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STABILIZATION OF SEMILINEAR WAVE EQUATIONS WITH TIME-DEPENDENT VARIABLE COEFFICIENTS AND MEMORY

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Abstract. In this article, we study the stabilization of semilinear wave equations with time-dependent variable coefficients and memory in the nonlinear boundary feedback. We obtain the energy decay rate of the solution by an equivalent energy approach in the framework of Riemannian geometry.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded domain with a smooth boundary Γ of class C^2. We assume $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$, where Γ_0 and Γ_1 are closed and disjoint. We consider semilinear wave equations with time-dependent variable coefficients and memory on the boundary:

$$\begin{gathered} u_{tt}(x,t) + \mu(t)\mathcal{A}u(x,t) + h(\nabla u) + f(u) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,+\infty), \\ \mu(t)\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) + \int_0^t g(t-s)u_s(x,s)\,ds + l(u_t) = 0, \quad (x,t) \in \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where

$$\mathcal{A}u = -\operatorname{div} A(x)\nabla u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad x \in \mathbb{R}^n. \tag{1.2}$$

$A(x) = (a_{ij}(x))$ $(i,j = 1,2,\dots,n)$ is a symmetric and positive matrix with the functions $a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^n)$ satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^n \xi_i^2, \quad \forall x \in \Omega, \quad 0 \neq \xi = (\xi_1,\xi_2,\dots,\xi_n)^T \in \mathbb{R}^n, \tag{1.3}$$

for some positive constant λ. $\nu = (\nu_1,\nu_2,\dots,\nu_n)$ be the unit normal vector of Γ pointing toward the exterior of Ω, $\nu_{\mathcal{A}} = A\nu$, $\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \sum_{i=1}^n a_{ij}\frac{\partial u}{\partial x_j}\nu_i$. $\mu : (0,+\infty) \to (0,+\infty)$ is a continuous non-increasing function. $f, l : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuous nonlinear functions satisfying some hypotheses (see (H3)–(H5) below). $g : [0,+\infty) \to (0,+\infty)$ is a C^2-function.

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The stabilization of wave equations has been widely investigated; wee [2, 12, 13, 19, 30] and their references. For the constant coefficient case ($a_{ij} = \delta_{ij}$, $\mu(t) = 1$) and $g(t) = 0$, a classical semilinear wave equation

$$\begin{gathered} u_{tt} - \Delta u + h(\nabla u) + f(u) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ u = 0, \quad (x,t) \in \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} + l(u_t) = 0, \quad (x,t) \in \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \end{gathered} \tag{1.4}$$

was considered in [1, 7]. The existence of strong (and weak) solution and uniform stabilization of the system (1.4) were established.

Variable-coefficients wave equations are mathematical models arisen in solid mechanics, electromagnetics, fluid flow in porous media, etc. In the case of variable coefficients, the main tool is the Riemannian geometry method which was introduced by Yao [28] to obtain boundary exact controllability for the wave equation in the form

$$u_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0, \quad (x,t) \in \Omega \times (0,T).$$

This method was applied to achieve the controllability and stabilization of PDEs with variable coefficients in [6, 9, 20, 21]. In 2009, Guo and Shao [8] considered the semilinear wave equation with variable coefficients

$$u_{tt} - \Delta_g u + h(\nabla u) + f(u) = 0, \quad (x,t) \in \Omega \times (0,+\infty). \tag{1.5}$$

This was done under the nonlinear boundary feedback

$$\frac{\partial u}{\partial \mu} + l(u_t) = 0, \quad (x,t) \in \Gamma_1 \times (0,+\infty),$$

where Δ_g is the Beltrami-Laplace operator of Riemannian metric g. Here, μ is the normal vector field on Γ in terms of Riemannian metric g. The existence of both strong and weak solutions to (1.5) was proven by Faedo-Galerkin method and denseness argument. The exponential stability of this equation was obtained by introducing an equivalent energy functional and using the energy multiplier method on Riemannian manifold.

Variable coefficients depend not only on space but also on time. In 2019, Liu [15] dealt with the boundary exact controllability for the wave equation with variable coefficients in time and space

$$u_{tt} - \mu(t) \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0, \quad (x,t) \in \Omega \times (0,T),$$

which was subject to Dirichlet or Neumann boundary controls. In 2021, Ha [10] explored the time-dependent variable coefficients wave equation with damping and supercritical source terms

$$u_{tt} + \mu(t)\mathcal{A}u + g(u_t) = |u|^{\rho}u, \quad (x,t) \in \Omega \times (0,+\infty),$$

where ρ is a constant. He proved the existence of solutions and energy decay rate.

When waves propagate in viscous and elastic materials, some properties of the materials might change. Meanwhile, the state at each moment would be affected by the previous state in the propagation, that is called the memory effect. Many papers have studied the viscoelastic wave equations, see [3, 4, 17, 18, 22]. In 2004, Chai and Guo [5] established the boundary stabilization of wave equations with variable coefficients and memory

$$
\begin{gathered}
u_{tt}(x,t) - \operatorname{div} A(x)\nabla u(x,t) = 0, \quad (x,t) \in \Omega \times (0,+\infty),\\
u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,+\infty),\\
\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) + \int_0^t g(t-s,x)u_s(x,s)\,ds + a(x)l(u_t) = 0, \quad (x,t) \in \Gamma_1 \times (0,+\infty),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega
\end{gathered}
$$

by Riemannian geometry method and sharp trace regularity. In 2009, Park and Ha [24] considered energy decay for non-dissipative distributed systems with source terms

$$
u_{tt}(x,t) - \Delta u(x,t) + h(\nabla u) = |u|^\rho u, \quad (x,t) \in \Omega \times (0,+\infty).
$$

And it had the nonlinear boundary condition

$$
\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) + \int_0^t g(t-s,x)u_s(x,s)\,ds + a(x)l(u_t) = 0, \quad (x,t) \in \Gamma_1 \times (0,+\infty).
$$

In 2010, Wu et al. [25] showed the exponential decay of energy for the system

$$
\begin{gathered}
u_{tt}(x,t) - \operatorname{div} A(x)\nabla u(x,t) + f(u) = 0, \quad (x,t) \in \Omega \times (0,+\infty),\\
u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,+\infty),\\
\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) = -\int_0^t g(t-s,x)u_s(x,s)\,ds - l(u_t), \quad (x,t) \in \Gamma_1 \times (0,+\infty),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega.
\end{gathered}
$$

In 2018, the stabilization of a wave equation with variable coefficients and internal memory in an open bounded domain

$$
u_{tt} + \mathcal{A}u + a(x)\Big(\mu_1 u_t(x,t) + \mu_2 \int_0^{+\infty} g(s)u_t(x,t-s)\,ds\Big) = 0,
$$

for $(x,t) \in \Omega \times (0,+\infty)$ was considered by Ning and Yang in [23]. Later, some scholars studied the energy decay rate of wave systems with variable coefficients combining the memory boundary condition and acoustic boundary condition, see Jeong et al. [11], Liu [16] and Wu et al. [26].

Motivated by the above work, we explore semilinear wave equations with time-dependent variable coefficients and memory on the boundary. Compared with previous articles on this subject, the highlights of this article are the time-dependent variable coefficients in the principal part and nonlinear terms with the memory boundary condition. Such a mathematical model can more accurately reflect the actual situations of wave propagation in materials.

In this article, we study the stabilization of system (1.1) by equivalent energy approach and Riemannian geometry method. The Riemannian method is a powerful tool to deal with variable coefficients PDEs. Several multiplier identities, which have been built for constant coefficient wave equations (see Lions [14]), are generalized to the variable coefficients case by geometric multiplier identities subject to a different geometric condition. Besides that, it is interesting that some factors cause the energy to be non-dissipative in the system (1.1), however, the energy decays exponentially.

This article is organized as follows. In Section 2, we present some notations needed for our work and state the main result. In Section 3, we show the energy decay rate.

2. Preliminaries and main results

In this section, we introduce some notation and assumptions that will be used in the following content. All definitions and notations related with Riemannian geometry are standard and classical in the [27].

A couple $(\mathbb{R}^n, g)$ represents a Riemannian manifold with metric g. $G(x) = (g_{ij}(x)) = A^{-1}(x)$, $x \in \mathbb{R}^n$, where $A(x)$ is defined in (1.2). For each $x \in \mathbb{R}^n$, we denote the inner product and norm with Riemannian metric g over the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ as

$$g(X,Y) = \langle X, Y\rangle_g = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j, \quad |X|_g = \langle X, X\rangle_g^{1/2},$$
$$\forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x.$$

We define the usual dot product and norm in Euclidean space $\mathbb{R}^n$ by

$$X \cdot Y = \sum_{i=1}^n \alpha_i\beta_i, \quad |X| = \langle X, X\rangle^{1/2}, \quad \forall X, Y \in \mathbb{R}^n_x.$$

And the divergence of X in Euclidean metric is

$$\operatorname{div} X = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i}, \quad \forall x \in \mathbb{R}^n.$$

We denote the Levi-Civita connection in Riemannian metric g by D. Let H be a vector field on $(\mathbb{R}^n, g)$, then the covariant differential DH of H determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$, defined by

$$DH(X,Y) = g(D_Y H, X) = \langle D_Y H, X\rangle_g, \quad \forall X, Y \in \mathbb{R}^n_x,$$

where $D_Y H$ is covariant derivative of the vector field H with respect to Y. If $f \in C^1(\mathbb{R}^n)$, we denote gradients of f by ∇ and ∇_g in Euclidean metric and in Riemannian metric g, respectively. It follows from [28, Lemma 2.1] that

$$\nabla_g f = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x)\frac{\partial f}{\partial x_j}\Big)\frac{\partial}{\partial x_i},$$
$$|\nabla_g f|_g^2 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}.$$

It is easy to verify that

$$\nabla_g f = A(x)\nabla f,$$

and via the Riesz representation theorem, we have

$$X(f) = \langle \nabla_g f, X\rangle_g,$$

where X is any vector field on Riemannian manifold $(\mathbb{R}^n, g)$. For more details, we refer to [28, 29].

To obtain the stabilization of problem (1.1), we assume the following hypotheses:

(H1) There exists a vector field H on Riemannian manifold $(\mathbb{R}^n, g)$ such that
$$DH(X,X) \geq \sigma |X|_g^2, \quad \forall x \in \overline{\Omega},\ X \in \mathbb{R}^n_x, \tag{2.1}$$
for some constant $\sigma > 0$. The divergence of H satisfies
$$\operatorname{div} H > \frac{r}{r-1}\sigma, \quad \forall x \in \overline{\Omega}, \tag{2.2}$$
where r is given in (2.7). Futhermore, we suppose that the vector field H satisfies
$$H \cdot \nu \leq 0, \quad \text{on } \Gamma_0, \tag{2.3}$$
$$H \cdot \nu \geq \delta > 0, \quad \text{on } \Gamma_1, \tag{2.4}$$
where δ is a constant.

(H2) The function $\mu \in C^1(0, +\infty)$ is non-increasing and satisfies
$$\mu(t) \geq \mu_0 > 0, \quad \forall t > 0, \tag{2.5}$$
where μ_0 is a constant.

(H3) $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function deriving from a potential:
$$F(s) := \int_0^s f(\tau)d\tau \geq 0, \quad \forall s \in \mathbb{R}, \tag{2.6}$$
and satisfies
$$|f(s)| \leq b_1|s|^\rho + b_2, \quad |f'(s)| \leq b_3|s|^{\rho-1} + b_4,$$
where b_i $(i = 1, 2, 3, 4)$ are positive constants and the parameter ρ satisfies
$$1 \leq \rho \leq \begin{cases} 2, & n \leq 3, \\ \frac{n}{n-2}, & n \geq 4. \end{cases}$$
Also F and f have the following relationship:
$$2rF(s) \leq sf(s), \quad \forall s \in \mathbb{R}, \quad \text{for some constant } \ r > 1. \tag{2.7}$$
Example. A function satisfying (H3) is given in [8] as
$$f(s) = \gamma|s|^{\rho-1}s, \text{ for some constants } \gamma > 0,\ 1 \leq \rho \leq \begin{cases} 2, & n \leq 3, \\ \frac{n}{n-2}, & n \geq 4. \end{cases}$$

(H4) $h : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function and there exist two constants $\beta > 0$ and $L > 0$ such that
$$|h(\xi)| \leq \beta\sqrt{\lambda}|\xi|, \quad \forall \xi \in \mathbb{R}^n, \tag{2.8}$$
$$|\nabla h(\xi)| \leq L, \quad \forall \xi \in \mathbb{R}^n. \tag{2.9}$$
Here,
$$\beta < \min\Big\{\frac{\sqrt{\lambda}\sigma\mu_0}{4M + 2R(C_\Omega + 1)}, \frac{\varepsilon C_2}{C_1}\Big\}, \tag{2.10}$$
where ε is from (3.18). The constants involved in (2.10) can be found in the text, and we do not repeat them here.

(H5) $l : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^1-function and there exist two positive constants c_1 and c_2 such that
$$c_1|s|^2 \leq l(s)s \leq c_2|s|^2, \quad \forall s \in \mathbb{R}. \tag{2.11}$$

(H6) $g : [0, +\infty) \to (0, +\infty)$ is a non-increasing C^2-function satisfying $g(0) > 0$, and there exist constants $\zeta_1,\ \zeta_2 > 0$ such that

$$g'(t) \le -\zeta_1 g(t), \quad \forall t \ge 0, \tag{2.12}$$

$$g''(t) \ge -\zeta_2 g'(t), \quad \forall t \ge 0. \tag{2.13}$$

We denote

$$g \circ u(t) := \int_0^t g(t-s)|u(x,t) - u(x,s)|^2 ds. \tag{2.14}$$

We define the energy corresponding to the solution of problem (1.1) by

$$\begin{aligned} E(t) := &\frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx + \int_\Omega F(u)\,dx - \frac{1}{2}\int_{\Gamma_1} g' \circ u(t)\,d\Gamma \\ &+ \frac{1}{2}g(t)\int_{\Gamma_1} |u(t) - u_0|^2\,d\Gamma, \end{aligned} \tag{2.15}$$

and denote

$$E_0(t) := \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx + \int_\Omega F(u)\,dx. \tag{2.16}$$

Set

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega),\ u|_{\Gamma_0} = 0\} \quad \text{and} \quad V = H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega).$$

Proposition 2.1 (Well-posedness)**.** *Let us assume* (H1)-(H6)*, and let the initial values* $(u_0, u_1) \in V \times V$ *satisfy the compatibility condition*

$$\mu(0)\frac{\partial u_0}{\partial \nu_{\mathcal{A}}} + l(u_1) = 0, \quad \text{on } \Gamma_1\,.$$

Then problem (1.1) *admits a unique solution* u *such that*

$$u \in L^\infty(0,\infty; V), \quad u_t \in L^\infty(0,\infty; H^1_{\Gamma_0}(\Omega)), \quad u_{tt} \in L^\infty(0,\infty; L^\infty(\Omega)).$$

Moreover, if $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$*, then problem* (1.1) *possesses at least a weak solution in the class*

$$u \in C([0,\infty); H^1_{\Gamma_0}(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

The above proposition can be proved using the Faedo-Galerkin method and a denseness argument (see [8] for details). We omit it here.

Theorem 2.2. *Let* u *be a solution to problem* (1.1)*. Suppose that* (H1)–(H6) *hold. Then, the energy* $E(t)$ *associated with* (1.1) *decays exponentially. That is to say, there exist two positive constants* γ *and* ω *independent of initial values such that*

$$E(t) \le \gamma E(0) e^{-\omega t}, \quad \forall t \ge 0. \tag{2.17}$$

3. Proof of main result

Lemma 3.1 ([28, 29])**.** *Let* $u, v \in C^1(\overline{\Omega})$ *and* H *be a vector field on* $(\mathbb{R}^n, g)$*. Then, the following formulae hold:*

(i) *divergence theorem:*

$$\operatorname{div}(uH) = u \operatorname{div} H + H(u), \tag{3.1}$$

$$\int_\Omega \operatorname{div} H\,dx = \int_\Gamma H \cdot \nu\,d\Gamma. \tag{3.2}$$

(ii) *Green's formula:*

$$\int_\Omega v\mathcal{A}u\,dx = \int_\Omega \langle \nabla_g u, \nabla_g v\rangle_g\,dx - \int_\Gamma v\frac{\partial u}{\partial \nu_{\mathcal{A}}}\,d\Gamma. \tag{3.3}$$

(iii)

$$\langle \nabla_g u, \nabla_g\left(H(u)\right)\rangle_g = DH(\nabla_g u, \nabla_g u) + \frac{1}{2}\operatorname{div}(|\nabla_g u|_g^2 H) - \frac{1}{2}|\nabla_g u|_g^2 \operatorname{div} H. \tag{3.4}$$

To simplify computations, we integrate by parts using the boundary condition on Γ_1 of problem (1.1). This means

$$\begin{aligned}
\int_0^t g(t-s)u_s(x,s)\,ds &= g(t-s)u(x,s)\big|_0^t + \int_0^t g'(t-s)u(x,s)\,ds\\
&= g(0)u(x,t) - g(t)u(x,0) + \int_0^t g'(t-s)\left(u(x,s)-u(x,t)\right)\,ds\\
&\quad + u(x,t)\int_0^t g'(t-s)\,ds\\
&= \int_0^t g'(t-s)\left(u(x,s)-u(x,t)\right)\,ds + g(t)\left(u(x,t)-u_0(x)\right).
\end{aligned}$$

Thus, problem (1.1) is transformed into the problem

$$\begin{gathered}
u_{tt}(x,t) + \mu(t)\mathcal{A}u(x,t) + h(\nabla u) + f(u) = 0, \quad (x,t)\in\Omega\times(0,+\infty),\\
u(x,t) = 0, \quad (x,t)\in\Gamma_0\times(0,+\infty),\\
\mu(t)\frac{\partial u}{\partial \nu_{\mathcal{A}}}(x,t) + \int_0^t g'(t-s)\left(u(x,s)-u(x,t)\right)\,ds\\
+ g(t)\left(u(x,t)-u_0(x)\right) + l(u_t) = 0, \quad (x,t)\in\Gamma_1\times(0,+\infty),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in\Omega.
\end{gathered} \tag{3.5}$$

Proposition 3.2. *Under hypotheses* (H1)–(H6)*, the energy* (2.15) *associated with system* (1.1) *satisfies*

$$\begin{aligned}
\frac{d}{dt}E(t) &\le \beta C_1 E(t) - c_1\int_{\Gamma_1} u_t^2\,d\Gamma - \frac{1}{2}\int_{\Gamma_1} g''\circ u(t)\,d\Gamma\\
&\quad + \frac{1}{2}g'(t)\int_{\Gamma_1} |u(t)-u_0|^2\,d\Gamma,
\end{aligned} \tag{3.6}$$

where β and c_1 are defined in (2.8) *and* (2.11)*, $C_1>0$ is a constant.*

Proof. Differentiating the energy $E(t)$ in system (3.5) induces

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_\Omega u_t u_{tt}\,dx + \mu(t)\int_\Omega \langle\nabla_g u, \nabla_g u_t\rangle_g\,dx + \frac{1}{2}\mu'(t)\int_\Omega |\nabla_g u|_g^2\,dx\\
&\quad + \int_\Omega f(u)u_t\,dx - \frac{1}{2}\int_{\Gamma_1} g''\circ u(t)\,d\Gamma + g(t)\int_{\Gamma_1} u_t(t)(u(t)-u_0)\,d\Gamma\\
&\quad + \frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma - \int_{\Gamma_1}\int_0^t g'(t-s)u_t(t)(u(t)-u(s))\,ds\,d\Gamma\\
&= \int_\Omega u_t u_{tt}\,dx + \mu(t)\int_\Omega u_t\mathcal{A}u\,dx + \int_\Omega f(u)u_t\,dx + \frac{1}{2}\mu'(t)\int_\Omega |\nabla_g u|_g^2\,dx\\
&\quad + \mu(t)\int_{\Gamma_1}\frac{\partial u}{\partial\nu_{\mathcal{A}}}u_t\,d\Gamma - \frac{1}{2}\int_{\Gamma_1} g''\circ u(t)\,d\Gamma + g(t)\int_{\Gamma_1} u_t(t)(u(t)-u_0)\,d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma-\int_{\Gamma_1}\int_0^t g'(t-s)u_t(t)(u(t)-u(s))\,ds\,d\Gamma\\
&=-\int_\Omega h(\nabla u)u_t\,dx-\int_{\Gamma_1} l(u_t)u_t\,d\Gamma-\frac{1}{2}\int_{\Gamma_1} g''\circ u(t)\,d\Gamma\\
&\quad+\frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma+\frac{1}{2}\mu'(t)\int_\Omega|\nabla_g u|_g^2\,dx.
\end{aligned}
$$

In terms of (1.3), (2.5) and (2.8), we have

$$
-\int_\Omega h(\nabla u)u_t\,dx\le\frac{\beta}{2}\int_\Omega u_t^2\,dx+\frac{\beta}{2\mu_0}\mu(t)\int_\Omega|\nabla_g u|_g^2\,dx.
$$

The monotonicity of $\mu(t)$ and (2.11) lead to

$$
\begin{aligned}
\frac{d}{dt}E(t)&\le\frac{\beta}{2}\int_\Omega u_t^2\,dx+\frac{\beta}{2\mu_0}\mu(t)\int_\Omega|\nabla_g u|_g^2\,dx-c_1\int_{\Gamma_1}u_t^2\,d\Gamma-\frac{1}{2}\int_{\Gamma_1}g''\circ u(t)\,d\Gamma\\
&\quad+\frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma\\
&\le\beta C_1E(t)-c_1\int_{\Gamma_1}u_t^2\,d\Gamma-\frac{1}{2}\int_{\Gamma_1}g''\circ u(t)\,d\Gamma+\frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma,
\end{aligned}
$$

where $C_1=\max\{1,\frac{1}{\mu_0}\}>0$. □

Suppose that H is a vector field on $\overline{\Omega}$, we construct a functional

$$
P(t):=\int_\Omega u_t\Big(H(u)+\frac{\operatorname{div}H-\sigma}{2}u\Big)\,dx, \tag{3.7}
$$

where the vector field H and the constant σ satisfy (2.1) and (2.2).

Remark 3.3. Here, $H(u)=H\cdot\nabla u$, where u is a continuous and differentiable function. We can see [28, 29] for more details regarding the existence and examples of vector field H. If $a_{ij}=\delta_{ij}$ in (1.2), we choose $H=x-x_0$ for fixed $x_0\in\mathbb{R}^n$. The inequality (2.1) can take the equal sign and $\sigma=1$.

Proposition 3.4. *Under hypotheses* (H1)–(H6). *If β in* (2.8) *conforms to* (2.10), *the functional $P(t)$ satisfies*

$$
\begin{aligned}
\frac{d}{dt}P(t)&\le-C_2E_0(t)-2C_3\int_{\Gamma_1}g'\circ u(t)\,d\Gamma+C_3g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma\\
&\quad+\big(\frac{M}{2}+C_4\big)\int_{\Gamma_1}u_t^2\,d\Gamma,
\end{aligned}\tag{3.8}
$$

where $E_0(t)$ is defined in (2.16), *C_2,C_3,C_4,M are the positive constants independent of initial values.*

Proof. Direct calculations yield

$$
\begin{aligned}
&\frac{d}{dt}P(t)\\
&=\int_\Omega u_t\Big(H(u_t)+\frac{\operatorname{div}H-\sigma}{2}u_t\Big)\,dx+\int_\Omega u_{tt}\Big(H(u)+\frac{\operatorname{div}H-\sigma}{2}u\Big)\,dx\\
&=\int_\Omega u_t\Big(H(u_t)+\frac{\operatorname{div}H-\sigma}{2}u_t\Big)\,dx-\mu(t)\int_\Omega\mathcal{A}u\Big(H(u)+\frac{\operatorname{div}H-\sigma}{2}u\Big)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\Omega h(\nabla u)\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx - \int_\Omega f(u)\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{gathered}
I_1 := \int_\Omega u_t\Big(H(u_t) + \frac{\operatorname{div} H - \sigma}{2} u_t\Big)\, dx, \\
I_2 := -\mu(t) \int_\Omega \mathcal{A}u\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx, \\
I_3 := - \int_\Omega h(\nabla u)\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx, \\
I_4 := - \int_\Omega f(u)\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx.
\end{gathered}
$$

Now, we estimate I_i $(i = 1, 2, 3, 4)$, respectively. Set $M = \sup_{x\in\overline{\Omega}} |H|$. Using the formulae (3.1) and (3.2), we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_\Omega H(u_t^2)\, dx + \int_\Omega \frac{\operatorname{div} H - \sigma}{2} u_t^2\, dx \\
&= \int_\Omega \Big(\frac{1}{2}\operatorname{div}(u_t^2 H) - \frac{1}{2} u_t^2 \operatorname{div} H\Big)\, dx + \int_\Omega \frac{\operatorname{div} H - \sigma}{2} u_t^2\, dx \\
&= \frac{1}{2}\int_\Gamma u_t^2 H\cdot\nu\, d\Gamma - \frac{\sigma}{2}\int_\Omega u_t^2\, dx \\
&\le \frac{M}{2}\int_{\Gamma_1} u_t^2\, d\Gamma - \frac{\sigma}{2}\int_\Omega u_t^2\, dx.
\end{aligned} \tag{3.9}
$$

Next, we estimate I_2. Since $u|_{\Gamma_0} = 0$, it follows that

$$
\frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) = |\nabla_g u|_g^2 H\cdot\nu.
$$

This together with (2.1) and (3.4) yields

$$
\begin{aligned}
I_2 &= \mu(t)\int_\Omega \operatorname{div} A(x)\nabla u\Big(H(u) + \frac{\operatorname{div} H - \sigma}{2} u\Big)\, dx \\
&= \mu(t)\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u)\, d\Gamma - \mu(t)\int_\Omega \langle \nabla_g u, \nabla_g H(u)\rangle_g\, dx \\
&\quad + \mu(t)\int_\Gamma \frac{\operatorname{div} H - \sigma}{2}\frac{\partial u}{\partial \nu_{\mathcal{A}}} u\, d\Gamma - \mu(t)\int_\Omega \frac{\operatorname{div} H - \sigma}{2}|\nabla_g u|_g^2\, dx \\
&= \mu(t)\int_{\Gamma_0} |\nabla_g u|_g^2 H\cdot\nu\, d\Gamma + \mu(t)\int_{\Gamma_1}\Big(\frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) + \frac{\operatorname{div} H - \sigma}{2}\frac{\partial u}{\partial \nu_{\mathcal{A}}} u\Big)\, d\Gamma \\
&\quad - \mu(t)\int_\Omega DH(\nabla_g u, \nabla_g u)\, dx - \frac{1}{2}\mu(t)\int_\Omega \operatorname{div}\big(|\nabla_g u|_g^2 H\big)\, dx \\
&\quad + \frac{1}{2}\mu(t)\int_\Omega |\nabla_g u|_g^2 \operatorname{div} H\, dx - \mu(t)\int_\Omega \frac{\operatorname{div} H - \sigma}{2}|\nabla_g u|_g^2\, dx \\
&\le \mu(t)\int_{\Gamma_1}\Big(\frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) + \frac{\operatorname{div} H - \sigma}{2}\frac{\partial u}{\partial \nu_{\mathcal{A}}} u - \frac{1}{2}|\nabla_g u|_g^2 H\cdot\nu\Big)\, d\Gamma \\
&\quad - \frac{\sigma}{2}\mu(t)\int_\Omega |\nabla_g u|_g^2\, dx.
\end{aligned} \tag{3.10}
$$

Let $R = \sup_{x\in\overline{\Omega}}(\operatorname{div} H - \sigma)/2$. Using the Cauchy inequality with $\eta = \frac{\lambda\sigma}{4\tilde{C}_\Omega R} > 0$ and the trace theorem

$$\int_{\Gamma_1} |u|^2 \, d\Gamma \le \tilde{C}_\Omega \int_\Omega |\nabla u|^2 \, dx \le \frac{\tilde{C}_\Omega}{\lambda} \int_\Omega |\nabla_g u|_g^2 \, dx, \tag{3.11}$$

for some constant $\tilde{C}_\Omega > 0$ depending on Ω under condition (1.3), we have

$$\begin{aligned}
&\mu(t) \int_{\Gamma_1} \Big(\frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) + \frac{\operatorname{div} H - \sigma}{2} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u - \frac{1}{2} |\nabla_g u|_g^2 H \cdot \nu \Big) \, d\Gamma \\
&\le \mu(t) \int_{\Gamma_1} \Big(\frac{\delta}{2} |\nabla_g u|_g^2 + \frac{M^2}{2\delta\lambda} \big| \frac{\partial u}{\partial \nu_{\mathcal{A}}} \big|^2 + \frac{R}{4\eta} \big| \frac{\partial u}{\partial \nu_{\mathcal{A}}} \big|^2 + R\eta u^2 - \frac{\delta}{2} |\nabla_g u|_g^2 \Big) \, d\Gamma \\
&\le \Big(\frac{M^2}{2\delta\lambda} + \frac{\tilde{C}_\Omega R^2}{\lambda\sigma} \Big) \mu(t) \int_{\Gamma_1} \big| \frac{\partial u}{\partial \nu_{\mathcal{A}}} \big|^2 \, d\Gamma + \frac{\sigma}{4} \mu(t) \int_\Omega |\nabla_g u|_g^2 \, dx,
\end{aligned}$$

where $\delta > 0$ is from (2.4). On the other hand, in light of the Cauchy inequality, Hölder inequality, trace theorem (3.11), and (2.11), we deduce that

$$\begin{aligned}
&\mu(t) \int_{\Gamma_1} \big| \frac{\partial u}{\partial \nu_{\mathcal{A}}} \big|^2 \, d\Gamma \\
&\le \frac{1}{\mu_0} \int_{\Gamma_1} \big| \mu(t) \frac{\partial u}{\partial \nu_{\mathcal{A}}} \big|^2 \, d\Gamma \\
&= \frac{1}{\mu_0} \int_{\Gamma_1} \Big| - \int_0^t g'(t-s) \, (u(s) - u(t)) \, ds - g(t)(u(t) - u_0) - l(u_t) \Big|^2 \, d\Gamma \\
&\le \frac{3}{\mu_0} \int_{\Gamma_1} \int_0^t -g'(t-s) \, ds \int_0^t -g'(t-s) |u(s) - u(t)|^2 \, ds \, d\Gamma \\
&\quad + \frac{3}{\mu_0} \int_{\Gamma_1} g^2(t) |u(t) - u_0|^2 \, d\Gamma + \frac{3}{\mu_0} \int_{\Gamma_1} l^2(u_t) \, d\Gamma \\
&\le - \frac{6g(0)}{\mu_0} \int_{\Gamma_1} g' \circ u(t) \, d\Gamma + \frac{3g(0)}{\mu_0} g(t) \int_{\Gamma_1} |u(t) - u_0|^2 \, d\Gamma + \frac{3c_2^2}{\mu_0} \int_{\Gamma_1} u_t^2 \, d\Gamma.
\end{aligned}$$

Then

$$\begin{aligned}
&\mu(t) \int_{\Gamma_1} \Big(\frac{\partial u}{\partial \nu_{\mathcal{A}}} H(u) + \frac{\operatorname{div} H - \sigma}{2} \frac{\partial u}{\partial \nu_{\mathcal{A}}} u - \frac{1}{2} |\nabla_g u|_g^2 H \cdot \nu \Big) \, d\Gamma \\
&\le -2C_3 \int_{\Gamma_1} g' \circ u(t) \, d\Gamma + C_3 g(t) \int_{\Gamma_1} |u(t) - u_0|^2 \, d\Gamma + C_4 \int_{\Gamma_1} u_t^2 \, d\Gamma \\
&\quad + \frac{\sigma}{4} \mu(t) \int_\Omega |\nabla_g u|_g^2 \, dx,
\end{aligned} \tag{3.12}$$

where $C_3 = \frac{3M^2 g(0)}{2\delta\lambda\mu_0} + \frac{3\tilde{C}_\Omega R^2 g(0)}{\lambda\sigma\mu_0} > 0$, $C_4 = \frac{3M^2 c_2^2}{2\delta\lambda\mu_0} + \frac{3\tilde{C}_\Omega R^2 c_2^2}{\lambda\sigma\mu_0} > 0$. Substituting (3.12) into (3.10), we arrive at

$$\begin{aligned}
I_2 \le& -\frac{\sigma}{4} \mu(t) \int_\Omega |\nabla_g u|_g^2 \, dx - 2C_3 \int_{\Gamma_1} g' \circ u(t) \, d\Gamma + C_3 g(t) \int_{\Gamma_1} |u(t) - u_0|^2 \, d\Gamma \\
&+ C_4 \int_{\Gamma_1} u_t^2 \, d\Gamma.
\end{aligned} \tag{3.13}$$

We estimate I_3 by the Cauchy-Schwarz inequality and Poincaré inequality under conditions (H4) and (1.3),

$$\int_\Omega u^2\,dx \le C_\Omega \int_\Omega |\nabla u|^2\,dx \le \frac{C_\Omega}{\lambda}\int_\Omega |\nabla_g u|_g^2\,dx, \tag{3.14}$$

where $C_\Omega > 0$ is the Poincaré constant depending on Ω. This implies

$$\begin{aligned} I_3 &\le \beta\sqrt{\lambda}M\int_\Omega |\nabla u|^2\,dx + \beta\sqrt{\lambda}R\int_\Omega |\nabla u||u|\,dx \\ &\le \frac{\beta M}{\sqrt{\lambda}\mu_0}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx + \frac{\beta R}{2\sqrt{\lambda}\mu_0}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx + \frac{\beta\sqrt{\lambda}R}{2}\int_\Omega u^2\,dx \\ &\le \frac{\beta\left(2M + R(C_\Omega+1)\right)}{2\sqrt{\lambda}\mu_0}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx. \end{aligned} \tag{3.15}$$

Now we consider I_4. Because $u|_{\Gamma_0} = 0$, we deduce that $F(u) = 0$ on Γ_0. By (H1), (H3), and formulae in Lemma 3.1, we have

$$\begin{aligned} I_4 &= -\int_\Omega H\left(F(u)\right)\,dx - \int_\Omega \frac{\operatorname{div} H - \sigma}{2} f(u)u\,dx \\ &\le -\int_\Omega \operatorname{div}\left(F(u)H\right)\,dx + \int_\Omega F(u)\operatorname{div} H\,dx - 2r\int_\Omega \frac{\operatorname{div} H - \sigma}{2}F(u)\,dx \\ &= -\int_{\Gamma_1} F(u)H\cdot\nu\,d\Gamma - \int_\Omega \left[(r-1)\operatorname{div} H - r\sigma\right]F(u)\,dx \\ &\le -C_5\int_\Omega F(u)\,dx, \end{aligned} \tag{3.16}$$

where $C_5 = \inf_{x\in\overline{\Omega}}[(r-1)\operatorname{div} H - r\sigma] > 0$.

Using (3.9), (3.13), (3.15), and (3.16), we have

$$\begin{aligned} \frac{d}{dt}P(t) &\le -\frac{\sigma}{2}\int_\Omega u_t^2\,dx - \Big(\frac{\sigma}{4} - \frac{\beta\left(2M + R(C_\Omega+1)\right)}{2\sqrt{\lambda}\mu_0}\Big)\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx \\ &\quad - C_5\int_\Omega F(u)\,dx - 2C_3\int_{\Gamma_1} g'\circ u(t)\,d\Gamma + C_3 g(t)\int_{\Gamma_1} |u(t)-u_0|^2\,d\Gamma \\ &\quad + \big(\frac{M}{2} + C_4\big)\int_{\Gamma_1} u_t^2\,d\Gamma \\ &\le -C_2E_0(t) - 2C_3\int_{\Gamma_1} g'\circ u(t)\,d\Gamma + C_3 g(t)\int_{\Gamma_1} |u(t)-u_0|^2\,d\Gamma \\ &\quad + \big(\frac{M}{2} + C_4\big)\int_{\Gamma_1} u_t^2\,d\Gamma, \end{aligned}$$

where $C_2 = \min\big\{\frac{\sigma}{2} - \frac{\beta(2M+R(C_\Omega+1))}{\sqrt{\lambda}\mu_0},\ C_5\big\} > 0$. □

Let us introduce a new energy functional,

$$E_\varepsilon(t) := E(t) + \varepsilon P(t). \tag{3.17}$$

Here, ε is a suitable small positive constant satisfying

$$\varepsilon < \min\big\{\frac{2c_1}{M + 2C_4}, \frac{\zeta_2}{C_2 + 4C_3}, \frac{\zeta_1}{C_2 + 2C_3}\big\}. \tag{3.18}$$

Through calculations we obtain

$$
\begin{aligned}
\varepsilon^{-1}|E_\varepsilon(t)-E(t)| = |P(t)| &= \Big|\int_\Omega u_t\Big(H(u)+\frac{\operatorname{div} H-\sigma}{2}u\Big)\Big| \\
&\le \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{M^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{R^2}{2}\int_\Omega u^2\,dx \\
&\le \int_\Omega u_t^2\,dx + \frac{M^2+R^2C_\Omega}{2\lambda\mu_0}\mu(t)\int_\Omega |\nabla_g u|_g^2\,dx \\
&\le cE(t),
\end{aligned}
$$

where $c=\max\big\{2,\ \frac{M^2+R^2C_\Omega}{\lambda\mu_0}\big\}>0$. We show that $E_\varepsilon(t)$ and $E(t)$ are equivalent. Next, we prove the main theorem.

Proof of Theorem 2.2. It follows from estimates (3.6), (3.8) and applying (2.12), (2.13), that

$$
\begin{aligned}
\frac{d}{dt}E_\varepsilon(t) &= \frac{d}{dt}E(t)+\varepsilon\frac{d}{dt}P(t) \\
&\le \beta C_1E(t) - c_1\int_{\Gamma_1} u_t^2\,d\Gamma - \frac{1}{2}\int_{\Gamma_1} g''\circ u(t)\,d\Gamma \\
&\quad + \frac{1}{2}g'(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma - \varepsilon C_2E_0(t) - 2\varepsilon C_3\int_{\Gamma_1} g'\circ u(t)\,d\Gamma \\
&\quad + \varepsilon C_3 g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma + \varepsilon\big(\frac{M}{2}+C_4\big)\int_{\Gamma_1} u_t^2\,d\Gamma \\
&\le \beta C_1E(t) - c_1\int_{\Gamma_1} u_t^2\,d\Gamma + \frac{\zeta_2}{2}\int_{\Gamma_1} g'\circ u(t)\,d\Gamma \\
&\quad - \frac{\zeta_1}{2}g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma - \varepsilon C_2E(t) - \frac{\varepsilon C_2}{2}\int_{\Gamma_1} g'\circ u(t)\,d\Gamma \\
&\quad + \frac{\varepsilon C_2}{2}g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma - 2\varepsilon C_3\int_{\Gamma_1} g'\circ u(t)\,d\Gamma \\
&\quad + \varepsilon C_3 g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma + \varepsilon\big(\frac{M}{2}+C_4\big)\int_{\Gamma_1} u_t^2\,d\Gamma \\
&\le -(\varepsilon C_2-\beta C_1)E(t) - \big[c_1-\varepsilon\big(\frac{M}{2}+C_4\big)\big]\int_{\Gamma_1} u_t^2\,d\Gamma \\
&\quad + \big[\frac{\zeta_2}{2}-\varepsilon\left(\frac{C_2}{2}+2C_3\right)\big]\int_{\Gamma_1} g'\circ u(t)\,d\Gamma \\
&\quad - \big[\frac{\zeta_1}{2}-\varepsilon\big(\frac{C_2}{2}+C_3\big)\big]g(t)\int_{\Gamma_1}|u(t)-u_0|^2\,d\Gamma,
\end{aligned}
$$

where the positive constants $\zeta_1,\ \zeta_2$ are given in (2.12) and (2.13). From (2.10) and (3.18), we know that $\varepsilon C_2-\beta C_1,\ c_1-\varepsilon(\frac{M}{2}+C_4),\ \frac{\zeta_2}{2}-\varepsilon(\frac{C_2}{2}+2C_3)$, and $\frac{\zeta_1}{2}-\varepsilon(\frac{C_2}{2}+C_3)>0$. Then, recalling that g is a positive and non-increasing function, and noting the equivalence of $E_\varepsilon(t)$ and $E(t)$, we can find a positive constant ω such that

$$
\frac{d}{dt}E(t)\le -\omega E(t). \tag{3.19}
$$

Hence, we obtain the desired inequality (2.17) and complete the proof. □

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