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MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS CHOQUARD EQUATIONS

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Abstract. In this article, we consider the multiple solutions for the nonhomogeneous Choquard equations

$$-\Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u + h(x), \quad x \in \mathbb{R}^N,$$

and

$$-\Delta u = \Big(\frac{1}{|x|^\alpha} * |u|^{2_\alpha^*}\Big)|u|^{2_\alpha^*-2}u + h(x), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $0 < \alpha < N$, $2 - \frac{\alpha}{N} < p < 2_\alpha^* = \frac{2N-\alpha}{N-2}$. Under suitable assumptions on h, we obtain at least two solutions on the subcritical case $2 - \frac{\alpha}{N} < p < 2_\alpha^*$ and on the critical case $p = 2_\alpha^*$.

1. Introduction and main results

In this article, we consider the nonhomogeneous Choquard equation

$$- \Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u + h(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $0 < \alpha < N$ and $2 - \frac{\alpha}{N} < p \leq 2_\alpha^* = \frac{2N-\alpha}{N-2}$.

A special case of (1.1) is the Choquard equation

$$-\Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^2\Big)u, \quad x \in \mathbb{R}^N,$$

which was proposed by Choquard in 1976, can be described as an approximation to Hartree-Fock theory of a one-component plasma [22, 23]. It was also proposed by Moroz, Penrose and Tod [26] as a model for the self-gravitational collapse of a quantum mechanical wave function. In this context, Choquard equation is usually called the nonlinear Schrödinger-Newton equation. For more details on the physical aspects of the problem we refer the readers to [11, 12, 13, 14, 24, 26, 30] and the references therein.

Recently, the nonlinear Choquard equations has been widely studied. When $h \equiv 0$, the existence and multiplicity results of system (1.1) have been discussed in many papers. Take for instance, Lieb [22] proved the existence and uniqueness of the ground state to (1.1) by using symmetric decreasing rearrangement inequalities.

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Later, Lions [24] showed the existence of infinitely many radially symmetric solutions to (1.1). Gao and Yang [18] established some existence results for the Brezis-Nirenberg type problem for the nonlinear Choquard equation with critical exponent. Further results for related problems may be found in [1, 2, 3, 4, 15, 19, 25, 27, 29] and the references therein.

Next, we consider the nonhomogeneous case, that is $h \not\equiv 0$. In [35], Xie, Xiao and Wang proved the following Choquard equation

$$-\Delta u + V(x)u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u + h(x), \quad x \in \mathbb{R}^N,$$

had two nontrivial solutions if $2 - \frac{\alpha}{N} < p < \frac{2N-\alpha}{N-2}$ satisfies the compactness condition:

- (A1) $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ is coercive, that is $\lim_{|x|\to+\infty} V(x) = +\infty$.

Zhang, Xu and Zhang [37] also considered the bound and ground states for nonhomogeneous Choquard equations under the condition

- (A2) $\inf_{\mathbb{R}^N} V > 0$, and there exists a constant $r > 0$ such that, for any $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N, |x - y| \le r, V(x) \le M\} \to 0$ as $|y| \to \infty$, where meas stands for the Lebesgue measure.

Under condition(A1) or (A2), they define a new Hilbert space

$$E := \big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx < \infty\big\}$$

with the inner product

$$\langle u, v\rangle_E = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx$$

and the norm $\|u\|_E = \langle u, u\rangle^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous, for any $s \in [2, 2^*]$. Consequently, for each $s \in [2, 2^*]$, there exists a constant $d_s > 0$ such that

$$|u|_s \le d_s\|u\|_E, \quad \forall u \in E. \tag{1.2}$$

Furthermore, it follows from condition(A1) (or (A2)) that the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ are compact for any $s \in [2, 2^*)$(See [5]). Other related results about nonhomogeneous equations can be found in [9, 10, 16, 17, 21, 31, 32, 33, 34, 36] and the references therein.

Motivated by the works above, in this paper we study the existence of multiple solutions to the nonhomogenous Choquard equation with the critical exponent

$$-\Delta u = \Big(\frac{1}{|x|^\alpha} * |u|^{2^*_\alpha}\Big)|u|^{2^*_\alpha-2}u + h(x), \quad x \in \mathbb{R}^N \tag{1.3}$$

and the subcritical exponent

$$-\Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u + h(x), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $N \ge 3$, $0 < \alpha < N$ and $2 - \frac{\alpha}{N} < p < 2^*_\alpha$.

Before giving our main results, we give some notation. Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard scalar and norm

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u\nabla v + uv)\, dx, \quad \|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx.$$

$D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_D^2 := \|u\|_{D^{1,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

The norm on $L^s = L^s(\mathbb{R}^N)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^N} |u|^s dx$.

We use the following assumptions:

(A3) $\|h\|_{H^{-1}} < C_{2_\alpha^*} S^{2_\alpha^*/(2(2_\alpha^*-1))}{}_{H,L}$, where H^{-1} is the dual space of $D^{1,2}(\mathbb{R}^N)$, $S_{H,L}$ is the best constant defined by

$$S_{H,L} = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}\, dx\, dy\right)^{\frac{N-2}{2N-\alpha}}}$$

and

$$C_{2_\alpha^*} = \frac{2(2_\alpha^*-1)}{(2\cdot 2_\alpha^*-1)^{\frac{2\cdot 2_\alpha^*-1}{2\cdot 2_\alpha^*-2}}};$$

(A4) $h \in L^{\frac{2Np}{2N(p-1)+\alpha}}(\mathbb{R}^N)$, $h(x) \geq 0$ and $h \not\equiv 0$;

(A5) $|h|_{\frac{2Np}{2N(p-1)+\alpha}} < \varepsilon = \varepsilon(N, p, \alpha, d_{\frac{2Np}{2N-\alpha}})$, where ε is a positive constant, $d_{\frac{2Np}{2N-\alpha}}$ is defined in Lemma 4.1 below.

To our best knowledge, in the nonhomogeneous case, this is the first result involving critical exponent, so that we think this type of problem is worth to consider. We mentioned here that the basic idea of this paper follows from that of [34]. Our main results read as follows:

Theorem 1.1. *Assume $h \not\equiv 0$ and (A3) hold. Then (1.3) has at least two solutions. One of which is a local minimum solution with the ground state energy, and the other one has the energy which is strictly bigger than the least energy. If additionally, we assume $h > 0$ holds, then the two solutions are positive.*

Theorem 1.2. *Assume $h \not\equiv 0$, (A4) and (A5) hold. Then (1.4) has a local minimum solution with the ground state energy. If additionally, if $h > 0$, then (1.4) has at least two positive solutions.*

Remark 1.3. There is a standard method for obtaining two solutions of a nonhomogeneous system. Usually it is not difficult to obtain a negative local minimum and a positive mountain-pass value of the energy functional. But because of the lack of the compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), p \in (2, 2^*)$, the Palais-Smale condition no longer holds. Especially, many authors avoid the lack of compactness by some coercive assumptions on the potential or by restricting the problem to the radially symmetric subspace of $H^1(\mathbb{R}^N)$. But in this paper, these methods are not adopted. To overcome this difficulty, we use the Brezis-Nirenberg method [7, 8], which preserve the compactness except some fixed bad energy level. And then by estimating the asymptotic behavior of the local minimum solution, we obtain the second solution.

Throughout this paper, the letters C_0, d, c_i, $i = 1, 2, 3\dots$ will be used to denote various positive constants which may vary from line to line and are not essential to the problem. We denote by $\rightharpoonup$ weak convergence and by $\to$ strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

This article is organized as follows. In Section 2, we introduce the variational setting for the problem and give some related preliminaries. In Section 3, we manage to give the existence of the solutions for the critical case. In Section 4, we give the proof of Theorem 1.2.

2. Variational setting and compactness condition

First we give the well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [23]). *Assume $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)||g(y)|}{|x-y|^\alpha}\,dx\,dy \le C(p,q,\alpha)|f|_p|g|_q, \tag{2.1}$$

where $1<p,q<\infty$, $0<\alpha<N$, $\frac{1}{p}+\frac{1}{q}+\frac{\alpha}{N}=2$.

If $p=q=2^*_\alpha$, then

$$C(p,q,\alpha)=C(N,\alpha)=\pi^{\alpha/2}\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(N-\frac{\alpha}{2})}\Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{\frac{\alpha}{N}-1}.$$

The equality in (2.1) *holds if and only if f is a constant times g and*

$$g(x)=\frac{A_1}{(C+|x-a|^2)^{(2N-\alpha)/2}}$$

for some $A_1\in\mathbb{C}$, $0\ne C\in\mathbb{R}$ and $a\in\mathbb{R}^N$.

By the Hardy-Littlewood-Sobolev inequality, the integral

$$B(u)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^q}{|x-y|^\alpha}\,dx\,dy$$

is well defined if $|u|^p\in L^s(\mathbb{R}^N)$ for some $s>1$ satisfying

$$\frac{2}{s}+\frac{\alpha}{N}=2.$$

Therefore, for $u\in H^1(\mathbb{R}^N)$, by Sobolev embedding Theorem, we know that

$$2\le sp\le\frac{2N}{N-2};$$

that is

$$\frac{2N-\alpha}{N}\le p\le\frac{2N-\alpha}{N-2}.$$

Thus, $\frac{2N-\alpha}{N}$ is called the lower critical exponent and $2^*_\alpha=\frac{2N-\alpha}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

For all $u\in D^{1,2}(\mathbb{R}^{\mathbb{N}})$, by the Hardy-Littlewood-Sobolev inequality, we have

$$\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dx\,dy\Big)^{\frac{N-2}{2N-\alpha}}\le C(N,\alpha)^{\frac{N-2}{2N-\alpha}}|u|^2_{2^*},$$

$C(N,\alpha)$ is defined in Lemma 2.1. We use $S_{H,L}$ to denote best constant defined in (A3).

Lemma 2.2 ([18]). *The constant $S_{H,L}$ defined in* (A3) *is achieved if and only if*

$$U(x) = C\Big(\frac{b}{b^2+|x-a|^2}\Big)^{N-2/2},$$

where $C>0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0,+\infty)$ are parameters. We can also obtain

$$S_{H,L} = \frac{S}{C(N,\alpha)^{N-2/2N-\alpha}},$$

where S is the best Sobolev constant. In particular, let

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$$

be a minimizer for S, S is the best Sobolev constant, then

$$W(x) = S^{\frac{(N-\alpha)(2-N)}{4(N-\alpha+2)}} C(N,\alpha)^{\frac{2-N}{2(N-\alpha+2)}} U(x)$$

is the unique minimizer for $S_{H,L}$ and satisfies

$$-\Delta u = \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy |u|^{2^*_\alpha - 2} u \quad in \quad \mathbb{R}^N.$$

Moreover,

$$\|W\|_D = \int_{\mathbb{R}^N} \frac{|W(x)|^{2^*_\alpha}|W(y)|^{2^*_\alpha}}{|x-y|^\alpha}\, dx\, dy = S_{H,L}^{\frac{2N-\alpha}{N-\alpha+2}}.$$

To prove the problem by variational methods, we define the energy functional associated with (1.3) by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2\cdot 2^*_\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x) u\, dx, \tag{2.2}$$

for $u \in D^{1,2}(\mathbb{R}^N)$.

By the Hardy-Littlewood-Sobolev inequality, we know that $I \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned}\langle I'(u), v\rangle = &\int_{\mathbb{R}^N} |\nabla u||\nabla v|\, dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)v(y)}{|x-y|^\alpha}\, dx\, dy \\ &- \int_{\mathbb{R}^N} h(x) v\, dx\end{aligned} \tag{2.3}$$

for all $v \in C_0^\infty(\mathbb{R}^N)$. And so u is a weak solution of (1.3) if and only if u is a critical point of function I. We will constrain the functional I on the Nehari manifold

$$\Lambda = \{u \in D^{1,2}(\mathbb{R}^N), \langle I'(u), u\rangle = 0\}.$$

Denote $\Phi(u) = \langle I'(u), u\rangle$, so we know that

$$\langle I'(u), u\rangle = \|u\|_D^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x) u\, dx,$$

and

$$\langle \Phi'(u), u\rangle = 2\|u\|_D^2 - 2\cdot 2^*_\alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x) u\, dx.$$

Notice that, when u_0 is a local minimum solution of I, it holds

$$\langle I'(u_0), u_0\rangle = 0, \quad \langle \Phi'(u_0), u_0\rangle \geq 0,$$

which leads us to consider the following manifolds:

$$\begin{aligned}
\Lambda &= \{u \in D^{1,2}(\mathbb{R}^N) : \langle I'(u), u\rangle = 0\},\\
\Lambda^+ &= \{u \in \Lambda : \langle \Phi'(u), u\rangle > 0\},\\
\Lambda^- &= \{u \in \Lambda : \langle \Phi'(u), u\rangle < 0\},\\
\Lambda^0 &= \{u \in \Lambda : \langle \Phi'(u), u\rangle = 0\}.
\end{aligned}$$

Obviously, only Λ^0 contains the element 0. Furthermore, it is easy to see that $\Lambda^0 \cup \Lambda^+$ and $\Lambda^0 \cup \Lambda^-$ are both closed subsets of $D^{1,2}(\mathbb{R}^N)$.

To simplify calculations, for $u \in D^{1,2}(\mathbb{R}^N)$, we denote

$$\begin{gathered}
A = A(u) = \|u\|_D^2,\\
B = B(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dx\,dy,\\
C = C(u) = \int_{\mathbb{R}^N} h(x)u\,dx.
\end{gathered}$$

Define the fibering map

$$\varphi_u(t) = I(tu) = \frac{A}{2}t^2 - \frac{B}{2\cdot 2^*_\alpha}t^{2\cdot 2^*_\alpha} - Ct, \quad t > 0. \tag{2.4}$$

Therefore,

$$\begin{aligned}
\varphi_u'(t) &= At - Bt^{2\cdot 2^*_\alpha - 1} - C,\\
\varphi_u''(t) &= A - (2\cdot 2^*_\alpha - 1)Bt^{2\cdot 2^*_\alpha - 2}.
\end{aligned} \tag{2.5}$$

Obviously, $tu \in \Lambda$ with $t > 0$ if and only if $\varphi_u'(t) = 0$. By the sign of $\varphi_u''(t)$, the stationary points of $\varphi_u(t)$ can be classified into three types, namely local minimum, local maximum and turning point. Moreover, the set Λ is a natural constraint for the functional I. This is means that if the infimum is attained by $u \in \Lambda$, then u is a solution of (1.3). However, in our case, the global maximum point of $\varphi_u(t)$ is not unique. This leads us to partition the set Λ according to the critical points of $\varphi_u(t)$. This kind of idea was first introduced by Tarantello in [34]. Later, many mathematicians apply this idea to study other problems; for instance, see [6, 31, 37, 36] and the references therein. Now we give some properties of $\Lambda^\pm$ and Λ^0.

Lemma 2.3. *(i) Assume that $h \not\equiv 0$ for $u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}$, there is a unique $t^- = t^-(u) > 0$ such that $t^-u \in \Lambda^-$. If additionally we assume $\int_{\mathbb{R}^N} hu\,dx > 0$, then there exists a unique $0 < t^+ = t^+(u) < t^-$ satisfying $t^+u \in \Lambda^+$. Moreover,*

$$\begin{gathered}
I(t^-u) = \max_{t\ge 0} I(tu) \quad \textit{for } \int_{\mathbb{R}^N} hu\,dx \le 0;\\
I(t^-u) = \max_{t\ge t^+} I(tu), \quad I(t^+u) = \min_{0\le t\le t^-} I(tu) \quad \textit{for } \int_{\mathbb{R}^N} hu\,dx > 0.
\end{gathered}$$

Proof. Define $\varphi_u(t) = \frac{A}{2}t^2 - \frac{B}{2\cdot 2^*_\alpha}t^{2\cdot 2^*_\alpha} - Ct$ for all $t > 0$. In the case $\int_{\mathbb{R}^N} hu\,dx \le 0$, there is a unique $t^- > 0$ such that $\varphi_u'(t^-) = 0$ and $\varphi_u''(t^-) < 0$. So that

$$\begin{gathered}
\langle I'(t^-u), t^-u\rangle = 0,\\
\|t^-u\|_D^2 - (2\cdot 2^*_\alpha - 1)B(u)(t^-)^{2^*_\alpha - 2} < 0.
\end{gathered}$$

Thus, $t^-u \in \Lambda^-$ and $I(t^-u) = \max_{t\ge 0} I(tu)$.

In the case $\int_{\mathbb{R}^N} hu\,dx > 0$, for $t_0 = t_0(u) = \big[\frac{A}{(2\cdot 2^*_\alpha - 1)B}\big]^{\frac{1}{2\cdot 2^*_\alpha - 2}} > 0$, we have

$$\begin{aligned}
\max_{t\ge 0}\varphi'_u(t) &\ge At_0 - Bt_0^{2\cdot 2^*_\alpha - 1} - C \\
&= \Big[\frac{\|u\|_D^2}{(2\cdot 2^*_\alpha - 1)B}\Big]^{\frac{1}{2\cdot 2^*_\alpha - 2}} \cdot \frac{2\cdot 2^*_\alpha - 2}{2\cdot 2^*_\alpha - 1}\|u\|_D^2 - \int_{\mathbb{R}^N} hu\,dx \\
&\ge S_{H,L}^{2^*_\alpha/2(2^*_\alpha - 1)} \frac{2(2^*_\alpha - 1)}{(2\cdot 2^*_\alpha - 1)^{2\cdot 2^*_\alpha - 1/2(2^*_\alpha - 1)}}\|u\|_D - \|h\|_{H^{-1}}\|u\|_D \\
&= S_{H,L}^{2^*_\alpha/2(2^*_\alpha - 1)} C^*_{2_\alpha}\|u\|_D - \|h\|_{H^{-1}}\|u\|_D > 0.
\end{aligned}$$

From $\varphi'_u(0) = -C < 0$ and $\varphi'_u(t) \to -\infty$ as $t \to +\infty$, we know that there exist unique $0 < t^+ < t_0 < t^-$ such that $\varphi'_u(t^-) = \varphi'_u(t^+) = 0, \varphi''_u(t^-) < 0 < \varphi''_u(t^+)$. Equivalently, $t^+u \in \Lambda^+$ and $t^-u \in \Lambda^-$. Moreover, since $\frac{d}{dt}I(tu) = \varphi'_u(t)$, we can easily see that $I(t^-u) = \max_{t\ge t^+} I(tu)$ and $I(t^+u) = \min_{0\le t\le t^-} I(tu)$. The proof is complete. $\square$

Lemma 2.4. *Assume $h \not\equiv 0$ and (A3) hold. Then*
 (i) $\Lambda^0 = \{0\}$.
 (ii) $\Lambda^\pm \ne \emptyset$ *and* Λ^- *is closed.*

Proof. (i) To prove $\Lambda^0 = \{0\}$, we need to prove that, for $u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$, $\varphi_u(t)$ has no critical point that is a turning point. Set $\|u\|_D = 1$, define

$$\kappa(t) = At - Bt^{2\cdot 2^*_\alpha - 1}. \tag{2.6}$$

Then $\varphi'_u(t) = \kappa(t) - C, \kappa''(t) = -B(2\cdot 2^*_\alpha - 1)(2\cdot 2^*_\alpha - 2)t^{2\cdot 2^*_\alpha - 3} < 0$ for $t > 0$. So $\kappa(t)$ is strictly concave. If $\kappa'(t_0) = 0$, $t_0 = (\frac{1}{(2\cdot 2^*_\alpha - 1)B})^{1/(2\cdot 2^*_\alpha - 2)} > 0$, for $2^*_\alpha > 2 - \frac{\alpha}{N} > 1$. Moreover, $\lim_{t\to 0^+}\kappa(t) = 0, \lim_{t\to+\infty}\kappa(t) = -\infty$ and $\kappa(t) > 0$ for $t > 0$ small. Therefore, we have that $\kappa(t)$ has a unique global maximum point t_0 and

$$\kappa(t_0) = \frac{2(2^*_\alpha - 1)}{2\cdot 2^*_\alpha - 1}\Big(\frac{1}{(2\cdot 2^*_\alpha - 1)B}\Big)^{1/(2\cdot 2^*_\alpha - 2)} := \kappa_0.$$

By (2.4) and (2.5), we infer that if $0 < C < \kappa_0$, the equation $\varphi'_u(t) = 0$ has exactly two points t_1, t_2 satisfying $t_1 < t_0 < t_2$. If $C \le 0$, the equation $\varphi'_u(t) = 0$ has one roots $t_3 > t_0$. Since $\varphi''_u(t) = A - (2\cdot 2^*_\alpha - 1)Bt^{2\cdot 2^*_\alpha - 2}$, it follows that $\varphi''_u(t_1) > 0, \varphi''_u(t_2) < 0$ and $\varphi''_u(t_3) < 0$. It follows that $t_1u \in \Lambda^+, t_2u \in \Lambda^-$ if $0 < C < \kappa_0$ and $t_3u \in \Lambda^-$ if $C \le 0$. Since $\{u \in D^{1,2}(\mathbb{R}^N) : \|u\|_D = 1, 0 < C < \kappa_0\}$ and $\{u \in D^{1,2}(\mathbb{R}^N) : \|u\|_D = 1, C \le 0\}$ are nonempty, we can infer that $\Lambda^\pm$ are nonempty. This implies $\Lambda^0 = \{0\}$.

It is suffices to prove $\kappa_0 > C$. By (A3), Lemma 2.3 and the definition of $S_{H,L}$ we have

$$\begin{aligned}
\kappa_0 - C = k(t_0) - C &= At_0 - Bt_0^{2\cdot 2^*_\alpha - 1} - C \\
&= t_0[1 - t_0^{2\cdot 2^*_\alpha - 2}B] - \int_{\mathbb{R}^N} hu\,dx \\
&\ge S_{H,L}^{2^*_\alpha/2(2^*_\alpha - 1)} \frac{2(2^*_\alpha - 1)}{(2\cdot 2^*_\alpha - 1)^{2\cdot 2^*_\alpha - 1/2(2^*_\alpha - 1)}} - \|h\|_{H^{-1}} > 0.
\end{aligned}$$

(ii) Let $u \in \Lambda^-$, denote $\tilde u = \frac{u}{\|u\|_D}$, then $\|\tilde u\|_D = 1$. By (i), we know that $C(\tilde u) < \kappa_0 = \frac{2(2^*_\alpha - 1)}{2\cdot 2^*_\alpha - 1}\Big(\frac{1}{(2\cdot 2^*_\alpha - 1)B}\Big)^{1/2(2^*_\alpha - 1)}$ with $B := B(\tilde u)$. Furthermore, if $0 < C(\tilde u) < \kappa_0$, the equation $\varphi'_{\tilde{u}}(t)=0$ has exactly two roots $\tilde{t}_1,\tilde{t}_2$ satisfying $0<\tilde{t}_1<t_0<\tilde{t}_2$ such that $\tilde{t}_1\tilde{u}\in\Lambda^+,\tilde{t}_2\tilde{u}\in\Lambda^-$. Then $\tilde{t}_2\tilde{u}=u$ and so $\|u\|_D=\tilde{t}_2>t_0$. If $C\le 0$, the equation $\varphi'_{\tilde{u}}(t)=0$ has exactly one roots $\tilde{t}_3>t_0$. Then $\tilde{t}_3\tilde{u}=u\in\Lambda^-$ and so $\|u\|_D=\tilde{t}_3>t_0$. In other words,

$$\|u\|_D>t_0>0,\quad u\in\Lambda^-.$$

So there exists $\tau>0$ such that

$$\|u\|_D>\tau>0,\quad \forall u\in\Lambda^-. \tag{2.7}$$

Therefore, $0\notin cl(\Lambda^-)$, where $cl(\Lambda^-)$ is the closure of Λ^-. On the other hand, by (i),

$$cl(\Lambda^-)\subset\Lambda^-\cup\Lambda^0=\Lambda^-\cup\{0\}.$$

Hence, $cl(\Lambda^-)=\Lambda^-$ and Λ^- is closed. The proof is complete. □

Lemma 2.5. *Under assumption* (A3)*, for $u\in\Lambda\setminus\{0\}$, there exists $\epsilon>0$ and a differential function $t=t(w)>0, w\in D^{1,2}(\mathbb{R}^N), \|w\|<\epsilon$ such that*

1. *$t(0)=1$;*
2. *$t(w)(u-w)\in\Lambda$, for all $w\in B_\epsilon(0)$;*
3. $\langle t'(0),w\rangle=\frac{2\int_{\mathbb{R}^N}\nabla u\nabla w\,dx-2\cdot 2^*_\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\alpha}|u(x)|^{2^*_\alpha-2}u(x)w(x)}{|x-y|^\alpha}\,dx\,dy-\int_{\mathbb{R}^N}hw\,dx}{\|u\|_D^2-(2\cdot 2^*_\alpha-1)B(u)}$.

Proof. We define $F:\mathbb{R}\times D^{1,2}\to\mathbb{R}$ by

$$\begin{aligned}F(t,w)&=t\|u-w\|_D^2-t^{2\cdot 2^*_\alpha-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u-w)(x)|^{2^*_\alpha}|(u-w)(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dx\,dy\\&\quad-\int_{\mathbb{R}^N}h(u-w).\end{aligned}$$

Obviously, $F(1,0)=0$, $F'_t(1,0)=\varphi''_u(1)\neq 0$. According to the implicit function theorem at point $(1,0)$, we get that there exist $\epsilon=\epsilon(u)>0$ and differentiable function $t:B_\epsilon(0)\to\mathbb{R}^+$ such that: (1) $t(0)=1$; (2) $t(w)(u-w)\in\Lambda$, for all $w\in B_\epsilon(0)$; and (3) $\langle t'(0),w\rangle=-\frac{\langle\frac{\partial F}{\partial w}|_{(1,0)},w\rangle}{\frac{\partial F}{\partial t}|_{(1,0)}}$. The proof is complete. □

3. Local minimum solution

Now we can define

$$c_0=\inf_{u\in\Lambda}I(u),\quad c_1=\inf_{u\in\Lambda^-}I(u),\quad c^+=\inf_{u\in\Lambda^+}I(u).$$

Proposition 3.1. *Under assumption* (A3)*, equation* (1.3) *has a local minimum solution with the least energy $c_0=\inf_\Lambda I(u)$.*

Proof. Firstly, we will show that $\|u\|_D$ is bounded from both above and below. For any $u\in\Lambda$,

$$\begin{aligned}I(u)&=\frac12\|u\|_D^2-\frac{1}{2\cdot 2^*_\alpha}B(u)-C(u)\\&=\Big(\frac12-\frac{1}{2\cdot 2^*_\alpha}\Big)\|u\|_D^2-\Big(1-\frac{1}{2\cdot 2^*_\alpha}\Big)\int_{\mathbb{R}^N}hu\,dx\\&\ge\frac{2^*_\alpha-1}{2\cdot 2^*_\alpha}\|u\|_D^2-\frac{2\cdot 2^*_\alpha-1}{2\cdot 2^*_\alpha}\|h\|_{H^{-1}}\|u\|_D\\&\ge-\frac{(2^*_\alpha-1)^2}{8\cdot 2^*_\alpha(2^*_\alpha-1)}\|h\|_{H^{-1}}^2.\end{aligned}\tag{3.1}$$

Thus,

$$c_0 \geq -\frac{(2\cdot 2^*_\alpha - 1)^2}{8\cdot 2^*_\alpha(2^*_\alpha - 1)}\|h\|^2_{H^{-1}}.$$

By the proof of Lemma 2.4, we have that if $u \in D^{1,2}(\mathbb{R}^N)$ and $\|u\|_D = 1$, then

$$0 < C < \kappa_0 = \frac{2(2^*_\alpha - 1)}{2\cdot 2^*_\alpha - 1}\Big(\frac{1}{(2\cdot 2^*_\alpha - 1)B}\Big)^{1/2(2^*_\alpha - 1)}.$$

So that equation $\varphi'_u(t) = 0$ has a positive solution t_1 satisfying $0 < t_1 < t_0$ and $t_1 u \in \Lambda^+$. Since $\varphi'_u(t) = t - Bt^{2^*_\alpha - 1} - C$, we know that $\lim_{t\to 0}\varphi'_u(t) = -C < 0$, $\varphi''_u(t) > 0$ for all $t \in (0, t_0)$. From $\varphi'_u(t_1) = 0$ we have that $\varphi_u(t_1) < \lim_{t\to 0^+}\varphi_u(t) = 0$ and $\varphi_u(t_1) = I(t_1 u) \geq c^+$. Thus $c^+ < 0$ and $c_0 = \inf_\Lambda I(u) \leq \inf_{\Lambda^+} I(u) = c^+ < 0$.

By using the Ekeland's variational principle on Λ, we get a minimizing sequence $\{u_n\} \subset \Lambda$ which satisfies

$$\begin{gathered} I(u_n) < c_0 + \frac{1}{n}, \\ I(w) \geq I(u_n) - \frac{1}{n}\|u - w\|_D, \quad w \in \Lambda. \end{gathered} \tag{3.2}$$

Since $\{u_n\} \subset \Lambda$, it follows that $\|u_n\|^2_D = B(u_n) + C(u_n)$. Furthermore, we infer from (3.2) that

$$\begin{aligned} c_0 + \frac{1}{n} \geq I(u_n) &= \Big(\frac{1}{2} - \frac{1}{2\cdot 2^*_\alpha}\Big)\|u_n\|^2_D - \Big(1 - \frac{1}{2\cdot 2^*_\alpha}\Big)\int_{\mathbb{R}^N} h(x)u_n\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{2\cdot 2^*_\alpha}\Big)\|u_n\|^2_D - \Big(1 - \frac{1}{2\cdot 2^*_\alpha}\Big)\|h\|_{H^{-1}}\|u_n\|_D. \end{aligned} \tag{3.3}$$

We know that $\{u_n\}$ is bounded. We claim that $\inf_n \|u_n\|_D \geq \sigma > 0$, which σ is a positive constant. Indeed, if not, by (3.3), $I(u_n)$ would converge to zero. We can infer that $c_0 \geq 0$ which is contradict with $c_0 < 0$. So we have

$$\sigma \leq \|u_n\|_D \leq \delta. \tag{3.4}$$

Secondly, we claim that, for a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), $\|\nabla I(u_n)\|_D \to 0$ as $n \to \infty$. In fact, if the claim were false, we could assume $\|\nabla I(u_n)\|_D \geq c > 0$ for n large enough. Consequently, according to Lemma 2.5, for u_n there exist ϵ_n and differentiable t_n satisfying

$$t_n(0) = 1, \quad t_n(w)(u_n - w) \in \Lambda, \quad \|w\|_D < \epsilon_n$$

and

$$\begin{aligned} &\langle t'_n(0), w\rangle \\ &= \frac{2\int_{\mathbb{R}^N}\nabla u_n\nabla w\,dx - 2\cdot 2^*_\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\alpha}|u_n(x)|^{2^*_\alpha - 2}u_n(x)w(x)}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} hw\,dx}{\|u_n\|^2_D - (2\cdot 2^*_\alpha - 1)B(u_n)}. \end{aligned}$$

We choose $w_n = \delta_n\nabla I(u_n)/\|\nabla I(u_n)\|_D$, $v_n = t_n(w_n)(u_n - w_n)$, where $0 < \delta_n < \epsilon_n$ is sufficiently small satisfying $\delta_n \to 0$, $t_n(w_n) \to 1$ as $n \to \infty$ and

$$\begin{gathered} \frac{\big|I(v_n) - I(u_n) - \langle I'(u_n), v_n - u_n\rangle\big|}{\|u_n - v_n\|_D} < \frac{1}{n}, \\ \frac{\big|t_n(w_n) - 1 - \langle t'_n(0), w_n\rangle\big|}{\|w_n\|_D} < 1. \end{gathered}$$

From (3.2), that $v_n \in \Lambda$ and the above, we deduce

$$\frac{1}{n}\|v_n - u_n\|_D \geq I(u_n) - I(v_n) \geq \langle I'(u_n), u_n - v_n\rangle - \frac{1}{n}\|u_n - v_n\|_D.$$

Thus, we have

$$\begin{aligned}
&\frac{2}{n}\|t_n(w_n)(u_n - w_n) - u_n\|_D \\
&\geq (1 - t_n(w_n))\langle I'(u_n), u_n\rangle + t_n(w_n)\delta_n \langle I'(u_n), \frac{\nabla I(u_n)}{\|\nabla I(u_n)\|_D}\rangle,
\end{aligned}$$

and

$$\frac{2}{n}\Big[(|\langle t_n'(0), w_n\rangle| + \|w_n\|_D)\|u_n\|_D + t_n(w_n)\|w_n\|_D\Big] \geq t_n(w_n)\delta_n\|\nabla I(u_n)\|_D.$$

Dividing by $\delta_n > 0$ on both left and right hand of the above inequality, we obtain

$$\frac{2}{n}\Big[(|\langle t_n'(0), \frac{\nabla I(u_n)}{\|\nabla I(u_n)\|_D}\rangle| + 1)\|u_n\|_D + t_n(w_n)\Big] \geq t_n(w_n)\|\nabla I(u_n)\|_D. \tag{3.5}$$

Now, if there exists $\lambda > 0$ such that

$$\big|\|u_n\|_D^2 - (2\cdot 2_\alpha^* - 1)B(u_n)\big| \geq \lambda\,,$$

we can get the claim. In fact,

$$\begin{aligned}
&\Big|\langle t_n'(0), h_n\rangle\Big| \\
&= \Big|\frac{\int_{\mathbb{R}^N}(2\nabla u_n \nabla h_n - h_n u_n)dx - 2\cdot 2_\alpha^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\alpha^*}|u_n(x)|^{2_\alpha^*-2}u_n(x)h_n(x)}{|x-y|^\alpha}\,dx\,dy}{\|u_n\|_D^2 - (2\cdot 2_\alpha^* - 1)B(u_n)}\Big| \\
&\leq \frac{C}{\lambda}.
\end{aligned}$$

Here, $h_n = \frac{\nabla I(u_n)}{\|\nabla I(u_n)\|_D}$ and we have used the uniformly boundedness of $\|u_n\|_D$. Consequently, as $n \to \infty$,

$$\frac{2}{n}\Big[(|\langle t_n'(0), h_n\rangle| + 1)\|u_n\|_D + t_n(w_n)\Big] \to 0.$$

So that, by passing to the limit as $n \to \infty$ in (3.5), we get a contradiction which implies the claim is true.

To show the existence of positive lower bound of $\Big|\|u_n\|_D^2 - (2\cdot 2_\alpha^* - 1)B(u_n)\Big|$, we argue indirectly and assume

$$\|u_n\|_D^2 - (2\cdot 2_\alpha^* - 1)B(u_n) = o(1), \quad n \to \infty.$$

Here, $\{u_n\}$ is a subsequence still denoted by the original symbol. Combining this and (3.3), similarly to the proof of Lemma 2.4(i), we can easily get a contradiction.

So that we conclude that, for a subsequence which we still denote by $\{u_n\}$,

$$I(u_n) \to c_0, \quad \|\nabla I(u_n)\|_D \to 0,$$

as $n \to \infty$. By (3.3) we know that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, and the weak limit of $\{u_n\}$ which we denote by u_0 is a weak solution of system (1.3). Obviously, $u_0 \in \Lambda$ and

$$c_0 \leq I(u_0) = \frac{1}{2}\|u_0\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}B(u_0) - C(u_0)$$

$$\begin{aligned}&= \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\alpha^*}\Big)\|u_0\|_D^2 - \Big(1 - \frac{1}{2\cdot 2_\alpha^*}\Big)C(u_0)\\ &\le \liminf I(u_n) = c_0.\end{aligned}$$

Therefore, u_0 is a least energy solution.

Now, we only need to show that u_0 is a local minimum solution. We apply Lemma 2.3 to u_0 and $|u_0|$. Since

$$\frac{d}{dt}I(tu_0) = \varphi'_{u_0}(t) > 0, \quad t \in (t^+(u_0), t^-(u_0)),$$

we know that $u_0 \in \Lambda^+$. Otherwise, $u_0 \in \Lambda^-$ and $c_0 \le I(t^+(u_0)u_0) < I(u_0) = c_0$ which is a contradiction. By Lemma 2.3 and $u_0 \in \Lambda^+$ we know that

$$1 = t^+(u_0) < t_0(u_0) = \Big[\frac{\|u_0\|_D^2}{(2\cdot 2_\alpha^* - 1)B(u_0)}\Big]^{1/(2\cdot 2_\alpha^*-2)}.$$

Therefore

$$1 < \Big[\frac{\|u_0 - w\|_D^2}{(2\cdot 2_\alpha^* - 1)B(u_0 - w)}\Big]^{1/(2\cdot 2_\alpha^*-2)}, \quad \|w\|_D < \varepsilon,$$

for ε small enough. Applying Lemma 2.5, we get a $t(w) > 0$ such that $t(w)(u_0-w) \in \Lambda$ for $\|w\| < \varepsilon$ small. Moreover, it holds $t(w) \to 1$ as $w \to 0$. Thus we can assume that, for $\|w\| < \varepsilon$ sufficiently small,

$$t(w) < \Big[\frac{\|u_0 - w\|_D^2}{(2\cdot 2_\alpha^* - 1)B(u_0 - w)}\Big]^{1/(2\cdot 2_\alpha^*-2)}, \quad t(w)(u_0 - w) \in \Lambda^+.$$

Then by Lemma 2.3, we conclude that

$$I(u_0) \le I(t(w)(u_0 - w)) \le I(t(u_0 - w)) \tag{3.6}$$

for $0 < t < [\frac{\|u_0-w\|_D^2}{(2\cdot 2_\alpha^*-1)B(u_0-w)}]^{1/(2\cdot 2_\alpha^*-2)}$. Taking $t = 1$ in (3.6) we have

$$I(u_0) \le I(u_0 - w), \quad \|w\|_D < \varepsilon,$$

which means u_0 is a local minimum solution.

Additionally, if we assume that $h > 0$,

$$\varphi'_{|u_0|}(t) < \varphi'_{u_0}(t) < 0, \quad t \in [0, 1).$$

Hence, $t^+(|u_0|) \ge 1$ and consequently,

$$c_0 \le I(t^+(|u_0|)|u_0|) \le I(|u_0|) \le I(u_0) = c_0.$$

Therefore, $t^+(|u_0|) = 1$ and $\int_{\mathbb{R}^N} h(x)|u_0|\,dx = \int_{\mathbb{R}^N} h(x)u_0\,dx$, which yields $u_0 \ge 0$. Then by the maximum principle, we know $u_0 > 0$. The proof is complete. $\square$

We remark that since $c_0 < 0$, any solution u_0 of system (1.3) with the least energy c_0 satisfies

$$\begin{aligned}c_0 = I(u_0) &= \frac{1}{2}\|u_0\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}B(u_0) - \int_{\mathbb{R}^N} h(x)u_0 dx\\ &= \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\alpha^*}\Big)\|u_0\|_D^2 - \Big(1 - \frac{1}{2\cdot 2_\alpha^*}\Big)\int_{\mathbb{R}^N} h(x)u_0\,dx < 0.\end{aligned}$$

Thus, $\int_{\mathbb{R}^N} h(x)u_0 dx > 0$ and consequently, $u_0 \in \Lambda^+$.

Proposition 3.2. *Assume $N \geq 3$, $0 < \alpha < N$ and* (A3)*. If $\{u_n\}$ is a $(PS)_c$ sequence of I with*
$$c < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}, \tag{3.7}$$
then $\{u_n\}$ has a convergent subsequence.

Proof. Obviously, $\|u_n\|_D$ is bounded. In fact
$$\begin{aligned}
c + \|u_n\|_D &\geq I(u_n) - \frac{1}{2\cdot 2_\alpha^*}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{2}\|u_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*} B(u_n) - C(u_n) \\
&\quad - \frac{1}{2\cdot 2_\alpha^*}\|u_n\|_D^2 + \frac{1}{2\cdot 2_\alpha^*} B(u_n) + \frac{1}{2\cdot 2_\alpha^*} C(u_n) \\
&\geq \frac{1}{2}(1 - \frac{1}{2_\alpha^*})\|u_n\|_D^2 - (1 - \frac{1}{2\cdot 2_\alpha^*})\|h\|_{H^{-1}}\|u_n\|_D.
\end{aligned}$$
Thus, there exists $w \in D^{1,2}(\mathbb{R}^N)$ which satisfies $u_n \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$ and solves (1.3). Therefore, $w \neq 0$ and $I(w) \geq c_0$. Let $u_n - w = v_n$, by Brezis-Lieb lemma [7] and [18, Lemmas 2.1 and 2.2], we deduce that
$$\|u_n\|_D^2 = \|v_n\|_D^2 + \|w\|_D^2 + o(1), \quad n \to \infty,$$
and
$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^{2_\alpha^*}|v_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)|^{2_\alpha^*}|w(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy + o_n(1)
\end{aligned}$$
as $n \to \infty$. So we obtain
$$\begin{aligned}
&c \leftarrow I(u_n) \\
&= \frac{1}{2}\|u_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)u_n\,dx \\
&= \frac{1}{2}\|v_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^{2_\alpha^*}|v_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)v_n\,dx \\
&\quad + \frac{1}{2}\|w\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)|^{2_\alpha^*}|w(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)w\,dx + o_n(1) \\
&= I(w) + \frac{1}{2}\|v_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*} B(v_n) + o_n(1).
\end{aligned}$$
As a result, for n large we have
$$\frac{1}{2}\|v_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*} B(v_n) + o_n(1) < \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}. \tag{3.8}$$
On the other hand,
$$o(1) = \langle I'(u_n), u_n\rangle = \langle I'(w), w\rangle + \|v_n\|_D^2 - B(v_n) + o(1),$$
which implies
$$\|v_n\|_D^2 - B(v_n) = o(1). \tag{3.9}$$

If we can show that $\{v_n\}$ has subsequence converging strong to 0, we have the result. Therefore, arguing indirectly, we assume $\|v_n\|_D \geq c > 0$ for n large. According to (3.9), the definition of $S_{H,L}$ and $\frac{2^*_\alpha}{2^*_\alpha-1} = \frac{2N-\alpha}{N+2-\alpha}$, we have

$$\|v_n\|_D^2 = B(v_n) + o(1) \leq \frac{\|v_n\|_D^{2\cdot 2^*_\alpha}}{S_{H,L}^{2^*_\alpha}}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{N+2-\alpha}{2N-\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}} &= \frac{1}{2}\frac{2^*_\alpha - 1}{2^*_\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}} \\
&\leq \frac{1}{2}\frac{2^*_\alpha - 1}{2^*_\alpha}\|v_n\|_D^2 \\
&= \frac{1}{2}\|v_n\|_D^2 - \frac{1}{2}\frac{1}{2^*_\alpha} B(v_n) + o_n(1) \\
&< \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}},
\end{aligned}$$

which is a contradiction. The proof is complete. ◻

Remark 3.3. *To find the second solution of* (1.3)*, the only we need to show is that*

$$c_0 < c_1 = \inf_{\Lambda^-} I(u) < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

By the continuity of I and Lemma 2.2, we know that there exists $\delta > 0$ such that

$$I(u_0 + tW) < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}, \quad 0 \leq t < \delta,$$

where u_0 is the positive local minimum solution we get in Proposition 3.1. For $t \geq \delta$, a directly computation shows us

$$\begin{aligned}
I(u_0 + tW) &= \frac{1}{2}\|u_0 + tW\|_D^2 - \frac{1}{2\cdot 2^*_\alpha} B(u_0 + tW) - \int_{\mathbb{R}^N} h(u_0 + tW)dx \\
&= \frac{1}{2}\|u_0\|_D^2 + t\int_{\mathbb{R}^N} \nabla u_0 \nabla W\, dx + \frac{t^2}{2}\|W\|_D^2 - \frac{1}{2\cdot 2^*_\alpha} B(u_0) \\
&\quad + \frac{1}{2\cdot 2^*_\alpha}[B(u_0) + B(tW) - B(u_0 + tW)] - \frac{1}{2\cdot 2^*_\alpha} B(tW) \\
&\quad - \int_{\mathbb{R}^N} h u_0 dx - \int_{\mathbb{R}^N} htW\, dx \\
&= I(u_0) + \frac{t^2}{2}[\|W\|_D^2 - \frac{t^{2(2^*_\alpha - 1)}}{2\cdot 2^*_\alpha} B(W)] + \frac{1}{2\cdot 2^*_\alpha}[B(u_0) + B(tW) \\
&\quad - B(u_0 + tW) + 2\cdot 2^*_\alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_\alpha}|u_0(y)|^{2^*_\alpha - 2}u_0(y)}{|x-y|^\alpha}\, dx\, dy] \\
&< c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.
\end{aligned}$$

Here, we use that $\langle I'(u_0), tW\rangle = 0$ and W is a minimizer of $S_{H,L}$.

Proof of Theorem 1.1. Firstly, we show that

$$c_0 < c_1 = \inf_{\Lambda^-} I(u) < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

We observe that for every $u \in D^{1,2}(\mathbb{R}^N)$ with $\|u\|_D = 1$, there exists a unique $t^-(u) > 0$ such that (see Lemma 2.3)

$$t^-(u)u \in \Lambda^-.$$

Similar to Lemma 2.5, we know that $t^-(u)$ is a continuous function of u. And consequently the manifold Λ^- disconnects $D^{1,2}(\mathbb{R}^N)$ in exactly two connected components U_1 and U_2, where

$$U_1 = \Big\{u \in D^{1,2}(\mathbb{R}^N) : u = 0 \text{ or } \|u\|_D < t^-\big(\frac{u}{\|u\|_D}\big)\Big\},$$
$$U_2 = \Big\{u \in D^{1,2}(\mathbb{R}^N) : \|u\|_D > t^-\big(\frac{u}{\|u\|_D}\big)\Big\}.$$

Obviously, $D^{1,2}(\mathbb{R}^N) = \Lambda^- \cup U_1 \cup U_2$. In particular, $u_0 \in \Lambda^+ \subset U_1$. Since

$$t^-\Big(\frac{u_0 + tW}{\|u_0 + tW\|_D}\Big)\frac{u_0 + tW}{\|u_0 + tW\|_D} \in \Lambda,$$

we have

$$0 < t^-\Big(\frac{u_0 + tW}{\|u_0 + tW\|_D}\Big) < C_0$$

uniformly for $t \in \mathbb{R}$.

On the other hand, there exists $\tilde{t} > 0$ such that

$$\|u_0 + tW\|_D \geq t\|W\|_D - \|u_0\|_D \geq C_0, \quad t \geq \tilde{t}.$$

So that we can fix a $t_0 > 0$ such that $\|u_0 + t_0W\|_D > t^-(\frac{u_0+t_0W}{\|u_0+t_0W\|_D})$. Thus, $u_0 + t_0W \in U_2$. Combining this and the fact $u_0 \in U_1$, we know that

$$u_0 + t_1W \in \Lambda^-,$$

for some $0 < t_1 < t_0$. Consequently, by Remark 3.3, we have

$$c_1 = \inf_{\Lambda^-} I(u) \leq \max_{0\leq t\leq t_0} I(u_0 + tW) < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

Next, we show that c_1 is a critical value of I and satisfies $c_1 > c_0$. Similarly to the proof of Proposition 3.1, we apply Ekeland's variational principle and get a minimizing sequence $\{u_n\} \subset \Lambda^-$ such that

$$I(u_n) < c_1 + \frac{1}{n};$$
$$I(w) \geq I(u_n) - \frac{1}{n}\|u - w\|_D, \quad w \in \Lambda^-.$$

So that

$$\begin{aligned} c_1 + 1 > I(u_n) &= \frac{1}{2}\|u_n\|_D^2 - \frac{1}{2\cdot 2_\alpha^*}B(u_n) - \int_{\mathbb{R}^N} h(x)u_n\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{2\cdot 2_\alpha^*}\Big)\|u_n\|_D^2 - \Big(1 - \frac{1}{2\cdot 2_\alpha^*}\Big)\|h\|_{H^{-1}}\|u_n\|_D, \end{aligned}$$

which implies $\|u_n\|$ has a upper bound. Moreover, from $\{u_n\} \subset \Lambda^-$, we know that

$$\|u_n\|_D^2 \leq (2\cdot 2_\alpha^* - 1)\frac{\|u_n\|_D^{2_\alpha^*}}{S_{H,L}^{2_\alpha^*}}.$$

Thus, $\|u_n\|_D$ has a uniform positive lower bound. Then, analogously to the proof of Proposition 3.1, we know that

$$I(u_n) \to c_1, \quad I'(u_n) \to 0 \quad in \quad H^{-1}.$$

By Proposition 3.1 and $c_1 < c_0 + \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^{\frac{2N-\alpha}{N+2-\alpha}}$, we can conclude that there exists a subsequence of $\{u_n\}$ such that $u_n \to u_1$ strongly in $D^{1,2}(\mathbb{R}^N)$. Therefore u_1 is a critical point of I and $I(u_1) = c_1$. Noting that Λ^- is closed, we have $u_1 \in \Lambda^-$. To show $c_1 > c_0$, arguing indirectly, we assume $c_1 = c_0$. Thus by Remark 3.3 we have

$$\int_{\mathbb{R}^N} h(x)u_1 dx > 0 \quad and \quad u_1 \in \Lambda^+,$$

which leads to a contradiction.

Finally, we consider the case $h > 0$. Applying Lemma 2.3 to u_1 and $|u_1|$, we know that there exist a $t^-(|u_1|)$ such that

$$t^-(|u_1|)|u_1| \in \Lambda^-.$$

Moreover,

$$t^-(|u_1|) \geq t_0(|u_1|) = t_0(u_1) = \Big[\frac{A(u_1)}{(2^*_\alpha - 1)B(u_1)}\Big]^{\frac{1}{2^*_\alpha - 2}}$$

Thus in both cases $\int_{\mathbb{R}^N} h(x)u\,dx > 0$ and $\int_{\mathbb{R}^N} h(x)u\,dx \leq 0$, we can deduce that

$$c_1 = I(u_1) \geq I(t^-(u_1)u_1) \geq I(t^-(|u_1|)|u_1|) = t_0(u_1) \geq c_1.$$

Therefore, $\int_{\mathbb{R}^N} h(x)u_1 dx = \int_{\mathbb{R}^N} h(x)|u_1|dx$, which implies $u_1 \geq 0$. According to the maximum principle we get $u_1 > 0$. The proof is complete. $\square$

4. Proof of Theorem 1.2

We define the energy functional associated with

$$-\Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u + h(x), \quad x \in \mathbb{R}^N, \tag{4.1}$$

where $N \geq 3$, $0 < \alpha < N$ and $2 - \frac{\alpha}{N} < p < 2^*_\alpha$, by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)u\,dx, \tag{4.2}$$

for $u \in H^1(\mathbb{R}^N)$. By the Hardy-Littlewood-Sobolev inequality of Lemma 2.1, we know that $J \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned}\langle J'(u), v\rangle = &\int_{\mathbb{R}^N} (|\nabla u||\nabla v| + uv)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^{p-2}u(y)v(y)}{|x-y|^\alpha}\,dx\,dy \\ &- \int_{\mathbb{R}^N} h(x)v\,dx.\end{aligned}$$

We will constrain the functional J on the Nehari manifold

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N), \langle J'(u), u\rangle = 0\}. \tag{4.3}$$

Denote $\Psi(u) = \langle J'(u), u\rangle$, so we know that

$$\langle J'(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)u\,dx,$$

$$\langle \Psi'(u), u\rangle = 2\|u\|^2 - 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} h(x)u\,dx.$$

Note that, when u_0 is a local minimum solution of J, it holds

$$\langle J'(u), u\rangle = 0, \quad \langle \Psi'(u), u\rangle \geq 0,$$

which leads us to consider the following manifolds:

$$\begin{gathered}
\mathcal{N} = \{u \in H^1(\mathbb{R}^N) : \langle J'(u), u\rangle = 0\}, \\
\mathcal{N}^+ = \{u \in \mathcal{N} : \langle \Psi'(u), u\rangle > 0\}, \\
\mathcal{N}^- = \{u \in \mathcal{N} : \langle \Psi'(u), u\rangle < 0\}, \\
\mathcal{N}^0 = \{u \in \mathcal{N} : \langle \Psi'(u), u\rangle = 0\}.
\end{gathered}$$

Moreover, we define

$$j_0 = \inf_{\mathcal{N}} J(u); \quad j_1 = \inf_{\mathcal{N}^-} J(u); \quad j^+ = \inf_{\mathcal{N}^+} J(u).$$

Obviously, only $\mathcal{N}^0$ contains the element 0. It is easy to see that $\mathcal{N}^0 \cup \mathcal{N}^+$ and $\mathcal{N}^0 \cup \mathcal{N}^-$ are both closed subsets of $H^1(\mathbb{R}^N)$.

To simplify the calculation, for $u \in H^1(\mathbb{R}^N)$, we denote

$$\begin{gathered}
\tilde{A} = \tilde{A}(u) = \|u\|^2, \\
\tilde{B} = \tilde{B}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy, \\
\tilde{C} = \tilde{C}(u) = \int_{\mathbb{R}^N} h(x)u\,dx.
\end{gathered}$$

Define the fibering map

$$\psi(t) = \psi_u(t) := J(tu) = \frac{\tilde{A}}{2}t^2 - \frac{\tilde{B}}{2p}t^{2p} - \tilde{C}t, \quad t > 0. \tag{4.4}$$

Therefore,

$$\begin{gathered}
\psi'(t) = \tilde{A}t - \tilde{B}t^{2p-1} - \tilde{C}, \\
\psi''(t) = \tilde{A} - (2p-1)\tilde{B}t^{2p-2}.
\end{gathered} \tag{4.5}$$

Obviously, $tu \in \mathcal{N}$ with $t > 0$ if and only if $\psi'(t) = 0$. By the sign of $\psi''(t)$, the stationary points of $\psi(t)$ can be classified into three types, namely local minimum, local maximum and turning point. Moreover, the set $\mathcal{N}$ is a natural constraint for the functional J. This is means that if the infimum is attained by $u \in \mathcal{N}$, then u is a solution of (1.4). However, in our case, the global maximum point of $\psi(t)$ is not unique. This leads us to partition the set $\mathcal{N}$ according to the critical points of $\psi(t)$. Now we give some properties of $\mathcal{N}^\pm$ and $\mathcal{N}^0$.

Lemma 4.1. *(i) Assume that $h \not\equiv 0$ for $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, there is a unique $\tilde{t}^- = \tilde{t}^-(u) > 0$ such that $\tilde{t}^-u \in \mathcal{N}^-$. If additionally we assume $\int_{\mathbb{R}^N} hu\,dx > 0$ and (A5), then there exists $\varepsilon = \varepsilon(N, p, \alpha, d_{\frac{2Np}{2N-\alpha}})$ and a unique $0 < \tilde{t}^+ = \tilde{t}^+(u) < \tilde{t}^-$ satisfying $\tilde{t}^+u \in \mathcal{N}^+$. Moreover,*

$$\begin{gathered}
J(\tilde{t}^-u) = \max_{t\geq 0} J(tu) \quad \text{for } \int_{\mathbb{R}^N} hu\,dx \leq 0; \\
J(\tilde{t}^-u) = \max_{t\geq \tilde{t}^+} J(tu), \quad J(\tilde{t}^+u) = \min_{0\leq t\leq \tilde{t}^-} J(tu) \quad \text{for } \int_{\mathbb{R}^N} hu\,dx > 0.
\end{gathered}$$

Proof. Define $\psi(t) = \frac{\tilde{A}}{2}t^2 - \frac{\tilde{B}}{2\cdot p}t^{2p} - \tilde{C}t$, for all $t > 0$. In the case $\int_{\mathbb{R}^N} hu\,dx \le 0$, there is a unique $\tilde{t}^- > 0$ such that $\psi'(\tilde{t}^-) = 0$ and $\psi''(\tilde{t}^-) < 0$. So that

$$\langle J'(\tilde{t}^- u), \tilde{t}^- u\rangle = 0;$$
$$\|\tilde{t}^- u\|^2 - (2p-1)\tilde{B}(u)(\tilde{t}^-)^{2p-2} < 0.$$

Thus, $\tilde{t}^- u \in \mathcal{N}^-$ and $J(\tilde{t}^- u) = \max_{t\ge 0} J(tu)$.

In the case $\int_{\mathbb{R}^N} hu\,dx > 0$, for $\|u\| = 1, \tilde{t}_0 = \tilde{t}_0(u) = \big[\frac{1}{(2p-1)\tilde{B}}\big]^{\frac{1}{2p-2}} > 0$ and (A5), we have

$$\begin{aligned}
\max_{t\ge 0} \psi'(t) &\ge t_0 - \tilde{B}t_0^{2p-1} - \tilde{C}\\
&= \Big[\frac{1}{(2p-1)\tilde{B}}\Big]^{\frac{1}{2p-2}} \cdot \frac{2p-2}{2p-1} - \int_{\mathbb{R}^N} hu\,dx\\
&\ge \Big[\frac{2p-2}{(2p-1)^{2p-1/(2p-2)}B_0^{1/2p-2}} - |h|_{\frac{2Np}{2N(p-1)+\alpha}}\, d_{\frac{2Np}{2N-\alpha}}\Big] > 0.
\end{aligned}$$

Here

$$\varepsilon(N,p,\alpha,d_{\frac{2Np}{2N-\alpha}}) = \frac{2p-2}{(2p-1)^{2p-1/(2p-2)}B_0^{1/2p-2}d_{\frac{2Np}{2N-\alpha}}},$$
$$B_0 = \sup_{\|u\|=1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,dx\,dy.$$

From $\psi'(0) = -\tilde{C} < 0$ and $\psi'(t) \to -\infty$ as $t \to +\infty$, we know that there exist unique $0 < \tilde{t}^+ < \tilde{t}_0 < \tilde{t}^-$ such that $\psi'(\tilde{t}^-) = \psi'(\tilde{t}^+) = 0, \psi''(\tilde{t}^-) < 0 < \psi''(\tilde{t}^+)$. Equivalently, $\tilde{t}^+ u \in \mathcal{N}^+$ and $\tilde{t}^- u \in \mathcal{N}^-$.

Moreover, since $\frac{d}{dt}J(tu) = \psi'(t)$, we can easily see that $J(\tilde{t}^- u) = \max_{t\ge\tilde{t}^+} J(tu)$ and $J(\tilde{t}^+ u) = \min_{0\le t\le\tilde{t}^-} J(tu)$. The proof is complete. □

Lemma 4.2. *Assume t* $h \not\equiv 0$, (A4) *and (A5)* hold. *Then*

(i) $\mathcal{N}^0 = \{0\}$*;*
(ii) $\mathcal{N}^\pm \ne \emptyset, \mathcal{N}^-$ *is closed.*

Proof. (i) To prove $\mathcal{N}^0 = \{0\}$, we need to prove that, for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $\tilde{\varphi}(t)$ has no critical point that is a turning point. Set $\|u\| = 1$, define

$$k(t) = \tilde{A}t - \tilde{B}t^{2p-1}. \tag{4.6}$$

Then $\psi'(t) = k(t) - \tilde{C}, k''(t) = -\tilde{B}(2p-1)(2p-2)t^{2p-3} < 0$ for $t > 0$. So $k(t)$ is strictly concave. If $k''(\tilde{t}_0) = 0$, $\tilde{t}_0 = (\frac{\tilde{A}}{(2p-1)\tilde{B}})^{1/(2p-2)} > 0$, for $p > 2 - \frac{\alpha}{N} > 1$. Moreover, $\lim_{t\to 0^+} k(t) = 0$, $\lim_{t\to+\infty} k(t) = -\infty$ and $k(t) > 0$ for $t > 0$ small. Therefore, $k(t)$ has a unique global maximum points t_0 and

$$k(t_0) = \frac{2\tilde{A}(2p-1)}{2p-1}\Big(\frac{\tilde{A}}{(2p-1)\tilde{B}}\Big)^{1/(2p-2)} := k_0.$$

By (4.4) and (4.5), we infer that if $0 < \tilde{C} < k_0$, the equation $\psi'(t) = 0$ has exactly two points $\tilde{t}_1, \tilde{t}_2$ satisfying $\tilde{t}_1 < \tilde{t}_0 < \tilde{t}_2$. If $\tilde{C} \le 0$, the equation $\psi'(t) = 0$ has one root $\tilde{t}_3 > \tilde{t}_0$. Since $\psi''(t) = \tilde{A} - (2p-1)\tilde{B}t^{2p-2}$, it follows that $\psi''(\tilde{t}_1) > 0, \psi''(\tilde{t}_2) < 0$ and $\psi''(\tilde{t}_3) < 0$. It follows that $\tilde{t}_1 u \in \mathcal{N}^+, \tilde{t}_2 u \in \mathcal{N}^-$ if $0 < \tilde{C} < k_0$ and $\tilde{t}_3 u \in \mathcal{N}^-$ if $\tilde{C} \le 0$. Since $\{u \in H^1(\mathbb{R}^N) : \|u\| = 1, 0 < \tilde{C} < k_0\}$ and $\{u \in H^1(\mathbb{R}^N) :$

$\|u\| = 1, \tilde{C} \le 0\}$ are nonempty, we can infer that $\mathcal{N}^{\pm}$ are nonempty. This implies $\mathcal{N}^0 = \{0\}$.

It is suffices to prove that $k_0 > \tilde{C}$. By (A4), (A5) and Lemma 4.1 we have

$$k_0 - \tilde{C} = k(t_0) - \tilde{C} = \tilde{A}t_0 - \tilde{B}t_0^{2p-1} - \tilde{C} > 0.$$

(ii) Let $u \in \mathcal{N}^-$ and denote $\tilde{u} = u/\|u\|$. Then $\|\tilde{u}\| = 1$. By (i), we know that

$$\tilde{C}(\tilde{u}) < k_0 = \frac{2(p-1)}{2p-1}\Big(\frac{1}{(2p-1)\tilde{B}}\Big)^{1/(2p-1)}$$

with $\tilde{B} := B(\tilde{u})$. Furthermore, if $0 < \tilde{C}(\tilde{u}) < \kappa_0$, the equation $\psi'(t) = 0$ has exactly two roots $\tilde{t}_1, \tilde{t}_2$ satisfying $0 < \tilde{t}_1 < t_0 < \tilde{t}_2$ such that $\tilde{t}_1 u \in \mathcal{N}^+, \tilde{t}_2 u \in \mathcal{N}^-$. Then $\tilde{t}_2\tilde{u} = u$ and so $\|u\| = \tilde{t}_2 > \tilde{t}_0$. If $\tilde{C} \le 0$, the equation $\psi'(t) = 0$ has exactly one root $\tilde{t}_3 > \tilde{t}_0$. Then $\tilde{t}_3\tilde{u} = u \in \mathcal{N}^-$ and so $\|u\| = \tilde{t}_3 > \tilde{t}_0$. In other words,

$$\|u\| > \tilde{t}_0 > 0, \quad u \in \mathcal{N}^-.$$

So there exists $\tau > 0$ such that

$$\|u\| > \tau > 0, \quad \forall u \in \mathcal{N}^-. \tag{4.7}$$

Therefore, $0 \notin cl(\mathcal{N}^-)$, where $cl(\mathcal{N}^-)$ is the closure of $\mathcal{N}^-$. On the other hand, by (i),

$$cl(\mathcal{N}^-) \subset \mathcal{N}^- \cup \mathcal{N}^0 = \mathcal{N}^- \cup \{0\}.$$

Hence, $cl(\mathcal{N}^-) = \mathcal{N}^-$ and $\mathcal{N}^-$ is closed. The proof is complete. $\square$

Lemma 4.3. *Under assumption* (A4) *and* (A5)*, for $u \in \mathcal{N} \setminus \{0\}$, there exists $\epsilon > 0$ and a differential function $\eta = \eta(w) > 0, w \in H^1(\mathbb{R}^N), \|w\| < \epsilon$ such that*

(1) $\eta(0) = 1$;
(2) $\eta(w)(u - w) \in \mathcal{N}$, *for all* $w \in B_\epsilon(0)$;
(3)

$$\begin{aligned}\langle \eta'(0), w\rangle = \Big(2\int_{\mathbb{R}^N} (\nabla u \nabla w + uw)dx \\ &- 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2p}|u(x)|^{p-2}u(x)w(x)}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} hw\,dx\Big) \\ &\div \big(\|u\|^2 - (2p-1)\tilde{B}(u)\big).\end{aligned}$$

Proposition 4.4. *Assume* (A4) *and* (A5) *hold. Then* (1.4) *has a local minimum solution with the least energy $j_0 = \inf_{\mathcal{N}} J(u)$.*

Proof. Firstly, we show that $\|u\|$ is bounded from both above and below: For any $u \in \mathcal{N}$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2p}\tilde{B}(u) - \tilde{C}(u) \\ &= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u\|^2 - \Big(1 - \frac{1}{2p}\Big)\int_{\mathbb{R}^N} h(x)u\,dx \\ &\ge \frac{p-1}{2p}\|u\|^2 - \frac{2p-1}{2p}|h|_{\frac{2Np}{2N(p-1)+\alpha}} d_{\frac{2Np}{2N-\alpha}}\|u\| \\ &\ge -\frac{(2p-1)^2}{8p(p-1)}\|h\|^2_{\frac{2Np}{2N-\alpha}}. \end{aligned} \tag{4.8}$$

Thus,

$$j_0 \geq -\frac{(2p-1)^2}{8p(p-1)}\|h\|^2_{\frac{2Np}{2N-\alpha}}.$$

Similar to the proof of Proposition 3.1, we can prove $j_0 < 0$. By using the Ekeland's variational principle on $\mathcal{N}$, we get a minimizing sequence $\{u_n\} \subset \mathcal{N}$ which satisfies

$$\begin{gathered} J(u_n) < j_0 + \frac{1}{n}, \\ J(w) \geq J(u_n) - \frac{1}{n}\|u - w\|, \quad w \in \mathcal{N}. \end{gathered} \tag{4.9}$$

Since $\{u_n\} \subset \mathcal{N}$, it follows that $\|u_n\|^2 = \tilde{B}(u_n) + \tilde{C}(u_n)$. Furthermore, we infer from (4.7) that

$$\begin{aligned} j_0 + \frac{1}{n} \geq J(u_n) &= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2 - \Big(1 - \frac{1}{2p}\Big)\int_{\mathbb{R}^N} h(x)u_n\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2 - \Big(1 - \frac{1}{2p}\Big)|h|_{\frac{2Np}{2N(p-1)+\alpha}}\, d_{\frac{2Np}{2N-\alpha}}\|u_n\|. \end{aligned} \tag{4.10}$$

We know that $\{u_n\}$ is bounded. We claim that $\inf_n \|u_n\| \geq \sigma_1 > 0$, which σ_1 is a positive constant. Indeed, if not, by (4.10), $J(u_n)$ would converge to zero. We can infer that $j_0 \geq 0$ which is contradict with $j_0 < 0$. So we have

$$\sigma_1 \leq \|u_n\| \leq \delta_1. \tag{4.11}$$

Secondly, we claim that, for a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), $\|\nabla J(u_n)\| \to 0$ as $n \to \infty$.

In fact, if the claim were false, we could assume $\|\nabla J(u_n)\| \geq d > 0$ for n large enough. Consequently, according to Lemma 4.3, for u_n there exist ϵ_n and differentiable η_n satisfying

$$\eta_n(0) = 1, \quad \eta_n(w)(u_n - w) \in \mathcal{N}, \quad \|w\| < \epsilon_n$$

and

$$\begin{aligned} \langle \eta_n'(0), w\rangle = \Big(&2\int_{\mathbb{R}^N}(\nabla u_n \nabla w + u_n w)dx \\ &- 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^p|u_n(x)|^{p-2}u_n(x)w(x)}{|x-y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N} hw\,dx\Big) \\ &\div \big(\|u_n\|^2 - (2p-1)\tilde{B}(u_n)\big). \end{aligned}$$

We choose $w_n = \delta_n \frac{\nabla J(u_n)}{\|\nabla J(u_n)\|}$, $v_n = \eta_n(w_n)(u_n - w_n)$, where $0 < \delta_n < \epsilon_n$ is sufficiently small satisfying $\delta_n \to 0$, $\eta_n(w_n) \to 1$ as $n \to \infty$ and

$$\begin{gathered} \frac{\big|J(v_n) - J(u_n) - \langle J'(u_n), v_n - u_n\rangle\big|}{\|u_n - v_n\|} < \frac{1}{n}, \\ \frac{\big|\eta(w_n) - 1 - \langle \eta_n'(0), w_n\rangle\big|}{\|w_n\|} < 1. \end{gathered}$$

From (4.6), that $v_n \in \mathcal{N}$ and the above, we deduce that

$$\frac{1}{n}\|v_n - u_n\| \geq J(u_n) - J(v_n) \geq \langle J'(u_n), u_n - v_n\rangle - \frac{1}{n}\|u_n - v_n\|$$

Thus, we have

$$\frac{2}{n}\|\eta_n(w_n)(u_n - w_n) - u_n\|$$

$$\geq (1-\eta_n(w_n))\langle J'(u_n), u_n\rangle + \eta_n(w_n)\delta_n\langle J'(u_n), \frac{\nabla J(u_n)}{\|\nabla J(u_n)\|}\rangle,$$

and

$$\frac{2}{n}\Big[(|\langle \eta_n'(0), w_n\rangle| + \|w_n\|)\|u_n\| + \eta_n(w_n)\|w_n\|\Big] \geq \eta_n(w_n)\delta_n\|\nabla J(u_n)\|.$$

Dividing by $\delta_n > 0$ on both left and right hand of the above inequality, we get

$$\frac{2}{n}\Big[(|\langle \eta_n'(0), \frac{\nabla J(u_n)}{\|\nabla J(u_n)\|}\rangle| + 1)\|u_n\| + \eta_n(w_n)\Big] \geq \eta_n(w_n)\|\nabla J(u_n)\|. \tag{4.12}$$

Now, if there exists $\lambda > 0$ such that

$$\big|\|u_n\|^2 - (2p-1)\tilde{B}(u_n)\big| \geq \lambda,$$

we can get the claim. In fact,

$$\begin{aligned}
&\Big|\langle \eta_n'(0), h_n\rangle\Big| \\
&= \Big|\frac{2(u_n, h_n) - \int_{\mathbb{R}^N} h_n u_n - 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^p|u_n(x)|^{p-2}u_n(x)h_n(x)}{|x-y|^\alpha}\,dx\,dy}{\|u_n\|^2 - (2p-1)\tilde{B}(u_n)}\Big| \\
&\leq \frac{\tilde{C}}{\lambda}.
\end{aligned}$$

Here, $h_n = \nabla J(u_n)/\|\nabla J(u_n)\|$ and we have used the uniformly boundedness of $\|u_n\|$. Consequently, as $n \to \infty$,

$$\frac{2}{n}\Big[(|\langle \eta_n'(0), h_n\rangle| + 1)\|u_n\| + \eta_n(w_n)\Big] \to 0.$$

So that, by passing to the limit as $n \to \infty$ in (4.12), we get a contradiction which implies the claim is true.

To show the existence of positive lower bound of $\big|\|u_n\|^2 - (2p-1)\tilde{B}(u_n)\big|$, we argue indirectly again and assume

$$\|u_n\|^2 - (2p-1)\tilde{B}(u_n) = o(1) \quad n \to \infty.$$

Here, $\{u_n\}$ is a subsequence still denoted by itself. Combining this and (4.10), similarly to the proof of Lemma 4.2(ii), we can easily get a contradiction.

So that we conclude that, for a subsequence still denoted by $\{u_n\}$,

$$I(u_n) \to j_0, \quad \|\nabla J(u_n)\| \to 0,$$

as $n \to \infty$. By (4.10) we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and the weak limit of $\{u_n\}$ which we denote by u_0 is a weak solution of system (1.4). Obviously, $u_0 \in \mathcal{N}$ and

$$\begin{aligned}
j_0 \leq J(u_0) &= \frac{1}{2}\|u_0\|^2 - \frac{1}{2p}\tilde{B}(u_0) - \tilde{C}(u_0) \\
&= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_0\|^2 - \Big(1 - \frac{1}{2p}\Big)\tilde{C}(u_0) \\
&\leq \liminf J(u_n) = j_0.
\end{aligned}$$

Therefore, u_0 is a least energy solution.

Now, we only need to show that u_0 is a local minimum solution. We apply Lemma 4.1 to u_0 and $|u_0|$. Since

$$\frac{d}{dt}J(tu_0) = \psi'(t) > 0, \quad t \in (\tilde{t}^+(u_0), \tilde{t}^-(u_0)),$$

we know that $u_0 \in \mathcal{N}^+$. Otherwise, $u_0 \in \mathcal{N}^-$ and $j_0 \le J(\tilde{t}^+(u_0)u_0) < J(u_0) = j_0$ which is a contradiction. By Lemma 4.1 and $u_0 \in \mathcal{N}^+$ we know that

$$1 = \tilde{t}^+(u_0) < \tilde{t}_0(u_0) = \Big[\frac{\|u_0\|^2}{(2p-1)\tilde{B}(u_0)}\Big]^{1/(2p-2)}.$$

Therefore

$$1 < \Big[\frac{\|u_0 - w\|^2}{(2p-1)\tilde{B}(u_0 - w)}\Big]^{1/(2p-2)}, \quad \|w\| < \varepsilon,$$

for ε small enough. Applying Lemma 4.3, we get a $\eta(w) > 0$ such that $\eta(w)(u_0 - w) \in \mathcal{N}$ for $\|w\| < \varepsilon$ small. Moreover, it holds $\eta(w) \to 1$ as $w \to 0$. Thus we can assume that, for $\|w\| < \varepsilon$ sufficiently small,

$$\eta(w) < \Big[\frac{\|u_0 - w\|^2}{(2p-1)\tilde{B}(u_0 - w)}\Big]^{1/(2p-2)}, \quad \eta(w)(u_0 - w) \in \mathcal{N}^+.$$

Then by Lemma 4.1, we conclude that

$$J(u_0) \le J(\eta(w)(u_0 - w)) \le J(t(u_0 - w)) \tag{4.13}$$

for $0 < \eta < \big[\frac{\|u_0 - w\|^2}{(2p-1)\tilde{B}(u_0 - w)}\big]^{1/(2p-2)}$. Taking $\eta = 1$ in (4.13) we have

$$J(u_0) \le J(u_0 - w), \quad \|w\| < \varepsilon,$$

which means u_0 is a local minimum solution.

Additionally, if we assume that $h > 0$, then

$$\psi'_{|u_0|}(t) < \psi'_{u_0}(t) < 0, \quad t \in [0, 1).$$

Hence, $\tilde{t}^+(|u_0|) \ge 1$ and consequently,

$$j_0 \le J(\tilde{t}^+(|u_0|)|u_0|) \le J(|u_0|) \le J(u_0) = j_0.$$

Therefore, $\tilde{t}^+(|u_0|) = 1$ and $\int_{\mathbb{R}^N} h(x)|u_0|\,dx = \int_{\mathbb{R}^N} h(x)u_0\,dx$, which yield $u_0 \ge 0$. Then by the maximum principle, we know $u_0 > 0$. The proof is complete. $\square$

Consider the nonlinear Schrödinger equation

$$-\Delta u + u = \Big(\frac{1}{|x|^\alpha} * |u|^p\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{4.14}$$

By [28, Proposition 2.], we know that (4.14) has positive smooth solution $V(x)$, which is also a minimizer of

$$S_{\alpha,p} = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx}{(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha}\,dx\,dy)^{1/p}}.$$

If V is a positive solution of (4.14) if and only if V is a critical point of the energy functional

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha}\,dx\,dy.$$

We know that

$$\|V\| = \tilde{B}(V) = S_{\alpha,p}^{\frac{p}{p-1}}.$$

From the fact that the Sobolev embedding

$$H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \quad (2 \le q \le 2^*)$$

is not compact, the variational functional $J(u)$ fails to satisfy the (PS) condition. Such a failure brings us some difficulties in applying the variational approach. In order to overcome the lack of compactness, we introduce the following proposition which plays a key role in our argument. We remark here that since $j_0 < 0$, any solution u_0 of system (1.4) with the least energy j_0 satisfies

$$\begin{aligned} j_0 = J(u_0) &= \frac{1}{2}\|u_0\|^2 - \frac{1}{2p}\tilde{B}(u_0) - \int_{\mathbb{R}^N} h(x)u_0 dx \\ &= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_0\|^2 - \Big(1 - \frac{1}{2p}\Big)\int_{\mathbb{R}^N} h(x)u_0\, dx < 0. \end{aligned}$$

Thus, $\int_{\mathbb{R}^N} h(x)u_0 dx > 0$ and consequently, $u_0 \in \mathcal{N}^+$.

Proposition 4.5. *Let $N \geq 3$, $0 < \alpha < N$,* (A4) *and* (A5) *hold. If $\{u_n\}$ be a $(PS)_c$ sequence of J with*

$$c < j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}, \tag{4.15}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Obviously, $\|u_n\|$ is bounded. Thus, there exists $w \in H^1(\mathbb{R}^N)$ which satisfies $u_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$ and solves (1.4). Therefore, $v \neq 0$ and $J(v) \geq j_0$. Let $u_n - v = w_n$, by Brezis-Lieb lemma and [18, Lemmas 2.1 and 2.2], we deduce

$$\|u_n\|^2 = \|w_n\|^2 + \|v\|^2 + o(1), \quad n \to \infty,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^\alpha}\, dx\, dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\alpha}\, dx\, dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)|^p|w(y)|^p}{|x-y|^\alpha}\, dx\, dy + o_n(1) \end{aligned}$$

as $n \to \infty$. So we obtain

$$\begin{aligned} c \leftarrow J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x)u_n\, dx \\ &= \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)|^p|w_n(y)|^p}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x)w_n\, dx \\ &\quad + \frac{1}{2}\|v\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^p|v(y)|^p}{|x-y|^\alpha}\, dx\, dy - \int_{\mathbb{R}^N} h(x)v\, dx + o_n(1) \\ &= J(v) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1). \end{aligned}$$

As a result, for n large, we have

$$\frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1) < \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}. \tag{4.16}$$

On the other hand,

$$o(1) = \langle J'(u_n), u_n\rangle = \langle J'(v), v\rangle + \|w_n\|^2 - \tilde{B}(w_n) + o(1),$$

which implies

$$\|w_n\|^2 - \tilde{B}(w_n) = o(1). \tag{4.17}$$

If we can show that $\{w_n\}$ has subsequence converging strong to 0, we have the result. Therefore, arguing indirectly, we assume $\|w_n\| \geq C > 0$ for n large. By (4.17), we have

$$\|w_n\|^2 = \tilde{B}(w_n) \leq \frac{\|w_n\|^{2p}}{S_{\alpha,p}^p}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{p-1}{p}S_{\alpha,p}^{\frac{p}{p-1}} &= \frac{1}{2}(1-\frac{1}{p})S^{\frac{p}{p-1}}\\
&\le \frac{1}{2}(1-\frac{1}{p})\|w_n\|^2\\
&= \frac{1}{2}\|w_n\|^2 - \frac{1}{2p}\tilde{B}(w_n) + o_n(1)\\
&< \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}
\end{aligned}$$

which is a contradiction. The proof is complete. □

To prove Theorem 1.2, the only we need to show that

$$j_0 < j_1 = \inf_{\mathcal{N}^-} J(u) < j_0 + \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}.$$

Consider $V(x)$ is a minimizer for both $S_{\alpha,p}$. Let u_0 be the positive local minimum solution we get above. By the continuity of J, we know that there exists $\gamma > 0$ such that

$$J(u_0+tV) < j_0 + \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}, \quad 0 \le t < \gamma.$$

$$\begin{aligned}
J(u_0+tV) &= \frac{1}{2}\|u_0+tV\|^2 - \frac{1}{2p}\tilde{B}(u_0+tV) - \int_{\mathbb{R}^N} h(u_0+tV)dx\\
&= J(u_0) + \frac{t^2}{2}[\|V\|^2 - \frac{t^{p-2}}{p}\tilde{B}(V)] + \tilde{B}(u_0) + \tilde{B}(tV) - \tilde{B}(u_0+tV)\\
&< j_0 + \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}.
\end{aligned}$$

For $t \ge \gamma$, a directly computation shows that

$$\begin{aligned}
J(u_0+tV) &= \frac{1}{2}\|u_0+tV\|^2 - \frac{1}{2p}\tilde{B}(u_0+tV) - \int_{\mathbb{R}^N} h(u_0+tV)dx\\
&= \frac{1}{2}\|u_0\|^2 + t\int_{\mathbb{R}^N} \nabla u_0\nabla V + u_0 V dx + \frac{t^2}{2}\|V\|^2 - \frac{1}{2p}\tilde{B}(u_0)\\
&\quad + \frac{1}{2p}[\tilde{B}(u_0) + \tilde{B}(tV) - \tilde{B}(u_0+tV)] - \frac{1}{2p}\tilde{B}(tV)\\
&\quad - \int_{\mathbb{R}^N} hu_0 dx - \int_{\mathbb{R}^N} htV dx\\
&= J(u_0) + \frac{t^2}{2}[\|V\|^2 - \frac{t^{2(p-1)}}{2p}\tilde{B}(V)] + \frac{1}{2p}[\tilde{B}(u_0) + \tilde{B}(tV)\\
&\quad - \tilde{B}(u_0+tV) + 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)|^p|u_0(y)|^{p-2}u_0(y)}{|x-y|^\alpha}\,dx\,dy]\\
&< j_0 + \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}.
\end{aligned}$$

Here, we use that $\langle J'(u_0), tV\rangle = 0$ and $V(x)$ is a solution of (4.14).

Proof of Theorem 1.2. To show that

$$j_0 < j_1 = \inf_{\mathcal{N}^-} J(u) < j_0 + \frac{p-1}{2p}S_{\alpha,p}^{\frac{p}{p-1}}.$$

Firstly, we observe that for every $u \in H^1(\mathbb{R}^N)$ with $\|u\| = 1$, there exists a unique $\eta^-(u) > 0$ such that (see Lemma 4.3)

$$t^-(u) \in \mathcal{N}^-.$$

By Lemma 4.3, we know that $\eta^-(u)$ is a continuous function of u. Consequently the manifold $\mathcal{N}^-$ disconnects $H^1(\mathbb{R}^N)$ in exactly two connected components U_1 and U_2, where

$$U_1 = \Big\{u \in H^1(\mathbb{R}^N) : u = 0 \text{ or } \|u\| < t^-\big(\frac{u}{\|u\|}\big)\Big\},$$
$$U_2 = \Big\{u \in H^1(\mathbb{R}^N); \|u\| > t^-\big(\frac{u}{\|u\|}\big)\Big\}.$$

Obviously, $H^1(\mathbb{R}^N) = \mathcal{N}^- \cup U_1 \cup U_2$. In particular, $u_0 \in \mathcal{N}^+ \subset U_2$. Since

$$t^-\Big(\frac{u_0 + tV}{\|u_0 + tV\|}\Big)\frac{u_0 + tV}{\|u_0 + tV\|} \in \mathcal{N},$$

we have

$$0 < t^-\Big(\frac{u_0 + tV}{\|u_0 + tV\|}\Big) < C_0$$

uniformly for $t \in \mathbb{R}$.

On the other hand, there exists $\tilde{t} > 0$ such that

$$\|u_0 + tU\| \geq t\|V\| - \|u_0\| \geq C_0, \quad t \geq \tilde{t}.$$

So that we can fix a $t_0 > 0$ such that $\|u_0 + t_0V\| > t^-(\frac{u_0+t_0V}{\|u_0+t_0V\|})$. Thus, $u_0 + t_0V \in U_2$. Combining this and the fact $u_0 \in U_1$, we know that

$$u_0 + t_1V \in \Lambda^-,$$

for some $0 < t_1 < t_0$. Consequently, by Remark 3.3, we have

$$j_1 = \inf_{\Lambda^-} J(u) \leq \max_{0\leq t\leq t_0} J(u_0 + tV) < j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}.$$

Next, we show that j_1 is a critical value of J and satisfies $j_1 > j_0$. Similarly to the proof of Proposition 4.4, we apply Ekeland's variational principle and get a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ such that

$$J(u_n) < j_1 + \frac{1}{n},$$
$$J(w) \geq J(u_n) - \frac{1}{n}\|u - w\|, \quad w \in \mathcal{N}^-.$$

So that we have

$$\begin{aligned} j_1 + 1 > J(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2p}\tilde{B}(u_n) - \int_{\mathbb{R}^N} h(x)u_n\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|u_n\|^2 - \Big(1 - \frac{1}{2p}\Big)|h|_{\frac{2Np}{2N(p-1)+\alpha}} d_{\frac{2Np}{2N-\alpha}}\|u_n\| \end{aligned}$$

which applies $\|u_n\|$ has a upper bound. Moreover, from $\{u_n\} \subset \mathcal{N}^-$, we know that

$$\|u_n\|^2 \leq (2p-1)\frac{\|u_n\|^p}{S_{\alpha,p}^p}.$$

Thus, $\|u_n\|$ has a uniform positive lower bound. Then, analogously to the proof of Proposition 4.4, we know that

$$J(u_n) \to j_1, \quad J'(u_n) \to 0 \quad \text{in } H^{-1}.$$

From Proposition 4.5 and $j_1 < j_0 + \frac{p-1}{2p} S_{\alpha,p}^{\frac{p}{p-1}}$, we can conclude that there exists a subsequence of $\{u_n\}$ such that $u_n \to u_1$ strongly in $H^1(\mathbb{R}^N)$. Therefore u_1 is a critical point of J and $J(u_1) = j_1$. Noting that $\mathcal{N}^-$ is closed, we have $u_1 \in \mathcal{N}^-$. To show $j_1 > j_0$, arguing indirectly, we assume $j_1 = j_0$. Thus by Remark 3.3 we have

$$\int_{\mathbb{R}^N} h(x) u_1 dx > 0 \quad \text{and} \quad u_1 \in \mathcal{N}^+,$$

which leads to a contradiction.

Finally, we consider the case $h > 0$. Applying Lemma 4.3 to u_1 and $|u_1|$, we know that there exist a $\eta^-(|u_1|)$ such that $\eta^-(|u_1|)|u_1| \in \mathcal{N}^-$. Moreover,

$$\eta^-(|u_1|) \geq \eta_0(|u_1|) = \eta_0(u_1) = \Big[\frac{\tilde{A}(u_1)}{(p-1)\tilde{B}(u_1)}\Big]^{\frac{1}{p-2}}$$

Thus both in the case $\int_{\mathbb{R}^N} h(x)u\,dx > 0$ and $\int_{\mathbb{R}^N} h(x)u\,dx \leq 0$, we can deduce that

$$j_1 = J(u_1) \geq J(\eta^-(u_1)u_1) \geq J(\eta^-(|u_1|)|u_1|) = \eta_0(u_1) \geq j_1.$$

Therefore, $\int_{\mathbb{R}^N} h(x)u_1 dx = \int_{\mathbb{R}^N} h(x)|u_1|dx$, which implies $u_1 \geq 0$. By the maximum principle we get $u_1 > 0$. The proof is complete. $\square$

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