \in \mathbb{R}^N$, then $I_{\mu,\varepsilon}(u) = I_\varepsilon(u)$. Here no limit process $\mu \to 0$ is needed for the existence of critical point of the original problem, and for small μ and ε, $\Gamma_{\mu,\varepsilon}$ shares critical points with I_ε, resulting in solutions of original equation for small μ and ε.

The article is organized as follows. In Section 2 we collect elementary properties of the auxiliary functions involved in the perturbed functionals and prove some technical results. In Section 3 we construct critical values of $\Gamma_{\mu,\varepsilon}$ by the method of invariant sets with respect to the descending flow. In Section 4 we prove the uniform bound for the gradient of the approximate sign-changing solutions obtained in Section 3 and complete the proof of Theorem 1.1.

Also we fix some notations $c, c_0, c_1, \dots$ denote possibly different positive constants, and $c(\mu)$, if necessary, denotes constants depending on μ. In a given Banach space, $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively.

2. Properties of auxiliary functions

In this section,we first recall some elementary properties and some estimates on the auxiliary functions involved in the perturbations of the functionals, and the following three lemmas whose proofs are quite the same as that of the results in [6] and omit it here.

Lemma 2.1. *For $s > 0$, $z \in \mathbb{R}$, $x \in \mathbb{R}^N$, $p = (p_i) \in \mathbb{R}^N$, $\xi = (\xi_i) \in \mathbb{R}^N$, the following statements hold:*
(1) $0 \le b_\mu(x, s) \le \frac{m_\mu(x,s)}{s} \le 1$.
(2) $m_\mu(x, s) = s$, *if* $s < \mu^{-1} \exp\{-\operatorname{dist}(\mu x, M)\}$;

$$\mu^{-1} \exp\{-\operatorname{dist}(\mu x, M)\} \le m_\mu(x, s) \le c\mu^{-1} \exp\{-\operatorname{dist}(\mu x, M)\},$$

if $\mu^{-1} \exp\{-\operatorname{dist}(\mu x, M)\} \le s \le 2\mu^{-1} \exp\{-\operatorname{dist}(\mu x, M)\}$;

$$m_\mu(x, s) = c\mu^{-1} \exp\{-\operatorname{dist}(\mu x,\, M)\},$$

if $s \ge 2\mu^{-1} \exp\{-\operatorname{dist}(\mu x,\, M)\}$, *where* $c = \int_0^\infty \varphi(\tau)\, d\tau$.
(3) *We define* $f_\mu(p) = \frac{1}{2}\sigma\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-2}|p|^2$. *Then*
(3.1) $c_1(1 + \mu^{m-2}|p|^{m-2})|p|^2 \le f_\mu(p) \le c_2(1 + \mu^{m-2}|p|^{m-2})|p|^2$;
(3.2) $2f_\mu(p) \le \nabla_p f_\mu(p) \cdot p \le |\nabla f_\mu(p)| \cdot |p| \le m f_\mu(p)$;
(3.3) $\sum_{i,j=1}^N \frac{\partial^2}{\partial p_i \partial p_j} f_\mu(p) \xi_i \xi_j \ge \sigma\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-2} |\xi|^2 \ge c(1 + \mu^{m-2}|p|^{m-2})|\xi|^2$;
(3.4) $|\frac{\partial^2}{\partial p_i \partial p_j} f_\mu(p)| \le c\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-2} \le c(1 + \mu^{m-2}|p|^{m-2})$.

(4) *We define* $k_\varepsilon(x, z) = \frac{1}{2}\sigma\big(\frac{z}{m_\varepsilon(x,z)}\big)^{m-2} z^2$. *Then*

(4.1)
$$\begin{aligned} &c_1(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,M)\}|z|^{m-2})z^2 \le k_\varepsilon(x,z)\\ &\le c_2(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,M)\}|z|^{m-2})z^2\,; \end{aligned}$$

(4.2) $2k_\varepsilon(x,z)\le \frac{\partial}{\partial z}k_\varepsilon(x,z)z\le mk_\varepsilon(x,z)$;

(4.3)
$$\begin{aligned} \frac{\partial^2}{\partial z^2}k_\varepsilon(x,z) &\ge \sigma\left(\frac{z}{m_\mu(x,z)}\right)^{m-2}\\ &\ge c(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,M)\}|z|^{m-2})\,; \end{aligned}$$

(4.4)
$$\begin{aligned} 0\le \frac{\partial^2}{\partial z^2}k_\varepsilon(x,z) &\le c\left(\frac{z}{m_\mu(x,z)}\right)^{m-2}\\ &\le c(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,M)\}|z|^{m-2})\,. \end{aligned}$$

(5) *We define* $h_\mu(z,p)=\frac12\sigma\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}z^2|p|^2$. *Then*

(5.1) $c_1(1+\mu^{m-4}|p|^{m-4})z^2|p|^2\le h_\mu(z,p)\le c_2(1+\mu^{m-4}|p|^{m-4})z^2|p|^2$;

(5.2)
$$\begin{aligned} 4h_\mu(z,p) &\le \nabla_p h_\mu(z,p)p+\frac{\partial}{\partial z}h_\mu(z,p)z\\ &\le |\nabla_p h_\mu(z,p)|\,|p|+\big|\frac{\partial}{\partial z}h_\mu(z,p)\big|\,|z|\\ &\le mh_\mu(z,p)\,; \end{aligned}$$

(5.3)
$$\begin{gathered} \begin{aligned} \sum_{i,j=1}^N\frac{\partial^2}{\partial p_i\partial p_j}h_\mu(z,p)\xi_i\xi_j &\ge \sigma\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}z^2|\xi|^2\\ &\ge c(1+\mu^{m-4}|p|^{m-4})z^2|\xi|^2, \end{aligned}\\ \frac{\partial^2}{\partial z^2}h_\mu(z,p)=\sigma\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}|p|^2\ge c(1+\mu^{m-4}|p|^{m-4})|p|^2\,; \end{gathered}$$

(5.4) $\big|\frac{\partial^2}{\partial p_i\partial p_j}h_\mu(z,p)\big|\le c\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}z^2\le c(1+\mu^{m-4}|p|^{m-4})z^2$,
$\big|\frac{\partial^2}{\partial z^2}h_\mu(z,p)\big|\le c\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}|p|^2\le c(1+\mu^{m-4}|p|^{m-4})|p|^2$,
$$\begin{aligned} \big|\nabla_p\frac{\partial}{\partial z}h_\mu(z,p)\big| &= \big|\frac{\partial}{\partial z}\nabla_p h_\mu(z,p)\big|\\ &\le c\big(\frac{|p|}{m_\mu(|p|)}\big)^{m-4}|z|\,|p|\\ &\le c(1+\mu^{m-4}|p|^{m-4})|z|\,|p|\,. \end{aligned}$$

Lemma 2.2. *For* $x\in\mathbb{R}^N$, $p,\overline{p}\in\mathbb{R}^N$, *and* $z,\overline{z}\in\mathbb{R}^N$, *the following three properties hold:*

(1)
$$\begin{aligned} \langle\nabla_p f_\mu(p)-\nabla_p f_\mu(\overline{p}),p-\overline{p}\rangle &\ge c(1+\mu^{m-2}(|p|^{m-2}-|\overline{p}|^{m-2}))|p-\overline{p}|^2\\ &\ge c|p-\overline{p}|^2+c\mu^{m-2}|p-\overline{p}|^m, \end{aligned}$$

$$|\nabla_p f_\mu(p) - \nabla_p f_\mu(\overline{p})| \le c(1 + \mu^{m-2}(|p|^{m-2} - |\overline{p}|^{m-2}))|p - \overline{p}|.$$

(2)

$$\begin{aligned}
&\Big(\frac{\partial}{\partial z}k_\varepsilon(x,z) - \frac{\partial}{\partial z}k_\varepsilon(x,\overline{z})\Big)(z-\overline{z}) \\
&\ge c\big(1 + \varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}(|z|^{m-2} + |\overline{z}|^{m-2})\big)|z-\overline{z}|^2 \\
&\ge c|z-\overline{z}|^2 + c\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|z-\overline{z}|^{m-2}, \\
&\big|\frac{\partial}{\partial z}k_\varepsilon(x,z) - \frac{\partial}{\partial z}k_\varepsilon(x,\overline{z})\big| \\
&\le c\big(1 + \varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}(|z|^{m-2} + |\overline{z}|^{m-2})\big)|z-\overline{z}|.
\end{aligned}$$

(3)

$$\begin{aligned}
&\langle \nabla_p h_\mu(z,p) - \nabla_p h_\mu(\overline{z},\overline{p}), p - \overline{p}\rangle + \big(\frac{\partial}{\partial z}h_\mu(z,p) - \frac{\partial}{\partial z}h_\mu(\overline{z},\overline{p})\big)(z-\overline{z}) \\
&\ge c\left(|p|^2 + |\overline{p}|^2 + \mu^{m-4}(|p|^{m-2} + |\overline{p}|^{m-2})\right)|z-\overline{z}|^2 \\
&\quad - \nu\left(1 + \mu^{m-2}(|p|^{m-2} + |\overline{p}|^{m-2})\right)|p-\overline{p}|^2 - c_\nu \mu^{-2}(|z|^{m-2} + |\overline{z}|^{m-2})|z-\overline{z}|^2, \\
&\left|\nabla_p h_\mu(z,p) - \nabla_p h_\mu(\overline{z},\overline{p})\right| \\
&\le c(1 + \mu^{m-4}(|p|^{m-4} + |\overline{p}|^{m-4}))\left((z^2 + \overline{z}^2)|p-\overline{p}| + (|z| + |\overline{z}|)(|p| + |\overline{p}|)|z-\overline{z}|\right), \\
&\big|\frac{\partial}{\partial z}h_\mu(z,p) - \frac{\partial}{\partial z}h_\mu(\overline{z},\overline{p})\big| \\
&\le c(1 + \mu^{m-4}(|p|^{m-4} + |\overline{p}|^{m-4}))\left((|z| + |\overline{z}|)(|p| + |\overline{p}|)|p-\overline{p}| + (|p|^2 + |\overline{p}|^2)|z-\overline{z}|\right),
\end{aligned}$$

where $\nu > 0$ is any small constant, and c_ν depends on ν.

Lemma 2.3. *Let $J_{\mu,\varepsilon}$ be the functional defined on X_ε by*

$$\begin{aligned}
J_{\mu,\varepsilon}(u) =& \frac{1}{2}\sigma\int_{\mathbb{R}^N}\Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-2}|\nabla u|^2 dx + \frac{1}{2}\sigma\int_{\mathbb{R}^N}\Big(\frac{u}{m_\varepsilon(x,u)}\Big)^{m-2}u^2 dx \\
&+ \frac{1}{2}\sigma\int_{\mathbb{R}^N}\Big(\frac{|\nabla u|}{m_\mu(|\nabla u|)}\Big)^{m-4}u^2|\nabla u|^2 dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}\sum_{i,j=1}^N \beta_{ij}(u)D_i u D_j u dx + \frac{1}{2}\int_{\mathbb{R}^N}E(\varepsilon x)u^2 dx\,.
\end{aligned} \tag{2.1}$$

Then for $u, v, \varphi \in X_\varepsilon$, we have:

(1)

$$\begin{aligned}
&\langle DJ_{\mu,\varepsilon}(u) - DJ_{\mu,\varepsilon}(v), u - v\rangle \\
&\ge c\mu^{m-2}\int_{\mathbb{R}^N}|\nabla u - \nabla v|^m dx + c\varepsilon^{m-2}\int_{\mathbb{R}^N}\exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u-v|^m\, dx \\
&\quad + c\int_{\mathbb{R}^N}|\nabla u - \nabla v|^2\, dx - c\mu^{-2}\int_{\mathbb{R}^N}(|u|^{m-2} + |v|^{m-2})(u-v)^2\, dx \\
&\quad - c\mu^{-2}\int_{\mathbb{R}^N}(u-v)^2\, dx \\
&\ge c_{\mu,\varepsilon}\|u-v\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)} + c\|u-v\|^2_{H^1(\mathbb{R}^N)} \\
&\quad - c\mu^{-2}\int_{\mathbb{R}^N}(|u|^{m-2} + |v|^{m-2})(u-v)^2\, dx - c\mu^{-2}\int_{\mathbb{R}^N}(u-v)^2 dx\,,
\end{aligned}$$

(2)

$$
\begin{aligned}
&|\langle DJ_{\mu,\varepsilon}(u) - DJ_{\mu,\varepsilon}(v), \varphi\rangle| \\
&\le c\|u-v\|_{H^1(\mathbb{R}^N)}\|\varphi\|_{H^1(\mathbb{R}^N)} \\
&\quad + c\left(\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)} + \|v\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\right)\|u-v\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\|\varphi\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}.
\end{aligned}
$$

3. Construction of critical points of $\Gamma_{\mu,\varepsilon}$

In this section, we will adopt the method of invariant sets of descending flow developed in [4] to obtain multiple sign-changing critical points of the perturbed functional $\Gamma_{\mu,\varepsilon}$. For the reader's convenience, we first give an abstract critical point theorem, which has been proved in [9].

Let X be a Banach space, f be an even C^1-functional on X. Let P_j, Q_j, $j = 1,\dots,k$ be a family of open convex sets of $X, Q_j = -P_j, j = 1,\dots,k$. Set

$$W = \cup_{j=1}^k (P_j \cup Q_j), \quad \Sigma = \cap_{j=1}^k (\partial P_j \cap \partial Q_j).$$

Assume

- (A7) f satisfies the Palais-Smale condition,
- (A8) $c^* = \inf_{x\in\Sigma} f(x) > 0$,

and assume there exists an odd continuous map $A: X \to X$ satisfying

- (A9) For $c_0, b_0 > 0$, there exists $b = b(c_0, b_0) > 0$ such that if $\|Df(x)\| \ge b_0$, $|f(x)| \le c_0$, then
$$\langle Df(x), x - Ax\rangle \ge b\|x - Ax\| > 0\,.$$
- (A10) $A(\partial P_j) \subset P_j$, $A(\partial Q_j) \subset Q_j$, $j = 1,\dots,k$.

We define

$$
\begin{aligned}
\Gamma_j &= \{E \subset X : E \text{ is compact}, \ -E = E,\ \gamma(E\cap \eta^{-1}(\Sigma)) \ge j \text{ for } \eta\in\Lambda\}, \\
\Lambda &= \Big\{\eta \in C(X, X) : \eta \text{ is odd}, \eta(P_j)\subset P_j,\ \eta(Q_j)\subset Q_j,\ j = 1,\dots,k, \\
&\qquad \eta(x) = x \text{ if } f(x) < 0\Big\}
\end{aligned}
$$

where γ is the genus of symmetric sets,

$$\gamma(E) = \inf\big\{n : \text{ there exists an odd map } \eta : E \to \mathbb{R}^n\backslash\{0\}\big\}\,.$$

We define the assumption

- (A11) Γ_j is nonempty,

and the notation

$$
\begin{gathered}
c_j = \inf_{A\in\Gamma_j} \sup_{x\in A\backslash W} f(x), \quad j = 1, 2, \dots\,, \\
K_c = \{x : Df(x) = 0,\ f(x) = c\}, \quad K_c^* = K_c \setminus W\,.
\end{gathered}
$$

Theorem 3.1. *Assume* (A7)–(A11) *hold. Then*

1. $c_j \ge c^*$, $K^*_{c_j} \ne \emptyset$.
2. $c_j \to \infty$, *as* $j \to \infty$.
3. *If* $c_j = c_{j+1} = \dots = c_{j+k-1} = c$, *then* $\gamma(K_c^*) \ge k$.

In the following we verify that the functional $\Gamma_{\mu,\varepsilon}$ satisfies all the assumptions of Theorem 3.1. First we prove that the functional $\Gamma_{\mu,\varepsilon}$ satisfies the Palais-Smale condition, i.e. assumption (A7).

Lemma 3.2. $\Gamma_{\mu,\varepsilon}$ *is differentiable and satisfies the Palais-Smale condition.*

Proof. For $u, \varphi \in X_\varepsilon$, we have

$$
\begin{aligned}
&\langle D\Gamma_{\mu,\varepsilon}(u), \varphi\rangle \\
&= \int_{\mathbb{R}^N} \nabla_p f_\mu(\nabla u)\nabla\varphi dx + \int_{\mathbb{R}^N} \frac{\partial}{\partial z} k_\varepsilon(x,u)\varphi\, dx \\
&\quad + \int_{\mathbb{R}^N} \Big(\nabla_p h_\mu(u,\nabla u)\nabla\varphi + \frac{\partial}{\partial z} h_\mu(u,\nabla u)\varphi\Big)\, dx \\
&\quad + \int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(\beta_{ij}(u)D_i u D_j\varphi + \frac{1}{2} D_z\beta_{ij}(u) D_i u D_j u\varphi\Big) dx + \int_{\mathbb{R}^N} E(\varepsilon x) u\varphi\, dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\, dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x) u\varphi\, dx - \lambda \int_{\mathbb{R}^N} |u|^{q-2} u\varphi\, dx.
\end{aligned}
$$

Since the imbedding from $W^{1,m}_\varepsilon(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ is compact, there exists $c > 0$ such that

$$
\|u\|_{L^q(\mathbb{R}^N)} \le c\|u\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}.
$$

Let $\{u_n\} \subset X_\varepsilon$ be a Palais-Smale sequence of $\Gamma_{\mu,\varepsilon}$, namely, there exists $L > 0$ such that $|\Gamma_{\mu,\varepsilon}(u_n)| \le L$ and $D\Gamma_{\mu,\varepsilon}(u_n) \to 0$ as $n \to \infty$. By Lemma 2.1 and assumption (A3), we deduce

$$
\begin{aligned}
&\Gamma_{\mu,\varepsilon}(u_n) \\
&= \int_{\mathbb{R}^N} f_\mu(\nabla u_n)\, dx + \int_{\mathbb{R}^N} k_\varepsilon(x,u_n)\, dx + \int_{\mathbb{R}^N} h_\mu(u_n,\nabla u_n)\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \beta_{ij}(u_n) D_i u_n D_j u_n\, dx + \frac{1}{2}\int_{\mathbb{R}^N} E(\varepsilon x) u_n^2\, dx \\
&\quad + \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u_n^2 dx - 1\Big)_+^{\beta} - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u_n|^q\, dx \\
&\ge c\Big\{\mu^{m-2}\int_{\mathbb{R}^N} |\nabla u_n|^m dx + \varepsilon^{m-2}\int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u_n|^m\, dx \\
&\quad + \mu^{m-4}\int_{\mathbb{R}^N} |\nabla u_n|^{m-2}u_n^2\, dx\Big\} + c\Big(\int_{\mathbb{R}^N} (1+u_n^2)|\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2\, dx\Big) \qquad (3.1)\\
&\quad + c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u_n^2\, dx - 1\Big)_+^{\beta} - c\int_{\mathbb{R}^N} u_n^q\, dx \\
&\ge c\Big\{\mu^{m-2}\int_{\mathbb{R}^N} |\nabla u_n|^m\, dx + \varepsilon^{m-2}\int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u_n|^m\, dx \\
&\quad + \mu^{m-4}\int_{\mathbb{R}^N} |\nabla u_n|^{m-2}u_n^2 dx\Big\} + c\Big(\int_{\mathbb{R}^N} (1+u_n^2)|\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2 dx\Big) \\
&\quad + c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u_n^2\, dx - 1\Big)_+^{\beta} \\
&\quad - c\Big\{\int_{\mathbb{R}^N} |\nabla u_n|^m\, dx + \int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u_n|^m\, dx\Big\}^{q/m}
\end{aligned}
$$

which implies that $\{u_n\}$ is bounded in X_ε and $\big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u_n^2 dx - 1\big)_+^{\beta}$ is bounded. Assume $u_n \rightharpoonup u$ in X_ε and $u_n \to u$ in $L^s(\mathbb{R}^N), 2 \le s < 2\cdot 2^*$. By Lemma 2.3, we have

$$
\begin{aligned}
o(1) &= \langle D\Gamma_{\mu,\varepsilon}(u_n) - D\Gamma_{\mu,\varepsilon}(u_m), u_n - u_m\rangle \\
&= \langle DJ_{\mu,\varepsilon}(u_n) - DJ_{\mu,\varepsilon}(u_m), u_n - u_m\rangle \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n(u_n - u_m)\,dx \\
&\quad - \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_m^2\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n(u_n - u_m)\,dx \\
&\quad - \int_{\mathbb{R}^N} (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m)(u_n - u_m)\,dx \\
&\geq c\Big(\mu^{m-2}\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^m\,dx \\
&\quad + \varepsilon^{m-2}\int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, M)\}|u_n - u_m|^m\,dx\Big) \\
&\quad + c\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^2\,dx - c\mu^{-2}\Big(\int_{\mathbb{R}^N} \big(|u_n|^{m-2} + |u_m|^{m-2}\big)|u_n - u_m|^2\,dx \\
&\quad + \int_{\mathbb{R}^N} |u_n - u_m|^2\,dx\Big) + o(1) \\
&\geq c\|u_n - u_m\|^m_{W_\varepsilon^{1,m}(\mathbb{R}^N)} + c\|u_n - u_m\|^2_{H^1(\mathbb{R}^N)} + o(1).
\end{aligned}
$$

So $\{u_n\}$ is a Cauchy sequence in X_ε, hence a convergent sequence. □

We define the operator $A : X_\varepsilon \to X_\varepsilon$. Given $u \in X_\varepsilon$, for a suitable constant $c_\mu > 0$, we define $v = Au \in X_\varepsilon$:

$$
\begin{aligned}
&\langle DJ_{\mu,\varepsilon}(v), \varphi\rangle + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)v\varphi\,dx \\
&+ c_\mu \int_{\mathbb{R}^N} (|v|^{m-2}v + v)\varphi\,dx \\
&= \lambda \int_{\mathbb{R}^N} |u|^{q-2}u\varphi\,dx + c_\mu \int_{\mathbb{R}^N} (|u|^{m-2}u + u)\varphi\,dx, \quad \text{for } \varphi \in X_\varepsilon
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
J_{\mu,\varepsilon}(u) &= \int_{\mathbb{R}^N} \big(f_\mu(\nabla u) + k_\varepsilon(x, \nabla u) + h_\mu(u, \nabla u)\big)\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \beta_{ij}(u)D_iuD_ju\,dx + \frac{1}{2}\int_{\mathbb{R}^N} E(\varepsilon x)u^2\,dx, \quad \text{for } u \in X_\varepsilon.
\end{aligned}
\tag{3.3}
$$

In view of [6, Lemma 4.1], we know that for sufficiently large $c_\mu > 0$ the operator A is well-defined and continuous. And similar to [6], we can prove the following lemmas 3.3–3.6.

Lemma 3.3. *There exist constants $D > 0$ and $\alpha \in (\frac{2}{q}, 1)$ such that*

$$
\int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(\beta_{ij}(u) + \frac{1}{2}uD_z\beta_{ij}(u)\Big)D_iuD_ju\,dx + \int_{\mathbb{R}^N} E(\varepsilon x)u^2\,dx \geq D\Big(\int_{\mathbb{R}^N} |u|^q\,dx\Big)^\alpha.
$$

Now we define

$$
\begin{aligned}
Q = Q_\delta = \Big\{ u \in X_\varepsilon : \frac{1}{2} D \Big(\int_{\mathbb{R}^N} u_+^q \, dx \Big)^\alpha + \frac{m-1}{m} c_\mu \int_{\mathbb{R}^N} u_+^m \, dx \\
+ \frac{1}{2} c_\mu \int_{\mathbb{R}^N} u_+^2 \, dx < \delta \Big\}, \\
P = -Q = \Big\{ u \in X_\varepsilon : \frac{1}{2} D \Big(\int_{\mathbb{R}^N} u_-^q \, dx \Big)^\alpha + \frac{m-1}{m} c_\mu \int_{\mathbb{R}^N} u_-^m \, dx \\
+ \frac{1}{2} c_\mu \int_{\mathbb{R}^N} u_-^2 \, dx < \delta \Big\}.
\end{aligned}
$$

Lemma 3.4. *There exists $\delta_0 = \delta_0(\mu)$ such that for $\delta \le \delta_0$*

$$
A(\partial P) \subset P, \quad A(\partial Q) \subset Q.
$$

Lemma 3.5. *There exist $\delta_0 = \delta_0(\mu), c^* = c^*(\delta, \mu)$ such that*

$$
\Gamma_{\mu,\varepsilon}(u) \ge c^* \quad \text{for } u \in \partial P \cap \partial Q.
$$

Lemma 3.6. *Let $u \in X_\varepsilon$, $v = Au$, then it holds*
(1)

$$
\langle D\Gamma_{\mu,\varepsilon}(u), u - v \rangle \ge c \Big(\|u - v\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)} + \|u - v\|^2_{H^1(\mathbb{R}^N)} \Big).
$$

(2)

$$
\begin{aligned}
\langle D\Gamma_{\mu,\varepsilon}(u), \varphi \rangle &\le c \left(\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)} + \|v\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)} \right) \|u - v\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)} \|\varphi\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)} \\
&\quad + c \Big(1 + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2 dx - 1 \Big)_+^{\beta-1} \Big) \|u - v\|_{H^1(\mathbb{R}^N)} \|\varphi\|_{H^1(\mathbb{R}^N)}
\end{aligned}
$$

for all $\varphi \in X_\varepsilon$.

Lemma 3.7. *Let $u \in X_\varepsilon$, $v = Au$. Assume $|\Gamma_{\mu,\varepsilon}(u)| \le c_0$, $\|D\Gamma_{\mu,\varepsilon}(u)\| \ge b_0$. Then there exists $b = b(c_0, b_0)$ such that*

$$
\langle D\Gamma_{\mu,\varepsilon}(u), \, u - v \rangle \ge b \|u - v\|_{X_\varepsilon} > 0.
$$

Proof. By Lemma 2.1, we have

$$
\begin{aligned}
&\Gamma_{\mu,\varepsilon}(u) - \frac{1}{2q} \langle DJ_{\mu,\varepsilon}(u) - DJ_{\mu,\varepsilon}(v), u \rangle \\
&= \Gamma_{\mu,\varepsilon}(u) - \frac{1}{2q} \langle DJ_{\mu,\varepsilon}(u), u \rangle + \frac{1}{2q} \langle DJ_{\mu,\varepsilon}(v), u \rangle \\
&= \int_{\mathbb{R}^N} \Big(f_\mu(\nabla u) - \frac{1}{2q} \nabla_p f_\mu(\nabla u) \nabla u \Big) dx + \int_{\mathbb{R}^N} \Big(k_\varepsilon(x, u) - \frac{1}{2q} \frac{\partial}{\partial z} k_\varepsilon(x, u) u \Big) dx \\
&\quad + \int_{\mathbb{R}^N} \Big(h_\mu(u, \nabla u) - \frac{1}{2q} \big(\nabla_p h_\mu(u, \nabla u) \nabla u + \frac{\partial}{\partial z} h_\mu(u, \nabla u) u \big) \Big) dx \\
&\quad + \int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(\frac{1}{2} \beta_{ij}(u) - \frac{1}{2q} \big(\beta_{ij}(u) + \frac{1}{2} u D_z \beta_{ij}(u) \big) \Big) D_i u D_j u \, dx \\
&\quad + \big(\frac{1}{2} - \frac{1}{2q} \big) \int_{\mathbb{R}^N} E(\varepsilon x) u^2 \, dx \\
&\quad + \frac{1}{2\beta} \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2 \, dx - 1 \Big)_+^\beta - \frac{1}{2q} \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2 \, dx - 1 \Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x) u v \, dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2q}c_\mu\int_{\mathbb{R}^N}\big(|u|^{m-2}u-|v|^{m-2}v\big)u\,dx+\frac{1}{2q}c_\mu\int_{\mathbb{R}^N}(u-v)udx-\frac{\lambda}{2q}\int_{\mathbb{R}^N}|u|^q\,dx\\
&\geq c\Big(\|u\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u\|^2_{H^1(\mathbb{R}^N)}\Big)-c+c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_+\\
&\quad-c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(u-v)^2\,dx\Big)^\beta_++\frac{1}{2q}c_\mu\int_{\mathbb{R}^N}\big(|u|^{m-2}u-|v|^{m-2}v\big)u\,dx\\
&\quad+\frac{1}{2q}c_\mu\int_{\mathbb{R}^N}(u-v)u\,dx.
\end{aligned}
$$

The above estimate holds since

$$
\begin{aligned}
&\frac{1}{2\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_+-\frac{1}{2q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^{\beta-1}_+\int_{\mathbb{R}^N}\chi_\varepsilon(x)uv\,dx\\
&=\frac{1}{2\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_++\frac{1}{2q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^{\beta-1}_+\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx\\
&\quad-\frac{1}{2q}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^{\beta-1}_+\int_{\mathbb{R}^N}\chi_\varepsilon(x)u(u-v)\,dx\\
&\geq(\frac{1}{2\beta}-\frac{\eta}{2q})\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_+\\
&\quad+(\frac{1}{2q}-\frac{\eta}{2q})\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^{\beta-1}_+\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx\\
&\quad-c_\eta\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(u-v)^2\,dx\Big)^\beta_+\\
&\geq c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_+-c-c_\eta\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(u-v)^2\,dx\Big)^\beta_+,
\end{aligned}
$$

where $0<\eta<1$, c_η is a constant. By Lemma 2.3 (2), we obtain

$$
\begin{aligned}
&\|u\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u\|^2_{H^1(\mathbb{R}^N)}+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)^\beta_+\\
&\leq c(1+|\Gamma_{\mu,\varepsilon}(u)|)+c|\langle DJ_{\mu,\varepsilon}(u)-DJ_{\mu,\varepsilon}(v),u\rangle|+c\Big|\int_{\mathbb{R}^N}\big(|u|^{m-2}-|v|^{m-2}v\big)\,u\,dx\Big|\\
&\quad+c\Big|\int_{\mathbb{R}^N}(u-v)u\,dx\Big|+c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(u-v)^2\,dx\Big)^\beta_+\\
&\leq c(1+|\Gamma_{\mu,\varepsilon}(u)|)+c\Big(\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\Big)\|u-v\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\|u\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\\
&\quad+c\|u-v\|_{H^1(\mathbb{R}^N)}\|u\|_{H^1(\mathbb{R}^N)}+c\|u-v\|^{2\beta}_{L^2(\mathbb{R}^N)}\\
&\leq c\Big(1+|\Gamma_{\mu,\varepsilon}(u)|+\|u-v\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u-v\|^2_{H^1(\mathbb{R}^N)}+\|u-v\|^{2\beta}_{L^2(\mathbb{R}^N)}\Big)\\
&\quad+\tau\big(\|u\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u\|^2_{H^1(\mathbb{R}^N)}\big)
\end{aligned}
$$

for $\tau\in(0,1)$. Therefore

$$
\begin{aligned}
&\|u\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u\|^2_{H^1(\mathbb{R}^N)}+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2dx-1\Big)^\beta_+\\
&\leq c\Big(1+|\Gamma_{\mu,\varepsilon}(u)|+\|u-v\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u-v\|^2_{H^1(\mathbb{R}^N)}\Big).
\end{aligned}
\tag{3.4}
$$

By Lemma 3.6 (2), we obtain

$$\|D\Gamma_{\mu,\varepsilon}(u)\|$$

$$
\begin{aligned}
&\le c\Big(\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\Big)\|u-v\|_{W^{1,m}_\varepsilon(\mathbb{R}^N)}\\
&\quad+c\Big(1+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2dx-1\Big)_+^{\beta-1}\Big)\|u-v\|_{H^1(\mathbb{R}^N)}\\
&\le c\Big(\|u\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+1+\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2dx-1\big)_+^{\beta-1}\Big)\|u-v\|_{X_\varepsilon}\\
&\le c\Big(\|u\|^{m}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|v\|^{m}_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+1+\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2dx-1\big)_+^{\beta}\Big)\|u-v\|_{X_\varepsilon}\\
&\le c\Big(1+|\Gamma_{\mu,\varepsilon}(u)|+\|u-v\|^m_{W^{1,m}_\varepsilon(\mathbb{R}^N)}+\|u-v\|^2_{H^1(\mathbb{R}^N)}\\
&\quad+\|u-v\|^{2\beta}_{L^2(\mathbb{R}^N)}\Big)\|u-v\|_{X_\varepsilon}\\
&\le c\left(1+|\Gamma_{\mu,\varepsilon}(u)|+\|u-v\|^\alpha_{X_\varepsilon}\right)\|u-v\|_{X_\varepsilon}\\
&\le c\left(1+|\Gamma_{\mu,\varepsilon}(u)|+\|u-v\|_{X_\varepsilon}\right)^\alpha\|u-v\|_{X_\varepsilon}
\end{aligned}
$$

for $\alpha=\max\{m,2,2\beta\}$. Lemma 3.7 follows from the above inequality and Lemma 3.6 (1). □

Now we consider the assumption (A11).

Lemma 3.8. *Assume $B=\{x\in\mathbb{R}^N:|x|\le r\}\subset M$. Let $\{e_k\}_{k=1}^\infty$ be a family of linearly independent functions in $C_0^\infty(B)$. There exist $R_k,d_k>0$ such that for $\lambda>d_k$,*

$$J_0(u)<0 \quad \text{for } u\in E_k,\ \|u\|=R_k$$

where $E_k=\operatorname{span}\{e_1,\dots,e_k\}$ and

$$
\begin{aligned}
J_0(u)&=\sigma\int_{\mathbb{R}^N}\Big(|\nabla u|^m+e^{(m-2)|x|}|u|^m+|u|^{m-2}|\nabla u|^2\Big)\,dx\\
&\quad+\frac12\int_{\mathbb{R}^N}\sum_{i,j=1}^N b_{ij}(u)D_iuD_ju\,dx+\frac12\int_{\mathbb{R}^N}V_\infty u^2\,dx-\frac{\lambda}{q}\int_{\mathbb{R}^N}|u|^q\,dx
\end{aligned}
$$

and $V_\infty=\sup_{x\in\mathbb{R}^N}V(x)$.

Proof. Since E_k is finite-dimensional, all norms are equivalent, there exist constants $c_k,d_k>0$ such that

$$
\begin{gathered}
\frac1q\int_{\mathbb{R}^N}|u|^q\,dx=c_k,\\
J_0(u)+\frac{\lambda}{q}\int_{\mathbb{R}^N}|u|^q\,dx<b_k
\end{gathered}
$$

for $u\in H_k,\|u\|=R_k$. It is easy to see that $d_k=b_k/c_k$ satisfies the condition. □

We define $\varphi_k\in C(B_k,\,C_0^\infty(B))$ as

$$\varphi_k(t)=R_k\sum_{i=1}^k t_ie_i,\quad t=(t_1,\dots,t_k)\in B_k=\{t\,|\,t\in\mathbb{R}^N,|t|\le1\}.$$

Let

$$
\begin{gathered}
\Gamma_j=\{E\subset X_\varepsilon:E\text{ is compact }-E=E,\,\gamma(E\cap\eta^{-1}(\Sigma))\ge j\text{ for }\eta\in\Lambda\}\\
\Lambda=\{\eta\in C(X_\varepsilon,X_\varepsilon):\eta\text{ is odd }\eta(p)\subset P,\,\eta(Q)\subset Q,\,\eta(u)=u\text{ if }\Gamma_{\mu,\varepsilon}(u)\le0\}.
\end{gathered}
$$

Moreover, we define $\Lambda_k=\max_{1\le i\le k+1}\{d_i\}$. Then we have the following result.

Lemma 3.9. *For $\lambda \geq \Lambda_k$, $E_j = \varphi_{j+1}(B_{j+1}) \subset \Gamma_j, j = 1, \dots, k$, so Γ_j is nonempty.*

Proof. It is obviously that

$$\varphi_{j+1}(-t) = -\varphi_{j+1}(t), \quad \varphi_{j+1}(0) = 0 \in W.$$

For $t \in B_{j+1}$, $u = \varphi_{j+1}(t)$, $\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\right)_+^\beta = 0$, we have

$$\Gamma_{\mu,\varepsilon}(u) \leq J_0(u), \quad u \in \varphi_{j+1}(B_{j+1}), \quad \mu, \varepsilon \in (0,1].$$

By Lemmas 3.6 and 3.7, we obtain

$$\begin{gathered}\varphi(t) \notin W, \quad \text{if } t \in \partial B_{j+1}, \\ \sup_{t \in \partial B_{j+1}} \Gamma_{\mu,\varepsilon}(\varphi_{j+1}(t)) \leq 0\,.\end{gathered}$$

By [5, Lemma 4.2], we have $\gamma(E_j \cup \eta^{-1}(\Sigma)) \geq j$ for all $\eta \in \Lambda$, hence $E_j \subset \Gamma_j$. □

All the assumptions of Theorem 3.1 are satisfied, and we have the following existence theorem.

Theorem 3.10. *Assume* (A1)–(A6). *Then for any positive integer k, the functional $\Gamma_{\mu,\varepsilon}(\mu, \varepsilon \in (0,1])$ has k sign-changing critical points, the corresponding critical values are defined as*

$$c_j(\mu,\varepsilon) = \inf_{E \in \Gamma_j} \sup_{u \in E \setminus M} \Gamma_{\mu,\varepsilon}(u), \quad j = 1, \dots, k. \tag{3.5}$$

Moreover
(1) there exist $m_j, j = 1, \dots, k$, independent of μ, ε such that

$$c_j(\mu,\varepsilon) \leq m_j, \quad j = 1, \dots, k. \tag{3.6}$$

(2) If $c_j(\mu,\varepsilon) = \dots = c_{j+k-1}(\mu,\varepsilon) = c$, then $\gamma(K^*(c)) \geq k$.

Proof. By Theorem 3.1, we know that $\Gamma_{\mu,\varepsilon}(\mu, \varepsilon \in (0,1])$ has k sign-changing critical points, so we only need to prove estimate (3.6). Since $E_j = \varphi_{j+1}(B_{j+1}) \in \Gamma_j$ and $\Gamma_{\mu,\varepsilon}(u) \leq J_0(u)$ for $u \in \varphi_{j+1}(B_{j+1})$ and $\mu, \varepsilon \in (0,1]$, , we have

$$c_j(\mu,\varepsilon) \leq m_j := \sup_{u \in E_j} J_0(u).$$

The second part of the theorem is the direct result of the Theorem 3.1. □

4. Proof of Theorem 1.1

In this section, we prove that the perturbed functionals shares critical points with the original problem for small parameters. Therefore, we first prove the uniform bound for the gradient of the sign-changing solutions obtained in Section 3. In order to prove the following Theorem 4.9, we need some propositions and lemmas.

Proposition 4.1. *Assume $\Gamma_{\mu,\varepsilon}(u) \leq L, D\Gamma_{\mu,\varepsilon}(u) = 0$. Then*

1. *There exists $K = K(L)$ such that $|u(x)| \leq K$ for $x \in \mathbb{R}^N$.*
2. *For any $\delta > 0$ there exists $c = c(\delta, L)$ such that $|u(x)| \leq c\varepsilon^3$ for $x \in \mathbb{R}^N \setminus (M_\varepsilon)^\delta$.*

Proof. We apply Moser's iteration to obtain the L^∞-bound.

(1) For $T>0$, let $u_T(x)=u(x)$ if $|u(x)|\le T$; $u_T(x)=\pm T$ if $\pm u(x)\ge T$. Take $\varphi=|u_T|^{2k-2}u$ as test function in $\langle D\Gamma_{\mu,\varepsilon}(u),\varphi\rangle=0$ where $k\ge 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\nabla_p f_\mu(\nabla u)\nabla\varphi\,dx+\int_{\mathbb{R}^N}\frac{\partial}{\partial z}k_\varepsilon(x,u)\varphi\,dx\\
&+\int_{\mathbb{R}^N}\Big(\nabla_p h_\mu(u,\nabla u)\nabla\varphi+\frac{\partial}{\partial z}h_\varepsilon(u,\nabla u)\varphi\Big)\,dx+\int_{\mathbb{R}^N}E(\varepsilon x)u\varphi\,dx\\
&\quad+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)u\varphi\,dx\ge 0.
\end{aligned}$$

Since $\langle D\Gamma_{\mu,\varepsilon}(u),\varphi\rangle=0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{i,j=1}^N\Big(\beta_{ij}(u)D_iuD_j\varphi+\frac12 D_z\beta_{ij}(u)D_iuD_ju\varphi\Big)\,dx+\int_{\mathbb{R}^N}E(\varepsilon x)u\varphi\,dx\\
&\le\lambda\int_{\mathbb{R}^N}|u|^{q-2}u\varphi\,dx.
\end{aligned}\tag{4.1}$$

For the right-hand side of (4.1), for any $\nu>0$ there exists $c_\nu>0$ such that

$$\int_{\mathbb{R}^N}|u|^{q-2}u\varphi\,dx\le\nu\int_{\mathbb{R}^N}u^2|u_T|^{2k-2}\,dx+c_\nu\int_{\mathbb{R}^N}(u^2|u_T|^{k-1})^2\,dx.\tag{4.2}$$

For the left-hand side of (4.1), by the conditions (A5) and (A3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{i,j=1}^N\Big(\beta_{ij}(u)D_iuD_j\varphi+\frac12 D_z\beta_{ij}(u)D_iuD_ju\varphi\Big)\,dx+\int_{\mathbb{R}^N}E(\varepsilon x)u\varphi\,dx\\
&\ge\int_{\mathbb{R}^N}\sum_{i,j=1}^N\Big(\beta_{ij}(u)+\frac12 uD_z\beta_{ij}(u)\Big)D_iuD_ju|u_T|^{2k-2}\,dx\\
&\quad+c_1\int_{\mathbb{R}^N}u^2|u_T|^{2k-2}\,dx\\
&\ge c\int_{\mathbb{R}^N}|\nabla u|^2u^2|u_T|^{2k-2}dx+c_1\int_{\mathbb{R}^N}u^2|u_T|^{2k-2}\,dx\\
&\ge\frac{c}{k^2}\int_{\mathbb{R}^N}|\nabla(u^2|u_T|^{k-1})|^2\,dx+c_1\int_{\mathbb{R}^N}u^2|u_T|^{2k-2}\,dx\\
&\ge\frac{c}{k^2}\Big(\int_{\mathbb{R}^N}(u^2|u_T|^{k-1})^{2^*}\,dx\Big)^{2/2^*}+c_1\int_{\mathbb{R}^N}u^2|u_T|^{2k-2}\,dx.
\end{aligned}\tag{4.3}$$

By (4.1)–(4.3), with $\lambda\nu<c_1$, we have

$$\Big(\int_{\mathbb{R}^N}(u^2|u_T|^{k-1})^{2^*}\,dx\Big)^{2/2^*}\le ck^2\int_{\mathbb{R}^N}(u^2|u_T|^{k-1})^2\,dx.\tag{4.4}$$

Assume $\int_{\mathbb{R}^N}(u^2|u_T|^{k-1})^2\,dx<+\infty$. Let $T\to\infty$ in (4.4) we obtain

$$\Big(\int_{\mathbb{R}^N}|u|^{(k+1)\cdot 2^*}\,dx\Big)^{2/2^*}\le ck^2\int_{\mathbb{R}^N}|u|^{2(k+1)}\,dx.\tag{4.5}$$

Denote $d=2^*/2$, then

$$\Big(\int_{\mathbb{R}^N}|u|^{(k+1)\cdot 2^*}\,dx\Big)^{\frac{1}{2^*(k+1)}}\le(ck^2)^{\frac{1}{2(k+1)}}\Big(\int_{\mathbb{R}^N}|u|^{2^*(k+1)\frac1d}\,dx\Big)^{\frac{d}{2^*(k+1)}}.$$

Let $2^*(1+k_1)\frac{1}{d}=2^*$, i.e., $k_1=\frac{2^*-2}{2}>0$. Starting from $k=k_1$, by iteration we have

$$\|u\|_{L^\infty(\mathbb{R}^N)}\le c\|u\|_{L^{2^*}(\mathbb{R}^N)}\le c,$$

since

$$\Big(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\Big)^{1/2^*}\le c\Big(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Big)^{1/2}$$

and by (3.1) we know that $\int_{\mathbb{R}^N}|\nabla u|^2\,dx$ is bounded.

(2) For $x_0\in\mathbb{R}^N, 0<\rho<R\le 1$. Let $\eta\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\eta(x)=1$ for $x\in B_\rho=B_\rho(x_0)$, $\eta(x)=0$ for $x\notin B_R=B_R(x_0)$ and $|\nabla\eta|\le\frac{c}{R-\rho}$. Take $\varphi=u|u|^{2k-2}\eta^m, k\ge 1$ as test function in $\langle D\Gamma_{\mu,\varepsilon}(u),\varphi\rangle=0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\nabla_p f_\mu(\nabla u)\nabla\varphi\,dx+\int_{\mathbb{R}^N}\frac{\partial}{\partial z}k_\varepsilon(x,u)\varphi\,dx\\
&+\int_{\mathbb{R}^N}\Big(\nabla_p h_\mu(u,\nabla u)\nabla\varphi+\frac{\partial}{\partial z}h_\mu(u,\nabla u)\varphi\Big)\,dx\\
&+\int_{\mathbb{R}^N}\sum_{i,j=1}^N\Big(\beta_{ij}(u)D_iuD_j\varphi+\frac12 D_z\beta_{ij}(u)D_iuD_ju\varphi\Big)\,dx+\int_{\mathbb{R}^N}E(\varepsilon x)u\varphi\,dx\\
&+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)u\varphi\,dx\\
&=\lambda\int_{\mathbb{R}^N}|u|^{q-2}u\varphi\,dx.
\end{aligned}\tag{4.6}$$

By the definition of φ, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{\partial}{\partial z}k_\varepsilon(x,u)\varphi\,dx+\int_{\mathbb{R}^N}\frac{\partial}{\partial z}h_\mu(u,\nabla u)\varphi\,dx+\int_{\mathbb{R}^N}E(\varepsilon x)u\varphi\,dx\\
&+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)u\varphi\,dx\ge 0.
\end{aligned}\tag{4.7}$$

And by Lemma 2.1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\nabla_p f_\mu(\nabla u)\nabla\varphi\,dx\\
&=(2k-1)\int_{\mathbb{R}^N}(\nabla f_\mu(\nabla u),\nabla u)|u|^{2k-2}\eta^m\,dx\\
&\quad+m\int_{\mathbb{R}^N}\nabla_p f_\mu(\nabla u)|u|^{2k-2}u\eta^{m-1}\nabla\eta dx\\
&\ge c\int_{\mathbb{R}^N}(1+\mu^{m-2}|\nabla u|^{m-2})|\nabla u|^2|u|^{2k-2}\eta^m\,dx\\
&\quad-c\int_{\mathbb{R}^N}(1+\mu^{m-2}|\nabla u|^{m-2})|\nabla u||u|^{2k-1}\eta^{m-1}|\nabla\eta|\,dx\\
&\ge -c\int_{\mathbb{R}^N}|u|^{2k}\eta^{m-2}|\nabla\eta|^2\,dx-c\mu^{m-2}\int_{\mathbb{R}^N}|u|^{2k-2}|u|^m|\nabla\eta|^m\,dx\\
&\ge-\frac{c}{(R-\rho)^m}\int_{B_R}|u|^{2k}\,dx.
\end{aligned}\tag{4.8}$$

Similarly, we estimate the integral $\int_{\mathbb{R}^N} \nabla_p h_\mu(u, \nabla u)\nabla\varphi\, dx$ and obtain

$$\int_{\mathbb{R}^N} \nabla_p h_\mu(u, \nabla u)\nabla\varphi\, dx \geq -\frac{c}{(R-\rho)^m}\int_{B_R} |u|^{2k}\, dx\,. \tag{4.9}$$

Also, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{i,j=1}^N \Big(\beta_{ij}(u)D_iuD_j\varphi + \frac{1}{2}D_z\beta_{ij}(u)D_iuD_ju\varphi\Big)\, dx \\
&\geq c\int_{\mathbb{R}^N} (1+u^2)|\nabla u|^2|u|^{2k-2}\eta^m dx \\
&\quad - c\int_{\mathbb{R}^N} (1+u^2)|\nabla u||\nabla\eta||u|^{2k-1}\eta^{m-1}\, dx \\
&\geq \frac{c}{k^2}\int_{\mathbb{R}^N} |\nabla(|u|^k\eta^{\frac{m}{2}})|^2 dx - c\int_{\mathbb{R}^N} |u|^{2k}|\nabla\eta|^2\, dx \\
&\geq \frac{c}{k^2}\Big(\int_{B_\rho} |u|^{k\cdot 2^*} dx\Big)^{2/2^*} - \frac{c}{(R-\rho)^m}\int_{B_R} |u|^{2k}\, dx.
\end{aligned} \tag{4.10}$$

For the right-hand side of (4.6) we have

$$\lambda\int_{\mathbb{R}^N} |u|^{q-2}u\varphi\, dx = \lambda\int_{\mathbb{R}^N} |u|^{q-2}|u|^{2k}\eta^m\, dx \leq c\int_{B_R} |u|^{2k}\, dx. \tag{4.11}$$

By (4.6)-(4.11), we have

$$\Big(\int_{B_\rho} |u|^{2^*\cdot k} dx\Big)^{2/2^*} \leq \frac{ck^2}{(R-\rho)^m}\int_{B_R} |u|^{2k} dx \quad \text{for } k \geq 1.$$

Applying iteration we obtain

$$\|u\|_{L^\infty_{R/2}(x)} \leq c\|u\|_{L^2_R(x)}.$$

Because

$$\int_{\mathbb{R}^N\setminus(M_\varepsilon)^\delta} u^2\, dx \leq c_\delta\varepsilon^6,$$

we have $|u(x)| \leq c_\delta\varepsilon^3$ for $x \in \mathbb{R}^N\setminus(M_\varepsilon)^\delta$. □

Lemma 4.2 (Profile decomposition)**.** *Fix μ and let $\varepsilon_n \to 0$. Assume $u_n \in X_{\varepsilon_n}$, $D\Gamma_{\mu,\varepsilon_n}(u_n) = 0$, $\Gamma_{\mu,\varepsilon_n}(u_n) \leq L$. Then there exist $U_k, r_n \in X = W^{1,m}(\mathbb{R}^N) \cup H^1(\mathbb{R}^N)$, $y_{n,k} \in \mathbb{R}^N$ such that*

$$u_n = \sum_k U_k(\cdot - y_{n,k}) + r_n\,. \tag{4.12}$$

1. $u_n(\cdot + y_{n,k}) \rightharpoonup U_k$ *in* X *as* $n \to \infty$.
2. $|y_{n,k} - y_{n,l}| \to \infty$ *as* $n \to \infty$ *for* $k \neq l$.
3. $\|u_n\|^2_{H^1(\mathbb{R}^N)} = \sum_k \|u\|^2_{H^1(\mathbb{R}^N)} + \|r_n\|_{H^1(\mathbb{R}^N)} + o(1)$,
   $\|u_n\|^m_{W^{1,m}(\mathbb{R}^N)} \geq \sum_k \|U_k\|^m_{W^{1,m}(\mathbb{R}^N)} + \|r_n\|^m_{W^{1,m}(\mathbb{R}^N)} + o(1)$.
4. $\|r_n\|_{L^s(\mathbb{R}^N)} \to 0$ *as* $n \to \infty, 2 < s < 2\cdot 2^*$,
   $\|u_n\|^s_{L^s(\mathbb{R}^N)} = \sum_k \|U_k\|^s_{L^s(\mathbb{R}^N)} + o(1)$ *as* $n \to \infty$.

Proof. By Lemma 3.2 we know that $\{u_n\}$ is bounded in X, so the result of the lemma follows from [8]. □

By Proposition 4.1 (2) $\lim_{n\to\infty} \operatorname{dist}(y_{n,k}, M_{\varepsilon_n}) < +\infty$. We denote

$$y_k^* = \lim_{n\to\infty} \varepsilon_n y_{n,k}.$$

Since $\operatorname{dist}(y_{n,k}, M_\varepsilon) = \varepsilon_n^{-1} \operatorname{dist}(\varepsilon_n y_{n,k}, M)$, we have

$$\operatorname{dist}(y_{n,k}^*, \overline{M}) = 0, \quad y_k^* \in \overline{M}. \tag{4.13}$$

Similar to the proof of [6, Lemma 3.2 and Corollary 3.1], we can obtain that the summation in the profile decomposition (4.12) has only finitely many terms and there exists $m > 0$ such that $\int_{\mathbb{R}^N} |U_n|^q\,dx \geq m$. Assume that the sequence $\{u_n\}$ has the profile decomposition (4.12). We denote

$$\Omega_R^{(n)} = \mathbb{R}^N \backslash \big\{ \cup_k B_R(y_{n,k}) \cup B_R(0) \big\}.$$

Proposition 4.3. *There exist c, α, independent of n, such that*

$$\int_{\Omega_R^{(n)}} F_{\mu,\varepsilon_n}(x, u_n, \nabla u_n)\,dx \leq c\exp\{-\alpha R\},\ \lambda \int_{\Omega_R^{(n)}} |u_n|^q\,dx \leq c\exp\{-\alpha R\}$$

where

$$\begin{gathered} F_{\mu,\varepsilon_n}(x,z,p) = f_\mu(p) + k_{\varepsilon_n}(x,z) + h_\mu(z,p) + \frac{1}{2}\sum_{i,j=1}^N \beta_{ij}(z)p_ip_j \\ + \frac{1}{2}E(\varepsilon_n x)z^2 + \frac{1}{2}\xi_n \chi_{\varepsilon_n}(x) z^2, \\ \xi_n = \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n^2 dx - 1 \Big)_+^{\beta - 1}. \end{gathered}$$

Proposition 4.4. *Assume the profile decomposition* (4.12) *holds, and denote $y_k^* = \lim_{n\to\infty} \varepsilon_n y_{n,k}$. Then $y_k^* \in \mathcal{A}$, i.e., y_k^* is a critical point of V in $\overline{M}$.*

The proofs of Propostions 4.3 and 4.4 are similar to the corresponding results in [6], and we omit them.

Lemma 4.5. *Assume $\Gamma_{\mu,\varepsilon}(u) \leq L$, $D\Gamma_{\mu,\varepsilon}(u) = 0$. Then there exist constants $\alpha = \alpha(\mu, L)$ and $c = c(\mu, L)$ such that*

$$|u(x)| \leq c\exp\{-\alpha \operatorname{dist}(x, M_\varepsilon)\}, \quad \text{for } x \in \mathbb{R}^N .$$

Proof. Assume $u_n \in X_{\varepsilon_n}$, $D\Gamma_{\mu,\varepsilon_n}(u_n) = 0$, $\Gamma_{\mu,\varepsilon_n}(u_n) \leq L$. By Lemma 3.2, $\{u_n\}$ is bounded in $X = W^{1,m}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Suppose the profile decomposition (4.12) holds,

$$u_n = \sum_{k=1}^{k_0} U_k(\cdot - y_{n,k}) + r_n\,.$$

By Proposition 4.3, there exist α, c such that

$$\int_{\mathbb{R}^N \backslash \{\cup_{k=1}^{k_0} B_R(y_{n,k}) \cup B_R(0)\}} u_n^2\,dx \leq c\exp\{-\alpha R\}\,.$$

By Moser's iteration,

$$|u_n(x)| \leq c\exp\{-\alpha R\} \quad \text{for } x \in \mathbb{R}^N \setminus \{\cup_{k=1}^{k_0} B_R(y_{n,k}) \cup B_R(0)\}\,.$$

Let $R_n(x) = \min\{|x|, |x - y_{n,k}|, k = 1, \dots, k_0\}$. Then

$$|u_n(x)| \leq c\exp\{-\alpha R_n(x)\}\,.$$

Since $\varepsilon_n y_{n,k} \to y_k^* \in \mathcal{A}$, $k = 1, \dots, k_0$, for any $\delta > 0$ there exists $\overline{\varepsilon}$ such that for $\varepsilon_n < \overline{\varepsilon}$, $\operatorname{dist}(\varepsilon_n y_{n,k}, \mathcal{A}) < \delta$; hence $R_n(x) \geq \operatorname{dist}(x, (\mathcal{A}^\delta)_{\varepsilon_n})$ and

$$|u_n(x)| \leq c \exp\{-\alpha \operatorname{dist}(x, (\mathcal{A}^\delta)_{\varepsilon_n})\} \leq c \exp\{-\alpha \operatorname{dist}(x, M_{\varepsilon_n})\} . \tag{4.14}$$

□

To prove Lemma 4.8, to apply the regularity theory for elliptic equations (see [7]), and write down the divergence form of the equation, which is satisfied by the critical points of the functional $I_{\mu,\varepsilon}$,

$$Qu = \operatorname{div} A(u, \nabla u) + B(x, u, \nabla u) = 0, \tag{4.15}$$

where

$$\begin{aligned} A^i(z,p) = A^i_\mu(z,p) &= \frac{\partial}{\partial p_i} f_\mu(p) + \frac{\partial}{\partial p_i} h_\mu(z,p) + \sum_{j=1}^N \beta_{ij}(z) p_j \\ &= \frac{1}{2}\sigma\Big(\frac{|p|}{m_\mu(|p|)}\Big)^{m-2}\Big(m - (m-2)\frac{|p| b_\mu(|p|)}{m_\mu(|p|)}\Big) p_i \\ &\quad + \frac{1}{2}\sigma\Big(\frac{|p|}{m_\mu(|p|)}\Big)^{m-4}\Big((m-2) - (m-4)\frac{|p| b_\mu(|p|)}{m_\mu(|p|)}\Big) z^2 p_i \\ &\quad + \sum_{j=1}^N \beta_{ij}(z) p_j \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &B(x,z,p) \\ &= B_\mu(x,z,p) = -\frac{\partial}{\partial z} h_\mu(z,p) - \frac{1}{2}\sum_{i,j=1}^N D_z\beta_{ij}(z) p_i p_j - V(\varepsilon x) z + \lambda |z|^{q-2} z \\ &= -\sigma\Big(\frac{|p|}{m_\mu(|p|)}\Big)^{m-4} z|p|^2 - \frac{1}{2}\sum_{i,j=1}^N D_z\beta_{ij}(z) p_i p_j - V(\varepsilon x) z + \lambda |z|^{q-2} z \,. \end{aligned} \tag{4.17}$$

For $\mu \in (0,1]$, we define

$$g_\mu(t) = \mu^{m-2} t^{m-1} + t, \; t > 0 \,. \tag{4.18}$$

Then

$$1 \leq \frac{g'_\mu(t) \cdot t}{g_\mu(t)} \leq m - 1 \,. \tag{4.19}$$

We apply the regularity theory for elliptic equations ([7]) to prove Lemma 4.8 and the theory needs the following two propositions, which are similar to the proof in [6].

Proposition 4.6. *It holds that*

1. $p \cdot A(z,p) \geq \phi(K) g_\mu(|p|)|p|$,
2. $|A(z,p)| \leq \Phi(K) g_\mu(|p|)$,
3. $|B(x,z,p)| \leq \Phi(K)(1 + g_\mu(|p|)|p|)$

for $x \in \mathbb{R}^N$, $z \in \mathbb{R}$, $|z| \leq K$, $p \in \mathbb{R}^N$, where ϕ, Φ are two functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that ϕ is decreasing and Φ is increasing.

Proposition 4.7. *Let*

$$a^{ij} = a_\mu^{ij} = \frac{\partial}{\partial p_j} A_\mu^i(z,p)\,.$$

Then

(1) $\sum_{i,j=1}^N a^{ij}\xi_i\xi_j \ge \phi(K)\frac{g_\mu(|p|)}{|p|}|\xi|^2$;
(2) $|a^{ij}(z,p)| \le \Phi(K)\frac{g_\mu(|p|)}{|p|}$;
(3) $|A(z,p) - A(w,p)| \le \Phi(K)|z-w|\, g_\mu(|p|)$;
(4) $|B(x,z,p)| \le \Phi(K)(1 + g_\mu(|p|)|p|)$.

Lemma 4.8. *Assume $I_{\mu,\varepsilon}(u) \le L$, $DI_{\mu,\varepsilon}(u) = 0$. Then there exists a constant $H = H(L)$ such that*

$$|\nabla u(x)| \le H \quad \textit{for } x \in \mathbb{R}^N\,.$$

Proof. By Proposition 4.1, there exists $K = K(L)$ such that if $DI_{\mu,\varepsilon}(u) = 0$ and $I_{\mu,\varepsilon}(u) \le L$, then u is bounded, $|u(x)| \le K$ for $x \in \mathbb{R}^N$. By [7, Corollary 1.5 and Theorem 1.7], Propositions 4.6 and 4.7, we have

$$\|u\|_{C^{1,\beta}(\mathbb{R}^N)} \le H \tag{4.20}$$

where $\beta = \beta(K) \in (0,1)$, $H = H(K)$, β, H are independent of μ, ε. □

Theorem 4.9. *Assume $\Gamma_{\mu,\varepsilon}(u) \le L, D\Gamma_{\mu,\varepsilon}(u) = 0$. Then there exists $\overline{\varepsilon} = \overline{\varepsilon}(\mu, L)$, $\overline{\mu} = \overline{\mu}(L)$ such that $\Gamma_{\mu,\varepsilon}(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$ if $0 < \varepsilon \le \overline{\varepsilon}$ and $0 < \mu \le \overline{\mu}$.*

Proof. Assume $D\Gamma_{\mu,\varepsilon}(u) = 0$, $\Gamma_{\mu,\varepsilon}(u) \le L$. By Lemma 4.5 there exist $c = c(\delta, L)$, $\alpha = \alpha(\delta, L)$ such that

$$|u(x)| \le c\exp\{-\alpha\ \mathrm{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \le c\exp\{-\alpha\ \mathrm{dist}(x, M_\varepsilon)\}. \tag{4.21}$$

Then for $\varepsilon \le \varepsilon(\mu)$, we have

$$|u(x)| \le \frac{1}{\varepsilon}\exp\{-\varepsilon\,\mathrm{dist}(x, M_\varepsilon)\} = \frac{1}{\varepsilon}\exp\{-\,\mathrm{dist}(\varepsilon x, M)\} \quad \text{for } x \in \mathbb{R}^N\,.$$

So

$$m_\varepsilon(x, u(x)) \equiv u(x) \quad \text{for } x \in \mathbb{R}^N\,. \tag{4.22}$$

Also, if we denote $d = \mathrm{dist}(\mathcal{A}^\delta,\, \partial M)$, then for $x \notin M_\varepsilon$, we obtain

$$\mathrm{dist}(x, (\mathcal{A}^\delta)_\varepsilon) \ge \mathrm{dist}(x, M_\varepsilon) + d\varepsilon^{-1}\,. \tag{4.23}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx &\le \varepsilon^{-6}\int_{\mathbb{R}^N\setminus M_\varepsilon} u^2\,dx \\
&\le c\varepsilon^{-6}\int_{\mathbb{R}^N\setminus M_\varepsilon} \exp\{-2\alpha\,\mathrm{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}\,dx \\
&\le c\varepsilon^{-6}\int_{\|x\|\ge d\varepsilon^{-1}} \exp\{-\alpha|x|\}\,dx \\
&\le c\varepsilon^{-N-5}\exp\{-\alpha d\varepsilon^{-1}\} \to 0 \quad \text{as } \varepsilon \to 0\,.
\end{aligned}$$

Moreover, for $\varepsilon \le \varepsilon(\mu)$, we have

$$\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+ = 0\,. \tag{4.24}$$

By (4.22) and (4.24), $I_{\mu,\varepsilon}(u) = \Gamma_{\mu,\varepsilon}(u)$ and $DI_{\mu,\varepsilon}(u) = D\Gamma_{\mu,\varepsilon}(u) = 0$. By Proposition 4.1 there exists $K = K(L)$ such that $|u(x)| \le K$ for $x \in \mathbb{R}^N$. By Lemma 4.8

there exist $\beta = \beta(K) \in (0,1)$, $H = H(K)$ such that $\|u\|_{C^{1,\beta}(\mathbb{R}^N)} \le H$. For $\mu \le \mu(K) := \frac{1}{H}$, we have $|\nabla u(x)| \le H \le \frac{1}{\mu}$ for $x \in \mathbb{R}^N$. Hence $m_\mu(|\nabla u|) = |\nabla u|$, $I_\varepsilon(u) = I_{\mu,\varepsilon}(u) = \Gamma_{\mu,\varepsilon}(u)$ and $DI_\varepsilon(u) = DI_{\mu,\varepsilon}(u) = D\Gamma_{\mu,\varepsilon}(u) = 0$. □

Proof of Theorem 1.1. Given an integer k, for $\lambda > \Lambda_k$, by Theorem 3.10, the functional $\Gamma_{\mu,\varepsilon}(\mu, \varepsilon \in (0,1])$ has k pairs of sign-changing critical points $\pm u_j(\mu,\varepsilon)$, $j = 1,\dots,k$, the corresponding critical values satisfy

$$0 < c_1(\mu,\varepsilon) \le \dots \le c_k(\mu,\varepsilon) \le m_k. \tag{4.25}$$

By Theorem 4.9 there exist $\mu_k = \mu_k(m_k)$ and $\varepsilon_k = \varepsilon_k(\overline{\mu}, m_k) > 0$ such that if $0 < \overline{\mu} < \mu_k$, $0 < \varepsilon < \varepsilon_k$, $\Gamma_{\overline{\mu},\varepsilon}(u) \le m_k$, and $D\Gamma_{\overline{\mu},\varepsilon}(u) = 0$, then $\Gamma_{\overline{\mu},\varepsilon}(u) = I_\varepsilon(u)$, and $DI_\varepsilon(u) = 0$.

For $0 < \varepsilon < \varepsilon_k$, $u_{j,\varepsilon} = u_j(\overline{\mu},\varepsilon)$, $j = 1,\dots,k$ are critical points of the functional I_ε. Also, by (4.14), for any $\delta > 0$ there exists $\overline{\varepsilon}_k(\delta)$ such that for $0 < \varepsilon < \overline{\varepsilon}_k(\delta)$ it holds

$$|u_{j,\varepsilon}(x)| \le c \exp\{-\alpha \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}, \quad x \in \mathbb{R}^N,$$

so

$$|v_{j,\varepsilon}(x)| \le c \exp\big\{ -\frac{\alpha}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\big\}, \quad x \in \mathbb{R}^N,$$

where $c = c_k(\overline{\mu}, m_k)$. □

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