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EXISTENCE AND CONCENTRATION OF GROUND STATE SOLUTIONS FOR A KIRCHHOFF TYPE PROBLEM

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Abstract. This article concerns the Kirchhoff type problem

$$-\Big(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V(x)u=K(x)|u|^{p-1}u,\quad x\in\mathbb{R}^3,$$
$$u\in H^1(\mathbb{R}^3),$$

where a,b are positive constants, $2<p<5$, $\varepsilon>0$ is a small parameter, and $V(x),K(x)\in C^1(\mathbb{R}^3)$. Under certain assumptions on the non-constant potentials $V(x)$ and $K(x)$, we prove the existence and concentration properties of a positive ground state solution as $\varepsilon\to 0$. Our main tool is a Nehari-Pohozaev manifold.

1. Introduction

In this article we study the Kirchhoff type problem

$$\begin{gathered}-\Big(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V(x)u=K(x)|u|^{p-1}u,\quad x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where a,b are positive constants, $2<p<5$, $\varepsilon>0$ is a small parameter, $V(x),K(x)\in C^1(\mathbb{R}^3)$. Such problems are often referred as being nonlocal because of the presence of the term $\big(\int_{\mathbb{R}^3}|\nabla u|^2dx\big)\Delta u$ which implies that (1.1) is no longer a point-wise equation. Problem (1.1) is related to the stationary analogue of the equation

$$u_{tt}-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u=f(x,u),\tag{1.2}$$

presented by Kirchhoff in [9] as an extension of classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.2), u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). We have to point out that nonlocal problems also appear in other fields such as biological systems, where u describes a process which depends on the average of itself (for example, population density, see [1, 5]).

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In recent years, there have been many works concerned with the existence of solutions to the problems similar to (1.1) via variational methods, see e.g. [2, 6, 7, 10, 13, 15, 22]. Also, there are some recent works considered the concentration property of solutions as $\varepsilon \to 0$, see for instance [8, 14, 18, 19, 20] and the references therein. Indeed, a typical way to deal with (1.1) is to use the mountain pass theorem. For this purpose, the most of the above results focused on the nonlinear model $|u|^{p-1}u$ with $3 < p < 5$ (6 is the critical Sobolev exponent) or similar conditions. Under such conditions, one easily sees that the energy functional associated with (1.1) possess a mountain-pass geometry around $0 \in H^1(\mathbb{R}^3)$ and a bounded (PS) sequence. Moreover, some further conditions are assumed to guarantee the compactness of the (PS) sequence.

A natural question now is whether problem (1.1) has nontrivial solutions for $1 < p \leq 3$. Recently, Li and Ye [11] studied (1.1) under the assumptions that $2 < p < 5$, $\varepsilon = 1$, $K(x) \equiv 1$ and $V(x)$ satisfies

- (A1') $V \in C(\mathbb{R}^3, \mathbb{R})$ is weakly differentiable and satisfies $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3)$ and $V(x) - \nabla V(x) \cdot x \geq 0$ a.e. $x \in \mathbb{R}^3$.
- (A2') for every $x \in \mathbb{R}^3$, $V(x) \leq \lim_{|x|\to\infty} V(x) := V_\infty < +\infty$ with a strict inequality in a subset of positive Lebesgue measure.
- (A3') there exists a $\overline{c} > 0$ such that

$$\overline{c} = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx}{\int_{\mathbb{R}^3} u^2 dx} > 0.$$

By using a monotonicity trick and constructing a new version of global compactness Lemma, they proved that (1.1) has a positive ground state solution. More recently, Ye [23] studied (1.1) under different conditions. On one hand, if $1 < p < 5$, $\varepsilon = 1$, $V(x)$ and $K(x)$ are constants, it was showed that (1.1) has a positive ground state solution. On the other hand, if $1 < p < 5$, $\varepsilon = 1$, $K(x) \equiv 1$ and $V(x)$ satisfies

- (A1") $V \in C^2(\mathbb{R}^3, \mathbb{R})$ and $\lim_{|x|\to\infty} V(x) := V_\infty > 0$.
- (A2") $\nabla V(x) \cdot x \leq 0$ for all $x \in \mathbb{R}^3$ and the inequality is strict in a subset of positive Lebesgue measure.
- (A3") $V(x) + \frac{\nabla V(x)\cdot x}{4} \geq V_\infty$ for all $x \in \mathbb{R}^3$.
- (A4") $\nabla V(x) \cdot x + \frac{xH(x)x}{4} \leq 0$ for all $x \in \mathbb{R}^3$, where H denotes the Hessian matrix of V.
- (A5") there exists a constant $T > 1$ which is defined in [23] such that

$$\sup_{x\in\mathbb{R}^3} V(x) \leq V_\infty + T.$$

Ye [23] proved that (1.1) has a high energy solution. However, to the best of our knowledge, for the case $2 < p \leq 3$ and $V(x), K(x)$ are not constants, there is no work concerning the existence and concentration property of positive ground state solutions of (1.1) as $\varepsilon \to 0$. In this paper, our purpose is to give an affirmative answer to this case. Since we consider the case $2 < p < 5$, the usual variational techniques, such as the Nehari manifold, do not work. Following [11, 16, 17, 20, 23], the main tool of our work is a Nehari-Pohozaev manifold. Moreover, as we consider the case that $K(x)$ and $V(x)$ are not constants, the Nehari-Pohozaev manifold for (1.1) becomes more complicated than in [11, 23], and thus the method used in [11, 23] can not be directly used in our work.

To state our main result, we assume

(A1) $V(x) \in C^1(\mathbb{R}^3,\mathbb{R})$ and $0 < V_{\min} := \inf_{x\in\mathbb{R}^3} V(x) \leq V(x) \leq V_\infty := \lim_{|x|\to\infty} V(x)$, $V(x) \not\equiv V_\infty$ for all $x \in \mathbb{R}^3$.
(A2) $\nabla V(x)\cdot x \in L^\infty(\mathbb{R}^3)$.
(A3) The map $s \mapsto s^{\frac{5}{4+p}} V(s^{\frac{1}{4+p}}x)$ is concave for any $x \in \mathbb{R}^3$.
(A4) There exists an $R_V > 0$ such that $\nabla V(x) \equiv 0$ for all $|x| \geq R_V$.
(A5) $K(x) \in C^1(\mathbb{R}^3,\mathbb{R})$ and $0 < K_\infty := \lim_{|x|\to\infty} K(x) \leq K(x)$, and $K(x) \not\equiv K_\infty$ for all $x \in \mathbb{R}^3$.
(A6) $\nabla K(x)\cdot x \in L^\infty(\mathbb{R}^3)$.
(A7) The map $s \mapsto s^{\frac{5}{4+p}} K(s^{\frac{1}{4+p}}x)$ is concave for any $x \in \mathbb{R}^3$.
(A8) There exists an $R_K > 0$ such that $\nabla K(x) \equiv 0$ for all $|x| \geq R_K$.

Remark 1.1. There are many examples of V and K that satisfy the hypotheses above. For example, define $\eta \in C^\infty(\mathbb{R}^3)$ by

$$\eta(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}), & \text{if } |x| < 1,\\ 0, & \text{if } |x| > 1, \end{cases}$$

where $C > 0$ is a constant. Then $V(x) = C - \eta(x)$ satisfies $(V_1) - (V_4)$ and $K(x) = \frac{C}{2} + \eta(x)$ satisfies $(K_1) - (K_4)$.

Clearly, the above assumptions imply that there exists an $\overline{x} \in \Omega_1$ such that $K(\overline{x}) \geq K(x)$ for all $|x| \geq R$ and some $R > 0$. Here, we denote

$$\Omega_1 := \{x \in \mathbb{R}^3; V(x) = V_{\min}\}, \Omega_2 := \{x \in \mathbb{R}^3; K(x) = K_{\max} := \max_{x\in\mathbb{R}^3} K(x)\},$$
$$\mathcal{H} := \{x \in \Omega_1; K(x) = K(\overline{x})\} \cup \{x \notin \Omega_1; K(x) > K(\overline{x})\}.$$

Remark 1.2. Obviously, $\mathcal{H} \neq \emptyset$ because $\overline{x} \in \mathcal{H}$. It is clear that $\mathcal{H} = \Omega_1 \cap \Omega_2$ when $\Omega_1 \cap \Omega_2 \neq \emptyset$. For example, let $V(x) = C - \eta(x)$ and $K(x) = \frac{C}{2} + \eta(x)$ as in Remark 1.1, then $\Omega_1 = \{0\}, \Omega_2 = \{0\}$ and $\mathcal{H} = \{0\}$. If we set $V(x) = C - \eta(x - x_0)$ and $K(x) = \frac{C}{2} + \eta(x)$ and $x_0 \neq 0$, we can easily see that $\Omega_1 = \{x_0\}, \Omega_2 = \{0\}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. We obtain that $\mathcal{H} = \{x; |x| \leq |x_0|\}$.

The main result of this article reads as follows.

Theorem 1.3. (I) *Assume* (A1)–(A3), (A5)–(A7 *hold. Then* (1.1) *possesses a positive ground state solution u_ε for all $\varepsilon > 0$.*

(II) *Suppose* (A1), (A3), (A4), (A5), (A7), (A8) *are satisfied. Then*

(1) *u_ε possesses one maximum point x_ε such that, up to a subsequence, $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{H}) = 0$, $\omega_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of*

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V(x_0)u = K(x_0)|u|^{p-1}u, \quad x \in \mathbb{R}^3.$$

In particular, if $\Omega_1 \cap \Omega_2 \neq \emptyset$, then $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \Omega_1 \cap \Omega_2) = 0$ and ω_ε converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V_{\min}u = K_{\max}|u|^{p-1}u, \ x \in \mathbb{R}^3.$$

(2) *There exist $C_1, C_2 > 0$ such that*

$$u_\varepsilon(x) \leq C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|}.$$

Remark 1.4. Note that (A1) and (A4) imply (A2). Also (A5) and (A8) imply (A6).

This article is organized as follows. In Section 2, we establish some preliminary results. Section 3 is to prove the existence of ground states. Section 4 is devoted to the proof of Theorem 1.3. Throughout this paper we denote by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. The letters $C, C_1, C_2, \dots$ will be repeatedly used to denote various positive constants whose exact values are irrelevant.

2. Preliminaries

Throughout this article by $|\cdot|_r$ we denote the L^r-norm. On the space $H^1(\mathbb{R}^3)$ we consider the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx\Big)^{1/2}.$$

Without loss of generality, we may assume that $\varepsilon = 1$, then (1.1) becomes

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u + V(x)u = K(x)|u|^{p-1}u,\ x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{gathered} \tag{2.1}$$

At this step, we see that (2.1) is variational and its weak solutions are the critical points of the functional given by

$$\begin{aligned} J(u) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)u^2dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 \\ - \frac{1}{p+1}\int_{\mathbb{R}^3}K(x)|u|^{p+1}dx. \end{aligned}$$

For $2 < p \le 3$, the path $\gamma(t) := J(tu)$ may not intersect with the Nehari manifold $N := \{u \in H^1(\mathbb{R}^3)\backslash\{0\}; J'(u)u = 0\}$ for a unique t. Thus, following the idea from [11, 16, 17, 20, 23], we will define a Nehari-Pohozaev manifold to replace the Nehari manifold. First of all, let us introduce the Pohozaev identity in the following Lemma.

Lemma 2.1. *Assume that* (A1), (A2), (A5), (A6) *are satisfied. Let* $u \in H^1(\mathbb{R}^3)$ *be a weak solution to* (2.1) *and* $p \in (1,5)$*, then we have the Pohozaev identity*

$$\begin{aligned} P(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}V(x)u^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\nabla V(x)\cdot xu^2dx \\ + \frac{b}{2}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{3}{p+1}\int_{\mathbb{R}^3}K(x)|u|^{p+1}dx \\ - \frac{1}{p+1}\int_{\mathbb{R}^3}\nabla K(x)\cdot x|u|^{p+1}dx = 0. \end{aligned}$$

The proof of the above lemma is standard (see e.g. [3, 4]), so we omit it here. Let us introduce the map

$$T : \mathbb{R}^+ \to H^1(\mathbb{R}^3), \quad t \mapsto u_t(x) = tu(t^{-1}x).$$

It is clear that $t \mapsto u_t$ is indeed a continuous curve in $H^1(\mathbb{R}^3)$ by using Brezis-Lieb Lemma (see [21]). Then we define

$$f_u(t) := J(u_t)$$

$$
\begin{aligned}
&= \frac{at^3}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{t^5}{2}\int_{\mathbb{R}^3}V(x)u^2dx + \frac{bt^6}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2\\
&\quad - \frac{t^{4+p}}{p+1}\int_{\mathbb{R}^3}K(tx)|u|^{p+1}dx.
\end{aligned}
$$

Obviously, $f_u(t)$ attains its maximum since $2<p<5$. (A2) and (A6) imply that $f_u(t)$ is continuously differentiable and

$$
\begin{aligned}
f_u'(t) :&= \frac{3at^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{5t^4}{2}\int_{\mathbb{R}^3}V(tx)u^2dx + \frac{t^4}{2}\int_{\mathbb{R}^3}\nabla V(tx)txu^2dx\\
&\quad + \frac{3bt^5}{2}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{4+p}{p+1}t^{3+p}\int_{\mathbb{R}^3}K(tx)|u|^{p+1}dx\\
&\quad - \frac{t^{3+p}}{p+1}\int_{\mathbb{R}^3}\nabla K(tx)tx|u|^{p+1}dx.
\end{aligned}
$$

Denote $G(u) := f_u'(1)$, i.e.

$$
\begin{aligned}
G(u) &= \frac{3a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{5}{2}\int_{\mathbb{R}^3}V(x)u^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\nabla V(x)xu^2dx\\
&\quad + \frac{3b}{2}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{4+p}{p+1}\int_{\mathbb{R}^3}K(x)|u|^{p+1}dx\\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^3}\nabla K(x)x|u|^{p+1}dx.
\end{aligned}
$$

So we define the Nehari-Pohozaev manifold

$$
M = \{u \in H^1(\mathbb{R}^3)\backslash\{0\}; G(u) = 0\}.
$$

It is clear that

$$
G(u) = P(u) + J'(u)u.
$$

Then, all solutions of (2.1) belong to M. Moreover, we have the following results.

Lemma 2.2. *Assume that* (A1)–(A3), (A5)–(A7) *hold. Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, then there is a unique $t = t_u > 0$ such that $f_u'(t) = 0$, $f_u(\cdot)$ is increasing for $(0, t_u)$ and decreasing for (t_u, ∞). That is, there is a unique t_u such that $u_{t_u} \in M$.*

Proof. By making the change of variable $s = t^{4+p}$, we obtain

$$
\begin{aligned}
f_u(s) &= \frac{a}{2}s^{\frac{3}{4+p}}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{s^{\frac{5}{4+p}}}{2}\int_{\mathbb{R}^3}V(s^{\frac{1}{4+p}}x)u^2dx\\
&\quad + \frac{bs^{\frac{6}{4+p}}}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \frac{s}{p+1}\int_{\mathbb{R}^3}K(s^{\frac{1}{4+p}}x)|u|^{p+1}dx.
\end{aligned}
$$

By (A3) and (A7), $f_u(s)$ is a concave function. We already know that attains its maximum. Let t_u be the unique point at which this maximum is achieved. Then t_u is the unique critical point of f_u and $f_u(t_u)$ is positive and $f_u(\cdot)$ is increasing for $0 < t < t_u$ and decreasing for $t > t_u$. In particular, for any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, $t_u \in \mathbb{R}$ is the unique value such that u_{t_u} belongs to M, and $J(u_t)$ reaches global maximum for $t = t_u$. This completes the proof. $\square$

Set

$$
m := \inf_{u\in M} J(u), \quad m^* := \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \max_{t>0} J(u_t).
$$

By Lemma 2.2, we have $m = m^* \geq 0$.

Lemma 2.3. *There holds $m > 0$.*

Proof. Let us define
$$\overline{J}(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V_{\min} u^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K_{\max}|u|^{p+1} dx.$$
Obviously, $\overline{J}(u) \le J(u)$, and this implies that
$$\overline{m} := \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \max_{t>0} \overline{J}(u_t) \le \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \max_{t>0} J(u_t) = m.$$
It suffices to show that $\overline{m} > 0$. Define
$$\overline{M} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\}; g_u'(1) = 0\},$$
where $g_u(t) = \overline{J}(u_t)$. For any $u \in \overline{M}$,
$$C\|u\|_{H^1}^2 \le \frac{3a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{5}{2}\int_{\mathbb{R}^3} V_{\min} u^2 dx \le \frac{4+p}{p+1}\int_{\mathbb{R}^3} K_{\max}|u|^{p+1} dx \le C\|u\|_{H^1}^{p+1}.$$
Thus we obtain $C \le \|u\|_{H^1}^{p-1}$. Consequently,
$$\begin{aligned}
\overline{J}(u) &= \overline{J}(u) - \frac{1}{p+4} g_u'(1)\\
&= \frac{(p+1)a}{2(p+4)}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{p-1}{2(p+4)}\int_{\mathbb{R}^3} V_{\min} u^2 dx + \frac{(p-2)b}{4(p+4)}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2\\
&\ge C\|u\|_{H^1}^2 \ge C > 0.
\end{aligned}$$
□

Lemma 2.4. *There exists $C > 0$ such that for any $u \in M$,*
$$J(u) \ge C\|u\|_{H^1}^2.$$

Proof. Fix $t \in (0,1)$. Then there exist $\delta, \gamma > 0$ such that
$$\begin{gathered}
V(tx) \ge V_{\min} \ge \delta V_\infty \ge \delta V(x),\\
K(tx) \le K_{\max} \le \gamma K_\infty \le \gamma K(x)
\end{gathered}$$
for all $x \in \mathbb{R}^3$. For $u \in M$, we compute
$$\begin{aligned}
&J(u_t) - t^{\lambda+4} J(u)\\
&= \Big(\frac{t^3}{2} - \frac{t^{\lambda+4}}{2}\Big) a\int_{\mathbb{R}^3} |\nabla u|^2 dx + \Big(\frac{t^6}{4} - \frac{t^{\lambda+4}}{4}\Big) b\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2\\
&\quad + \int_{\mathbb{R}^3} \Big(\frac{t^5}{2} V(tx) - \frac{t^{\lambda+4}}{2} V(x)\Big) u^2 dx + \int_{\mathbb{R}^3} \Big(\frac{t^{\lambda+4}}{p+1} K(x) - \frac{t^{p+4}}{p+1}\Big)|u|^{p+1} dx,
\end{aligned}$$
where $2 < \lambda < p$. By choosing a smaller t, if necessary, there exists $\varepsilon_0 > 0$ such that
$$\begin{gathered}
\frac{t^5}{2} V(tx) - \frac{t^{\lambda+4}}{2} V(x) \ge \Big(\delta\frac{t^5}{2} - \frac{t^{\lambda+4}}{2}\Big) V(x) \ge \varepsilon_0,\\
\frac{t^{\lambda+4}}{p+1} K(x) - \frac{t^{p+4}}{p+1} K(tx) \ge \Big(t^{\lambda+4} - \gamma t^{p+4}\Big)\frac{K(x)}{p+1} \ge 0.
\end{gathered}$$

From these two inequalities and Lemma 2.2, taking a smaller $\varepsilon_0 > 0$ if necessary, we obtain

$$(1 - t^{\lambda+4})J(u) \geq J(u_t) - t^{\lambda+4}J(u) \geq \varepsilon_0 \|u\|_{H^1}^2.$$

Taking $C = \varepsilon_0/(1 - t^{\lambda+4})$, we complete the proof. □

3. Existence result

In this section, we combine the Nehari-Pohozaev manifold with the concentration compactness principle to prove the existence of a ground state solution for (2.1). Initially, we give the following concentration-compactness principle.

Lemma 3.1 ([4, Lemma 1.1]). *Let $\{\rho_n\}$ be a sequence of nonnegative L^1 functions on $\mathbb{R}^N$ satisfying $\lim_{n\to\infty} \int_{\mathbb{R}^N} \rho_n dx = c_0 > 0$. There exists a subsequence, still denoted by $\{\rho_n\}$ satisfying one of the following three possibilities:*

(i) *(Vanishing) for all $R > 0$,*
$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y_n)} \rho_n dx = 0;$$

(ii) *(compactness) there exists $\{y_n\} \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists an $R > 0$ satisfying*
$$\lim_{n\to\infty} \inf \int_{B_R(y_n)} \rho_n dx \geq c_0 - \varepsilon;$$

(iii) *(Dichotomy) there exists an $\alpha \in (0, c_0)$ and $\{y_n\} \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists an $R > 0$, for all $r \geq R$ and $r' \geq R$,*
$$\lim_{n\to\infty} \sup \Big(\big|\alpha - \int_{B_r y_n} \rho_n \, dx\big| + \big|(c_0 - \alpha) - \int_{\mathbb{R}^N \setminus B_{r'}(y_n)} \rho_n \, dx\big| \Big) < \varepsilon;$$

Lemma 3.2 ([21, Lemma 1.21]). *Let $r > 0$ and $2 \leq q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*
$$\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^q dx \to 0, \text{ as } n \to +\infty,$$
then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 3.3. *Let $\{u_n\} \subset M$ be a minimizing sequence for m. Then there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists an $R > 0$ satisfying*
$$\int_{\mathbb{R}^3 \setminus B_R(y_n)} (|\nabla u_n|^2 + |u_n|^2) dx \leq \varepsilon.$$

Proof. First, we claim that $\int_{\mathbb{R}^3} |u_n|^{p+1} dx \nrightarrow 0$, as $n \to \infty$. Indeed, since $m > 0$, it is easy to obtain that $\|u_n\|_{H^1} \nrightarrow 0$ by the Sobolev embedding theorem. By Lemma 2.2, for any $t > 1$,

$$\begin{aligned}
m \leftarrow & J(u_n) \geq J((u_n)_t) && (3.1)\\
&= \frac{at^3}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{t^5}{2} \int_{\mathbb{R}^3} V(x) u_n^2 dx + \frac{bt^6}{4} \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \Big)^2 && (3.2)\\
&\quad - \frac{t^{4+p}}{p+1} \int_{\mathbb{R}^3} K(tx) |u_n|^{p+1} dx && (3.3)\\
&\geq \frac{t^3}{2} \int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V_{\min} u_n^2) dx - \frac{t^{p+4}}{p+1} K_{\max} \int_{\mathbb{R}^3} |u_n|^{p+1} dx && (3.4)
\end{aligned}$$

$$\geq \frac{t^3}{2}\sigma - \frac{t^{p+4}}{p+1} K_{\max} \int_{\mathbb{R}^3} |u_n|^{p+1} dx, \tag{3.5}$$

where σ is a fixed constant. It suffices to take $t > 1$ so that $\frac{t^3\sigma}{2} > 2m$ to get a lower bound for $\int_{\mathbb{R}^3} |u_n|^{p+1} dx$.

Let us assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{p+1} dx \to A \in (0, +\infty). \tag{3.6}$$

By Lemma 3.2, we obtain that there exist $\delta > 0$ and $\{x_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(x_n)} |u_n|^{p+1} dx > \delta > 0.$$

Take $R > \max\{1, \varepsilon^{-1}\}$, $\phi_R(t)$ a smooth function such that

- $\phi_R(t) = 1$ for $0 \leq t \leq R$.
- $\phi_R(t) = 0$ for $t \geq 2R$.
- $\phi_R'(t) \leq 2/R$.

Write

$$u_n(x) = \phi_R(|x - x_n|)u_n(x) + (1 - \phi_R(|x - x_n|))u_n(x) := v_n + \omega_n.$$

Then

$$\lim_{n\to\infty} \int_{B_R(x_n)} |v_n|^{p+1} dx \geq \delta. \tag{3.7}$$

To complete the proof, we only need to prove that there exist constants $C > 0$ independent of ε and $n_0 = n_0(\varepsilon)$ such that $\|\omega_n\|_{H^1} \leq C\varepsilon$ for all $n \geq n_0$.

Define $z_n = u_n(\cdot + x_n)$, and then $z_n \rightharpoonup z$ weakly in $H^1(\mathbb{R}^3)$. By taking a larger R, if necessary, we can assume that $\int_{A_0(R,2R)} |z|^{p+1} dx < \varepsilon$, where $A_0(R, 2R)$ denotes the annulus centered in 0 with radii R and $2R$. Then, for n large enough, we have

$$\Big| \int_{\mathbb{R}^3} K(tx)(|u_n|^{p+1} - |v_n|^{p+1} - |\omega_n|^{p+1}) dx \Big| \leq C\varepsilon. \tag{3.8}$$

Since $|\nabla z_n|^2$ is uniformly bounded in $L^1(\mathbb{R}^3)$, up to a subsequence, $|\nabla z_n|^2$ converges (in the sense of measure) to a certain positive measure μ with $\mu(\mathbb{R}^3) < +\infty$. By enlarging R necessary, we can assume that $\mu(A_0(R, 2R)) < \varepsilon$. Then, for n large enough,

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \phi_R(|x - x_n|)(1 - \phi_R(|x - x_n|)) dx < \varepsilon.$$

Taking this into account, direct calculations show that for n large enough,

$$\Big| \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx \Big| = \Big| 2 \int_{\mathbb{R}^3} \nabla v_n \nabla \omega_n dx \Big| \leq C\varepsilon, \tag{3.9}$$

and thus

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \Big)^2 \\
&= \Big(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx + C\varepsilon \Big)^2 \\
&= \Big(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \Big)^2 + \Big(\int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx \Big)^2 + 2 \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx + C\varepsilon \\
&\geq \Big(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \Big)^2 + \Big(\int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx \Big)^2 + C\varepsilon.
\end{aligned}$$

Arguing as before, for R large enough, we obtain

$$\Big| \int_{\mathbb{R}^3} V(tx)u_n^2 dx - \int_{\mathbb{R}^3} V(tx)v_n^2 dx - \int_{\mathbb{R}^3} V(tx)\omega_n^2 dx \Big| \leq C\varepsilon. \tag{3.10}$$

Putting together (3.8)-(3.10) we obtain that for n sufficient large and $t > 0$,

$$J((u_n)_t) \geq J((v_n)_t) + J((\omega_n)_t) - C\varepsilon. \tag{3.11}$$

Now let us denote with t_{v_n} and t_{ω_n} the positive values which maximize $f_{v_n}(t)$ and $f_{\omega_n}(t)$ respectively, namely,

$$J((v_n)_{t_{v_n}}) = \max_{t>0} J((v_n)_t) \text{ and } J((\omega_n)_{t_{\omega_n}}) = \max_{t>0} J((\omega_n)_t).$$

Let us assume that $t_{v_n} \leq t_{\omega_n}$(the other case will be treated later). Then

$$J((\omega_n)_t) \geq 0 \text{ for } t \leq t_{v_n}.$$

We claim that there exist $0 < \widetilde{t} < 1 < \bar{t}$ independent of ε such that $t_{v_n} \in (\widetilde{t}, \bar{t})$. Indeed, take $\bar{t} = (2(p+1)(K_{\max}A)^{-1}B)^{\frac{1}{p-2}}$, where A comes from (3.6) and B is large enough such that $\bar{t} > 1$ and moreover,

$$B \geq a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V_\infty |u_n|^2 dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2.$$

Then

$$\begin{aligned}
J((u_n)_{\bar{t}}) &\leq \frac{\bar{t}^6}{2}\Big(a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V_\infty |u_n|^2 dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 \\
&\quad - \frac{\bar{t}^{p-2}}{p+1}\int_{\mathbb{R}^3} K_{\max}|u_n|^{p+1} dx\Big) \\
&\leq -B\frac{\bar{t}^6}{2} < 0.
\end{aligned}$$

Taking a smaller ε in (3.11), we obtain

$$J((v_n)_{\bar{t}}) + J((\omega_n)_{\bar{t}}) < 0.$$

Then $J((v_n)_{\bar{t}}) < 0$ or $J((\omega_n)_{\bar{t}}) < 0$. In any case, Lemma 2.2 implies that $t_{v_n} < \bar{t}$ (recall that we are assuming $t_{v_n} \leq t_{\omega_n}$).

For the lower bound, take $\widetilde{t} = \big(\frac{m}{B}\big)^{\frac{1}{3}}$. Let us point out that $\widetilde{t} < 1$. For any $t \leq \widetilde{t}$,

$$J((u_n)_t) \leq \frac{\widetilde{t}^3}{2}\Big(a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} V_\infty |u_n|^2 dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2\Big) \leq \frac{m}{2}.$$

Since

$$m \leftarrow J(u_n) \geq J((u_n)_{t_{v_n}}) \geq J((v_n)_{t_{v_n}}) + J((\omega_n)_{t_{v_n}}) - c\varepsilon \geq m - C\varepsilon \tag{3.12}$$

and the right hand side can be made greater than $\frac{m}{2}$ by choosing a small ε, we conclude that $t_{v_n} > \widetilde{t}$ and the claim is proved.

Using (3.12) we deduce, for n large, $J((\omega_n)_t) \leq 2C\varepsilon$ for all $t \in (0, t_{v_n})$. Moreover, for any $t \in (0, \widetilde{t})$, we have

$$\begin{aligned}
2C\varepsilon &\geq J((\omega_n)_t) \\
&\geq \frac{t^6}{4}\Big(a\int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx + \int_{\mathbb{R}^3} V_{\min}\omega_n^2 dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{t^{p+4}}{p+1} \int_{\mathbb{R}^3} K_{\max} |\omega_n|^{p+1} dx \\
&\geq \frac{t^6}{4} q_n - D t^{p+4},
\end{aligned}
$$

where

$$
q_n = a \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx + \int_{\mathbb{R}^3} V_{\min} \omega_n^2 dx + b \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \Big)^2
$$

and $D > A$. Observe that

$$
\frac{t^6}{4} q_n - D t^{p+4} = \frac{(p+2)D}{2} \Big(\frac{q_n}{2(p+4)D} \Big)^{p+4} \quad \text{for } t = \Big(\frac{q_n}{2(p+4)D} \Big)^{\frac{1}{p-2}}.
$$

By taking a large D we can assume that $\big(\frac{q_n}{2(p+4)D}\big)^{\frac{1}{p-2}} \leq \widetilde{t}$. With this choice of t, we obtain

$$
2C\varepsilon \geq J((\omega_n)_t) \geq \frac{(p+2)D}{2} \Big(\frac{q_n}{2(p+4)D} \Big)^{p+4} \geq C q_n^{p+4}.
$$

Thus we have

$$
\|\omega_n\|_{H^1} \leq C\varepsilon \quad \text{for some } C > 0. \tag{3.13}
$$

In the case $t_{v_n} > t_{\omega_n}$, we can assume analogously to conclude that $\|v_n\|_{H^1} \leq C\varepsilon$ for some $C > 0$. But, choosing small ε, this contradicts (3.7), so (3.13) holds. This completes the proof. □

Lemma 3.4. *The value m is achieved at some $u \in M$.*

Proof. Recall that $z_n \rightharpoonup z$ in $H^1(\mathbb{R}^3)$, we have $z_n \to z$ in $L^q_{loc}(\mathbb{R}^3)$ for $1 < q < 6$. Thus, by (3.7), we obtain

$$
\delta < \liminf_{n\to\infty} \int_{\mathbb{R}^3} |v_n|^{p+1} dx \leq \lim_{n\to\infty} \int_{B_{2R}} |z_n|^{p+1} dx = \int_{B_{2R}} |z|^{p+1} dx.
$$

Recall also that $u_n = v_n + \omega_n$ with $\|\omega_n\|_{H^1} \leq C\varepsilon$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |u_n^2 - v_n^2| dx &\leq \int_{\mathbb{R}^3} |\omega_n| (|u_n| + |v_n|) dx \\
&\leq \Big(\int_{\mathbb{R}^3} \omega_n^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^3} (|u_n| + |v_n|)^2 dx \Big)^{1/2} \leq C\varepsilon.
\end{aligned}
$$

On the other hand,

$$
\int_{\mathbb{R}^3} v_n^2 dx \leq \int_{B_{2R}} z_n^2 dx \to \int_{B_{2R}} z^2 dx \leq \int_{\mathbb{R}^3} z^2 dx.
$$

Then we obtain

$$
\liminf_{n\to\infty} \int_{\mathbb{R}^3} z_n^2 dx = \liminf_{n\to\infty} \int_{\mathbb{R}^3} u_n^2 dx \leq \int_{\mathbb{R}^3} z^2 dx + C\varepsilon.
$$

Since ε is arbitrary, we obtain that $z_n \to z$ in $L^2(\mathbb{R}^3)$ and, by interpolation, $z_n \to z$ in $L^q(\mathbb{R}^3)$ for all $q \in [2, 6)$. We discuss two cases:

Case 1: $\{x_n\}$ is bounded. Assume, passing to a subsequence, that $x_n \to x_0$. In this case $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u$ strongly in $L^q(\mathbb{R}^3)$ for any $q \in [2, 6)$, where $u = z(\cdot - x_0)$. Recall the expression of $J((u_n)_t)$, we have

$$
m = \lim_{n\to\infty} J(u_n) \geq \liminf_{n\to\infty} J((u_n)_t) \geq J(u_t), \quad \text{for any } t > 0.
$$

Therefore, $\max_{t\geq 0} J(u_t) = m$ and $u_n \to u$ in $H^1(\mathbb{R}^3)$. In particular, $u \in M$ is a minimizer of $J|_M$.

Case 2: $\{x_n\}$ is unbounded. In this case, by Lebesgue convergence Theorem and (A1), we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^3} V(tx)(u_n(x))_t^2 dx &= \lim_{n\to\infty}\int_{\mathbb{R}^3} V(t(x+x_n))(z_n(x))_t^2 dx \\
&= V_\infty \int_{\mathbb{R}^3} z_t^2 dx \geq \int_{\mathbb{R}^3} V(tx) z_t^2 dx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^3} V(tx))(z_n(x))_t^2 dx
\end{aligned}$$

for any $t>0$ fixed. Moreover,

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^3} K(tx)|u_n(x))_t|^{p+1} dx &= \lim_{n\to\infty}\int_{\mathbb{R}^3} K(t(x+x_n))|(z_n(x))_t|^{p+1} dx \\
&= K_\infty \int_{\mathbb{R}^3} |z_t|^{p+1} dx \\
&\leq \lim_{n\to\infty}\int_{\mathbb{R}^3} K(tx))|(z_n(x))_t|^{p+1} dx
\end{aligned}$$

for any $t>0$ fixed. Therefore,

$$m = \lim_{n\to\infty} J(u_n) \geq \liminf_{n\to\infty} J((z_n)_t) \geq J(z_t), \quad \text{for any } t>0.$$

So, taking t_z so that $f_z(t) = J(z_t)$ reaches its maximum, we obtain that $z_{t_z} \in M$ and is a minimizer for $J|_M$. $\square$

Theorem 3.5. *The minimizer u of $J|_M$ is a positive ground state solution of* (2.1).

Proof. Let $u \in M$ be a minimizer of the functional $J|_M$. We will prove that u is a positive ground state solution of (P) in the following. Recall that, by Lemma 2.2,

$$J(u) = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} J(u_t) = m.$$

We argue by contradiction. Suppose that u is not a weak solution of (2.1). Then we can choose $\phi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{aligned}
\langle J'(u),\phi\rangle = a\int_{\mathbb{R}^3}\nabla u\nabla\phi dx + \int_{\mathbb{R}^3} V(x)u\phi dx + b\int_{\mathbb{R}^3}|\nabla u|^2 dx\int_{\mathbb{R}^3}\nabla u\nabla\phi dx \\
- \int_{\mathbb{R}^3} K(x)|u|^{p-1}u\phi dx < -1.
\end{aligned}$$

We fix $\varepsilon>0$ sufficiently small such that

$$\langle J'(u_t+\sigma\phi),\phi\rangle \leq -\frac{1}{2}, \quad \forall |t-1|, |\sigma| \leq \varepsilon.$$

and introduce a cutoff function $0\leq\eta\leq 1$ such that $\eta(t)=1$ for $|t-1|\leq\frac{\varepsilon}{2}$ and $\eta(t)=0$ for $|t-1|\geq\varepsilon$. Set

$$\gamma(t) = \begin{cases} u_t, & \text{if } |t-1|\geq\varepsilon, \\ u_t+\varepsilon\eta(t)\phi, & \text{if } |t-1|<\varepsilon. \end{cases}$$

Note that $\gamma(t)$ is a continuous curve in $H^1(\mathbb{R}^3)$ and, eventually choosing a smaller ε, we obtain that $\|\gamma(t)\|_{H^1} > 0$ for $|t-1|<\varepsilon$.

We claim $\sup_{t\geq 0} J(\gamma(t)) < m$. Indeed, if $|t-1| \geq \varepsilon$, then $J(\gamma(t)) = J(u_t) < J(u) = m$. If $|t-1| < \varepsilon$, by using the mean value theorem to the C^1 map $[0,\varepsilon] \ni \sigma \mapsto J(u_t + \varepsilon\eta(t)\phi) \in \mathbb{R}$, we find, for a suitable $\overline{\sigma} \in (0,\varepsilon)$,

$$J(u_t + \varepsilon\eta(t)\phi) = J(u_t) + \langle J(u_t + \overline{\sigma}\varepsilon\eta(t)\phi), \eta(t)\phi\rangle \leq J(u_t) - \frac{1}{2}\eta(t) < m.$$

Observe that $G(\gamma(1-\varepsilon)) > 0$ and $G(\gamma(1+\varepsilon)) < 0$, there exists $t_0 \in (1-\varepsilon, 1+\varepsilon)$ such that $G(\gamma(t_0)) = 0$, i.e., $\gamma(t_0) = u_{t_0} + \varepsilon\eta(t_0)\phi \in M$ and $J(\gamma(t_0)) < m$, this gives the desired contradiction. We have proved that the minimizer of $J|_M$ is a solution. Since any solution of (2.1) belongs to M, the minimizer is a ground state.

Moreover, consider $u \in M$ is a minimizer for $J|_M$. Then $|u| \in M$ is also a minimizer, and hence a solution. By the maximum principle, $|u| > 0$. □

4. Concentration behavior

In this section, we study the concentration behavior of the ground state solutions u_ε as $\varepsilon \to 0$. From now on, we assume (A1), (A3), (A4), (A5), (A7), (A8) are satisfied. Introducing the re-scaled transformation $x \mapsto \varepsilon x$ we can rewrite (1.1) as

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V(\varepsilon x)u = K(\varepsilon x)|u|^{p-1}u, \; x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{gathered} \tag{4.1}$$

Let

$$\begin{aligned} J_\varepsilon(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)u^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 \\ - \frac{1}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x)|u|^{p+1} dx \end{aligned}$$

be the associated energy functional, $P_\varepsilon(u)$,

$$M_\varepsilon := \{u \in H^1(\mathbb{R}^3); G_\varepsilon(u) = P_\varepsilon(u) + \langle J_\varepsilon'(u), u\rangle = 0\}$$

and $m_\varepsilon = \inf_{u\in M_\varepsilon} J_\varepsilon(u)$ be the corresponding Pohozaev identity, the Nehari-Pohozaev manifold and the least energy, respectively. We need the following constant coefficients problem

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + \lambda u = \mu|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{gathered} \tag{4.2}$$

where $\lambda, \mu > 0$. In the same way, we use the notations $J_{\lambda\mu}, P_{\lambda\mu}, M_{\lambda\mu}, G_{\lambda\mu}$ and $m_{\lambda\mu}$. In a similar way to Section 3, there exists some $u \in M_{\lambda\mu}$ such that $J_{\lambda\mu}(u) = m_{\lambda\mu}$.

Lemma 4.1. *Suppose $\lambda_1 \geq \lambda_2$ and $\mu_2 \geq \mu_1$. Then $m_{\lambda_1\mu_1} \geq m_{\lambda_2\mu_2}$ is achieved at some $u \in M$.*

Proof. Let $u \in M_{\lambda_1\mu_1}$ be such that $m_{\lambda_1\mu_1} = J_{\lambda_1\mu_1}(u) = \max_{t>0} J_{\lambda_1\mu_1}(u_t)$. Then there exists a unique $t_{\lambda_2\mu_2}$ such that $u_{t_{\lambda_2\mu_2}} \in M_{\lambda_2\mu_2}$, and hence

$$\begin{aligned} m_{\lambda_1\mu_1} &= J_{\lambda_1\mu_1}(u) \\ &\geq J_{\lambda_1\mu_1}(u_{t_{\lambda_2\mu_2}}) \\ &= J_{\lambda_2\mu_2}(u_{t_{\lambda_2\mu_2}}) + \frac{(\lambda_1-\lambda_2)(t_{\lambda_2\mu_2})^5}{2}\int_{\mathbb{R}^3} |u_{t_{\lambda_2\mu_2}}|^2 dx \end{aligned}$$

$$
\begin{aligned}
&+ \frac{(\mu_1 - \mu_2)(t_{\lambda_2\mu_2})^{p+4}}{p+1} \int_{\mathbb{R}^3} |u_{t_{\lambda_2\mu_2}}|^{p+1} dx \\
&\geq m_{\lambda_1\mu_1}.
\end{aligned}
$$

□

Without loss of generality, up to translation, we assume that

$$
K(\overline{x}) = \max_{x \in \Omega_1} K(x) \quad \text{and} \quad \overline{x} = 0 \in \Omega_1.
$$

Thus

$$
V(0) = V_{\min} \quad \text{and} \quad k := K(0) \geq K(x) \quad \text{for all } |x| \geq R.
$$

Lemma 4.2. *There exists $C > 0$ independent of ε such that $m_\varepsilon \geq C$. On the other hand,* $\limsup_{\varepsilon\to 0} m_\varepsilon \leq m_{V_{\min}k}$.

Proof. Since $m_\varepsilon \geq m_{V_{\min}K_{\max}} > 0$, we only need to prove the second part. Take $u \in M_{V_{\min}k}$ satisfying $J_{V_{\min}k}(u) = m_{V_{\min}k}$. By Lemma 2.2, we know that there is a unique $t_\varepsilon > 0$ such that $u_{t_\varepsilon} \in M_\varepsilon$ and

$$
\begin{aligned}
m_\varepsilon &\leq \max_{t>0} J_\varepsilon(u_t) \\
&= \frac{at_\varepsilon^3}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t_\varepsilon^5}{2} \int_{\mathbb{R}^3} V(t_\varepsilon \varepsilon x) u^2 dx \\
&\quad + \frac{bt_\varepsilon^6}{4} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^2 - \frac{t_\varepsilon^{4+p}}{p+1} \int_{\mathbb{R}^3} K(t_\varepsilon \varepsilon x) |u|^{p+1} dx.
\end{aligned} \tag{4.3}
$$

This combining with $m_\varepsilon > 0$, we have $\{t_\varepsilon\}$ is bounded with respect to ε. For each $\varepsilon > 0$, there exists an $R > 0$ such that

$$
\Big| \int_{|x|>R} (V(t_\varepsilon \varepsilon x) - V_{\min}) u^2 dx \Big| < \varepsilon. \tag{4.4}
$$

Since $0 \in \Omega_1$, we obtain

$$
\lim_{\varepsilon\to 0} \Big| \int_{|x|\leq R} (V(t_\varepsilon \varepsilon x) - V_{\min}) u^2 dx \Big| = 0. \tag{4.5}
$$

Similarly, there holds

$$
\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} (K(t_\varepsilon \varepsilon x) - k) |u|^{p+1} dx = 0. \tag{4.6}
$$

From (4.3)-(4.6), we can draw the conclusion that

$$
m_\varepsilon \leq J_\varepsilon(u_{t_\varepsilon}) = J_{V_{\min}k}(u_{t_\varepsilon}) + o(1) \leq J_{V_{\min}k}(u) + o(1) = m_{V_{\min}k} + o(1).
$$

Thus

$$
\limsup_{\varepsilon\to 0} m_\varepsilon \leq m_{V_{\min}k}.
$$

□

Let v_ε be the ground state solution of (4.1).

Lemma 4.3. *There exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, there exist $y_\varepsilon \in \mathbb{R}^3$ and $R, C > 0$ such that*

$$
\int_{B_R(y_\varepsilon)} v_\varepsilon^2 dx > C.
$$

Proof. Suppose by contradiction that there is a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that for all $R > 0$,

$$\lim_{\varepsilon\to 0} \sup_{y\in\mathbb{R}^3} \int_{B_R(y_\varepsilon)} v_\varepsilon^2 dx = 0.$$

From Lemma 3.2, we can deduce that $v_{\varepsilon_n} \to 0$ in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$. Since

$$\begin{aligned} m_{\varepsilon_n} &= J_{\varepsilon_n}(v_{\varepsilon_n}) - \frac{1}{2}\langle J'_{\varepsilon_n}(v_{\varepsilon_n}), v_{\varepsilon_n}\rangle \\ &= -\frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_n}|^2 dx\Big)^2 + \Big(\frac{1}{2} - \frac{1}{p+1}\int_{\mathbb{R}^3} K(\varepsilon_n x)|v_{\varepsilon_n}|^{p+1} dx\Big). \end{aligned}$$

Letting $n \to \infty$, we have

$$0 < \liminf_{\varepsilon\to 0} m_{\varepsilon_n} = -\liminf_{\varepsilon\to 0} \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla v_{\varepsilon_n}|^2 dx\Big)^2 \le 0.$$

Which is absurd. $\square$

We denote

$$\omega_\varepsilon(x) := v_\varepsilon(x + y_\varepsilon) = u_\varepsilon(\varepsilon x + \varepsilon y_\varepsilon).$$

So ω_ε is a positive ground state solution to

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V(\varepsilon x + \varepsilon y_\varepsilon)u = K(\varepsilon x + \varepsilon y_\varepsilon)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{gathered} \tag{4.7}$$

Denote the corresponding energy functional by Φ_ε. Set $\phi(\omega_\varepsilon) = \Phi'_\varepsilon((\omega_\varepsilon)_t)|_{t=1}$. Thus

$$\begin{aligned} \phi(\omega_\varepsilon) = &\frac{3a}{2}\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2 dx + \frac{5}{2}\int_{\mathbb{R}^3} V(\varepsilon x + \varepsilon y_\varepsilon)\omega_\varepsilon^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \nabla V(\varepsilon x + \varepsilon y_\varepsilon)\varepsilon x\omega_\varepsilon^2 dx \\ &+ \frac{3b}{2}\Big(\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2 dx\Big)^2 - \frac{4+p}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x + \varepsilon y_\varepsilon)|\omega_\varepsilon|^{p+1} dx \\ &- \frac{1}{p+1}\int_{\mathbb{R}^3} \nabla K(\varepsilon x + \varepsilon y_\varepsilon)\varepsilon x|\omega_\varepsilon|^{p+1} dx = 0. \end{aligned}$$

Lemma 4.4. *The sequence $\{\varepsilon y_\varepsilon\}$ is bounded.*

Proof. It is easy to know that $\{\omega_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$. We may assume that

$$\omega_\varepsilon \rightharpoonup \omega_0 \ge 0 \quad \text{in } H^1(\mathbb{R}^3).$$

It follows from Lemma 4.3 that $\omega_0 \not\equiv 0$.

Suppose to the contrary that, after passing to a subsequence,

$$|\varepsilon y_\varepsilon| \to \infty.$$

Clearly, we have $V(\varepsilon y_\varepsilon) \to V_\infty$ and $K(\varepsilon y_\varepsilon) \to K_\infty$ as $\varepsilon \to 0$. Thus ω_0 is a solution of

$$-(a + bA)\Delta u + V_\infty u = K_\infty|u|^{p-1}u, \quad x \in \mathbb{R}^3, \tag{4.8}$$

where $A = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2 dx$. Similarly as Lemma 2.1, we have the Pohozaev identity

$$P_{A,\infty}(\omega_0) := \frac{a+bA}{2}\int_{\mathbb{R}^3} |\nabla\omega_0|^2 dx - \frac{3K_\infty}{p+1}\int_{\mathbb{R}^3} |\omega_0|^{p+1} dx + \frac{3V_\infty}{2}\int_{\mathbb{R}^3} |\omega_0|^2 dx = 0.$$

Let us define

$$g_{\omega_0}(t) : = I_\infty((\omega_0)_t)$$

$$
\begin{aligned}
&= \frac{a+bA}{2}t^3\int_{\mathbb{R}^3}|\nabla\omega_0|^2dx + \frac{t^5}{2}\int_{\mathbb{R}^3}V_\infty\omega_0^2dx - \frac{t^{4+p}}{p+1}\int_{\mathbb{R}^3}\nabla K_\infty|\omega_0|^{p+1}dx\\
&= 0,
\end{aligned}
$$

where I_∞ is the energy functional associated to (4.8). Obviously, $g_{\omega_0}(t)$ attains its unique maximum since $2<p<5$. Moreover,

$$
g'_{\omega_0}(1) = P_{A,\infty}(\omega_0) + \langle I'_\infty(\omega_0),\omega_0\rangle = 0.
$$

Recall the definition of $M_{V_\infty K_\infty}$ and $\int_{\mathbb{R}^3}|\nabla\omega_0|^2dx \le A$, it is easy to obtain that there exists a unique $t_0\le 1$ such that $(\omega_0)_{t_0}\in M_{V_\infty K_\infty}$. It follows from (A4), (A8) and the Lebesgue's dominated theorem that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} m_\varepsilon\\
&= \limsup_{\varepsilon\to 0}\Phi_\varepsilon(\omega_\varepsilon) - \frac{1}{p+4}\phi(\omega_\varepsilon)\\
&= \limsup_{\varepsilon\to 0}\frac{p+1}{2(p+4)}a\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx + \frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V(\varepsilon x+\varepsilon y_\varepsilon)\omega_\varepsilon^2dx\\
&\quad + \frac{p-2}{4(p+4)}b\Big(\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx\Big)^2 - \frac{1}{2(p+4)}\int_{\mathbb{R}^3}\nabla V(\varepsilon x+\varepsilon y_\varepsilon)\varepsilon x\omega_\varepsilon^2dx\\
&\quad + \frac{1}{(p+1)(p+4)}\int_{\mathbb{R}^3}\nabla K(\varepsilon x+\varepsilon y_\varepsilon)\varepsilon x|\omega_\varepsilon|^{p+1}dx\\
&\ge \liminf_{\varepsilon\to 0}\frac{p+1}{2(p+4)}a\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx + \frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V(\varepsilon x+\varepsilon y_\varepsilon)\omega_\varepsilon^2dx\\
&\quad + \frac{p-2}{4(p+4)}b\Big(\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx\Big)^2 \qquad (4.9)\\
&\ge \liminf_{\varepsilon\to 0}t_0^3\frac{p+1}{2(p+4)}a\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx + t_0^5\frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V_\infty\omega_\varepsilon^2dx\\
&\quad + t_0^6\frac{p-2}{4(p+4)}b\Big(\int_{\mathbb{R}^3}|\nabla\omega_\varepsilon|^2dx\Big)^2\\
&\ge t_0^3\frac{p+1}{2(p+4)}a\int_{\mathbb{R}^3}|\nabla\omega_0|^2dx + t_0^5\frac{p-1}{2(p+4)}\int_{\mathbb{R}^3}V_\infty\omega_0^2dx\\
&\quad + t_0^6\frac{p-2}{4(p+4)}b\left(\int_{\mathbb{R}^3}|\nabla\omega_0|^2dx\right)^2\\
&= J_{V_\infty K_\infty}((\omega_0)_{t_0}) - \frac{1}{p+4}G_{V_\infty K_\infty}((\omega_0)_{t_0})\\
&= J_{V_\infty K_\infty}((\omega_0)_{t_0}) \ge m_{V\infty K_\infty}.
\end{aligned}
$$

Therefore,

$$
m_{V_{mink}} < m_{V\infty K_\infty} \le \limsup_{\varepsilon\to 0} m_\varepsilon \le m_{V_{mink}}.
$$

This is a contradiction. Thus $\{\varepsilon y_\varepsilon\}$ is bounded. □

For the rest of this article, we assume that

$$
\varepsilon y_\varepsilon \to x_0\in\mathbb{R}^3.
$$

Lemma 4.5. *We have*

$$
\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon y_\varepsilon,\mathcal{H}) = 0.
$$

Proof. It suffices to show that $x_0 \in \mathcal{H}$. Suppose to the contrary that $x_0 \notin \mathcal{H}$. Denote

$$\mathcal{A} := \{x \in \Omega_1; K(x) = \max_{x \in \Omega_1} K(x)\}, \quad \mathcal{B} := \{x \notin \Omega_1; K(x) > K(\overline{x})\}.$$

We see that $x_0 \in (\Omega_1 \backslash \mathcal{A}) \cup (\Omega_1^c \backslash \mathcal{B})$. As mentioned early, we may assume $\overline{x} = 0$ and $K(0) = \max_{x \in \Omega_1} K(x) = k$. When $x_0 \in \Omega_1 \backslash \mathcal{A}$, then $V(x_0) = V_{\min}$ and $K(x_0) < k$, so we obtain that $m_{V_{\min} k} < m_{V(x_0)K(x_0)}$. Similarly, for $x_0 \in \Omega_1^c \backslash \mathcal{B}$, we can have the same results. Using the same proof of (4.9) implies that

$$\limsup_{\varepsilon \to 0} m_\varepsilon \le m_{V_{\min} k} < m_{V(x_0)K(x_0)} \le \limsup_{\varepsilon \to 0} m_\varepsilon,$$

which is impossible. □

Lemma 4.6. *We have $\omega_\varepsilon \to \omega_0$ in $H^1(\mathbb{R}^3)$.*

Proof. Using a proof similar the one of Lemma 4.2, we can obtain $\limsup_{\varepsilon \to 0} m_\varepsilon \le m_{V(x_0)K(x_0)}$. Moreover, the same as the proof of Lemma 4.4 shows that there exists $0 < t_0 \le 1$ such that $(\omega_0)_{t_0} \in M_{V(x_0)K(x_0)}$. Therefore, we have

$$\begin{aligned}
m_{V(x_0)K(x_0)} &\le J_{V(x_0)K(x_0)}((\omega_0)_{t_0}) \\
&= J_{V(x_0)K(x_0)}((\omega_0)_{t_0}) - \frac{1}{p+4} G_{V(x_0)K(x_0)}((\omega_0)_{t_0}) \\
&\ge t_0^3 \frac{p+1}{2(p+4)} a \int_{\mathbb{R}^3} |\nabla \omega_0|^2 dx + t_0^5 \frac{p-1}{2(p+4)} \int_{\mathbb{R}^3} V(x_0) \omega_0^2 dx \\
&\quad + t_0^6 \frac{p-2}{4(p+4)} b \Big(\int_{\mathbb{R}^3} |\nabla \omega_0|^2 dx \Big)^2 \\
&\le \frac{p+1}{2(p+4)} a \int_{\mathbb{R}^3} |\nabla \omega_0|^2 dx + \frac{p-1}{2(p+4)} \int_{\mathbb{R}^3} V(x_0) \omega_0^2 dx \\
&\quad + \frac{p-2}{4(p+4)} b \Big(\int_{\mathbb{R}^3} |\nabla \omega_0|^2 dx \Big)^2 \\
&\le \liminf_{\varepsilon \to 0} \frac{p+1}{2(p+4)} a \int_{\mathbb{R}^3} |\nabla \omega_\varepsilon|^2 dx + \frac{p-1}{2(p+4)} \int_{\mathbb{R}^3} V(x_0) \omega_\varepsilon^2 dx \\
&\quad + \frac{p-2}{4(p+4)} b \Big(\int_{\mathbb{R}^3} |\nabla \omega_\varepsilon|^2 dx \Big)^2 \\
&\le \liminf_{\varepsilon \to 0} \Phi_\varepsilon(\omega_\varepsilon) - \frac{1}{p+4} \phi(\omega_\varepsilon) \\
&\le \limsup_{\varepsilon \to 0} m_\varepsilon \le m_{V(x_0)K(x_0)}.
\end{aligned}$$

Consequently, the above inequalities must be equalities, and hence

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{p+1}{2(p+4)} a \int_{\mathbb{R}^3} |\nabla \omega_\varepsilon|^2 dx + \frac{p-1}{2(p+4)} \int_{\mathbb{R}^3} V(x_0) \omega_\varepsilon^2 dx \\
&= \frac{p+1}{2(p+4)} a \int_{\mathbb{R}^3} |\nabla \omega_0|^2 dx + \frac{p-1}{2(p+4)} \int_{\mathbb{R}^3} V(x_0) \omega_0^2 dx.
\end{aligned}$$

The proof is complete. □

Using almost the same argument as that of [14, Lemma 4.5] we can show the following result.

Lemma 4.7. *There exist constants $C_1, C_2 > 0$ such that*
$$\omega_\varepsilon(x) \le C_1 e^{-C_2|x|}.$$
for all $x \in \mathbb{R}^3$.

Proof of Theorem 1.3. Let δ_ε be the global maximum of ω_ε. By Lemma 4.7, we see that $\delta_\varepsilon \in B_R(0)$ for some $R > 0$. Thus the global maximum of v_ε, given by $z_\varepsilon = y_\varepsilon + \delta_\varepsilon$, satisfies $\varepsilon z_\varepsilon = \varepsilon y_\varepsilon + \varepsilon\delta_\varepsilon$. Note that $u_\varepsilon(x) = (x/\varepsilon)$, then we see that $u_\varepsilon(x)$ is positive ground state solution to (1.1) with $\varepsilon > 0$ and has a global maximum point $x_\varepsilon = \varepsilon z_\varepsilon$. Since $\{\delta_\varepsilon\}$ is bounded, it follows from (4.7) and Lemma 4.5 that $\varepsilon z_\varepsilon \to x_0$ and $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon z_\varepsilon, \mathcal{H}) = 0$. In particular, if $\Omega_1 \cap \Omega_2 \neq \emptyset$, then $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon z_\varepsilon, \Omega_1 \cap \Omega_2) = 0$. Moreover, since ω_ε is a $(PS)_{m_{V(x_0)K(x_0)}}$ sequence for $J_{m_{V(x_0)K(x_0)}}$ and $\omega_\varepsilon \to \omega_0$ in $H^1(\mathbb{R}^3)$, we deduce that ω_0 is a positive ground state solution of
$$\begin{gathered}
-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V(x_0)u = K(x_0)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\
u \in H^1(\mathbb{R}^3),
\end{gathered}$$
In particular, if $\Omega_1 \cap \Omega_2 \neq \emptyset$, we have $V(x_0) = V_{\min}$, $K(x_0) = K_{\max}$ and ω_0 is a positive ground state solution of
$$\begin{gathered}
-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V_{\min}u = K_{\max}|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\
u \in H^1(\mathbb{R}^3),
\end{gathered}$$
In view of the definition of v_ε, from Lemma 4.7 we obtain
$$u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon}) = \omega_\varepsilon(\varepsilon^{-1}x - y_\varepsilon) = \omega_\varepsilon(\varepsilon^{-1}x - \varepsilon^{-1}x_\varepsilon + \delta_\varepsilon) \le C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|}.$$
The proof is complete. ☐

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