Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 123, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NULL CONTROLLABILITY OF NONLINEAR CONTROL SYSTEMS GOVERNED BY COUPLED DEGENERATE PARABOLIC EQUATIONS

FENGDAN XU, JINGXUE YIN, MINGJUN ZHOU, QIAN ZHOU

Abstract. This article concerns the null controllability of a nonlinear control system governed by coupled degenerate parabolic equations. We first establish the Carleman estimate and the observability inequality for solutions to the conjugate problem. Then we can prove the observability inequality and the null controllability of the linear control system. Finally, the nonlinear control system is shown to be null controllable by a fixed point argument and compact estimates.

1. Introduction

In this article, we study the null controllability of the following nonlinear system governed by coupled degenerate parabolic equations

$$u_t - (x^{\alpha_1} u_x)_x + g_1(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.1}$$

$$v_t - (x^{\alpha_2} v_x)_x + g_2(x,t,v) = b(x,t)u, \quad (x,t) \in Q_T, \tag{1.2}$$

$$u(0,t) = 0 \text{ if } 0 < \alpha_1 < 1, \quad (x^{\alpha_1} u_x)(0,t) = 0 \text{ if } 1 \le \alpha_1 < 2, \quad t \in (0,T), \tag{1.3}$$

$$v(0,t) = 0 \text{ if } 0 < \alpha_2 < 1, \quad (x^{\alpha_2} v_x)(0,t) = 0 \text{ if } 1 \le \alpha_2 < 2, \quad t \in (0,T), \tag{1.4}$$

$$u(1,t) = v(1,t) = 0, \quad t \in (0,T), \tag{1.5}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1), \tag{1.6}$$

where $0 < \alpha_1, \alpha_2 < 2$ and $T > 0$ are constants, $Q_T = (0,1) \times (0,T)$, g_1 and g_2 are Lipschitz functions satisfying

$$g_1(x,t,0) = 0, \quad |g_1(x,t,u) - g_1(x,t,v)| \le K|u-v|, \quad (x,t) \in Q_T, u,v \in \mathbb{R}, \tag{1.7}$$

$$g_2(x,t,0) = 0, \quad |g_2(x,t,u) - g_2(x,t,v)| \le K|u-v|, \quad (x,t) \in Q_T, u,v \in \mathbb{R} \tag{1.8}$$

where $K > 0$ is a constant, $b \in L^\infty(Q_T)$, $u_0, v_0 \in L^2(0,1)$, h is a control function, ω is a nonempty open subset of $(0,1)$, and χ_ω is the characteristic function of ω. Since the null controllability of v is controlled by bu, it is reasonable to assume that there exists a nonempty set $\breve{\omega} \subset\subset \omega$ and a constant $b_0 > 0$ such that

$$\inf_{\breve{\omega} \times (0,T)} b \ge b_0 > 0 \quad \text{or} \quad \sup_{\breve{\omega} \times (0,T)} b \le -b_0 < 0. \tag{1.9}$$

2010 *Mathematics Subject Classification.* 93B05, 93C20, 35K65.
Key words and phrases. Null controllability; coupled degenerate parabolic equations; Carleman estimate.

Submitted November 11, 2018. Published November 21, 2019.

See [16] for details.

The coupled parabolic equations (1.1) and (1.2) are degenerate at the boundary $x = 0$, and they are some version of the following Volterra-Lotka model in mathematical biology

$$u_t = (a_1(x)u_x)_x + b_1 u + u f_1(x,t,u,v), \quad (x,t) \in Q_T, \tag{1.10}$$
$$v_t = (a_2(x)v_x)_x + b_2 v + v f_2(x,t,u,v), \quad (x,t) \in Q_T, \tag{1.11}$$

where a_1 and a_2 are positive functions in $(0,1)$ [11, 22]. Under suitable assumptions on b_1, b_2, f_1 and f_2, the equations (1.10) and (1.11) describe the time evolution of two competing species when space diffusion effects are taken into account. Here, u and v denote the population densities of the two species, respectively. Additionally, there are other mathematical applications that appear in mathematical biology and in a wide variety of physical situations [3, 4, 21, 25, 26, 27, 28].

Controllability theory has been widely investigated for nondegenerate parabolic equations for almost five decades and there have been a lot of results (see for instance [2, 12, 14]). The study on the degenerate parabolic equations just began ten years ago and a few results have been known. In particular, the following system governed by a single degenerate parabolic equation has been widely studied

$$w_t - (x^\alpha w_x)_x + k(x,t)w = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.12}$$
$$w(0,t) = 0 \text{ if } 0 < \alpha < 1, \quad (x^\alpha w_x)(0,t) = 0 \text{ if } \alpha \ge 1, \quad t \in (0,T), \tag{1.13}$$
$$w(1,t) = 0, \quad t \in (0,T), \tag{1.14}$$
$$w(x,0) = w_0(x), \quad x \in (0,1), \tag{1.15}$$

where $k \in L^\infty(Q_T)$. The system is null controllable if $0 < \alpha < 2$ ([1, 7, 20]), while not if $\alpha \ge 2$ ([6]). It is noted that the degeneracy of (1.12) is weak if $0 < \alpha < 1$ and strong if $\alpha \ge 1$. The null controllability of the system (1.12)–(1.15) for $0 < \alpha < 2$ is based on the Carleman estimate for solutions to its conjugate problem

$$-W_t - (x^\alpha W_x)_x + k(x,t)W = F(x,t), \quad (x,t) \in Q_T, \tag{1.16}$$
$$W(0,t) = 0 \text{ if } 0 < \alpha < 1, \quad (x^\alpha W_x)(0,t) = 0 \text{ if } 1 \le \alpha < 2, \quad t \in (0,T), \tag{1.17}$$
$$W(1,t) = 0, \quad t \in (0,T), \tag{1.18}$$
$$W(x,T) = W_T(x), \quad x \in (0,1). \tag{1.19}$$

Although the system (1.12)–(1.15) is not null controllable for $\alpha \ge 2$, it was proved in [6] and [23] that it is regional null controllability and approximate controllability for each $\alpha > 0$, respectively. Moreover, the controllability on a single degenerate parabolic equation with linear or semilinear lower order terms have also been studied in [10, 13, 24], while the null controllability of the system governed by a degenerate equation in nondivergence form was considered in [5]. The controllability for the nondegenerate coupled systems has been studied in [15, 16, 19]. There is also a few results concerning with the controllability of the system governed by coupled degenerate parabolic equations. In [8], Cannarsa and de Teresa studied the system

$$u_t - (x^\alpha u_x)_x + c_1(x,t)u = \xi + h\chi_{\omega_1}, \quad (x,t) \in Q_T, \tag{1.20}$$
$$v_t - (x^\alpha v_x)_x + c_2(x,t)v = u\chi_{\omega_2}, \quad (x,t) \in Q_T, \tag{1.21}$$

subject to the conditions (1.3)–(1.6) with $\alpha_1 = \alpha_2 = \alpha$, where $0 < \alpha < 2$, $c_1, c_2 \in L^\infty(Q_T)$, $\xi \in L^2(Q_T)$, and ω_1 and ω_2 are nonempty open subsets of $(0,1)$. It was shown that the system is null controllable if $\omega_1 \cap \omega_2 \neq \emptyset$. In [17], the authors proved the null controllability of the weakly degenerate system

$$
\begin{gathered}
u_t - (a_1(x)u_x)_x + F_1(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \\
v_t - (a_2(x)v_x)_x + F_2(x,t,u,v) = 0, \quad (x,t) \in Q_T, \\
u(0,t) = v(0,t) = 0, \quad u(1,t) = v(1,t) = 0, \quad t \in (0,T), \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in (0,1),
\end{gathered}
$$

where $a_1, a_2 \in C^1((0,1]) \cap C([0,1])$ satisfying $a_1(0) = a_2(0) = 0$ and

$$
a_1(x) > 0, \quad a_2(x) > 0, \quad xa_1'(x) \le \kappa a_1(x), \quad xa_2'(x) \le \kappa a_2(x), \quad x \in (0,1]
$$

with some constant $\kappa \in (0,1)$. Note that in this paper, $0 < \alpha_1, \alpha_2 < 1$ if

$$
a_1(x) = x^{\alpha_1}, \quad a_2(x) = x^{\alpha_2}, \quad x \in [0,1].
$$

In [18], the authors studied the null controllability of the linear system

$$
u_t - (x^{\alpha_1}u_x)_x + c_1(x,t)u + c_2(x,t)v = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.22}
$$

$$
v_t - (x^{\alpha_2}v_x)_x + c_3(x,t)u + c_4(x,t)v = 0, \quad (x,t) \in Q_T, \tag{1.23}
$$

subject to the conditions (1.3)–(1.6) with $0 < \alpha_1$, $\alpha_2 < 1$ and $c_j \in L^\infty(Q_T)$ $(1 \le j \le 4)$. They proved that the system (1.22), (1.23), (1.3)–(1.6) is null controllable. It is noted that $0 < \alpha_1, \alpha_2 < 1$. That is to say, equations (1.22) and (1.23) are weakly degenerate. Recently, Du and Wang [9] proved the null controllability of the system

$$
\begin{gathered}
u_t - (x^\alpha u_x)_x = a_1(x,t)u + b_1(x,t)v + h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \\
v_t - (x^\alpha v_x)_x = a_2(x,t)u + b(x,t)v, \quad (x,t) \in Q_T
\end{gathered}
$$

subject to the conditions (1.3)–(1.6), where $0 < \alpha < 2$, $a_1, a_2, b_1, b \in L^\infty(Q_T)$ and b satisfies (1.9).

In this paper, we first study the null controllability of the linear control system

$$
u_t - (x^{\alpha_1}u_x)_x + c_1(x,t)u = h(x,t)\chi_\omega, \quad (x,t) \in Q_T, \tag{1.24}
$$

$$
v_t - (x^{\alpha_2}v_x)_x + c_2(x,t)v = b(x,t)u, \quad (x,t) \in Q_T \tag{1.25}
$$

with (1.3)–(1.6), where $0 < \alpha_1, \alpha_2 < 2$ and $c_1, c_2, b \in L^\infty(Q_T)$. This null controllability is based on the Carleman estimate for solutions to its conjugate problem

$$
-y_t - (x^{\alpha_1}y_x)_x + c_1(x,t)y = b(x,t)z, \quad (x,t) \in Q_T, \tag{1.26}
$$

$$
-z_t - (x^{\alpha_2}z_x)_x + c_2(x,t)z = 0, \quad (x,t) \in Q_T, \tag{1.27}
$$

$$
y(0,t) = 0 \text{ if } 0 < \alpha_1 < 1, \quad (x^{\alpha_1}y_x)(0,t) = 0 \text{ if } 1 \le \alpha_1 < 2, \quad t \in (0,T), \tag{1.28}
$$

$$
z(0,t) = 0 \text{ if } 0 < \alpha_2 < 1, \quad (x^{\alpha_2}z_x)(0,t) = 0 \text{ if } 1 \le \alpha_2 < 2, \quad t \in (0,T), \tag{1.29}
$$

$$
y(1,t) = z(1,t) = 0, \quad t \in (0,T), \tag{1.30}
$$

$$
y(x,T) = y_T(x), \quad z(x,T) = z_T(x), \quad x \in (0,1). \tag{1.31}
$$

Then we can prove the observability inequality and the null controllability of the system (1.1)–(1.6). Using a fixed point argument and many compact estimates, we can show that the nonlinear system (1.1)–(1.6) is null controllable.

This article is organized as follows. In §2, we recall the well-posedness and the Carleman estimates for the problem of the single degenerate parabolic equation. Then, we establish the Carleman estimate for solutions to the problem (1.26)–(1.31) in §3. In §4, we prove the observability inequality and the null controllability of the linear system (1.24), (1.25), (1.3)–(1.6). Subsequently, the null controllability of the nonlinear system (1.1)–(1.6) is shown in §5.

2. Recall of results on a single degenerate parabolic equation

In this section, we recall the well-posedness and the Carleman estimates for the problem of the single degenerate parabolic equation. For $0 < \alpha < 2$, consider the equation

$$w_t - (x^\alpha w_x)_x + k(x,t)w = f(x,t), \quad (x,t) \in Q_T, \tag{2.1}$$

subject to the conditions (1.13)–(1.15), where $k \in L^\infty(Q_T)$, $f \in L^2(Q_T)$. In order to define solutions to the problem, the following Hilbert space is introduced (see [1, 7, 20])

$$H_\alpha(0,1) = \begin{cases} \big\{w \in L^2(0,1) : w \text{ is absolutely continuous in } [0,1], \\ x^{\alpha/2} w_x \in L^2(0,1) \text{ and } w(0) = w(1) = 0\big\}, & \text{if } 0 < \alpha < 1, \\ \big\{w \in L^2(0,1) : w \text{ is locally absolutely continuous} \\ \text{in } (0,1], x^{\alpha/2} w_x \in L^2(0,1) \text{ and } w(1) = 0\big\}, & \text{if } 1 \le \alpha < 2. \end{cases}$$

Definition 2.1. A function w is called to be a solution to the problem (2.1), (1.13)–(1.15), if $w \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H_\alpha(0,1))$ satisfies (2.1) in the distribution sense and satisfies (1.15) in the common sense.

The well-posedness of problem (2.1), (1.13)–(1.15) was established in [1, 7, 20], by the semigroup method.

Lemma 2.2. *For any $f \in L^2(Q_T)$ and $w_0 \in L^2(0,1)$, problem (2.1), (1.13)–(1.15) admits a unique solution w. Furthermore, w satisfies*

$$\|w\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha/2} w_x\|_{L^2(Q_T)} \le C(\|w_0\|_{L^2(0,1)} + \|f\|_{L^2(Q_T)}),$$

and for any $0 < \tau < T$,

$$\|w_t\|_{L^2((0,1)\times(\tau,T))} + \|x^{\alpha/2} w_x\|_{L^\infty(\tau,T;L^2(0,1))} \le C_\tau(\|w_0\|_{L^2(0,1)} + \|f\|_{L^2(Q_T)}),$$

where C and C_τ are positive constants depending only on α, $\|k\|_{L^\infty(Q_T)}$ and T, while C_τ also on τ. Moreover, if $w_0 \in H_\alpha(0,1)$ additionally, then $x^{\alpha/2} w_x \in L^\infty(0,T;L^2(0,1))$ and $w_t \in L^2(Q_T)$.

Next, we recall the Carleman estimate for problem (1.16)–(1.19). For $\tilde\omega = (x_0, x_1)$ with $\tilde\omega \subset\subset \omega$, let $\xi \in C^2(\mathbb{R})$ satisfy $0 \le \xi \le 1$ in $\mathbb{R}$ and

$$\xi(x) = \begin{cases} 1, & \text{if } x \in (0, (2x_0 + x_1)/3), \\ 0, & \text{if } x \in ((x_0 + 2x_1)/3, 1). \end{cases}$$

Define

$$\psi(x) = \begin{cases} \kappa x^{2-\alpha} - \lambda, & 0 \le \alpha < 2,\ \alpha \ne 1, \\ \kappa \mathrm{e}^x - \lambda, & \alpha = 1, \end{cases} \quad x \in [0,1],$$

$$\Psi(x) = \mathrm{e}^{r\zeta(x)} - \mathrm{e}^{2r\zeta(0)} \quad \text{with } \zeta(x) = \frac{1 - x^{\alpha/2}}{1 - \alpha/2}, \quad x \in [0,1],$$

$$\theta(t) = \frac{1}{(t(T-t))^4}, \quad t \in (0,T),$$

where κ, λ are positive constants such that $\psi < 0$ on $[0,1]$ and r is a suitably large constant. Set

$$\begin{aligned}
&\phi(x,t) = \psi(x)\theta(t), \quad \Phi(x,t) = \Psi(x)\theta(t), \quad x \in [0,1],\, t \in (0,T),\\
&\varphi(x,t) = \xi(x)\phi(x,t) + (1-\xi(x))\Phi(x,t), \quad x \in [0,1],\, t \in (0,T).
\end{aligned}$$

One has the following Carleman estimate.

Lemma 2.3. *There exist two positive constants M_0 and R_0 depending only on α, T, $\|k\|_{L^\infty(Q_T)}$ and $\tilde\omega$, such that for any $F \in L^2(Q_T)$ and $W_T \in L^2(0,1)$, the solution W to problem* (1.16)–(1.19) *satisfies that for each $R \ge R_0$,*

$$\begin{aligned}
&\iint_{Q_T} (R\theta x^\alpha W_x^2 + R^3\theta^3 x^{2-\alpha} W^2)\mathrm{e}^{2R\varphi}\,dx\,dt\\
&\le M_0\Big(\iint_{Q_T} F^2 \mathrm{e}^{2R\varphi}\,dx\,dt + \iint_{\tilde\omega_T} R^3\theta^3 W^2 \mathrm{e}^{2R\varphi}\,dx\,dt\Big),
\end{aligned}$$

where $\tilde\omega_T = \tilde\omega \times (0,T)$.

Lemma 2.3 was proved in [8] by combining a Carleman estimate for a degenerate parabolic equation (see also [1, 20]) and a classical Carleman estimate for a nondegenerate parabolic equation.

3. Carleman estimate

In this section, we establish the Carleman estimate for solutions to problem (1.26)–(1.31).

Definition 3.1. A pair of functions (y,z) is called to be a solution to (1.26)–(1.31), if $y \in C([0,T];L^2(0,1)) \cap L^2(0,T;H_{\alpha_1}(0,1))$ and $z \in C([0,T];L^2(0,1)) \cap L^2(0,T;H_{\alpha_2}(0,1))$ satisfy (1.26) and (1.27) in the distribution sense, and satisfy (1.31) in the common sense.

Similarly to Lemma 2.2, one can prove the following result.

Lemma 3.2. *For any $y_T, z_T \in L^2(0,1)$, problem* (1.26)–(1.31) *admits a unique solution (y,z). Furthermore, the solution satisfies*

$$\begin{aligned}
&\|y\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_1/2}y_x\|_{L^2(Q_T)} + \|z\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_2/2}z_x\|_{L^2(Q_T)}\\
&\le C(\|y_T\|_{L^2(0,1)} + \|z_T\|_{L^2(0,1)}),
\end{aligned}$$

and for any $0<\tau<T$,

$$\begin{aligned}
&\|y_t\|_{L^2((0,1)\times(0,T-\tau))} + \|x^{\alpha_1/2}y_x\|_{L^\infty(0,T-\tau;L^2(0,1))} + \|z_t\|_{L^2((0,1)\times(0,T-\tau))}\\
&+ \|x^{\alpha_2/2}z_x\|_{L^\infty(0,T-\tau;L^2(0,1))}\\
&\le C_\tau(\|y_T\|_{L^2(0,1)} + \|z_T\|_{L^2(0,1)}),
\end{aligned}$$

where C and C_τ are positive constants depending only on α_1, α_2, T, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, while C_τ also on τ. Moreover, if $y_T \in H_{\alpha_1}(0,1)$ and $z_T \in H_{\alpha_2}(0,1)$ additionally, then $x^{\alpha_1/2}y_x, x^{\alpha_2/2}z_x \in L^\infty(0,T;L^2(0,1))$ and $y_t, z_t \in L^2(Q_T)$.

For $j=1,2$, we define

$$\psi_j(x) = \begin{cases} \kappa_j x^{2-\alpha_j} - \lambda_j, & 0<\alpha_j<2,\ \alpha_j \ne 1,\\ \kappa_j \mathrm{e}^x - \lambda_j, & \alpha_j = 1, \end{cases} \qquad x \in [0,1],$$

$$
\begin{gathered}
\Psi_j(x) = \mathrm{e}^{r_j\zeta_j(x)} - \mathrm{e}^{2r_j\zeta_j(0)} \quad \text{with } \zeta_j(x) = \frac{1-x^{\alpha_j/2}}{1-\alpha_j/2}, \quad x \in [0,1], \\
\phi_j(x,t) = \psi_j(x)\theta(t), \quad \Phi_j(x,t) = \Psi_j(x)\theta(t), \quad (x,t) \in [0,1] \times (0,T), \\
\varphi_j(x,t) = \xi(x)\phi_j(x,t) + (1-\xi(x))\Phi_j(x,t), \quad (x,t) \in [0,1] \times (0,T),
\end{gathered}
$$

where κ_1, λ_1, κ_2, λ_2 are positive constants such that

$$\psi_1(x) < \psi_2(x) < 0, \quad x \in [0,1], \tag{3.1}$$

and r_1, r_2 are suitably large constants and satisfy

$$\Psi_1(x) < \Psi_2(x) < 0, \quad x \in [0,1]. \tag{3.2}$$

Lemma 3.3. *There exist two positive constants M_1 and R_1, depending only on α_1, α_2, T, $\tilde{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, such that for any $y_T, z_T \in L^2(0,1)$, the solution (y,z) to (1.26)–(1.31) satisfies that for each $R \geq R_1$,*

$$
\begin{aligned}
&\iint_{Q_T} (R\theta x^{\alpha_1} y_x^2 + R^3\theta^3 x^{2-\alpha_1} y^2)\mathrm{e}^{2R\varphi_1}\,dx\,dt \\
&+ \iint_{Q_T} (R\theta x^{\alpha_2} z_x^2 + R^3\theta^3 x^{2-\alpha_2} z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \\
&\leq M_1 \iint_{\tilde{\omega}_T} (R^3\theta^3 y^2 \mathrm{e}^{2R\varphi_1} + R^3\theta^3 z^2 \mathrm{e}^{2R\varphi_2})\,dx\,dt.
\end{aligned} \tag{3.3}
$$

Proof. It follows from Lemma 2.3 that there exist two positive constants R_0 and M_0 depending on α_1, α_2, T, $\tilde{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, such that for any $R > R_0$,

$$
\begin{aligned}
&\iint_{Q_T} (R\theta x^{\alpha_1} y_x^2 + R^3\theta^3 x^{2-\alpha_1} y^2)\mathrm{e}^{2R\varphi_1}\,dx\,dt \\
&\leq M_0\Big(\iint_{Q_T} z^2\mathrm{e}^{2R\varphi_1}\,dx\,dt + \iint_{\tilde{\omega}_T} R^3\theta^3 y^2 \mathrm{e}^{2R\varphi_1}\,dx\,dt\Big),
\end{aligned}
$$

and

$$\iint_{Q_T} (R\theta x^{\alpha_2} z_x^2 + R^3\theta^3 x^{2-\alpha_2} z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \leq M_0 \iint_{\tilde{\omega}_T} R^3\theta^3 z^2 \mathrm{e}^{2R\varphi_1}\,dx\,dt.$$

It follows from (3.1) and (3.2) that

$$
\begin{aligned}
&\iint_{Q_T} (R\theta x^{\alpha_1} y_x^2 + R^3\theta^3 x^{2-\alpha_1} y^2)\mathrm{e}^{2R\varphi_1}\,dx\,dt \\
&+ \iint_{Q_T} (R\theta x^{\alpha_2} z_x^2 + R^3\theta^3 x^{2-\alpha_2} z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \\
&\leq M_0\Big(\iint_{Q_T} z^2\mathrm{e}^{2R\varphi_2}\,dx\,dt + \iint_{\tilde{\omega}_T} R^3\theta^3 (y^2\mathrm{e}^{2R\varphi_1} + z^2\mathrm{e}^{2R\varphi_2})\,dx\,dt\Big).
\end{aligned} \tag{3.4}
$$

Set

$$p(x,t) = z(x,t)\mathrm{e}^{R\varphi_2(x,t)}, \quad (x,t) \in Q_T.$$

Since $p(1,t)=0$ and $0\le\alpha_2<2$, we have

$$
\begin{aligned}
\iint_{Q_T} p^2(x,t)\,dx\,dt &= \iint_{Q_T} \Big(\int_s^1 p_x(x,t)dx\Big)^2 ds dt \\
&\le \iint_{Q_T} \Big(\int_s^1 x^{3/2}p_x^2(x,t)dx\Big)\Big(\int_s^1 x^{-3/2}dx\Big)ds dt \\
&\le 2\iint_{Q_T} \Big(\int_s^1 x^{3/2}p_x^2(x,t)dx\Big)s^{-1/2}ds dt \\
&= 2\iint_{Q_T} x^{3/2}p_x^2(x,t)\Big(\int_0^x s^{-1/2}ds\Big)\,dx\,dt \\
&\le 4\iint_{Q_T} x^2p_x^2(x,t)\,dx\,dt \\
&\le 4\iint_{Q_T} x^{\alpha_2}p_x^2(x,t)\,dx\,dt.
\end{aligned}
\tag{3.5}
$$

It follows from the definition of φ_2 and ξ that

$$
\begin{aligned}
\iint_{Q_T} x^{\alpha_2}p_x^2\,dx\,dt &= \iint_{Q_T} x^{\alpha_2}(z_x^2+R^2(\varphi_2)_x^2z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \\
&\le \int_0^T\int_0^{(2x_0+x_1)/3}(x^{\alpha_2}z_x^2+\kappa_2^2R^2x^{2-\alpha_2}\theta^2z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \\
&\quad + C\int_0^T\int_{(2x_0+x_1)/3}^1 (z_x^2+R^2\theta^2z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt,
\end{aligned}
\tag{3.6}
$$

where $C>0$ is a constant depending only on x_0 and x_1. Thus, it follows from (3.5) and (3.6) that

$$
M_0\iint_{Q_T} z^2\mathrm{e}^{2R\varphi_2}\,dx\,dt \le \frac{1}{2}\iint_{Q_T}(R\theta x^{\alpha_2}z_x^2+R^3\theta^3x^{2-\alpha_2}z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \tag{3.7}
$$

for suitably large R. Thanks to (3.4) and (3.7), one can get (3.3). The proof is complete. □

Theorem 3.4. *There exist two positive constants M_2 and R_2, depending only on α_1, α_2, b_0, T, $\tilde\omega$, $\check\omega$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, such that for any $y_T,z_T\in L^2(0,1)$, the solution (y,z) to the problem* (1.26)–(1.31) *satisfies that for each $R\ge R_2$,*

$$
\begin{aligned}
&\iint_{Q_T}(R\theta x^{\alpha_1}y_x^2+R^3\theta^3x^{2-\alpha_1}y^2)\mathrm{e}^{2R\varphi_1}\,dx\,dt \\
&+\iint_{Q_T}(R\theta x^{\alpha_2}z_x^2+R^3\theta^3x^{2-\alpha_2}z^2)\mathrm{e}^{2R\varphi_2}\,dx\,dt \\
&\le M_2\iint_{\omega_T} y^2\,dx\,dt,
\end{aligned}
$$

where $\omega_T=\omega\times(0,T)$.

Proof. Without loss of generality, we assume that b satisfies $\inf_{\check\omega\times(0,T)} b\ge b_0>0$. The case $\inf_{\check\omega\times(0,T)}(-b)\ge b_0>0$ is similar.

By Lemma 3.3, it suffices to prove the inequality

$$\iint_{\tilde{\omega}_T} R^3\theta^3 z^2 \mathrm{e}^{2R\varphi_2}\,dx\,dt \le C\iint_{\omega_T} y^2\,dx\,dt. \tag{3.8}$$

Here and below, we use C and $C(\varepsilon)$ to denote generic positive constants depending only on α_1, α_2, T, $\tilde{\omega}$, $\breve{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, while $C(\varepsilon)$ also on ε. By Lemma 3.2 and a standard compactness argument, we can assume additionally that $y_T \in H_{\alpha_1}(0,1)$ and $z_T \in H_{\alpha_2}(0,1)$ without loss of generality. Then, $x^{\alpha_1/2}y_x, x^{\alpha_2/2}z_x \in L^\infty(0,T;L^2(0,1))$ and $y_t, z_t \in L^2(Q_T)$.

Define a function $\rho \in C^\infty(0,1)$ satisfying

$$\rho(x) = 1,\quad x \in \tilde{\omega} \quad\text{and}\quad \rho(x) = 0,\quad x \in (0,1)\setminus\breve{\omega}.$$

Multiplying (1.26) by $R^3\theta^3 z\mathrm{e}^{2R\varphi_2}\rho$ yields

$$\begin{aligned}
&\iint_{Q_T} bR^3\theta^3 z^2\mathrm{e}^{2R\varphi_2}\rho\,dx\,dt\\
&= \iint_{Q_T} (-y_t R^3\theta^3 z\mathrm{e}^{2R\varphi_2}\rho)\,dx\,dt + \iint_{Q_T} (-(x^{\alpha_1}y_x)_x R^3\theta^3 z\mathrm{e}^{2R\varphi_2}\rho)\,dx\,dt\\
&\quad + \iint_{Q_T} c_1 y R^3\theta^3 z\mathrm{e}^{2R\varphi_2}\rho\,dx\,dt =: I_1 + I_2 + I_3.
\end{aligned} \tag{3.9}$$

Integrating by parts, we obtain that for any $\varepsilon > 0$,

$$\begin{aligned}
I_1 &= \iint_{Q_T} yz_t R^3\theta^3\mathrm{e}^{2R\varphi_2}\rho\,dx\,dt + \iint_{Q_T} yzR^3(\theta^3\mathrm{e}^{2R\varphi_2})_t\rho\,dx\,dt\\
&\le \varepsilon\iint_{Q_T} (R^{-1}\theta^{-1}|z_t|^2\mathrm{e}^{2R\varphi_2} + R^3\theta^3x^{2-\alpha_2}z^2\mathrm{e}^{2R\varphi_2})\,dx\,dt\\
&\quad + C(\varepsilon)\iint_{\omega_T} R^7\theta^7y^2\mathrm{e}^{2R\varphi_2}\,dx\,dt,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
I_2 &= \iint_{Q_T} x^{\alpha_1}y_xz_xR^3\theta^3\mathrm{e}^{2R\varphi_2}\rho\,dx\,dt + \iint_{Q_T} x^{\alpha_1}y_xzR^3\theta^3(\mathrm{e}^{2R\varphi_2}\rho)_x\,dx\,dt\\
&= -\iint_{Q_T} x^{\alpha_1-\alpha_2}y(x^{\alpha_2}z_x)_xR^3\theta^3\mathrm{e}^{2R\varphi_2}\rho\,dx\,dt\\
&\quad - \iint_{Q_T} x^{\alpha_2}yz_xR^3\theta^3(x^{\alpha_1-\alpha_2}\mathrm{e}^{2R\varphi_2}\rho)_x\,dx\,dt\\
&\quad - \iint_{Q_T} x^{\alpha_1}yz_xR^3\theta^3(\mathrm{e}^{2R\varphi_2}\rho)_x\,dx\,dt\\
&\quad - \iint_{Q_T} yzR^3\theta^3(x^{\alpha_1}(\mathrm{e}^{2R\varphi_2}\rho)_x)_x\,dx\,dt\\
&\le \varepsilon\iint_{Q_T} (R^{-1}\theta^{-1}|(x^{\alpha_2}z_x)_x|^2\mathrm{e}^{2R\varphi_2}\\
&\quad + R\theta x^{\alpha_2}z_x^2\mathrm{e}^{2R\varphi_2} + R^3\theta^3x^{2-\alpha_2}z^2\mathrm{e}^{2R\varphi_2})\,dx\,dt\\
&\quad + C(\varepsilon)\iint_{\omega_T} R^7\theta^7y^2\mathrm{e}^{2R\varphi_2}\,dx\,dt,
\end{aligned} \tag{3.11}$$

and

$$I_3 \le \varepsilon\iint_{Q_T} R^3\theta^3x^{2-\alpha_2}z^2\mathrm{e}^{2R\varphi_2}\,dx\,dt + C(\varepsilon)\iint_{\omega_T} R^3\theta^3y^2\mathrm{e}^{2R\varphi_2}\,dx\,dt. \tag{3.12}$$

By Lemma 2.3, there exists a constant C_1 depending on α_1, α_2, T, $\tilde{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, such that for a suitably large R,

$$
\begin{aligned}
&\iint_{Q_T} (R\theta x^{\alpha_2} z_x^2 + R^3\theta^3 x^{2-\alpha_2} z^2) \mathrm{e}^{2R\varphi_2}\, dx\, dt \\
&\le C_1 \iint_{\tilde{\omega}_T} R^3\theta^3 z^2 \mathrm{e}^{2R\varphi}\, dx\, dt \\
&\le \frac{3C_1\varepsilon}{b_0} \iint_{Q_T} (R^{-1}\theta^{-1}((x^{\alpha_2} z_x)_x)^2 + R^{-1}\theta^{-1} z_t^2 + R\theta x^{\alpha_2} z_x^2 \\
&\quad + R^3\theta^3 x^{2-\alpha_2} z^2) \mathrm{e}^{2R\varphi_2}\, dx\, dt + C(\varepsilon) \iint_{\omega_T} R^7\theta^7 y^2 \mathrm{e}^{2R\varphi_2}\, dx\, dt
\end{aligned}
$$

because of (1.9) and (3.9)–(3.12). Choosing $\varepsilon = \frac{b_0}{6C_1}$ yields

$$
\iint_{Q_T} (R^{-1}\theta^{-1}(R\theta x^{\alpha_2} z_x^2 + R^3\theta^3 x^{2-\alpha_2} z^2) \mathrm{e}^{2R\varphi_2}\, dx\, dt \le C \iint_{\omega_T} R^7\theta^7 y^2 \mathrm{e}^{2R\varphi_2}\, dx\, dt,
$$

which implies (3.8). □

4. Observability inequality and null controllability of linear system

In this section, we investigate the observability inequality for the problem (1.26)–(1.31) and deduce the null controllability of the linear system (1.24), (1.25), (1.3)–(1.6).

Theorem 4.1. *There exists a constant $M > 0$ depending only on α_1, α_2, b_0, T, $\tilde{\omega}$, $\check{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, such that for any $y_T, z_T \in L^2(0,1)$, the solution (y,z) to* (1.26)–(1.31) *satisfies*

$$
\|y(\cdot,0)\|_{L^2(0,1)}^2 + \|z(\cdot,0)\|_{L^2(0,1)}^2 \le M \iint_{\omega_T} y^2\, dx\, dt.
$$

Proof. By Lemma 3.2 and a standard compactness argument, we can assume additionally that $y_T \in H_{\alpha_1}(0,1)$ and $z_T \in H_{\alpha_2}(0,1)$ without loss of generality. Then, $x^{\alpha_1/2} y_x, x^{\alpha_2/2} z_x \in L^\infty(0,T; L^2(0,1))$ and $y_t, z_t \in L^2(Q_T)$. Multiplying (1.26) and (1.27) by y and z, respectively, and then integrating over $(0,1)$ with respect to x, one gets that

$$
\begin{gathered}
-\frac{1}{2}\frac{d}{dt}\int_0^1 y^2 dx + \int_0^1 x^{\alpha_1} y_x^2 dx + \int_0^1 c_1 y^2 dx = \int_0^1 byz\, dx, \quad t \in (0,T), \\
-\frac{1}{2}\frac{d}{dt}\int_0^1 z^2 dx + \int_0^1 x^{\alpha_2} z_x^2 dx + \int_0^1 c_2 z^2 dx = 0, \quad t \in (0,T).
\end{gathered}
$$

Hence

$$
-\frac{d}{dt}\int_0^1 (y^2 + z^2) dx \le 2\int_0^1 (c_1 y^2 + c_2 z^2 + byz) dx \le 2\Lambda \int_0^1 (y^2 + z^2) dx,
$$

for $t \in (0,T)$, where $\Lambda = \|c_1\|_{L^\infty(Q_T)} + \|c_2\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)}$. Hence

$$
\frac{d}{dt}\Big(\mathrm{e}^{2\Lambda t}\int_0^1 (y^2 + z^2) dx\Big) \ge 0, \quad t \in (0,T),
$$

which yields

$$\int_0^1 (y^2(x,0)+z^2(x,0))dx \le \mathrm{e}^{2\Lambda t}\int_0^1 (y^2(x,t)+z^2(x,t))dx, \quad t\in(0,T). \tag{4.1}$$

Integrating (4.1) over $(T/4,3T/4)$ leads to

$$\frac{T}{2}\int_0^1 (y^2(x,0)+z^2(x,0))dx \le \int_{T/4}^{3T/4}\int_0^1 \mathrm{e}^{2\Lambda t}(y^2+z^2)\,dx\,dt. \tag{4.2}$$

As in the proof of (3.5), we obtain

$$\int_{T/4}^{3T/4}\int_0^1 (y^2+z^2)\,dx\,dt \le C_0\int_{T/4}^{3T/4}\int_0^1 (x^{\alpha_1}y_x^2+x^{\alpha_2}z_x^2)\,dx\,dt \tag{4.3}$$

with some constant $C_0>0$ depending only on α_1 and α_2. Then, from (4.2), (4.3) and Theorem 3.4 it follows that

$$\begin{aligned}
&\int_0^1 (y^2(x,0)+z^2(x,0))dx\\
&\le \frac{2C_0}{T}\mathrm{e}^{3\Lambda T/2}\sup_{(T/4,3T/4)}\frac{\mathrm{e}^{-2R_1\varphi_1}}{\theta}\int_{T/4}^{3T/4}\int_0^1 (x^{\alpha_1}\theta y_x^2\mathrm{e}^{2R_1\varphi_1}+x^{\alpha_2}\theta z_x^2\mathrm{e}^{2R_1\varphi_2})\,dx\,dt\\
&\le \frac{2C_0M_1}{TR_1}\mathrm{e}^{3\Lambda T/2}\sup_{(T/4,3T/4)}\frac{\mathrm{e}^{-2R_1\varphi_1}}{\theta}\iint_{\omega_T} y^2\,dx\,dt,
\end{aligned}$$

which completes the proof. □

Solutions to problem (1.24), (1.25), (1.3)–(1.6) can be defined similarly to Definition 3.1. Furthermore, one can show its well-posedness for $u_0,v_0\in L^2(0,1)$ and $h\in L^2(Q_T)$.

Theorem 4.2. *For any $u_0,v_0\in L^2(0,1)$, there exists $h\in L^2(Q_T)$ such that the solution (u,v) to the problem (1.24), (1.25), (1.3)–(1.6) satisfies $u(\cdot,T)=v(\cdot,T)=0$ on $(0,1)$.*

Proof. To prove the null controllability of system (1.24), (1.25), (1.3)–(1.6), we first show the approximate controllability. For any $\varepsilon>0$, define the functional

$$\begin{aligned}
J_\varepsilon((y_T,z_T)) = \frac12\iint_{\omega_T} y^2\,dx\,dt + \varepsilon\Big(\int_0^1 (y_T^2(x)+z_T^2(x))dx\Big)^{1/2}\\
-\int_0^1 \Big(y(x,0)u_0(x)+z(x,0)v_0(x)\Big)dx,
\end{aligned}$$

for $(y_T,z_T)\in L^2(0,1)\times L^2(0,1)$, where (y,z) is the solution of (1.26)–(1.31). As the proof of approximate controllability in [23], one can prove that there exists a unique point $(\hat y_T,\hat z_T)\in L^2(0,1)\times L^2(0,1)$ such that J_ε achieves its minimum. Denote $(\hat y_\varepsilon,\hat z_\varepsilon)$ to be the solution to the problem (1.26)–(1.31) with $(y_T,z_T)=(\hat y_T,\hat z_T)$. Then take the control $h_\varepsilon=\chi_\omega\hat y_\varepsilon$ to get the solution $(u_\varepsilon,v_\varepsilon)$ to the problem (1.24), (1.25), (1.3)–(1.6) satisfying

$$\|u_\varepsilon(\cdot,T)\|_{L^2(0,1)}\le\varepsilon,\quad \|v_\varepsilon(\cdot,T)\|_{L^2(0,1)}\le\varepsilon. \tag{4.4}$$

From $J_\varepsilon(\hat y_T,\hat z_T)\le 0$, Hölder inequality and Theorem 4.1, we have

$$\frac12\iint_{\omega_T}\hat y_\varepsilon^2\,dx\,dt+\varepsilon\Big(\int_0^1(\hat y_T^2(x)+\hat z_T^2(x))dx\Big)^{1/2}$$

$$
\begin{aligned}
&\le \int_0^1 \hat{y}_\varepsilon(x,0)u_0(x)dx + \int_0^1 \hat{z}_\varepsilon(x,0)v_0(x)dx \\
&\le \Big(\int_0^1 \hat{y}_\varepsilon^2(x,0)dx\Big)^{1/2}\Big(\int_0^1 u_0^2(x)dx\Big)^{1/2} \\
&\quad + \Big(\int_0^1 \hat{z}_\varepsilon^2(x,0)dx\Big)^{1/2}\Big(\int_0^1 v_0^2(x)dx\Big)^{1/2} \\
&\le C\Big(\iint_{\omega_T} \hat{y}_\varepsilon^2\,dx\,dt\Big)^{1/2}\Big(\int_0^1 u_0^2(x)dx\Big)^{1/2} \\
&\quad + C\Big(\iint_{\omega_T} \hat{y}_\varepsilon^2\,dx\,dt\Big)^{1/2}\Big(\int_0^1 v_0^2(x)dx\Big)^{1/2} \\
&\le \frac{1}{4}\iint_{\omega_T} \hat{y}_\varepsilon^2\,dx\,dt + C\Big(\int_0^1 u_0^2(x)dx + \int_0^1 v_0^2(x)dx\Big),
\end{aligned}
$$

where $C > 0$ is a constant depending only on α_1, α_2, b_0, T, $\tilde{\omega}$, $\check{\omega}$, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$. Hence

$$
\iint_{\omega_T} h_\varepsilon^2\,dx\,dt + \varepsilon\Big(\int_0^1 (\hat{y}_T^2 + \hat{z}_T^2)dx\Big)^{1/2} \le C\Big(\int_0^1 u_0^2(x)dx + \int_0^1 v_0^2(x)dx\Big). \tag{4.5}
$$

From (4.5) and Lemma 2.2, there exist a strictly decreasing sequence $\{\varepsilon_n\}$ with $\lim_{n\to\infty}\varepsilon_n = 0$, and $h \in L^2(\omega_T)$ such that

$$
\begin{gathered}
h_{\varepsilon_n} \rightharpoonup h \text{ in } L^2(\omega_T), \quad u_{\varepsilon_n} \rightharpoonup u \text{ in } L^2(Q_T), \quad v_{\varepsilon_n} \rightharpoonup v \text{ in } L^2(Q_T), \\
u_{\varepsilon_n}(\cdot,T) \rightharpoonup u(\cdot,T) \text{ in } L^2(0,1), \quad v_{\varepsilon_n}(\cdot,T) \rightharpoonup v(\cdot,T) \text{ in } L^2(0,1),
\end{gathered}
$$

where (u,v) is the solution to (1.24), (1.25), (1.3)–(1.6). Then from (4.4) and (4.5) we obtain

$$
\begin{gathered}
u(x,T) = v(x,T) = 0, \quad x \in (0,1), \\
\iint_{\omega_T} h^2\,dx\,dt \le C\Big(\int_0^1 u_0^2(x)dx + \int_0^1 v_0^2(x)dx\Big).
\end{gathered}
$$

The proof is complete. □

5. Null controllability for the nonlinear system (1.1)–(1.6)

Definition 5.1. A pair of functions (u,v) is called a solution to problem (1.1)–(1.6), if $u \in C([0,T];L^2(0,1)) \cap L^2(0,T;H_{\alpha_1}(0,1))$ and $v \in C([0,T];L^2(0,1)) \cap L^2(0,T;H_{\alpha_2}(0,1))$ satisfy (1.1) and (1.2) in the distribution sense, and satisfy (1.6) in the common sense.

Using Lemma 2.2 and a fixed point argument, one can prove the following result.

Lemma 5.2. *For any $u_0, v_0 \in L^2(0,1)$ and $h \in L^2(Q_T)$, problem (1.1)–(1.6) admits a unique solution (u,v). Furthermore, the solution satisfies*

$$
\begin{aligned}
&\|u\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_1/2}u_x\|_{L^2(Q_T)} + \|v\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_2/2}v_x\|_{L^2(Q_T)} \\
&\le C(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)} + \|h\|_{L^2(Q_T)}),
\end{aligned}
$$

and for any $0 < \tau < T$,

$$
\begin{aligned}
&\|u_t\|_{L^2((0,1)\times(\tau,T))} + \|x^{\alpha_1/2}u_x\|_{L^\infty(\tau,T;L^2(0,1))} + \|v_t\|_{L^2((0,1)\times(\tau,T))} \\
&+ \|x^{\alpha_2/2}v_x\|_{L^\infty(\tau,T;L^2(0,1))}
\end{aligned}
$$

$$\le C_\tau(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)} + \|h\|_{L^2(Q_T)}),$$

where C and C_τ are positive constants depending only on α_1, α_2, T, K, $\|c_1\|_{L^\infty(Q_T)}$, $\|c_2\|_{L^\infty(Q_T)}$ and $\|b\|_{L^\infty(Q_T)}$, while C_τ also on τ. Moreover, if $u_0 \in H_{\alpha_1}(0,1)$ and $v_0 \in H_{\alpha_2}(0,1)$, then $x^{\alpha_1/2}u_x, x^{\alpha_2/2}v_x \in L^\infty(0,T;L^2(0,1))$ and $y_t, z_t \in L^2(Q_T)$.

The system (1.1)–(1.6) is null controllable.

Theorem 5.3. *For each $u_0, v_0 \in L^2(0,1)$, there exists $h \in L^2(Q_T)$ such that the solution (u,v) to (1.1)–(1.6) satisfies $u(\cdot,T) = v(\cdot,T) = 0$ in $(0,1)$.*

Proof. Give $\varepsilon > 0$. For any $(\varphi,\psi) \in L^1(Q_T) \times L^1(Q_T)$, we define

$$c_{1,\varphi}(x,t) = \begin{cases} \frac{g_1(x,t,\varphi(x,t))}{\varphi(x,t)}, & \varphi(x,t) \neq 0, \\ 0, & \varphi(x,t) = 0, \end{cases} \quad (x,t) \in Q_T,$$

$$c_{2,\psi}(x,t) = \begin{cases} \frac{g_2(x,t,\psi(x,t))}{\psi(x,t)}, & \psi(x,t) \neq 0, \\ 0, & \psi(x,t) = 0, \end{cases} \quad (x,t) \in Q_T.$$

It follows from (1.7) and (1.8) that $c_{1,\varphi}, c_{2,\psi} \in L^\infty(Q_T)$, and

$$\|c_{1,\varphi}\|_{L^\infty(Q_T)} \le K, \quad \|c_{2,\psi}\|_{L^\infty(Q_T)} \le K. \tag{5.1}$$

Let $(\hat y, \hat z)$ to be the solution to (1.26)–(1.31) with $c_1 = c_{1,\varphi}$, $c_2 = c_{2,\psi}$, $(y_T, z_T) = (\hat y_T, \hat z_T)$, where $(\hat y_T, \hat z_T)$ is the unique minimum point of J_ε in the proof of Theorem 4.2.

We define the operator $\mathcal{L} : L^1(Q_T) \times L^1(Q_T) \to L^1(Q_T) \times L^1(Q_T)$, by

$$\mathcal{L} : (\varphi,\psi) \mapsto (u,v),$$

where (u,v) is the solution to the problem (1.24), (1.25), (1.3)–(1.6) with $c_1 = c_{1,\varphi}$, $c_2 = c_{2,\psi}$ and $h = \chi_\omega \hat y$. Then, for any $(\varphi,\psi) \in L^1(Q_T) \times L^1(Q_T)$, it follows from (4.4) that

$$\|u(\cdot,T)\|_{L^2(0,1)} \le \varepsilon, \quad \|v(\cdot,T)\|_{L^2(0,1)} \le \varepsilon.$$

First, we show that $\mathcal{L}$ is continuous. Assume that $(\varphi_n,\psi_n) \to (\varphi,\psi)$ in $L^1(Q_T) \times L^1(Q_T)$ as $n \to \infty$. Set $(u_n,v_n) = \mathcal{L}((\varphi_n,\psi_n))$ and $(u,v) = \mathcal{L}((\varphi,\psi))$. Let $(\hat y_T^n, \hat z_T^n)$ and $(\hat y_T, \hat z_T)$ be the minimum points of J_ε with $c_1 = c_{1,\varphi_n}$, $c_2 = c_{2,\psi_n}$ and $c_1 = c_{1,\varphi}$, $c_2 = c_{2,\psi}$, respectively. And denote $(\hat y_n, \hat z_n)$, $(\hat y, \hat z)$ to be the solutions to the problem (1.26)–(1.31) with $c_1 = c_{1,\varphi_n}$, $c_2 = c_{2,\psi_n}$, $(y_T, z_T) = (\hat y_T^n, \hat z_T^n)$, and $c_1 = c_{1,\varphi}$, $c_2 = c_{2,\psi}$, $(y_T, z_T) = (\hat y_T, \hat z_T)$, respectively. It follows from (1.7), (1.8) and (4.5) that

$$\int_0^1 ((\hat y_T^n(x))^2 + (\hat z_T^n(x))^2)dx \le \frac{C}{\varepsilon}\Big(\int_0^1 u_0^2(x)dx + \int_0^1 v_0^2(x)dx\Big),$$

where C is a constant depending only on α_1, α_2, b_0, T, K, $\tilde\omega$, $\breve\omega$ and $\|b\|_{L^\infty(Q_T)}$. Then there exist four subsequences of $\{\hat y_T^n\}$, $\{\hat z_T^n\}$, $\{c_{1,\varphi_n}\}$, $\{c_{2,\psi_n}\}$, denoted by themselves for convenience, and $y_T^0, z_T^0 \in L^2(0,1)$, such that

$$\hat y_T^n \rightharpoonup y_T^0, \quad \hat z_T^n \rightharpoonup z_T^0 \quad \text{in } L^2(0,1),$$
$$c_{1,\varphi_n} \rightharpoonup c_{1,\varphi}, \quad c_{2,\psi_n} \rightharpoonup c_{2,\psi} \quad \text{weakly} * \text{ in } L^\infty(Q_T).$$

By Lemma 3.2, there exist two subsequences of $\{\hat y_n\}$ and $\{\hat z_n\}$, denoted by themselves for convenience, such that

$$\hat y_n \rightharpoonup y^0, \quad \hat z_n \rightharpoonup z^0 \quad \text{in } L^2(Q_T), \tag{5.2}$$
$$\hat y_n \to y^0, \quad \hat z_n \to z^0 \quad \text{in } L^2(\omega_T), \tag{5.3}$$

$$\hat{y}_n(\cdot,0) \to y^0(\cdot,0), \quad \hat{z}_n(\cdot,0) \to z^0(\cdot,0) \quad \text{in } L^2(0,1), \tag{5.4}$$

where (y^0, z^0) is the solution to the problem (1.26)–(1.31) with $(y_T, z_T) = (y_T^0, z_T^0)$. Then one can deduce from Lemma 3.2 that there exist two subsequences of $\{u_n\}$ and $\{v_n\}$, denoted by themselves for convenience, such that

$$u_n \rightharpoonup u^0, \quad v_n \rightharpoonup v^0 \text{ in } L^2(Q_T),$$

where (u^0, v^0) is the solution to (1.24), (1.25), (1.3)–(1.6) with $h = \chi_\omega z^0$. To prove $(u^0, v^0) = (u, v)$, it suffices to prove that $(y_T^0, z_T^0) = (\hat{y}_T, \hat{z}_T)$. For $(y_T, z_T) \in L^2(0,1) \times L^2(0,1)$, denote (y_n, z_n) the solution to (1.26)–(1.31) with $c_1 = c_{1,\varphi_n}$ and $c_2 = c_{2,\psi_n}$. By Lemma 3.2, there exist two subsequences of $\{y_n\}$ and $\{z_n\}$, denoted by themselves for convenience, such that

$$y_n \rightharpoonup y, \quad z_n \rightharpoonup z \quad \text{in } L^2(Q_T), \tag{5.5}$$

$$y_n \to y, \quad z_n \to z \quad \text{in } L^2(\omega_T), \tag{5.6}$$

$$y_n(\cdot,0) \to y(\cdot,0), \quad z_n(\cdot,0) \to z(\cdot,0) \quad \text{in } L^2(0,1), \tag{5.7}$$

where (y, z) is the solution to (1.26)–(1.31) with $c_1 = c_{1,\varphi}$, $c_2 = c_{2,\psi}$. Since $(\hat{y}_T^n, \hat{z}_T^n)$ is the minimum point of J_ε with $c_1 = c_{1,\varphi_n}$ and $c_2 = c_{2,\psi_n}$, it follows that

$$\begin{aligned}
&\frac{1}{2}\iint_{\omega_T} \hat{y}_n^2\,dx\,dt + \varepsilon\Big(\int_0^1 ((\hat{y}_T^n(x))^2 + (\hat{z}_T^n(x))^2)dx\Big)^{1/2} \\
&- \int_0^1 \Big(\hat{y}_n(x,0)u_0(x) + \hat{z}_n(x,0)v_0(x)\Big)dx \\
&\le \frac{1}{2}\iint_{\omega_T} y_n^2\,dx\,dt + \varepsilon\Big(\int_0^1 (y_T(x)^2 + z_T^2(x))dx\Big)^{1/2} \\
&\quad - \int_0^1 \Big(y_n(x,0)u_0(x) + z_n(x,0)v_0(x)\Big)dx.
\end{aligned}$$

Letting $n \to \infty$, from (5.2)–(5.7) and the weak lower semi-continuity of L^2 norm it follows that

$$\begin{aligned}
&\frac{1}{2}\iint_{\omega_T} (y^0)^2\,dx\,dt + \varepsilon\Big(\int_0^1 ((y_T^0(x))^2 + (z_T^0(x))^2)dx\Big)^{1/2} \\
&- \int_0^1 \Big(y^0(x,0)u_0(x) + z^0(x,0)v_0(x)\Big)dx \\
&\le \frac{1}{2}\iint_{\omega_T} y^2\,dx\,dt + \varepsilon\Big(\int_0^1 (y_T^2(x) + z_T^2(x))dx\Big)^{1/2} \\
&\quad - \int_0^1 \Big(y(x,0)u_0(x) + z(x,0)v_0(x)\Big)dx.
\end{aligned}$$

This means $J_\varepsilon(y_T^0, z_T^0) \le J_\varepsilon(y_T, z_T)$ with $c_1 = c_{1,\varphi}$ and $c_2 = c_{2,\psi}$ for each $(y_T, z_T) \in L^2(0,1) \times L^2(0,1)$. Hence $(y_T^0, z_T^0) = (\hat{y}_T, \hat{z}_T)$.

Next, we show that $\mathcal{L}$ is compact. Given $\varphi_n, \psi_n \in L^1(Q_T)$. By (5.1), there exist two subsequences of $\{\varphi_n\}$ and $\{\psi_n\}$, denoted by themselves for convenience, such that $c_{1,\varphi_n}, c_{2,\psi_n}$ converge weakly $*$ in $L^\infty(Q_T)$. By Lemma 3.2 there exists a subsequence of $\mathcal{L}(\varphi_n, \psi_n)$, denoted by itself for convenience, converges strongly in $L^1(Q_T) \times L^1(Q_T)$. Hence $\mathcal{L}$ is compact. It follows from the Schauder fixed point theorem that $\mathcal{L}$ admits a fixed point. That is to say, there exists $h_\varepsilon \in L^2(Q_T)$ such that the solution $(u_\varepsilon, v_\varepsilon)$ to problem (1.1)–(1.6) satisfies

$$\|u_\varepsilon(\cdot,T)\|_{L^2(0,1)} \le \varepsilon, \quad \|v_\varepsilon(\cdot,T)\|_{L^2(0,1)} \le \varepsilon.$$

Furthermore, from the proof of Theorem 4.2 we obtain

$$\iint_{\omega_T} h_\varepsilon^2\,dx\,dt \le \tilde{C}\Big(\int_0^1 u_0^2(x)dx + \int_0^1 v_0^2(x)dx\Big), \tag{5.8}$$

where $\tilde{C} > 0$ is a constant depending only on α_1, α_2, b_0, T, K, $\tilde{\omega}$, $\breve{\omega}$ and $\|b\|_{L^\infty(Q_T)}$. It follows from Lemma 5.2 and (5.8) that

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_1/2}(u_\varepsilon)_x\|_{L^2(Q_T)} \\ &+ \|v_\varepsilon\|_{L^\infty(0,T;L^2(0,1))} + \|x^{\alpha_2/2}(v_\varepsilon)_x\|_{L^2(Q_T)} \\ &\le \hat{C}(\|u_0\|_{L^2(0,1)} + \|v_0\|_{L^2(0,1)} + \|h_\varepsilon\|_{L^2(Q_T)}), \end{aligned} \tag{5.9}$$

where $\hat{C} > 0$ is a constant depending only on α_1, α_2, b_0, T, K, $\tilde{\omega}$, $\breve{\omega}$ and $\|b\|_{L^\infty(Q_T)}$. As in the proof of Theorem 4.2, from (5.8)–(5.9), system (1.1)–(1.6) is null controllable. $\square$

Acknowledgments. This research was supported by the National Natural Science Foundation of China (Nos. 11571137, 11601182, 11801211 and 11871133).

The authors would like to express their sincerely thanks to the referees and to the editor for their helpful comments on the original version of the paper.

References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., 6(2) (2006), 161–204.
[2] V. Barbu; *Controllability of parabolic and Navier-Stokes equations*, Sci. Math. Jpn., 56(1) (2002), 143–211.
[3] H. M. Byrne, M. R. Owen; *A new interpretation of the Keller-Segel model based on multiphase modelling*, J. Math. Biol., 49(6) (2004), 604–626.
[4] L. Corrias, B. Perthame, H. Zaag; *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan j. math., 72(1) (2004), 1–28.
[5] P. Cannarsa, G. Fragnelli, D. Rocchetti; *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evol. Equ., 8(4) (2008), 583–616.
[6] P. Cannarsa, P. Martinez, J. Vancostenoble; *Persistent regional controllability for a class of degenerate parabolic equations*, Commun. Pure Appl. Anal., 3(4) (2004), 607–635.
[7] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations*, Adv. Differential Equations, 10(2) (2005), 153–190.
[8] P. Cannarsa, L. de Teresa; *Controllability of 1-D coupled degenerate parabolic equations*, Electron. J. Differential Equations, 2009(73) (2009), 1–21.
[9] R. M. Du, C. P. Wang; *Null controllability of a class of systems governed by coupled degenerate equations*, Appl. Math. Lett., 26(1) (2013), 113–119.
[10] R. M. Du, C. P. Wang, Q. Zhou; *Approximate controllability of a semilinear system involving a fully nonlinear gradient term*, Appl. Math. Optim., 70(1) (2014), 165–183.
[11] Y. Du; *Effects of a degeneracy in the competition model: Part I. Classical and generalized steady-state solutions*, J. Differential Equatioins, 181(1) (2002), 92–132.
[12] E. Fernández-Cara, E. Zuazua; *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire, 17(5) (2000), 583–616.
[13] C. Flores, L. Teresa; *Carleman estimates for degenerate parabolic equations with first order terms and applications*, C. R. Math. Acad. Sci. Paris, 348(2010), 391–396.
[14] A. V. Fursikov, O. Y. Imanuvilov; *Controllability of evolution equations*, Lecture Notes Series 34, Seoul National University, Seoul, Korea, 1996.

[15] M. González-Burgos, R. Pérez-García; *Controllability of some coupled parabolic systems by one control force*, C. R. Acad. Sci. Paris, Ser. I, 340(2) (2005), 125–130.
[16] M. González-Burgos, R. Pérez-García; *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptotic Anal., 46(2) (2006), 123–162.
[17] E. Hassi, F. Khodja, A. Hajjaj, L. Maniar; *Null controllability of degenerate parabolic cascade systems*, Port. Math., 68(3) (2011), 345–367.
[18] E. Hassi, F. Khodja, A. Hajjaj, L. Maniar; *Carleman estimates and null controllability of coupled degenerate systems*, Evol. Equ. Control Theory, 2(3) (2013), 441–459.
[19] F. A. Khodja, A. Benabdallah, C. Dupaix, I. Kostin; *Null-controllability of some systems of parabolic type by one control force*, ESAIM: COCV, 11(3) (2005), 426–448.
[20] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., 6(2) (2006), 325–362.
[21] C. V. Pao; *Numerical Analysis of Coupled Systems of Nonlinear Parabolic Equations*, SIAM J. numer. Anal., 36(2) (1999), 393–416.
[22] A. Schiaffino, A. Tesei; *Competition Systems with Dirichlet Boundary Conditions*, J. Math. Biology, 15(1) (1982), 93–105.
[23] C. P. Wang; *Approximate controllability of a class of semilinear systems with boundary degeneracy*, J. Evol. Equ., 10(1) (2010), 163–193.
[24] C. P. Wang, R. M. Du; *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM J. Control Optim., 52(3) (2014), 1457–1480.
[25] C. P. Wang, Z. P. Xin; *On a degenerate free boundary problem and continuous subsonic-sonic flows in a convergent nozzle*, Arch. Ration. Mech. Anal., 208(3) (2013), 911–975.
[26] J. X. Yin, C. P. Wang; *Evolutionary weighted p-Laplacian with boundary degeneracy*, J. Differential Equations, 237(2) (2007), 421–445.
[27] J. N. Xu, Q. Zhou, Y. Y. Nie; *Null controllability of a coupled system of degenerate parabolic equations with lower order terms*, Electron. J. Differential Equations, 103 (2019), pp. 1–12.
[28] J. N. Xu, C.P. Wang, Y. Y. Nie; *Carleman estimate and null controllability of a cascade degenerate parabolic system with general convection terms*, Electron. J. Differential Equations, 195 (2018), pp. 1–20.

Fengdan Xu
School of Mathematics, Jilin University, Changchun 130012, China
Email address: 245952950@qq.com

Jingxue Yin
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China.
School of Mathematics, Jilin University, Changchun 130012, China
Email address: yjx@scnu.edu.cn

Mingjun Zhou (corresponding author)
School of Mathematics, Jilin University, Changchun 130012, China
Email address: zhoumingjun@jlu.edu.cn

Qian Zhou
School of Mathematics, Jilin University, Changchun 130012, China
Email address: zhouqian@jlu.edu.cn